Non-zero-sum optimal stopping game with continuous versus periodic observations

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Problem Formulation.

- Let X be a real-valued Markov process and N be an independent Poisson process with rate $\lambda > 0$, defined on a probability space $(\Omega, \mathcal{H}, \mathbb{P})$.
- Denote by $(T^{(n)})_{n \in \mathbb{N}}$ the sequence of jump times of N with inter-arrival times given by independent exponential random variables with parameter λ.
- We consider the following two players:
- Player C, who gets to observe X continuously (without delay) thanks to their full information access to the evolution of X. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ of player C is therefore the natural filtration of X given by $\mathcal{F}_t := \sigma(X_s \,|\, 0 \leq s \leq t)$.
- Player P, who gets to observe X periodically, only at the Poisson observation times $(T^{(n)})_{n \in \mathbb{N}}$. The filtration $\mathbb{G} = (\mathcal{G}_n)_{n \in \mathbb{N}}$ of player P is therefore given by $\mathcal{G}_n := \sigma(T^{(k)}, X_{T^{(k)}} \,|\, 1 \leq k \leq n)$.

Problem Formulation.

- Player C knows of the existence of competition with the partially informed player P, knows the opponent's rate of observations is $\lambda > 0$, but cannot know the actual (random) times $(T^{(n)})_{n\in\mathbb{N}}$ of the opponent's observations (they are not part of $\mathbb{F}$).
- On the other hand, player P also knows of the competition arising from the existence of player C.
- The aforementioned players compete against each other as follows:
- player C (resp., P) aims at maximising a discounted reward function $f_c : \mathbb{R} \to \mathbb{R}$ (resp., $f_p : \mathbb{R} \to \mathbb{R}$) by stopping the game before player P (resp., C), otherwise receives nothing.
- Both players discount their future gains with a constant discount rate $q > 0$.
- Even though they are both after the same asset, the additional information provided to player C (as opposed to player P) yields an additional fee for player C, if and when successfully stopping before player P.

Problem Formulation.

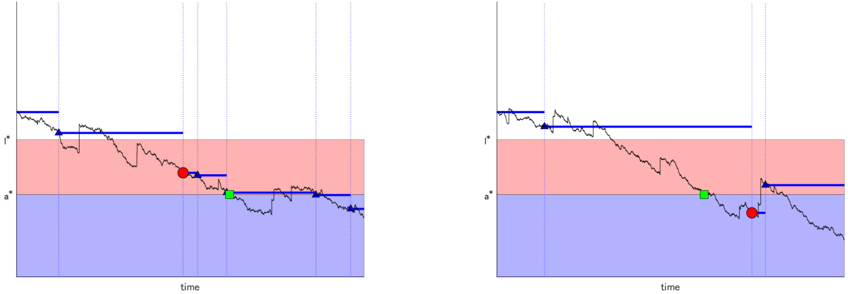
- A pair of stopping times (τ, σ) in this game consists of $\tau \in \mathcal{T}_c$ and $\sigma \in \mathcal{T}_p$, where $\mathcal{T}_c$ is the set of $\mathbb{F}$-stopping times and $\mathcal{T}_p := \{ T^{(M)} : M \text{ is a } \mathbb{G}\text{–stopping time}\}$.
- This means that while player C can stop in the "usual" way, player P can stop only at the Poisson observation times.
- Each player aims at maximising their expected discounted reward functions given by

$$V_c(\tau, \sigma; x) := \mathbb{E}_x\left[e^{-q\tau} f_c(X_\tau) 1_{\{\tau<\sigma\}}\right] \quad \text{and} \quad V_p(\tau, \sigma; x) := \mathbb{E}_x\left[e^{-q\sigma} f_p(X_\sigma) 1_{\{\sigma<\tau\}}\right].$$

The main aim is to obtain, for each $x \in \mathbb{R}$, a Nash equilibrium $(\tau^*, \sigma^*) \in \mathcal{T}_c \times \mathcal{T}_p$ such that

$$\begin{aligned} V_c(\tau^*, \sigma^*; x) &\geq V_c(\tau, \sigma^*; x), \quad \forall\, \tau \in \mathcal{T}_c, \\ V_p(\tau^*, \sigma^*; x) &\geq V_p(\tau^*, \sigma; x), \quad \forall\, \sigma \in \mathcal{T}_p. \end{aligned}$$

Problem Formulation.



