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AN AXIOMATIZATION FOR LINEAR FUNCTIONS OF COALESCENCE INTENSITY

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Power indices are used for measuring the influence of each agent in electoral bodies. The most used power indices do not take into account agents' preferences to coalesce. Power indices taking into account agents' preferences to coalesce were introduced in (Aleskerov, 2006). These indices are based on functions of intensity connections between agents, which can be divided into 2 groups – linear and nonlinear functions. We consider linear functions of intensity connections and introduce an axiomatization for them.

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Introduction

Power indices are used to study electoral bodies and an institutional balance of power in these bodies (Aleskerov, 2006; Brams, 1975; Felsenthal and Machover, 1998). As an example of power index consider the Banzhaf index (Bz), which was introduced in (Banzhaf, 1965).

Let N be a finite set of agents. Any subset of the set N is called a coalition. A coalition is called winning if the number of its votes is not less than a quota q. An agent i is called pivotal in winning coalition A if coalition $A\backslash\{i\}$ is not winning. The set of winning coalitions, where i is a pivotal agent is denoted as W_i.

The Banzhaf index (Bz) for agent i is defined as

$$Bz_i = \frac{|W_i|}{\sum_{j\in N}|W_j|}.$$

For instance, there are 3 parties in a legislative body – A, B, C. Party A has 30 seats, B has 45, C has 25 seats, q = 51 (simple majority rule). Thus, the Banzhaf index for A is $Bz_A = \frac{|\{\{A,B\},\{A,C\}\}|}{|\{\{B,A\},\{B,C\}\}|+|\{\{A,B\},\{A,C\}\}|+|\{\{C,B\},\{C,A\}\}|} = \frac{1}{3}$. For parties B, C there are $P_B = P_C = \frac{1}{3}$.

The Banzhaf and other most used indices are based on the idea that all coalitions can be formed. Going back to the previous example let us suppose that party B prefers not to coalesce with parties A and C. Thus, coalitions $\{B, A\}$, $\{B, C\}$ and $\{A, B, C\}$ are not possible. If we try to measure the Banzhaf index, we will get for party A $Bz_A = \frac{1}{2}$, for C $Bz_C = \frac{1}{2}$, for B $Bz_C = 0$. Party B has more seats than A and C, but the influence of party B in this legislative body is 0. Note that these values are not satisfied an axiomatization of the Banzhaf index (Dubey and Shapley, 1979; Laruelle and Valenciano, 2000). Such situations also occur in real political systems (Aleskerov et al., 2014).

Power indices taking into account agents' preferences to coalesce were introduced in (Aleskerov, 2006). These indices are constructed with the function of intensity of each agent. Sixteen functions of intensity were introduced in (Aleskerov, 2006). In that paper an axiomatization for one function was constructed and posed a problem of an axiomatization for other functions of intensity.

1. The statement of the problem

Power index for agent i taking into account agents' preferences is constructed with the functions $f(i,w)$ of intensity of i's connections with other agents from the coalition w. For agent $i \in w$, the value $f(i,w)$ is a real number, which is constructed by considering the preferences of members in coalition w. In other words, $f: N \times (2^N \backslash \emptyset) \to \mathbb{R}$.

In (Aleskerov, 2006) for each agent $i \in N$ it is defined the value

$$\chi_i = \sum_{w \in W_i} f(i,w).$$

In other words, the value χ_i is the aggregated intensity of connection of agent i with other agents in winning coalitions, where i is pivotal. The power index taking into account preferences of agent i is defined in (Aleskerov, 2006) similarly the Banzhaf index (modification for other power indices is introduced in (Sokolova, 2009)) as

$$\alpha_i = \frac{\chi_i}{\sum_{j \in N} \chi_j}.$$

It can be said that the power index α_i is the normalized value χ_i.

In (Aleskerov, 2006) it was proposed that the desire of agent i to coalesce with agent j is a real number $p_{ij} \in [0,1]$, $\forall\, i \in N \sum_{j \in N} p_{ij} = 1$. The value p_{ij} is called the intensity of connection i and j. Basically, it is allowed that $p_{ij} \neq p_{ji}$. Power indices based on values p_{ij} are called cardinal (Aleskerov, 2006).

