On the Stability of Random Matrix Product with Markovian Noise: Application to Linear Stochastic Approximation and TD Learning

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Abstract

This paper studies the exponential stability of random matrix products driven by a general (possibly unbounded) state space Markov chain. It is a cornerstone in the analysis of stochastic algorithms in machine learning (e.g. for parameter tracking in online-learning or reinforcement learning). The existing results impose strong conditions such as uniform boundedness of the matrix-valued functions and uniform ergodicity of the Markov chains. Our main contribution is an exponential stability result for the p-th moment of random matrix product, provided that (i) the underlying Markov chain satisfies a super-Lyapunov drift condition, (ii) the growth of the matrix-valued functions is controlled by an appropriately defined function (related to the drift condition). Using this result, we give finite-time p-th moment bounds for constant and decreasing stepsize linear stochastic approximation schemes with Markovian noise on general state space. We illustrate these findings for linear value-function estimation in reinforcement learning. We provide finite-time p-th moment bound for various members of temporal difference (TD) family of algorithms.

Keywords: stability of random matrix product, linear stochastic approximation, Markov chains, TD-learning

1. Introduction

Consider the following linear stochastic approximation (LSA) recursion: for $n \in \mathbb{N}$,

$$\theta_{n+1} = \theta_n + \alpha_{n+1}\{-\bar{A}(Z_{n+1})\theta_n + \bar{b}(Z_{n+1})\} , \tag{1}$$

where $(\alpha_i)_{i\in\mathbb{N}^*}$ is a sequence of positive step sizes, $\bar{A} : \mathsf{Z} \to \mathbb{R}^{d\times d}$, $\bar{b} : \mathsf{Z} \to \mathbb{R}^d$ are measurable functions on the state space Z, and $(Z_i)_{i\in\mathbb{N}^*}$ is a sequence of random variables on Z. The LSA recursion (1) encompasses a wide range of algorithms. LSA is central to the analysis of identification algorithms and control of linear systems. Early results have focused on these two applications and

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studied both the asymptotic behaviour of the sequence $(\theta_n)_{n\in\mathbb{N}}$ and the tracking error; see Eweda and Macchi (1983); Guo (1994); Guo and Ljung (1995b); Ljung (2002) and the references therein.

LSA is also a cornerstone in the analysis of linear value-function estimation (LVE) that are popular in reinforcement learning (Sutton, 1988; Bertsekas and Tsitsiklis, 1996). Seminal works on this topic (Bertsekas and Tsitsiklis, 1996; Tsitsiklis and Van Roy, 1997; Benveniste et al., 1990) established conditions for asymptotic convergence. Finite-time bound for LVE (and more generally LSA) has attracted a renewed interest. In the case when $(Z_i)_{i\in\mathbb{N}^*}$ is an i.i.d. sequence, (Lakshminarayanan and Szepesvari, 2018; Dalal et al., 2018) have investigated mean-squared error bounds for LSA. Recent developments (Bhandari et al., 2018; Srikant and Ying, 2019; Chen et al., 2020) have considered the setting that $(Z_i)_{i\in\mathbb{N}^*}$ is a Markov chain, and provided finite-time analysis. On a related subject, (Gupta et al., 2019; Xu et al., 2019; Doan, 2019; Kaledin et al., 2020) considered linear two-timescale stochastic approximation that involves coupled LSA recursions.

Most of the existing results on LSA are limited by strong conditions such as (i) uniform geometric ergodicity (UGE) on the Markov chain and/or (ii) uniformly bounded $\bar{A}, \bar{b}$, i.e. $\sup_{z\in\mathsf{Z}}\{\|\bar{A}(z)\| + \|\bar{b}(z)\|\} < +\infty$. These conditions are restrictive since the UGE condition typically requires the state space to be finite or compact and do not extend to general (unbounded) state space. This is of course a limitation because many applications involve general unbounded state space; see e.g. Ljung (2002) and (Bertsekas and Tsitsiklis, 1996, p. 305).

In this paper, we aim to provide high-order moment bounds on the LSA with Markovian noise. Our results are applicable under the relaxed conditions: (i) $(Z_i)_{i\in\mathbb{N}^*}$ is a Markov chain on a general (possibly unbounded) state-space satisfying a super-Lyapunov drift condition, and (ii) for some constant $C \geq 0$, for any $z \in \mathsf{Z}$, $\|\bar{A}(z)\| \leq C\mathrm{W}_1(z)$, $\|\bar{b}(z)\| \leq C\mathrm{W}_2(z)$, with $\mathrm{W}_1, \mathrm{W}_2 : \mathbb{R}_+ \to [1, +\infty)$ deduced from the drift condition in (i). They are strictly weaker than the conditions required in previously reported works. In particular, $\bar{A}, \bar{b}$ can be potentially unbounded.

