Lectures on relativistic gravity and cosmology.

Lectures 7-8

Alexei A. Starobinsky

Landau Institute for Theoretical Physics RAS

Faculty of Physics, National Research University
"Higher School of Economics"

Moscow, 30.04.2022

General properties of the Einstein equations

Gravitational field far away from bodies

Gravitational waves

Emission of gravitational waves

Number of equations and their structure

$$R_{ik} - \frac{1}{2} g_{ik} R = g_{ik}\Lambda + 8\pi G T_{ik} \ .$$

10 equations.

1. Due to the Bianchi identity $R^k_{i;k} = 0$, covariant differentiation with summation gives $T^k_{i;k} = 0$.
The equations for gravitational field already contain dynamical equations for matter in themselves.
2. No second time derivatives in $(0-0)$ and $(0-\alpha)$ equations $(\alpha = 1, 2, 3)$.
3. No second time derivatives of g_{00} and $g_{0\alpha}$ in all equations.

Number of 'physically different' variables in g_{ik}: $10 - 4 = 6$.
The synchronous reference system: $g_{00} = 1,\ g_{0\alpha} = 0$.
For a barotropic fluid or gas: $T_{ik} = (\rho + p)\, u_i u_k - p g_{ik}$, 4 more variables: $\rho,\ u^i$. The equations of state $p = p(\rho)$ have to be added separately.

Total active gravitational mass

For the spherically symmetric case ($\Lambda = 0$)

$$R_0^0 - \frac{1}{2}R = -e^{-\lambda}\left(-\frac{\lambda'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi G T_0^0,$$

$$\lambda = -\ln\left(1 - \frac{8\pi G}{r}\int_0^r T_0^0 r^2\, dr\right)$$

Outside the body ($T_0^0 = 0$ for $r > r_0$), $\lambda = -\ln\left(1 - \frac{2GM}{r}\right)$. Thus,

$$M = 4\pi \int_0^{r_0} T_0^0 r^2\, dr$$

Difference between $4\pi r^2\, dr$ and $dV = 4\pi r^2 e^{\lambda/2} dr$ - the gravitational mass defect.

Gravitational field far away from bodies

$$g_{ik} = \eta_{ik} + h_{ik}, \ \psi_i^k = h_i^k - \frac{1}{2}\delta_i^k$$

The Lorentz gauge: $\frac{\partial \psi_i^k}{\partial x^k} = 0$. Then

$$R_{ik} = -\frac{1}{2}\eta^{ik}\frac{\partial^2 h_{ik}}{\partial x^i \partial x^k} = -\frac{1}{2}\Box h_{ik}, \ \Box = \frac{\partial^2}{\partial t^2} - \triangle$$

.

The static case, order r^{-1}.

$$h_{00}^{(1)} = -\frac{r_g}{r}, \ h_{\alpha\beta}^{(1)} = -\frac{r_g}{r}\,\delta_{\alpha\beta}, \ h_{0\alpha}^{(1)} = 0$$

They are determined from the comparison with the Schwarzschild solution re-written in terms of the Cartesian coordinates x, y, z. Let $r = \rho\left(1 + \frac{r_g}{4\rho}\right)^2$ and $\rho^2 = x^2 + y^2 + z^2$.

$$ds^2 = \frac{\left(1 - \frac{r_g}{4\rho}\right)^2}{\left(1 + \frac{r_g}{4\rho}\right)^2} dt^2 - \left(1 + \frac{r_g}{4\rho}\right)^4 (dx^2 + dy^2 + dz^2)$$

As a result:

$$ds^2 = (1 + 2\varphi)\, dt^2 - (1 - 2\varphi)(dx^2 + dy^2 + dz^2),$$

where φ is the Newtonian gravitational potential.

In the next order r^{-2} ($\propto c^{-3}$ in usual units)

$$h_{00}^{(2)} = \frac{1}{2}\left(\frac{r_g}{r}\right)^2, \; h_{\alpha\beta}^{(2)} = \frac{3}{8}\left(\frac{r_g}{r}\right)^2 \delta_{\alpha\beta}, \; h_{0\alpha}^{(2)} = -2GL_{\alpha\beta}\frac{n_\beta}{r^2}$$

where $L_{\alpha\beta}$ is the total angular momentum 4-tensor.