For example, let $N = \{A, B, C\}$, B is in bad attitude toward A and in very good attitude toward C, A is in equally good attitude toward B and C. C prefers the agent A to B for coalescing. Thus, the values p_{ij} can be the following: $p_{BA} = 0, p_{BC} = 1,\ p_{AB} = p_{AC} = 0.5, p_{CA} = 0.7, p_{CB} = 0.3$.

Consider the following linear functions of intensity for cardinal indices (Aleskerov, 2006):

- average intensity of $i's$ connection with other agents of coalition w

$$f^+(i,w) = \frac{\sum_{j \in w} p_{ij}}{|w| - 1}$$

- average intensity of connection of other agents of coalition w with i

$$f^-(i,w) = \frac{\sum_{j\in w} p_{ji}}{|w|-1}$$

- average intensity for i

$$f_{avg}(i,w) = \frac{1}{2}(f^+(i,w) + f^-(i,w))$$

- average positive intensity in w

$$f^+(w) = \frac{\sum_{i\in w} f^+(i,w)}{|w|}$$

- average negative intensity in w

$$f^-(w) = \frac{\sum_{i\in w} f^-(i,w)}{|w|}$$

- average intensity in w

$$f_{avg}(w) = \frac{\sum_{i\in w} f_{avg}(i,w)}{|w|}.$$

Note that the last three functions do not depend on agent i. Thus, for these functions, each member in coalition w has an equal value with others.

Consider the function $f^+(i,w)$. Assume that there are 3 parties in a legislative body – A, B, C. Party A has 50 votes, B has 25 votes, C has 25 votes, quota q = 51. The values of intensity of connections are as follows: $p_{AB} = 0.4$, $p_{AC} = 0.6$, $p_{BA} = 0.5$, $p_{BC} = 0.5$, $p_{CA} = 0.7$, $p_{CB} = 0.3$. Thus, $f^+(A,\{A,B\}) = \frac{0.4}{1} = 0.4, f^+(A,\{A,C\}) = \frac{0.6}{1} = 0.6, f^+(A,\{A,B,C\}) = \frac{0.4+0.6}{2} = 0.5$, $f^+(B,\{A,B\}) = \frac{0.5}{1} = 0.5, f^+(B,\{B,C\}) = \frac{0.5}{1} = 0.5,\ f^+(B,\{A,B,C\}) = \frac{1}{2} = 0.5$, $f^+(C,\{A,C\}) = \frac{0.7}{1} = 0.7, f^+(C,\{B,C\}) = \frac{0.3}{1} = 0.3, f^+(C,\{A,B,C\}) = \frac{1}{2} = 0.5$.

For this case, the values χ_i for each agent are the following: $\chi_A = 0.4 + 0.6 + 0.5 = 1.5$, $\chi_B = 0.5$, $\chi_B = 0.7$. Thus, $\alpha_A = \frac{1.5}{1.5+0.5+0.7} = \frac{1.5}{2.7} = \frac{5}{9}, \alpha_B = \frac{0.5}{1.5+0.5+0.7} = \frac{0.5}{2.7} = \frac{5}{27}, \alpha_C = \frac{0.7}{1.5+0.5+0.7} = \frac{0.7}{2.7} = \frac{7}{27}$.

Despite the numbers of the agents' B and C votes are equaling, the value of power index α_B is higher than the value α_C. The cause of this is that party C has the higher value of the intensity of connection with coalition $\{A, C\}$ than the value of the intensity of connection B with coalition $\{A, B\}$.