For $m, n \in \mathbb{N}$, $m < n$ and $z_{m+1:n} = (z_{m+1}, \ldots, z_n) \in \mathsf{Z}^{n-m}$, we define

$$\Gamma_{m+1:n}(z_{m+1:n}) = \prod_{i=m+1}^{n}\{\mathrm{I}_d - \alpha_i \bar{A}(z_i)\} .$$

A key property used for deriving our bounds is an exponential stability result on the matrix product above, $\Gamma_{m+1:n}(Z_{m+1:n})$, for $m, n \in \mathbb{N}$, $m < n$. To motivate why this is relevant to LSA, suppose that the Markov chain $(Z_n)_{n\in\mathbb{N}^*}$ is ergodic so that, for all $z \in \mathsf{Z}$, the following limits $A = \lim_{n\to\infty}\mathbb{E}_z[\bar{A}(Z_n)]$, $b = \lim_{n\to\infty}\mathbb{E}_z[\bar{b}(Z_n)]$ exist. Assume in addition that the limiting matrix $-A$ is *Hurwitz*, i.e. the real parts of its eigenvalues are strictly negative, and denote by $\theta^\star$ the unique solution of the linear system $A\theta^\star = b$. The n-th error vector $\tilde{\theta}_n = \theta_n - \theta^\star$ may be expressed, for all $n \in \mathbb{N}$, by

$$\tilde{\theta}_n = \sum_{j=1}^{n} \alpha_j \Gamma_{j+1:n}(Z_{j+1:n})\bar{\varepsilon}(Z_j) + \Gamma_{1:n}(Z_{1:n})\tilde{\theta}_0 , \tag{2}$$

where $\bar{\varepsilon}(Z_j) = \bar{b}(Z_j) - b - \{\bar{A}(Z_j) - A\}\theta^\star$. Obtaining a bound on p-th moments for $\{\|\tilde{\theta}_n\|\}_{n\in\mathbb{N}}$ naturally requires that the sequence of random matrices $\{\bar{A}(Z_i)\}_{i\in\mathbb{N}^*}$ to be (V, q)-*exponentially stable*. Recall that for $q \geq 1$ and a function $\mathrm{V} : \mathsf{Z} \to [1, \infty)$, $\{\bar{A}(Z_i)\}_{i\in\mathbb{N}^*}$ is said to be (V, q)-exponentially stable if there exists $\mathsf{a}_q, \mathrm{C}_q > 0$ and $\alpha_{\infty,q} < \infty$ such that, for any sequence of positive step sizes $(\alpha_i)_{i\in\mathbb{N}^*}$ satisfying $\sup_{i\in\mathbb{N}^*}\alpha_i \leq \alpha_{\infty,q}$, $z \in \mathsf{Z}$, $m, n \in \mathbb{N}$, $m < n$,

$$\mathbb{E}_z[\|\Gamma_{m+1:n}(Z_{m+1:n})\|^q] \leq \mathrm{C}_q \exp\left(-\mathsf{a}_q \sum_{i=m+1}^{n} \alpha_i\right) \mathrm{V}(z) . \tag{3}$$

Intuitively, (V, q)-exponential stability means that the q-th moment of the product of random matrices $\Gamma_{m+1:n}(Z_{m+1:n})$ behaves similarly to that of the product of *deterministic* matrices $G_{m+1:n} = \prod_{i=m+1}^{n}(\mathrm{I}_d - \alpha_i A)$, under the assumption that $-A$ is Hurwitz.

Fix $p, q, r \in \mathbb{N}^*$ such that $p^{-1} = q^{-1} + r^{-1}$. Assume that the sequence $\{\bar{A}(Z_i)\}_{i\in\mathbb{N}^*}$ is (V, q)-exponentially stable for some $q > 1$, the r-th moments of the noise term $\|\bar{\varepsilon}(Z_n)\|$ and initialization error $\tilde{\theta}_0$ are bounded. Using (2), we can readily derive bounds for the p-th moment, $\mathbb{E}_z^{1/p}[\|\tilde{\theta}_n\|^p]$ by applying the Hölder's inequality. Note that the r-th moment bound for the "noise" terms may follow from classical Lyapunov drift conditions, which is implied by super-Lyapunov drift conditions.