Case 1 **Case 2**

Figure: Illustration of player C and player P's stopping strategies. The solid black trajectory shows the path of X and the piecewise horizontal blue lines show player P's most recent information on X; observation times are shown by dotted vertical lines. Given some $l^* > a^*$, player P stops at the first observation time of X below l^* (indicated by red circles) and player C stops at the classical hitting time below a^* (indicated by green squares).

Spectrally positive Lévy process.

- Let $X = (X(t); t \geq 0)$ be a spectrally positive Lévy process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Assume that its Laplace exponent $\psi : [0, \infty) \to \mathbb{R}$, i.e.

$$\mathbb{E}\left[\mathrm{e}^{-\theta X(t)}\right] =: \mathrm{e}^{\psi(\theta)t}, \qquad t, \theta \geq 0,$$

is given, by the *Lévy-Khintchine formula*

$$\psi(\theta) := \gamma\theta + \frac{\nu^2}{2}\theta^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-\theta z} - 1 + \theta x \mathbf{1}_{\{z<1\}}\right)\Pi(dz), \quad \theta \geq 0,$$

where $\gamma \in \mathbb{R}$, $\nu \geq 0$, and Π is a measure on $(0, \infty)$ called the Lévy measure of X that satisfies

$$\int_{(0,\infty)} (1 \wedge z^2)\Pi(dz) < \infty.$$

Scale functions

- Fix $q \geq 0$. Let $W^{(q)}$ be the scale function of X. Namely, this is a mapping from $\mathbb{R}$ to $[0, \infty)$ that takes the value zero on the negative half-line, while on the positive half-line it is a strictly increasing function that is defined by its Laplace transform:

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$

  where $\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$.
- We also define for $r > 0$,

$$Z^{(r)}(x; \theta) := e^{\theta x} \left(1 + (r - \psi(\theta)) \int_0^x e^{-\theta u} W^{(r)}(u) du\right), \quad x \in \mathbb{R}, \theta \geq 0,$$

- If X is of unbounded variation or the Lévy measure is atomless, it is known that $W^{(q)}$ is $C^1(\mathbb{R} \backslash \{0\})$.

Threshold strategies.

- We will prove that, for a large class of reward functions f_p and f_c satisfying only certain mild assumptions, a pair of threshold strategies leads to the Nash equilibrium.
- In particular, player C's optimal strategy will be to stop at the first down-crossing time of some level, while player P's optimal strategy will be to stop at the first Poisson time at which the process is below some other level. To this end, we further denote, for $b \in \mathbb{R}$, the random times

$$\tau_b^- := \inf\{t > 0 : X_t < b\} \in \mathcal{T}_c \quad \text{and} \quad T_b^- := \inf\{T^{(n)} : X_{T^{(n)}} < b\} \in \mathcal{T}_p,$$

  where we recall $(T^{(n)})_{n \in \mathbb{N}}$ are the jump times of an independent Poisson process with rate λ.
- For $x \in \mathbb{R}$ and $a \leq l$, we denote

$$v_c(x; a, l) := \mathbb{E}_x\left[e^{-q\tau_a^-} f_c(X_{\tau_a^-}) 1_{\{\tau_a^- < T_l^-\}}\right], \quad \text{and}$$
$$v_p(x; a, l) := \mathbb{E}_x\left[e^{-qT_l^-} f_p(X_{T_l^-}) 1_{\{T_l^- < \tau_a^-\}}\right].$$

- Note that $T_l^- \neq \tau_a^-$ a.s. thanks to the independence between X and N.