Note that

$$f^+(w) \equiv \frac{\sum_{i\in w} f^+(i,w)}{|w|} \equiv f^-(w) = \frac{\sum_{i\in w} f^-(i,w)}{|w|} \equiv f(w) = \frac{\sum_{i\in w} f_{avg}(i,w)}{|w|}.$$

Indeed,

$$\frac{\sum_{i\in w} f^+(i,w)}{|w|} = \frac{\sum_{i\in w} \frac{\sum_{j\in w} p_{ij}}{|w|-1}}{|w|} = \frac{\frac{\sum_{i,j\in w} p_{ij}}{|w|-1}}{|w|} = \frac{\sum_{j\in w} \frac{\sum_{i\in w} p_{ji}}{|w|-1}}{|w|} = \frac{\sum_{j\in w} f^-(i,w)}{|w|},$$

$$\frac{\sum_{i\in w} f^+(i,w)}{|w|} = \frac{\sum_{i\in w} \frac{\sum_{j\in w} p_{ij}}{|w|-1}}{|w|} = \frac{\frac{\sum_{i,j\in w} p_{ij}}{|w|-1}}{|w|} = \frac{\sum_{i\in w} \frac{1}{2}(\frac{\sum_{j\in w} p_{ij}}{|w|-1} + \frac{\sum_{j\in w} p_{ji}}{|w|-1})}{|w|}$$
$$= \frac{\sum_{i\in w} \frac{1}{2}(f^+(i,w) + f^-(i,w))}{|w|} = \frac{\sum_{i\in w} f_{avg}(i,w)}{|w|}.$$

Therefore, instead of all indices $f^+(w)$, $f^-(w)$, $f_{avg}(w)$ we consider the only one – $f_{avg}(w)$.

An axiomatization for function $f^+(i,w)$ was introduced in (Aleskerov, 2006).

- **Axiom 1.** For any m-tuple of values $(p_{i1}, \dots, p_{im})$ there exist a function $f(i,w)$, $0 \le f(i,w) \le 1$, where $f(i,w)$ – continuous differentiable function of each of its arguments.
- **Axiom 2.** If $p_{ij} = 0$ for any j, then $f(i,w) = 0$
- **Axiom 3 (Monotonicity).** A value of $f(i,w)$ increases (decreases) iff any value p_{ij} increases (decreases). Moreover,

$$\frac{\partial f(i,w)}{\partial p_{ij}} = \mu_i \text{ for any } j$$

and

$$\frac{\partial f(i,w)}{\partial p_{lj}} = 0 \text{ for any } l \neq i.$$

Theorem 1 (Aleskerov, 2006). $f(i,w)$ satisfies Axioms 1-3 iff $f(i,w) = f^{+}(i,w)$.

In this article, it is introduced an axiomatization for functions $f^{-}(i,w)$, $f_{avg}(i,w)$, $f_{avg}(w)$.

2. An axiomatization for linear functions of intensity

Consider the following axioms:

- **Axiom 1.** For any *m*-tuple of values $(p_{i1}, \dots, p_{im})$ there exist a function $f(i,w)$, $0 \leq f(i,w) \leq 1$, where $f(i,w)$ – continuous differentiable function of each argument. (The intensity of connection with other members of the coalition can be evaluated for each member i and for any i's preferences).
- **Axiom 1$\boldsymbol{a}$.** For any k-tuple of values $(p_{1i}, \dots, p_{ki})$ there exist a function $f(i,w)$, $0 \leq f(i,w) \leq 1$, where $f(i,w)$ – continuous differentiable function of each argument.
- **Axiom 1$\boldsymbol{b}$.** For any 2s-tuple of values $(p_{i1}, p_{1i} \dots, p_{is}, p_{si})$ there exist a function $f(i,w)$, $0 \leq f(i,w) \leq 1$, where $f(i,w)$ – continuous differentiable function of each argument.
- **Axiom 1$\boldsymbol{c}$.** For any *r*(*r*-1)-tuple of values $(p_{12}, \dots, p_{1r}, p_{21}, \dots, p_{r1,}, \dots, p_{r\,r-1})$ there exist a function $f(w)$, $0 \leq f(w) \leq 1$, where $f(w)$ – continuous differentiable function of each argument.
- **Axiom 2.** If $p_{ij} = 0$ for any p_{ij} from $(p_{i1}, \dots, p_{im})$, then $f(i,w) = 0$ (If member i is in bad attitude toward other members of the coalition, then the intensity of connection i with others is 0).
- **Axiom 2$\boldsymbol{a}$.** If $p_{ji} = 0$ for any p_{ji} from $(p_{1i}, \dots, p_{ki})$, then $f(i,w) = 0$ (If all members of the coalition are in bad attitude toward member i, then the intensity of connection i with them is 0).
- **Axiom 2$\boldsymbol{b}$.** If $p_{ij} = 0, p_{ji} = 0$ for any p_{ij}, p_{ji} from $(p_{i1}, p_{1i} \dots, p_{is}, p_{si})$, then $f(i,w) = 0$ (If member i of the coalition is in bad attitude toward other members and they are in bad attitude toward i, then the intensity of connection i with them is 0).