Contributions and Organization The contributions of this paper are three-fold:

- We establish (V, q)-exponential stability of the sequence of matrices $\{\bar{A}(Z_k)\}_{k\in\mathbb{N}^*}$, and provide explicit expression for constants appearing in (3); see Theorem 1. Compared to the prior works, our result can be applied to the settings where the function $\bar{A}(\cdot)$ is unbounded, not symmetric and $(Z_k)_{k\in\mathbb{N}^*}$ is a Markov chain on a general (unbounded) state-space not constrained to be uniformly geometrically ergodic. A discussion of how our results relax the restrictive conditions in previously reported works is given after the statement of Theorem 1.
- We provide finite-time bound and first-order expansion for the p-th moment of the error $(\tilde{\theta}_n)_{n\in\mathbb{N}^*}$ for LSA recursion (2). More precisely, we show that $\mathbb{E}_z^{1/p}[\|\tilde{\theta}_n\|^p] = \mathcal{O}(\alpha_n^{1/2})\,\mathrm{V}_p(z)$ both for constant $\alpha_n \equiv \alpha$ (where α is sufficiently small) or nonincreasing stepsizes under weak additional conditions including $\alpha_n = C/(n + n_0)^{\mathrm{t}}$, for any $\mathrm{t} \in (0, 1]$; see Theorem 3. From our analysis on the LSA error $\tilde{\theta}_n$, we identify a leading term, denoted $J_n^{(0)}$, which is a weighted additive linear functional of the error process $(\bar{\varepsilon}(Z_n))_{n\in\mathbb{N}^*}$. Furthermore, the leading term $J_n^{(0)}$ and its remainder $H_n^{(0)} = \tilde{\theta}_n - J_n^{(0)}$ admit a separation of scales. For example, when $\alpha_n = C/(n+n_0)$, the leading term has a p-th moment bound of $\mathcal{O}(n^{-1/2})\,\mathrm{V}_p(z)$, and the remainder has a p-th moment bound of $\mathcal{O}(n^{-1}\log(n))\,\mathrm{V}_p(z)$; see Theorem 4.
- Finally, we apply our results to TD-learning for LVE. We give sufficient conditions for a Markov Reward Process on general unbounded state space with unbounded reward and feature functions to satisfy the assumptions of Theorem 3 and Theorem 4. Therefore, the convergence bounds we derive hold for these algorithms.

The rest of this paper is organized as follows. Section 2 introduces the formal conditions required for (V, q)-exponential stability on $\{\bar{A}(Z_k)\}_{k\in\mathbb{N}^*}$ and states our main theorem. Section 2.1 outlines the major steps in the proof. We use this result in Section 3 to obtain upper bound on the p-th moments for the error vector (2); finally, we illustrate our results for LVE in TD learning framework.

Notations Denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Let $d \in \mathbb{N}^*$ and Q be a symmetric positive definite $d \times d$ matrix. Denote by I_d the d-dimensional identity matrix. For $x \in \mathbb{R}^d$, we denote $\|x\|_Q = \{x^\top Q x\}^{1/2}$. For brevity, we set $\|x\| = \|x\|_{\mathrm{I}_d}$. We denote $\|A\|_Q = \max_{\|x\|_Q=1} \|Ax\|_Q$, and the subscriptless norm $\|A\| = \|A\|_{\mathrm{I}}$ is the standard spectral norm. Let $A_1, \ldots, A_N$ be d-dimensional matrices. We denote $\prod_{\ell=i}^{j} A_\ell = A_j \ldots A_i$ if $i \leq j$ and with the convention $\prod_{\ell=i}^{j} A_\ell = \mathrm{I}_d$ if $i > j$.

Throughout this paper, we let Z be a Polish space equipped with sigma-algebra $\mathcal{Z}$ and fix a measurable function $V : \mathsf{Z} \to [1, \infty)$. For a measurable function $g : \mathsf{Z} \to \mathbb{R}$, we define its V-norm as $\|g\|_V = \sup_{z\in\mathsf{Z}} |g(z)|/V(z)$. Furthermore, L_∞^V denotes the set of all measurable functions $g : \mathsf{Z} \to \mathbb{R}$ satisfying $\|g\|_V < \infty$. Let $\mathrm{P} : \mathsf{Z} \times \mathcal{Z} \to \mathbb{R}_+$ be a Markov kernel and $V : \mathsf{Z} \to \mathbb{R}_+$ be a measurable function, the function $\mathrm{P}V : \mathsf{Z} \to \mathbb{R}_+$ is defined as $\mathrm{P}V(z) = \int_{\mathsf{Z}} V(z')\mathrm{P}(z, \mathrm{d}z')$. For a measure μ on $(\mathsf{Z}, \mathcal{Z})$ and a function $V : \mathsf{Z} \to \mathbb{R}_+$ we define $\|\mu\|_V = \sup_{f:\|f\|_V \leq 1} \int_{\mathsf{Z}} f(z)\mu(\mathrm{d}z)$. Let $m \in \mathbb{N}^*$, ν a probability on $\mathcal{Z}$ and ϵ. A set $\mathsf{C} \in \mathcal{Z}$ is said to be $(m, \epsilon\nu)$-*small* for P if for all $z \in \mathsf{C}$ and $\mathsf{A} \in \mathcal{Z}$, $\mathrm{P}^m(z, \mathsf{A}) \geq \epsilon\nu(\mathsf{A})$. A set $\mathsf{A} \in \mathcal{Z}$ is said to be *accessible* if for all $z \in \mathsf{Z}$, there exists $m \in \mathbb{N}^*$ such that $\mathrm{P}(z, \mathsf{A}) > 0$.