The Tolman formula for the total mass

Another expression for the total mass in the stationary case - the Tolman formula.

$$R_0^0 = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{-g} g^{i0} \Gamma_{0i}^\alpha \right)$$

Integrating over the 3-space, using the Gauss theorem and the formula for a metric far away from bodies at the remote 2-sphere, we get:

$$\int R_0^0 \sqrt{-g}\, dV = 2\pi r_g$$

$$M = \int (T_0^0 - T_1^1 - T_2^2 - T_3^3) \sqrt{-g}\, dV$$

Weak gravitational waves (GW) in flat space-time

In the Lorentz gauge $\psi^k_{i,k} = 0$,

$$\Box \psi^k_i = 0.$$

In flat space-time, GW propagate with the velocity of light. Recently confirmed by the observation of the GW170817 + GRB 170817A event - merging of two neutron stars at the distance 40 ± 10 Mpc. X-ray signal: 1.7 s after the peak of the GR one.

$$|v_{GW} - 1| < 3 \times 10^{-15}$$

The Lorentz gauge still does not fix the reference system unambiguously. The remaining freedom of coordinate transformations: $\tilde{x}^i = x^i + \xi^i$, $\Box \xi^i = 0$.

Consider a plane gravitational wave moving right along the $x^1 = x$ axis. ψ_i^k are functions of $t - x$. From the Lorentz gauge condition: $\dot{\psi}_i^1 = \dot{\psi}_i^0$.

$$\psi_1^1 = \psi_1^0,\ \psi_2^1 = \psi_2^0,\ \psi_3^1 = \psi_3^0,\ \psi_0^1 = \psi_0^0.$$

Additional transformation: $\tilde{x}^i = x^i + \xi^i(t - x)$. Used to make

$$\psi_1^0 = \psi_2^0 = \psi_3^0 = \psi_2^2 + \psi_3^3 = 0.$$

Then $\psi_1^1 = \psi_2^1 = \psi_3^1 = \psi_0^0 = \psi_i^i = 0$ and $\psi_i^k = h_i^k$.
Remaining non-zero components are $h_{22} = -h_{33}$ and h_{23}.
Thus, GW are transverse, traceless and have two polarization states. They have two degrees of freedom, and initial conditions for vacuum gravity at a space-like Cauchy hypersurface are given by 4 arbitrary functions of spatial coordinates. In the presence of a barotropic matter, 4 more arbitrary functions (energy density and spatial velocity) appear.

Weak high frequency GW in curved vacuum space-time

$$g_{ik} = g_{ik}^{(0)} + h_{ik}$$

Weak: $|h_i^k| \ll 1$.

High frequency: $\omega \gg L_R^{-1} = (R_{iklm}^{(0)} R^{iklm(0)})^{1/4}$ where $R_{iklm}^{(0)}$ is the background Riemann tensor, so that $|R_i^{k(1)}| \gg |R_i^{k(0)}|$.

$$\Gamma_{kl}^{i(1)} = \frac{1}{2}(h_{l;k}^i + h_{k;l}^i - h_{kl}{}^{;i}),$$

$$R_{ik}^{(1)} = \frac{1}{2}(h_{i;k;l}^l + h_{k;i;l}^l - h_{ik}{}^{;l}_{;l} - h_{;i;k}) \approx \frac{1}{2} g_{ik}^{(0)} g^{lm(0)} R_{lm}^{(1)} .$$

Imposing the generally covariant Lorentz gauge $\psi_{i;k}^k = 0$ and using the condition $\omega L_R \gg 1$, we get

$$\psi_{ik}{}^{;l}_{;l} = 0.$$

The Isaacson EMT of GW

The remaining freedom of coordinate transformations: $\tilde{x}^i = x^i + \xi^i$, $\xi^{i;k}{}_{;k} = 0$. Can be used to make $h \equiv h^i_i = 0$, then $h^k_{i;k} = 0$. After that the remaining admissible transformations should satisfy $\xi^i_{;i} = 0$.