- **Axiom $2c$.** If $p_{lm} = 0$ for any p_{lm} from $(p_{12}, \dots, p_{1r}, p_{21}, \dots, p_{r1,}, \dots, p_{r\,r-1})$, then $f(w) = 0$ (If all members of the coalition are in bad attitude toward each other, then for each member the intensity of connection with others is 0).
- **Axiom 3 (Monotonicity 1).** A value of $f(i,w)$ increases (decreases) iff any value p_{ij} increases (decreases). Moreover,

$$\frac{\partial f(i,w)}{\partial p_{ij}} = \mu_i \text{ for any } p_{ij} \text{ from } (p_{i1}, \dots, p_{im})$$

and

$$\frac{\partial f(i,w)}{\partial p_{lj}} = 0 \text{ for other cases.}$$

Equal changes in values of preferences lead to the same changes in the value of intensity.

- **Axiom $3a$ (Monotonicity 2).** A value of $f(i,w)$ increases (decreases) iff any value p_{ji} increases (decreases). Moreover,

$$\frac{\partial f(i,w)}{\partial p_{ji}} = \mu_i \text{ for any } p_{ji} \text{ from } (p_{1i}, \dots, p_{ki})$$

and

$$\frac{\partial f(i,w)}{\partial p_{jl}} = 0 \text{ for other cases.}$$

- **Axiom $3b$ (Monotonicity 3).** A value of $f(i,w)$ increases (decreases) iff any value p_{ij}, p_{ji} increases (decreases). Moreover,

$$\frac{\partial f(i,w)}{\partial p_{ij}} = \frac{\partial f(i,w)}{\partial p_{ji}} = \mu_i \text{ for any } p_{ij}, p_{ji} \text{ from } (p_{i1}, p_{1i} \dots, p_{is}, p_{si})$$

and

$$\frac{\partial f(i,w)}{\partial p_{il}} = \frac{\partial f(i,w)}{\partial p_{jl}} = 0 \text{ for other cases.}$$

- **Axiom 3c (Monotonicity 4).** A value of $f(w)$ increases (decreases) iff any value p_{lm} increases (decreases). Moreover,

$$\frac{\partial f(w)}{\partial p_{lm}} = \mu_w \text{ for any } p_{lm} \text{ from } (p_{12}, \dots, p_{1r}, p_{21}, \dots, p_{r1,}, \dots, p_{r\,r-1})$$

and

$$\frac{\partial f(w)}{\partial p_{kz}} = 0 \text{ for other cases.}$$

Theorem 2. 1. $f(i,w)$ satisfies Axioms 1, 2, 3 iff $f(i,w) = f^{+}(i,w)$;

2. $f(i,w)$ satisfies Axioms $1a$, $2a$, $3a$ iff $f(i,w) = f^{-}(i,w)$;

3. $f(i,w)$ satisfies Axioms $1b$, $2b$, $3b$ iff $f(i,w) = f_{avg}(i,w)$;

4. $f(w)$ satisfies Axioms $1c$, $2c$, $3c$ iff $f(w) = f_{avg}(w)$.

Proof. The proof of this theorem is based on the idea of proving the average rule for probabilistic social choice (Intriligator, 1973).