Threshold strategies.

Proposition.

For $l \geq a$ and any locally bounded measurable function f_c on $\mathbb{R}$, the function $v_c(x; a, l)$ is given by

$$v_c(x; a, l) = \begin{cases} f_c(a)\,\dfrac{Z^{(q+\lambda)}(l - x; \Phi(q))}{Z^{(q+\lambda)}(l - a; \Phi(q))}\,, & \text{for } x > a\,, \\ f_c(x)\,, & \text{for } x \leq a\,. \end{cases}$$

And,

Proposition

For $l \geq a$ and any locally bounded measurable function f_p on $\mathbb{R}$, the function $v_p(x; a, l)$ is given by

$$v_p(x; a, l) = \begin{cases} \lambda\left(\dfrac{Z^{(q+\lambda)}(l - x; \Phi(q))}{Z^{(q+\lambda)}(l - a; \Phi(q))}\Gamma(a; l) - \Gamma(x; l)\right), & \text{for } x > a, \\ 0, & \text{for } x \leq a, \end{cases}$$

where $\Gamma(x; l) := \int_0^{l-x} f_p(l - u) W^{(q+\lambda)}(l - x - u)du$, for all $x, l \in \mathbb{R}$.

Optimality over threshold strategies.

- First we consider a version of the game where admissible strategies are restricted to be of threshold-type.
- The objective is to find a Nash equilibrium $(a^*, l^*) \in \mathbb{R}^2$ satisfying simultaneously the following two equations:

$$v_c(x; a^*, l^*) = \max_{a \in \mathbb{R}} v_c(x; a, l^*),$$
$$v_p(x; a^*, l^*) = \max_{l \in \mathbb{R}} v_p(x; a^*, l).$$

Assumptions.

- The reward functions $f_c(\cdot)$ and $f_p(\cdot)$ satisfy the following properties:
  (i) We have $f_c(\cdot) < f_p(\cdot)$ on $\mathbb{R}$.
  (ii) For $i \in \{c, p\}$, the function $f_i(\cdot)$ is strictly decreasing, continuously differentiable and concave on $\mathbb{R}$ and admits a constant

$$\overline{x}_i \in \mathbb{R} \qquad \text{such that} \qquad f_i(x) \begin{cases} > 0, & x < \overline{x}_i, \\ < 0, & x > \overline{x}_i. \end{cases}$$

- Assumption (i) reflects the additional costs bared by player C for the additional information provided, if and when successfully stopping before player P.
- The decreasing reward functions in (ii) reflect the game's "optimal purchasing" nature, while the class of concave reward functions associates the present setting with the widely-used risk-averse or even risk-neutral utility maximisation theory.
- An important example for (ii) is the perpetual American option pricing driven by an exponential Lévy process.

Benchmark case: Single-player setting.

- Consider the special case when $\lambda = 0$, i.e. player P can never stop, as a benchmark.
- This involves only player C whose expected reward under a threshold strategy τ_a^- given by

$$v_c^o(x;a) := \mathbb{E}_x\left[e^{-q\tau_a^-} f_c(X_{\tau_a^-}) 1_{\{\tau_a^- < \infty\}}\right] = \begin{cases} f_c(x) & \text{for } x \leq a, \\ e^{\Phi(q)(a-x)} f_c(a) & \text{for } x > a. \end{cases}$$

- Straightforward differentiation gives

$$\frac{\partial}{\partial a} v_c^o(x;a) = \begin{cases} 0 & \text{for } x < a, \\ e^{\Phi(q)(a-x)} h_c^o(a) & \text{for } x > a, \end{cases}$$

where $h_c^o(x) := \Phi(q) f_c(x) + f_c'(x)$, for $x \in \mathbb{R}$.

Benchmark case: Single-player setting.