2. Main Results

Consider a Markov chain $(Z_k)_{k\in\mathbb{N}}$ with Markov kernel P. We assume without loss of generality that $(Z_k)_{k\in\mathbb{N}}$ is the canonical process corresponding to P on $(\mathsf{Z}^{\mathbb{N}}, \mathcal{Z}^{\otimes\mathbb{N}})$. We denote by $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ the corresponding probability distribution and expectation with initial distribution μ. By construction, for any $\mathsf{A} \in \mathcal{Z}$, $\mathbb{P}_\mu\left(Z_k \in \mathsf{A} \,|\, Z_{k-1}\right) = \mathrm{P}(Z_{k-1}, \mathsf{A})$, $\mathbb{P}_\mu$-a.s. In the case $\mu = \delta_z$, $z \in \mathsf{Z}$, $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ are denoted by $\mathbb{P}_z$ and $\mathbb{E}_z$. In addition, throughout this paper, we assume

UE1 *The Markov kernel $\mathrm{P} : \mathsf{Z} \times \mathcal{Z} \to \mathbb{R}_+$ is irreducible and aperiodic. There exist $c > 0, \mathrm{b} > 0, \delta \in (1/2, 1]$, $R_0 \geq 0$, and $V : \mathsf{Z} \to [\mathrm{e}, \infty)$ such that by setting $W = \log V$, $\mathsf{C}_0 = \{z : W(z) \leq R_0\}$, $\mathsf{C}_0^\complement = \{z : W(z) > R_0\}$, we have*

$$\mathrm{P}V(z) \leq \exp[-cW^\delta(z)]V(z)\mathbb{1}_{\mathsf{C}_0^\complement}(z) + \mathrm{b}\,\mathbb{1}_{\mathsf{C}_0}(z)\ . \tag{4}$$

In addition, for any $R \geq 1$, the level sets $\{z : W(z) \leq R\}$ are $(m_R, \varepsilon_R\nu)$-small for P, with $m_R \in \mathbb{N}^*$, $\varepsilon_R \in (0, 1]$ and ν being a probability measure on $(\mathsf{Z}, \mathcal{Z})$.

Since $(\mathsf{Z}, \mathcal{Z})$ is a general state-space, irreducibility here means that the Markov kernel P admits an accessible small set; see (Douc et al., 2018, Chapter 9). The drift condition (4) in UE 1 is referred to as a multiplicative or super-Lyapunov drift condition and plays a key role in studying the large deviations of additive functionals of Markov chains; see Varadhan (1984). Eq. (4) implies the classical Foster-Lyapunov drift condition, $\mathrm{P}V(z) \leq \lambda V(z) + \mathrm{b}\,\mathbb{1}_{\mathsf{C}_0}(z)$ with

$$\lambda = \exp(-c \inf_{\mathsf{C}_0^\complement} W^\delta) \leq \exp(-c) < 1\ . \tag{5}$$

It follows from (Douc et al., 2018, Theorem 15.2.4) that under UE 1 the Markov kernel P is V-uniformly geometrically ergodic and admits a unique stationary distribution π, i.e. there exists $\rho \in (0, 1)$ and $\mathrm{B}_V < \infty$ such that for each $z \in \mathsf{Z}$ and $n \in \mathbb{N}$,

$$\|\mathrm{P}^n(z, \cdot) - \pi\|_V \leq \mathrm{B}_V\rho^n V(z)\ . \tag{6}$$

UE 1 is a special case of condition **(DV3)** in Kontoyiannis and Meyn (2003, 2005) which plays a key role in multiplicative regularity of Markov chains. A key consequence of UE 1 is a bound for products (see Lemma 10 and (Kontoyiannis and Meyn, 2005, Theorem 1.2)): for any $z \in \mathsf{Z}$, $n \in \mathbb{N}$, and non-increasing sequence $(\alpha_i)_{i\in\mathbb{N}^*} \subset [0, 1]$, we get

$$\mathbb{E}_z[\exp\{c\textstyle\sum_{k=0}^{n-1} \alpha_k W^\delta(Z_k)\}] \leq \exp\{\tilde{\mathrm{b}}\textstyle\sum_{k=0}^{n-1} \alpha_k\}\exp\{\alpha_1 W(z)\}\ ,$$

where $\tilde{\mathrm{b}} = \log \mathrm{b} + \sup_{r\geq \mathrm{e}}\{cr^\delta - r\}$ and c is defined in (4). UE 1 is satisfied with $\delta = 1$ for Gaussian linear vector auto-regressive process and also non-linear auto-regressive process under exponential moment condition for innovation process, see e.g. Priouret and Veretenikov (1998).