1. It can be measured the value $f^{+}(i,w) = \frac{\sum_{j \in w} p_{ij}}{|w|-1}$ for each *m*-tuple $(p_{i1}, \dots, p_{im})$. Thus, $f^{+}(i,w)$ satisfies the Axiom 1. If in *m*-tuple $(p_{i1}, \dots, p_{im})$ all $p_{ij} = 0$, then $\frac{\sum_{j \in w} p_{ij}}{m-1} = \frac{0}{m-1} = 0$. Thus, $f^{+}(i,w)$ satisfies the Axiom 2. Since the function $f^{+}(i,w)$ is linear with respect to arguments $p_{i1}, \dots, p_{im}$, then this function satisfies the Axiom 3.

Consider an arbitrary coalition $w = \{1, \dots, m\}$, an agent $i \in w$, and *m*-tuple $(p_{i1}, \dots, p_{im})$. By the Axiom 1, it exists $f(i,w)$ for the tuple $(p_{i1}, \dots, p_{im})$, where $f(i,w)$ – continuous differentiable function of each argument, $0 \leq f(i,w) \leq 1$. Consider a total differential $df(i,w)$. By the Axiom 3, $df(i,w) = \mu_i \sum_{j \in w} dp_{ij} \Leftrightarrow \int df = \mu_i \sum_{j \in w} \int dp_{ij} \Leftrightarrow f(i,w) = C_i + \mu_i \sum_{j \in w} p_{ij}$. If all $p_{ij} = 0$, $j \in w$, then, by the Axiom 2, $f(i,w) = 0 = C_i$. Thus, $f(i,w) = \mu_i \sum_{j \in w} p_{ij}$. Expect that condition $\forall\, i \in N \sum_{j \in N} p_{ij} = 1$ is violated. If we prove the uniqueness of function $f^{+}(i,w)$ for this case, then this function is unique for the case with condition $\forall\, i \in N \sum_{j \in N} p_{ij} = 1$. If all $p_{ij} = 1$, $j \in w$, then, by monotonicity, $f(i,w) = 1$. Thus, $1 = \mu_i \sum_{j \in w} p_{ij} = \mu_i (m-1) \Rightarrow f(i,w) = \frac{\sum_{j \in w} p_{ij}}{|w|-1} = f^{+}(i,w)$.

2. It can be measured the value $f^-(i,w) = \frac{\sum_{j\in w} p_{ji}}{|w|-1}$ for each k-tuple $(p_{1i}, \dots, p_{ki})$. Thus, $f^-(i,w)$ satisfies the Axiom $1a$. If in k-tuple $(p_{1i}, \dots, p_{ki})$ all $p_{ji} = 0, j \in w$, then $\frac{\sum_{j\in w} p_{ji}}{m-1} = \frac{0}{m-1} = 0$. Thus, $f^-(i,w)$ satisfies the Axiom $2a$. Since the function $f^-(i,w)$ is linear with respect to arguments $p_{1i}, \dots, p_{ki}$, then this function satisfies the Axiom $3a$.

Consider an arbitrary coalition $w = \{1, \dots, k\}$, an agent $i \in w$, and k-tuple $(p_{1i}, \dots, p_{ki})$. By the Axiom $1a$, it exists $f(i,w)$ for the tuple $(p_{1i}, \dots, p_{ki})$, where $f(i,w)$ – continuous differentiable function of each argument, $0 \le f(i,w) \le 1$. Consider a total differential $df(i,w)$. By the Axiom $3a$, $df(i,w) = \mu_i \sum_{j\in w} dp_{ji} \Leftrightarrow \int df = \int \mu_i \sum_{j\in w} dp_{ji} = \mu_i \sum_{j\in w} \int dp_{ji} \Leftrightarrow f(i,w) = C_i + \mu_i \sum_{j\in w} p_{ji}$. If all $p_{ji} = 0$, $j \in w$, then, by the Axiom $2a$, $f(i,w) = 0 = C_i$. Thus, $f(i,w) = \mu_i \sum_{j\in w} p_{ji}$. If all $p_{ji} = 1$, $j \in w$, then, by monotonicity, $f(i,w) = 1$. Thus, $1 = \mu_i \sum_{j\in w} p_{ji} = \mu_i(k-1) \Rightarrow f(i,w) = \frac{\sum_{j\in w} p_{ji}}{|w|-1} = f^-(i,w)$.