- There exists

$$\underline{a} \in [-\infty, \overline{x}_c) \qquad \text{such that for } x \in \mathbb{R}, \qquad h_c^o(x) \begin{cases} > 0, & x < \underline{a}, \\ < 0, & x > \underline{a}. \end{cases}$$

- Hence by the results in Long and Zhang [1], we have

$$v_c^o(x; \underline{a}) = \max_{a \in \mathbb{R}} v_c^o(x; a) = \sup_{\tau \in \mathcal{T}_c} \mathbb{E}_x \left[e^{-q\tau} f_c(X_\tau) 1_{\{\tau < \infty\}} \right], \quad x \in \mathbb{R}.$$

  Instead, when $\underline{a} = -\infty$, an optimal stopping time does not exist.

- Notice that this provides the solution to the degenerate case in which $l \leq a$. Given that $\tau_a^- < T_l^-$ a.s.

$$v_c(x; a, l) = v_c^o(x; a) := \mathbb{E}_x \left[e^{-q\tau_a^-} f_c(X_{\tau_a^-}) 1_{\{\tau_a^- < \infty\}} \right] \quad \text{and} \quad v_p(x; a, l) = 0$$

  which boils down to a one-player maximisation problem for player C, while player P does not participate in the game under such a choice of $l \leq a$.

First order conditions.

Lemma

(i) For $a < l \wedge x$, define $I(a; l) := f_c'(a) + \Phi(q)\, f_c(a) + \lambda W^{(q+\lambda)}(l-a) v_c(l; a, l)$, then

$$\frac{\partial}{\partial a} v_c(x; a, l) = \frac{Z^{(q+\lambda)}(l-x; \Phi(q))}{Z^{(q+\lambda)}(l-a; \Phi(q))} I(a; l).$$

(ii) For $l > a$ define $J(l; a) := Z^{(q+\lambda)}(l-a; \Phi(q))\big(f_p(l) - v_p(l; a, l)\big)$ then for $x \geq a$ such that $l \neq x$,

$$\frac{\partial}{\partial l} v_p(x; a, l) = \lambda\left(\frac{Z^{(q+\lambda)}(l-x; \Phi(q))}{Z^{(q+\lambda)}(l-a; \Phi(q))} W^{(q+\lambda)}(l-a) - W^{(q+\lambda)}(l-x)\right)\frac{J(l; a)}{Z^{(q+\lambda)}(l-a; \Phi(q))}.$$

- The first-order condition $\frac{\partial}{\partial a} v_c(x; a, l) = 0$ for $x \geq a$, required for the optimality (best response to any given l) of the candidate threshold a, implies:

$$\mathbf{C}_a : I(a; l) = 0\,.$$

- The first-order condition $\frac{\partial}{\partial l} v_p(x; a, l) = 0$ for $x \geq a$, required for the optimality (best response to any given a) of the candidate threshold l, implies:

$$\mathbf{C}_l : J(l; a) = 0.$$

Construction of Nash equilibria.

- For any threshold l chosen by player P from an appropriate domain, player C chooses a *unique best response* $\tilde{a}(l)$ such that $\mathbf{C}_a$ holds, i.e. $I(\tilde{a}(l); l) = 0$.
- For any threshold a chosen by player C from an appropriate domain, player P chooses a *unique best response* $\tilde{l}(a)$ such that $\mathbf{C}_l$ holds, i.e. $J(\tilde{l}(a); a) = 0$.
- The Nash equilibrium will be given by a fixed point (a^*, l^*) satisfying

$$l^* = \tilde{l}(a^*) \quad \text{and} \quad a^* = \tilde{a}(l^*) \,.$$

Proposition (Existence of Nash Equilibria)

Fix $x \in \mathbb{R}$.

(i) There exists a root l^* to the equation $J(\cdot; \tilde{a}(\cdot)) = 0$.

(ii) For any root l^* in (i) and $a^* := \tilde{a}(l^*)$, the pair (a^*, l^*) is a Nash equilibrium.