We also impose some constraints on $\bar{A}$. For $\varepsilon \in (0, 1)$ consider the following assumptions

A1 (ε) *There exists $\mathrm{C}_A > 0$ such that for any $1 \leq i, j \leq d$, the (i, j)-th element of $\bar{A}$ satisfies $\left\|[\bar{A}]_{i,j}\right\|_{W^\beta} \leq \mathrm{C}_A$, where $\beta < \min(2\delta - 1, \delta/(1 + \varepsilon))$ and δ is given in UE 1.*

To simplify notations the dependence of constants C_A and β in ε is implicit. Whenever there is no ambiguity, we drop the dependence on ε in A 1.

A2 *The square matrix $-A = -\mathbb{E}_\pi[\bar{A}(Z_0)]$ is Hurwitz.*

A1, A2 are standard conditions on the parameter matrices in LSA. Under A2, there exists a positive definite matrix Q satisfying the Lyapunov equation [cf. Lemma 17]

$$A^{\top} Q + QA = \mathrm{I}_d, \quad \text{and we define} \quad \kappa_{\mathsf{Q}} = \lambda_{\mathsf{min}}^{-1}(Q)\lambda_{\mathsf{max}}(Q), \;\; a = \|Q\|^{-1}/2. \tag{7}$$

Consequently, we have $\|\mathrm{I} - \alpha A\|_Q \leq 1 - a\alpha/2$ for $\alpha \in [0, \|Q\|^{-1}\|A\|_Q^{-2}/2]$ [cf. Lemma 18].

Our aim is to establish (V, q)-exponential stability of the sequence $\{\bar{A}(Z_k)\}_{k\in\mathbb{N}^*}$; see Equation (3). The following example illustrates that, even if the function $\bar{A}(\cdot)$ is bounded, for the matrix product to be exponentially stable, it is necessary for the Markov chain $(Z_k)_{k\in\mathbb{N}}$ to be geometrically ergodic.

Example 1 *Set $\mathsf{Z} = \mathbb{N}^\star$ and consider the forward recurrence time chain on Z starting from $Z_0 = 1$ and defined based on an i.i.d. sequence $(\mathrm{Y}_i)_{i\in\mathbb{N}}$, $\mathrm{Y}_i \in \mathsf{Z}$ by $Z_{k+1} = Z_k - 1$, if $Z_k > 1$ and $Z_{k+1} = \mathrm{Y}_{k+1}$, if $Z_k = 1$. Douc et al. (2018, Proposition 8.1.5) shows that if $\mathbb{P}(\mathrm{Y}_1 = z) > 0$ for $z \in \mathsf{Z}$ and $m = \sum_{z\in\mathsf{Z}} z\mathbb{P}(\mathrm{Y}_1 = z) < +\infty$, then $(Z_k)_{k\in\mathbb{N}}$ admits a unique stationary distribution π. For any $\varepsilon > 0$, set $\bar{A}_\varepsilon(1) = 1$, and $\bar{A}_\varepsilon(z) = -\varepsilon$ for $z \in \mathsf{Z} \setminus \{1\}$. If $\varepsilon \in (0, \pi(1))$ then $\sum_{z\in\mathsf{Z}} \pi(z)A_\varepsilon(z) = \pi(1) - \epsilon\{1 - \pi(1)\} > 0$, so that both conditions A1, A2 are satisfied.*

Consider the sequence defined recursively as $\theta_{n+1}^\varepsilon = \{1 - \alpha\bar{A}_\varepsilon(Z_{n+1})\}\theta_n^\varepsilon$ with $\theta_0^\varepsilon > 0$. Assume that the distribution of Y_1 does not have exponential moments (i.e. , for all $\eta > 0$, $\mathbb{E}[(1+\eta)^{\mathrm{Y}_1}] = \infty$). We show in Appendix A that (i) $(Z_k)_{k\in\mathbb{N}}$ is not geometrically ergodic for any $\varepsilon \in (0, \pi(1))$ and $\alpha \in (0, 1)$ and (ii) the sequence $u_n = \mathbb{E}[|\theta_n^\varepsilon|] = \theta_0\mathbb{E}[\prod_{k=0}^{n-1}\{1 - \alpha\bar{A}_\varepsilon(Z_{k+1})\}]$ is not bounded.