3. It can be measured the value $\frac{1}{2}(f^+(i,w) + f^-(i,w))$ for each $2s$-tuple $(p_{i1}, p_{1i} \dots, p_{is}, p_{si})$. Thus, $f_{avg}(i,w)$ satisfies the Axiom $1b$. If in $2s$-tuple $(p_{i1}, p_{1i} \dots, p_{is}, p_{si})$ all $p_{ij} = p_{ji} = 0, j \in w$, then $\frac{1}{2}(f^+(i,w) + f^-(i,w)) = \frac{0}{2} = 0$. Thus, $f_{avg}(i,w)$ satisfies the Axiom $2b$. Since the function $f_{avg}(i,w)$ is linear with respect to arguments $p_{i1}, p_{1i} \dots, p_{is}, p_{si}$, then this function satisfies the Axiom $3b$.

Consider an arbitrary coalition $w = \{1, \dots, s\}$, an agent $i \in w$, and $2s$-tuple $(p_{i1}, p_{1i} \dots, p_{is}, p_{si})$. By the Axiom $1b$, it exists $f(i,w)$ for the tuple $(p_{i1}, p_{1i} \dots, p_{is}, p_{si})$, where $f(i,w)$ – continuous differentiable function of each argument, $0 \le f(i,w) \le 1$. Consider a total differential $df(i,w)$. By the Axiom $3b$, $df(i,w) = \mu_i \sum_{j\in w} d(p_{ji}) + \mu_i \sum_{j\in w} d(p_{ij}) \Leftrightarrow \int df = \int \mu_i(\sum_{j\in w} dp_{ji} + \sum_{j\in w} d(p_{ij})) \Leftrightarrow f = C_i + \mu_i(\sum_{j\in w} p_{ji} + \sum_{j\in w} p_{ij})$. If all $p_{ij} = p_{ji} = 0, j \in w$, then, by the Axiom $2b$, $f(i,w) = 0 = C_i$. Thus, $f(i,w) = \mu_i(\sum_{j\in w} p_{ji} + \sum_{j\in w} p_{ij})$. Expect that condition $\forall\, i \in N \sum_{j\in N} p_{ij} = 1$ is violated. If we prove the uniqueness of function $f(i,w)$ for this case, then this function is unique for the case with condition $\forall\, i \in N \sum_{j\in N} p_{ij} = 1$. If all $p_{ij} = p_{ji} = 1, j \in w$, then, by monotonicity, $f(i,w) = 1$. Thus, $1 = \mu_i(\sum_{j\in w} p_{ji} + \sum_{j\in w} p_{ij}) = \mu_i 2(s-1) \Rightarrow f(i,w) = \frac{\sum_{j\in w} p_{ji} + \sum_{j\in w} p_{ij}}{2(|w|-1)} = \frac{1}{2}\left(f^+(i,w) + f^-(i,w)\right)$.

4. It can be measured the value $\frac{\sum_{i\in w} f(i,w)}{|w|}$ for each $r(r$-1)-tuple $(p_{12},\dots,p_{1r},p_{21},\dots,p_{r1,},\dots,p_{r\,r-1})$. Thus, $f_{avg}(w)$ satisfies the Axiom $1c$. If in $r(r$-1)-tuple $(p_{12},\dots,p_{1r},p_{21},\dots,p_{r1,},\dots,p_{r\,r-1})$ all $p_{ij}=0$, $i,j\in w$, then $f_{avg}(w)=\frac{0}{r}=0$. Thus, $f_{avg}(w)$ satisfies the Axiom $2c$. Since the function $f_{avg}(w)$ is linear with respect to arguments $p_{12},\dots,p_{1r},p_{21},\dots,p_{r1,},\dots,p_{r\,r-1}$, then this function satisfies the Axiom $3c$.