Non-uniqueness and Pareto-superior Nash equilibria.

- If l^* satisfying $J(l^*; \tilde{a}(l^*)) = 0$ is unique, then the Nash equilibrium (a^*, l^*) is unique.
- However, there may be multiple l^* satisfying $J(l^*; \tilde{a}(l^*)) = 0$.
- Choosing the smallest (threshold) root, we can construct the unique Nash equilibrium that is Pareto-superior to any other Nash equilibrium.
- If both players are rational and intelligent enough, we can discard other (Pareto-dominated) equilibria, because all agents are strictly better-off if they switch to these Pareto-superior equilibrium strategies.
- To this end, we denote by l^*_{min} the minimum root, defined by

$$l^*_{min} := \min\{l \in (\underline{x}_c, \overline{x}_p) \ : \ J(l; \tilde{a}(l)) = 0\}.$$

- Then, with $a^*_{min} := \tilde{a}(l^*_{min})$, we conclude that (a^*_{min}, l^*_{min}) is a Nash equilibrium.

Non-uniqueness and Pareto-superior Nash equilibria.

Proposition

The Nash equilibrium (a^*_{min}, l^*_{min}) is Pareto-superior to any other Nash equilibrium (a^*, l^*) satisfying

$$v_c(x; a^*, l^*) = \max_{a \in \mathbb{R}} v_c(x; a, l^*),$$
$$v_p(x; a^*, l^*) = \max_{l \in \mathbb{R}} v_p(x; a^*, l).$$

In other words,

$$v_i(x; a^*, l^*) \leq v_i(x; a^*_{min}, l^*_{min}), \qquad \text{for both } i \in \{c, p\} \text{ and all } x \in \mathbb{R}.$$

In particular, if $x > a^*_{min}$, then the above inequality is strict.

Smoothness and convexity.

We have the following properties of smoothness and convexity of the functions $v_c(\cdot; a^*, l^*)$ and $v_p(\cdot; a^*, l^*)$.

Proposition

(I) Regarding the function $v_c(\cdot; a^*, l^*)$, we have the following:

(i) $v_c(\cdot; a^*, l^*)$ is continuous on $\mathbb{R}$ and C^2 (resp., C^1) on $(a^*, \infty)\backslash\{l^*\}$ when X is of unbounded (resp., bounded) variation.

(ii) $v_c(\cdot; a^*, l^*)$ is continuously differentiable at a^*.

(iii) $v_c(\cdot; a^*, l^*)$ is continuously differentiable at l^*, only when X is of unbounded variation.

(iv) $v_c(\cdot; a^*, l^*)$ is decreasing and convex on (a^*, ∞).

(II) Regarding the function $v_p(\cdot; a^*, l^*)$, we have the following:

(i) $v_p(\cdot; a^*, l^*)$ is continuous on $\mathbb{R}$ and twice continuously differentiable on $\mathbb{R}\backslash\{a^*, l^*\}$.

(ii) $v_p(\cdot; a^*, l^*)$ is continuously differentiable at l^*.

(iii) $v_p(\cdot; a^*, l^*)$ is twice continuously differentiable at l^*, only when X is of unbounded variation.

Optimality over all stopping times. Verification for player P.

- We define the infinitesimal generator $\mathcal{L}$ acting on $v_p(\cdot; a^*, l^*)$ as follows:

$$\mathcal{L}v_p(x; a^*, l^*) := -\gamma v_p'(x; a^*, l^*) + \frac{\nu^2}{2} v_p''(x; a^*, l^*) + \int_{(0,\infty)} [v_p(x+z; a^*, l^*) - v_p(x; a^*, l^*) - v_p'(x; a^*, l^*) z \mathbf{1}_{\{z<1\}}] \Pi(\mathrm{d}z).$$

Verification lemma for Player P.