The following theorem establishes the (V, p)-exponential stability of the sequence $\{\bar{A}(Z_k)\}_{k\in\mathbb{N}^*}$. For ease of notation, we simply denote $\Gamma_{m+1:n} = \Gamma_{m+1:n}(Z_{m+1:n})$.

Theorem 1 *For $\varepsilon \in (0, 1)$ assume UE1, A1(ε) and A2. Then for any $p \geq 1$, there exists $\alpha_{\infty,p} > 0$, given in (90), such that for any non-increasing sequence $(\alpha_k)_{k\in\mathbb{N}^*}$ satisfying $\alpha_1 \in (0, \alpha_{\infty,p})$, $z_0 \in \mathsf{Z}$ and $m, n \in \mathbb{N}$, $m < n$, it holds*

$$\mathbb{E}_{z_0}^{1/p}[\|\Gamma_{m+1:n}\|^p] \leq \mathrm{C}_{\mathsf{st},p}\,\mathrm{e}^{-(a/4)\sum_{\ell=m+1}^{n}\alpha_\ell}V^{1/2p}(z_0)\;, \tag{8}$$

where a, $\mathrm{C}_{\mathsf{st},p}$, and h are defined in (7), (92), and (89), respectively.

The theorem shows that provided $(\alpha_k)_{k\in\mathbb{N}^*}$ satisfies $\sum_{k\in\mathbb{N}^*}\alpha_k = +\infty$, $\mathbb{E}_z^{1/p}[\|\Gamma_{m+1:n}\|^p] \to 0$ as $(n - m) \to \infty$ for any $p \geq 1$. Specifically, it has a similar convergence rate as the deterministic matrix product $\|G_{m+1:n}\| = \|\prod_{i=m+1}^{n}(\mathrm{I}_d - \alpha_i A)\| \lesssim \mathrm{e}^{-a\sum_{\ell=m+1}^{n}\alpha_\ell}$.

Theorem 1 generalizes previously reported works. Guo (1994); Guo and Ljung (1995a) used a slightly different definitions allowing to consider non-Markovian processes satisfying more general mixing conditions (like ϕ- or β-mixing). As we will see later, when specialized to Markov chains, the results we obtain significantly improve those reported in these works. Priouret and Veretenikov (1998) established (V, q)-exponential stability for general state-space Markov chain under a super-Lyapunov drift condition (similar to UE1). However, the results in Priouret and Veretenikov (1998) assume constant stepsize and $\bar{A}(z)$ being symmetric and non-negative definite for any $z \in \mathsf{Z}$. Non-negative definiteness plays a key role in the arguments: in such case, for any $z \in \mathsf{Z}$, the spectral norm $\|\mathrm{I}_d - \alpha\bar{A}(z)\| \leq 1$ provided that $\|\bar{A}(z)\| \leq \alpha^{-1}$ for $\alpha > 0$ which is no longer true for general matrix-valued function $\bar{A}(z)$. Similar results, also under the condition that $\bar{A}(z)$ is symmetric for any $z \in \mathsf{Z}$, were obtained by Delyon and Juditsky (1999) based on perturbation theory for linear operators in Banach space and spectral theory. However, the bounds provided in Delyon and Juditsky (1999) are only qualitative and it is difficult to make these results quantitative because they are based on perturbation arguments of linear operators in Banach spaces. The restrictions imposed on these prior works have limited their applications to more general algorithms, in particular to most RL algorithms. As we will see below, the application to linear value-function estimation in temporal difference learning involve non-symmetric matrix function $\bar{A}$. In contrast, our result (cf. Theorem 1) can be applied to the setting where for some $z \in \mathsf{Z}$, $\bar{A}(z)$ is not necessary non-negative symmetric but only Hurwitz.

Notice that the case of uniformly geometric ergodic Markov chain is covered by UE 1. In this case the whole state-space Z is small and the drift function V can be chosen to be constant (Douc et al., 2018, Theorem 15.3.1). Together with the assumption of bounded $\bar{A}(\cdot)$, the exponential stability of product of random matrices has been implicitly established in (Srikant and Ying, 2019; Doan, 2019; Kaledin et al., 2020; Chen et al., 2020). In particular, their results on LSA can be applied on the recursion $y_0 = y$, $y_{n+1} = \{\mathrm{I}_d - \alpha_{n+1}\bar{A}(Z_{n+1})\}y_n$, $n \in \mathbb{N}$. Through studying the decomposition:

$$y_{n+1} = \{\mathrm{I}_d - \alpha_{n+1} A\}y_n - \alpha_{n+1}(\bar{A}(Z_{n+1}) - A)y_n,\ \forall\, n \in \mathbb{N}, \tag{9}$$

they derived bounds on $\mathbb{E}_{z_0}[\|y_{n+1}\|^p] = \mathbb{E}_{z_0}[\|\Gamma_{1:n+1}y\|^p]$. However, generalizing this approach for other classes of Markov chains (e.g., UE 1) or unbounded function appears to be impossible.