Consider an arbitrary coalition $w=\{1,\dots,r\}$, an agent $i\in w$, and $r(r$-1)-tuple $(p_{12},\dots,p_{1r},p_{21},\dots,p_{r1,},\dots,p_{r\,r-1})$. By the Axiom $1c$, it exists $f(w)$ for the tuple $(p_{12},\dots,p_{1r},p_{21},\dots,p_{r1,},\dots,p_{r\,r-1})$, where $f(w)$ – continuous differentiable function of each argument, $0\le f(i,w)\le 1$. Consider a total differential $df(w)$. By the Axiom $3c$, $df(w)=\mu_w\sum_{i,j\in w} d(p_{ij})\Leftrightarrow \int df=\int \mu_w\left(\sum_{i,j\in w} dp_{ij}\right)=\mu_w\sum_{i,j\in w}\int dp_{ij}\Leftrightarrow f=C_w+\mu_w\left(\sum_{i,j\in w} p_{ij}\right)$. If all $p_{ij}=0$, $i,j\in w$, then, by the Axiom $2c$, $f(w)=0=C_w$. Thus, $f(w)=\mu_w\sum_{i,j\in w} p_{ji}$. Expect that condition $\forall\, i\in N\sum_{j\in N} p_{ij}=1$ is violated. If we prove the uniqueness of function $f(w)$ for this case, then this function is unique for the case with condition $\forall\, i\in N\ \sum_{j\in N} p_{ij}=1$. If all $p_{ij}=1$, $i,j\in w$, then, by monotonicity, $f(w)=1$. Thus, $1=\mu_w\left(\sum_{i,j\in w} p_{ji}\right)=\mu_w r(r-1)\Rightarrow f(w)=\frac{\sum_{i,j\in w} p_{ji}}{|w|(|w|-1)}=\frac{\sum_{i\in w} f_{avg}(i,w)}{|w|}$.

Note that all considered linear functions of intensity can be presented by the axioms with slightly changed conditions. The list of conditions is given in Table 1, where “+” means that a function satisfies chosen axiom.

Table 1: List of conditions

Function/axiom	**1**	$\mathbf{1}\boldsymbol{a}$	$\mathbf{1}\boldsymbol{b}$	$\mathbf{1}\boldsymbol{c}$	**2**	$\mathbf{2}\boldsymbol{a}$	$\mathbf{2}\boldsymbol{b}$	$\mathbf{2}\boldsymbol{c}$	**3**	$\mathbf{3}\boldsymbol{a}$	$\mathbf{3}\boldsymbol{b}$	$\mathbf{3}\boldsymbol{c}$
$f^+(i,w)$	+				+				+			
$f^-(i,w)$		+				+				+		
$f_{avg}(i,w)$			+				+				+	
$f_{avg}(w)$				+				+				+

3. Example

Assume that $N = \{A, B, C, D\}$, $p_{AB} = 0.4, p_{AC} = 0.5$, $p_{AD} = 0.1$, $p_{BA} = 0.2, p_{BC} = 0.5, p_{BD} = 0.3$ $p_{CA} = 0.6, p_{CB} = 0.2$, $p_{CD} = 0.2$, $p_{DA} = 0.2, p_{DB} = 0.2, p_{DC} = 0.6$. Consider the coalition $\dot{w} = \{A, B, C\}$ and evaluate the values of linear functions for this coalition and each agent in $\dot{w}$.