Suppose that

(i) $(\mathcal{L} - q)v_p(x; a^*, l^*) = 0,$ for $x \geq l^*$,

(ii) $(\mathcal{L} - q)v_p(x; a^*, l^*) - \lambda\left(v_p(x; a^*, l^*) - f_p(x)\right) = 0,$ for $x \in (a^*, l^*]$,

(iii) $v_p(x; a^*, l^*) \geq f_p(x),$ for $x \geq l^*$,

(iv) $v_p(x; a^*, l^*) \leq f_p(x),$ for $x \in [a^*, l^*]$,

(v) $v_p(x; a^*, l^*) = 0,$ for $x \leq a^*$.

Then, $v_p(x; a^*, l^*) = \sup_{\sigma \in \mathcal{T}_p} V_p(\tau_{a^*}^-, \sigma; x)$, for $x \in \mathbb{R}$.

Optimality over all stopping times. Optimality for player P.

Proposition (Optimality for Player P)

With (a^*, l^*) satisfying $J(l^*; \tilde{a}(l^*)) = 0$ (and $a^* = \tilde{a}(l^*)$), we have

$$v_p(x; a^*, l^*) = \sup_{\sigma \in \mathcal{T}_p} V_p(\tau_{a^*}^-, \sigma; x), \quad x \in \mathbb{R}.$$

Optimality over all stopping times. Optimality for player C.

- To upgrade the optimality of threshold-type strategies over all stopping times for player C, we will not rely on an analytical verification theorem.
- Instead, we employ here a different methodology that is based on the use of an *average problem approach* (see e.g. Long and Zhang [1], Rodosthenous and Zhang [3], and Surya [4]) to prove the optimality of threshold type strategies over all stopping times.
- Using the definition of $T_{l^*}^-$ and the independence of the Poisson process N and Lévy process X, we can rewrite the current optimal stopping problem with random time-horizon $T_{l^*}^-$, in the form of

$$\sup_{\tau \in \mathcal{T}_c} V_c(\tau, T_{l^*}^-; x) = \sup_{\tau \in \mathcal{T}_c} \mathbb{E}_x \left[e^{-q\tau} f_c(X_\tau) 1_{\{\tau < T_{l^*}^-\}} \right] = \sup_{\tau \in \mathcal{T}_c} \mathbb{E}_x \left[e^{-A_\tau^X} f_c(X_\tau) 1_{\{\tau < \infty\}} \right],$$

where the latter is a perpetual optimal stopping problem with stochastic discounting given by the occupation time

$$A_t^X := qt + \lambda \int_0^t 1_{\{X_u < l^*\}} du, \quad \forall\, t \geq 0,$$

Optimality over all stopping times. Optimality for player C.

Hence, using the results in Long and Zhang [1].

Proposition (Optimality for Player C)

With (a^*, l^*) satisfying $J(l^*; \tilde{a}(l^*)) = 0$ (and $a^* = \tilde{a}(l^*)$), we have

$$v_c(x; a^*, l^*) = \sup_{\tau \in \mathcal{T}_c} V_c(\tau, T^-_{l^*}; x), \quad x \in \mathbb{R}.$$

Optimality over all stopping times.

By the previous results, we immediately get that the pair of strategies $(\tau_{a^*}^-, T_{l^*}^-)$ is a Nash equilibrium, when the strategy sets of both players are unrestricted (most general ones possible); this is formally stated in the following.