Let us average the space-time over scales much more than ω^{-1} but much less than L_R. Then

$$< R^{(2)}_{ik} > \approx -\frac{1}{4} < h^n_{m,i} h^m_{n,k} >$$

.

$$R^{(0)}_{ik} \approx - < R^{(2)}_{ik} > = 8\pi G T^{(GW)}_{ik},$$

$$T^{(GW)}_{ik} = \frac{1}{32\pi G} < h^n_{m,i} h^m_{n,k} > .$$

For a plane GW in flat space-time considered previously,

$$T^{01(GW)} = \frac{1}{16\pi G} \left[\dot{h}^2_{23} + \frac{1}{4}(\dot{h}_{22} - \dot{h}_{33})^2 \right].$$

Emission of GW by non-relativistic matter

$$\frac{1}{2}\Box\psi^i_k = -8\pi G\tau^k_i.$$

From the Lorentz gauge $\psi^k_{i,k} = 0$, it follows that $\tau^k_{i,k} = 0$. Solution in the form of retarded potentials:

$$\psi^k_i = -4G\int(\tau^k_i)_{t-R}\frac{dV}{R}.$$

Assuming that all matter velocities are small compared to the light velocity, we can write:

$$\psi^k_i = -\frac{4G}{R_0}\int(\tau^k_i)_{t-R_0}\,dV.$$

Calculation of the integral.

$$\frac{\partial \tau_{\alpha\gamma}}{\partial x^\gamma} = \frac{\partial \tau_{0\gamma}}{\partial x^0}, \ \frac{\partial \tau_{0\gamma}}{\partial x^\gamma} = \frac{\partial \tau_{00}}{\partial x^0}.$$

Multiplying the first equation by x^β, integrating over the volume and symmetrizing over $\alpha,\ \beta$, we get

$$\int \tau_{\alpha\beta}\, dV = -\frac{1}{2}\frac{\partial}{\partial x^0}\int (\tau_{\alpha 0}x^\beta + \tau_{\beta 0}x^\alpha)\, dV.$$

Multiplication of the second equation by $x^\alpha x^\beta$ and integration over volume leads to

$$\frac{\partial}{\partial x^0}\int \tau_{00}x^\alpha x^\beta\, dV = -\int (\tau_{\alpha 0}x^\beta + \tau_{\beta 0}x^\alpha)\, dV.$$

Combining these expression, we get

$$\int \tau_{\alpha\beta}\, dV = \frac{1}{2}\left(\frac{\partial}{\partial x^0}\right)^2 \int \tau_{00}x^\alpha x^\beta\, dV.$$

Since $\tau_{00} = \rho$,

$$\psi_{\alpha\beta} = -\frac{2G}{R_0}\left(\frac{\partial}{\partial t}\right)^2 \int \rho x^\alpha x^\beta \, dV.$$

Let us introduce the quadrupole moment of mass distribution:

$$D_{\alpha\beta} = \int \rho \left(3x^\alpha x^\beta - r^2 \delta_{\alpha\beta}\right) dV.$$

At large distances from the source and locally, the GW can be considered as a plane one. Along the x^1 axis,

$$h_{23} = -\frac{2G}{3R_0}\ddot{D}_{23}, \;\; h_{22} - h_{33} = -\frac{2G}{3R_0}(\ddot{D}_{22} - \ddot{D}_{33}).$$

The energy flux to the x^1 direction is

$$T^{01} = \frac{G}{36\pi R_0^2}\left[\left(\frac{\dddot{D}_{22} - \dddot{D}_{33}}{2}\right)^2 + \dddot{D}_{23}^2\right].$$

The polarization tensor $e_{\alpha\beta}$. Properties:

$$e_{\alpha\alpha} = 0, \;\; e_{\alpha\beta} n_\beta = 0, \;\; e_{\alpha\beta} e_{\alpha\beta} = 1.$$

The intensity of radiation into the solid angle do (restoring c):

$$dI = \frac{G}{72\pi c^5} (\dddot{D}_{\alpha\beta} e_{\alpha\beta})^2 \, do.$$

Summing over two polarizations of GW, we get:

$$dI = \frac{G}{36\pi c^5} \left[\frac{1}{4} (\dddot{D}_{\alpha\beta} n_\alpha n_\beta)^2 + \frac{1}{2} \dddot{D}^2_{\alpha\beta} - \dddot{D}_{\alpha\beta} \dddot{D}_{\alpha\gamma} n_\beta n_\gamma \right] do.$$

The total GW radiation flux I (luminosity L in GW):

$$I = \frac{d\mathcal{E}}{dt} = \frac{G}{45 c^5} \dddot{D}^2_{\alpha\beta}$$

The (hypothetical) limiting luminosity in GR:
$L \sim \frac{c^5}{G} = 3.63 \cdot 10^{59} \, \mathrm{erg/s}$ (compare to $L_\odot = 3.83 \cdot 10^{33} \, \mathrm{erg/s}$).