$$f^{+}(A,\dot{w}) = \frac{\sum_{j\in\dot{w}} p_{ij}}{|\dot{w}|-1} = \frac{p_{AB}+p_{AC}}{2} = \frac{0.9}{2} = 0.45$$

$$f^{-}(A,\dot{w}) = \frac{\sum_{j\in\dot{w}} p_{ji}}{|\dot{w}|-1} = \frac{p_{BA}+p_{CA}}{2} = \frac{0.8}{2} = 0.4$$

$$f^{+}(B,\dot{w}) = \frac{\sum_{j\in\dot{w}} p_{ij}}{|\dot{w}|-1} = \frac{p_{BA}+p_{BC}}{2} = \frac{0.7}{2} = 0.35$$

$$f^{-}(B,\dot{w}) = \frac{\sum_{j\in\dot{w}} p_{ji}}{|\dot{w}|-1} = \frac{p_{AB}+p_{CB}}{2} = \frac{0.6}{2} = 0.3$$

$$f^{+}(C,\dot{w}) = \frac{\sum_{j\in\dot{w}} p_{ij}}{|\dot{w}|-1} = \frac{p_{CA}+p_{CB}}{2} = \frac{0.8}{2} = 0.4$$

$$f^{-}(C,\dot{w}) = \frac{\sum_{j\in\dot{w}} p_{ji}}{|\dot{w}|-1} = \frac{p_{AC}+p_{BC}}{2} = \frac{1}{2} = 0.5$$

$$f_{avg}(A,\dot{w}) = \frac{1}{2}\left(f^{+}(A,\dot{w}) + f^{-}(A,\dot{w})\right) = \frac{1}{2}(0.45+0.4) = 0.425$$

$$f_{avg}(B,\dot{w}) = \frac{1}{2}\left(f^{+}(B,\dot{w}) + f^{-}(B,\dot{w})\right) = \frac{1}{2}(0.35+0.3) = 0.325$$

$$f_{avg}(C,\dot{w}) = \frac{1}{2}\left(f^{+}(C,\dot{w}) + f^{-}(C,\dot{w})\right) = \frac{1}{2}(0.4+0.5) = 0.45$$

$$f^{+}(\dot{w}) = f^{-}(\dot{w}) = f_{avg}(\dot{w}) = \frac{\sum_{i\in\dot{w}} f_{avg}(i,\dot{w})}{|\dot{w}|} = \frac{1.2}{3} = 0.4$$

4. Conclusion

In this article, we consider linear functions of intensity for cardinal indices. It is constructed the axiomatization for each linear function and proved the theorem. It is shown that the axiomatization for all linear functions can be defined by the axioms with slightly changed conditions.

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Literature

Aleskerov F. Power indices taking into account agents' preferences // Mathematics and democracy. Springer, Berlin, Heidelberg, 2006. P. 1–18.

Aleskerov F., Holler J.M., Kamalova R. Power distribution in the Weimar Reichstag in 1919–1933 // Annals of Operations Research. 2014. Vol. 215. No. 1. P. 25–37.

Banzhaf III, J.F. Weighted voting doesn't work: A mathematical analysis. Rutgers Law Review, 1965, 19, 317.

Brams S. Game Theory and Politics. N.Y.: The Free Press, 1975.

Dubey P., Shapley L.S. Mathematical Properties of the Banzhaf Power Index // Mathematics of Operations Research. 1979. P. 99–131.

Felsenthal D., Machover M. The Measurment of Voting Power: Theory and Practices, Problems and Paradoxes. Cheltenham, UK: Edgar Elgar Publishing House, 1998.

Intriligator M.D. A probabilistic model of social choice // The Review of Economic Studies. 1973. Vol. 40. Iss. 4. P. 553–560.

Laruelle A., Valenciano F. Shapley-Shubik and Banzhaf Indices Revisited // Mathematics of Operations Research. 2000. Vol. 26. No. 1. P. 89–104.

Sokolova A. Modified power indices taking into account agents' preferences to coalesce // Modelirovanie v social'no-politicheskoj sfere. 2009. No. 3. P. 41–46. (In Russian)

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Индексы влияния используются для оценки влияния участников в выборных органах. Наиболее известные индексы влияния не учитывают предпочтения участников по созданию коалиций. Индексы влияния, учитывающие предпочтения участников по созданию коалиций, были введены в (Aleskerov, 2006). Данные индексы основаны на функциях интенсивности связи участников, которые могут быть разделены на две группы: линейные и нелинейные функции. В данной работе рассматриваются линейные функции интенсивности и вводится аксиоматика для них.

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