Theorem

With (a^*, l^*) satisfying $J(l^*; \tilde{a}(l^*)) = 0$ (and $a^* = \tilde{a}(l^*)$), we have for all $x \in \mathbb{R}$,

$$\begin{cases} V_c(\tau_{a^*}^-, T_{l^*}^-; x) \geq V_c(\tau, T_{l^*}^-; x), & \forall\, \tau \in \mathcal{T}_c, \\ V_p(\tau_{a^*}^-, T_{l^*}^-; x) \geq V_p(\tau_{a^*}^-, \sigma; x), & \forall\, \sigma \in \mathcal{T}_p. \end{cases}$$

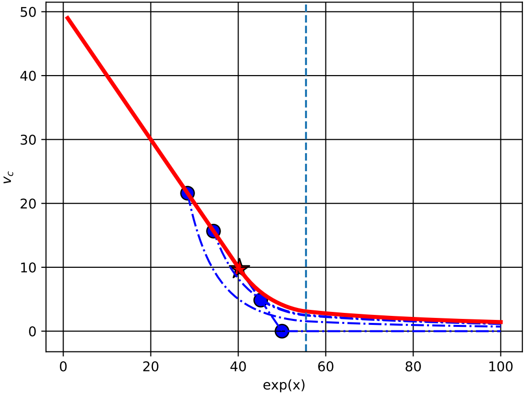
Numerical results.

- In this section, we confirm the analytical results focusing on a case study with put-type payoffs for both players. Suppose that $f_i(x) = K_i - e^x$, for $i \in \{c, p\}$, for some fixed $K_p > K_c$.
- As an underlying asset price e^X, we consider the case of X that is of the form

$$X_t = X_0 - \mu t + \nu B_t + \sum_{n=1}^{M_t} Z_n, \quad 0 \leq t < \infty,$$

  where $\mu > 0$ and $\nu \geq 0$ are constants, $B = (B_t : t \geq 0)$ is a standard Brownian motion, $M = (M_t : t \geq 0)$ is a Poisson process with arrival rate α, and $Z = (Z_n : n = 1, 2, \dots)$ is an i.i.d. sequence of exponential random variables with parameter β.
- The processes B, M, and Z are assumed mutually independent. In this model, the scale functions admit explicit expressions.
- We use the parameter values $\nu = 0.2$, $\alpha = 1$, $\beta = 2$, $q = 0.05$ and $\mu = 0.31333$ so that $(e^{-qt+X_t}; t \geq 0)$ becomes a martingale. In particular, we set $K_p = 60$ and $K_c = 50$, we also set $\lambda = 1$.

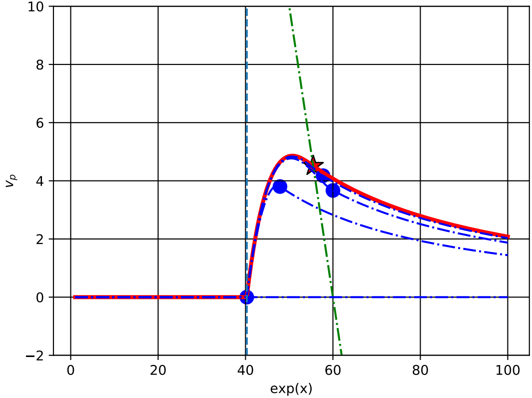
Numerical results.



$$e^x \mapsto v_c(x; a, l^*)$$

Figure: Plots of $e^x \mapsto v_c(x; a^*, l^*)$ in red, along with $e^x \mapsto v_c(x; a, l^*)$ in dotted blue for $e^a = e^{\underline{x}_c}, (e^{\underline{x}_c} + e^{a^*})/2, (e^{a^*} + K_c)/2, K_c$. The points at a and a^* are indicated by circles and a star, respectively. The value at l^* is indicated by the dotted vertical line.

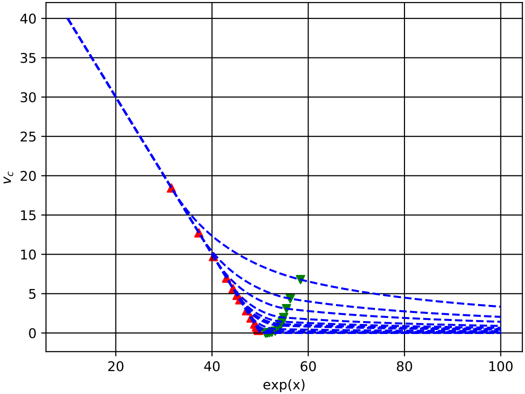
Numerical results.