2.1. Proof of Theorem 1

First note that for any $z_0 \in \mathsf{Z}$, by the Markov property,

$$\mathbb{E}_{z_0}[\|\Gamma_{m+1:n}\|^p] = \mathbb{E}_{z_0}[\|\Gamma_{m+1:n}(Z_{m+1:n})\|^p] = \mathbb{E}_{z_0}[\mathbb{E}_{Z_m}[\|\Gamma_{m+1:n}(Z_{1:n-m})\|^p]\ . \tag{10}$$

The first step is to fix some value $Z_m = z_m \in \mathsf{Z}$ and to derive a bound on $\mathbb{E}_{z_m}[\|\Gamma_{m+1:n}(Z_{1:n-m})\|^p]$. We denote by $\kappa = \kappa_{\mathsf{Q}}^{1/2}$ where κ_{Q} is defined in (7).

Step 1: Extracting the deterministic matrix product and a block decomposition Consider a block length $h \in \mathbb{N}$ [to be defined in (90)] and define the sequence $j_0 = m$, $j_{\ell+1} = \min(j_\ell + h, n)$ such that $j_{\ell+1} - j_\ell \le h$. Let $N = \lceil (n-m)/h \rceil$, where $\lceil \cdot \rceil$ is the ceiling function so that $j_\ell = j_N = n$ for any $\ell \ge N$. Then, we introduce the decomposition

$$\Gamma_{m+1:n}(Z_{1:n-m}) = \prod_{\ell=1}^{N} \overline{B}_\ell \quad \text{where} \quad \overline{B}_\ell := \prod_{i=j_{\ell-1}+1}^{j_\ell} (\mathrm{I}_d - \alpha_i \bar{A}(Z_{i-m})),\quad \ell \in \{0,\dots,N\}\ . \tag{11}$$

Using that $(Z_k)_{k\in\mathbb{N}}$ satisfies UE 1, it can be shown that if m is sufficiently large, then $\overline{B}_\ell$ is close in L^p to the deterministic matrix $B_\ell = \prod_{i=j_{\ell-1}+1}^{j_\ell}(\mathrm{I}_d - \alpha_i A)$. However, it is not sufficient to conclude because we need to deal with the product of these terms in (11). Therefore, we consider

$$\|\Gamma_{m+1:n}(Z_{1:n-m})\| \le \kappa \|\Gamma_{m+1:n}(Z_{1:n-m})\|_Q \le \kappa \textstyle\prod_{\ell=1}^{N}\{\|B_\ell\|_Q + \|B_\ell - \overline{B}_\ell\|_Q\}, \tag{12}$$

where the last inequality follows from $\overline{B}_\ell = B_\ell + (\overline{B}_\ell - B_\ell)$. Using A2, we have

$$\begin{aligned}\|\Gamma_{m+1:n}(Z_{1:n-m})\| &\overset{(a)}{\le} \kappa \textstyle\prod_{\ell=1}^{N}\{\prod_{i=j_{\ell-1}+1}^{j_\ell}(1-\alpha_i a/2) + \|B_\ell - \overline{B}_\ell\|_Q\} \\ &\overset{(b)}{\le} \kappa\{1 + \kappa\|B_N - \overline{B}_N\|\} \textstyle\prod_{\ell=1}^{N-1}\{(1-\alpha_{j_\ell} a/2)^h + \kappa\|B_\ell - \overline{B}_\ell\|\}\ ,\end{aligned}$$

where (a) is due to Lemma 18 and we assumed that $\sup_{i\in\mathbb{N}^*}\alpha_i \le \alpha_{\infty,p} \le (1/2)\|A\|_Q^{-2}\|Q\|^{-1}$, and (b) is due to the assumption $\alpha_{i+1} \le \alpha_i$. Assuming $a\alpha_{\infty,p} \le 1$ and $h\alpha_{\infty,p} \le 1$, we get $(1-a\alpha_{j_\ell}/2)^{-h} \le \mathrm{e}^a$ since for any $t \in [0,1/2]$, $(1-t)^{-1} \le 1+2t \le \mathrm{e}^{2t}$, therefore, we obtain

$$\|\Gamma_{m+1:n}(Z_{1:n-m})\| \le \kappa \prod_{\ell=1}^{N-1}(1-a\alpha_{j_\ell}/2)^h \prod_{\ell'=1}^{N}\{1+\kappa\mathrm{e}^a\|\overline{B}_{\ell'}-B_{\ell'}\|\} .$$