$$e^x \mapsto v_p(x; a^*, l)$$

Figure: Plots of $e^x \mapsto v_p(x; a^*, l^*)$ in red, along with $v_p(x; a^*, l)$ in dotted blue for $e^l = e^{a^*}, (e^{a^*} + e^{l^*})/2, (K_p + e^{l^*})/2, K_p$. The points at l and l^* are indicated by circles and a star, respectively. The value at a^* is indicated by the dotted vertical line and the green line depicts the (reward) mapping $e^x \mapsto K_p - e^x$.

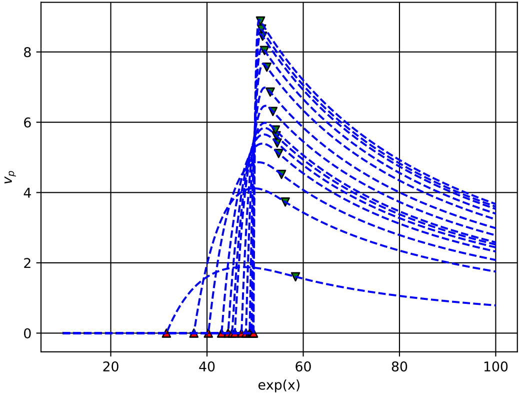
Numerical results.



$$e^x \mapsto v_c(x; a^*, l^*)$$

Figure: The value functions $e^x \mapsto v_c(x; a^*, l^*)$ for $\lambda = 0.1,\ 0.5,\ 1,\ 2,\ 3,\ 4,\ 5, 10, 20, 50, 100, 200, 300, 500.$

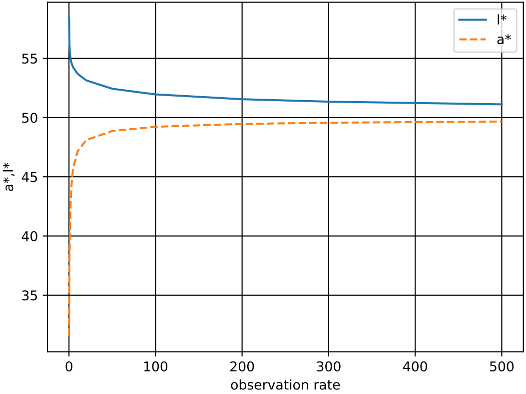
Numerical results.



$$e^x \mapsto v_p(x; a^*, l^*)$$

Figure: The value functions $e^x \mapsto v_p(x; a^*, l^*)$ for $\lambda = 0.1, 0.5, 1, 2, 3, 4, 5, 10, 20, 50, 100, 200, 300, 500$.

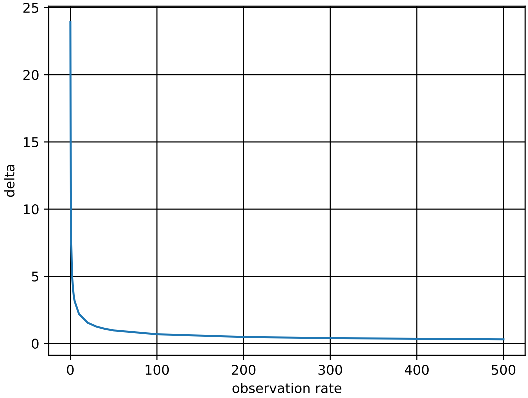
Numerical results.



$$\lambda \mapsto a^*, l^*$$

Figure: The barriers l^* and a^* as functions of λ.

Numerical results.



$$\lambda \mapsto \delta$$

Figure: Value of additional information $\delta := K_p - K_c$, such that $v_c(x; a^*, l^*) = v_p(x; a^*, l^*)$, as a function of the observation rate λ.

Conclusion

Thank for your attention!

Conclusion

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