Taking expectation leads to

$$\begin{aligned}\mathbb{E}_{z_m}^{1/p}[\|\Gamma_{m+1:n}(Z_{1:n-m})\|^p] &\le \kappa \prod_{\ell=1}^{N-1}(1-a\alpha_{j_\ell}/2)^h\, \mathbb{E}_{z_m}^{1/p}\Big[\prod_{\ell'=1}^{N}\{1+\kappa\mathrm{e}^a\|\overline{B}_{\ell'}-B_{\ell'}\|\}^p\Big] \\ &\le \kappa\mathrm{e}^{a\alpha_{\infty,p}h}\exp\left(-\frac{a}{2}\sum_{i=m+1}^{n}\alpha_i\right)\mathbb{E}_{z_m}^{1/p}\left[\prod_{\ell'=1}^{N}\{1+\kappa\mathrm{e}^a\|\overline{B}_{\ell'}-B_{\ell'}\|\}^p\right] , \end{aligned} \tag{13}$$

since $\prod_{\ell=1}^{N-1}(1-a\alpha_{j_\ell}/2)^h \le C\mathrm{e}^{-(a/2)\sum_{i=m+1}^n \alpha_i}$, with $C=\mathrm{e}^{a\alpha_{\infty,p}h}$, using $\sup_{i\in\mathbb{N}^*}\alpha_i \le \alpha_{\infty,p}$ and $(\alpha_i)_{i\in\mathbb{N}^*}$ is non-increasing. In order to complete the proof, our next step is to show that the last term in (13) grows in the order $\mathcal{O}(\mathrm{e}^{(a/4)\sum_{i=m+1}^n\alpha_i})$.

Step 2: Bounding the product of differences We now tackle the last term in (13). Note that for any sequence of square matrices $\{C_i\}_{i=1}^N$, $\prod_{i=1}^n\{\mathrm{I}+C_i\} = \sum_{r=0}^N\sum_{(i_1,\ldots,i_r)\in\mathsf{J}_r}\prod_{k=1}^r C_{i_k}$, where $\mathsf{J}_r = \{(i_1,\ldots,i_r)\in\{1,\ldots,N\}^r : i_1<\cdots<i_r\}$, with the convention $\prod_\emptyset = 1$. Using this expansion, we may therefore decompose the difference $\overline{B}_\ell - B_\ell$ as:

$$\overline{B}_\ell - B_\ell = S_\ell + R_\ell - \bar{R}_\ell , \tag{14}$$

where $S_\ell = \sum_{k=j_{\ell-1}+1}^{j_\ell}\alpha_k\{\bar{A}(Z_{k-m})-A\}$ is linear ($r=1$) and the remainders collect the higher-order terms ($r\ge 2$) in the products

$$\bar{R}_\ell = \sum_{r=2}^{h}(-1)^r\sum_{(i_1,\ldots,i_r)\in\mathsf{I}_r^\ell}\prod_{u=1}^r\alpha_{i_u}\bar{A}(Z_{i_u-m}),\ R_\ell = \sum_{r=2}^{h}(-1)^r\sum_{(i_1,\ldots,i_r)\in\mathsf{I}_r^\ell}\prod_{u=1}^r\alpha_{i_u}A^r, \tag{15}$$

where we have set $\mathsf{I}_r^\ell = \{(i_1,\ldots,i_r)\in\{j_{\ell-1}+1,\ldots,j_\ell\}^r : i_1<\cdots<i_r\}$. Since for $\{a_i\}_{i=1}^N \subset \mathbb{R}_+$, $(1+\sum_{i=1}^N a_i) \le \prod_{i=1}^N(1+a_i)$, the Hölder's inequality implies

$$\begin{aligned}&\mathbb{E}_{z_m}^{1/p}\Big[\prod_{\ell=1}^{N}\{1+\kappa\mathrm{e}^a\|\overline{B}_\ell-B_\ell\|\}^p\Big] \\ &\le \prod_{\ell=1}^N(1+\kappa\mathrm{e}^a\|R_\ell\|)\left\{\mathbb{E}_{z_m}\Big[\prod_{\ell=1}^N(1+\kappa\mathrm{e}^a\|\bar{R}_\ell\|)^{2p}\Big]\right\}^{1/(2p)}\left\{\mathbb{E}_{z_m}\Big[\prod_{\ell=1}^N(1+\kappa\mathrm{e}^a\|S_\ell\|)^{2p}\Big]\right\}^{1/(2p)} .\end{aligned} \tag{16}$$

Consider first the two terms involving $\{R_\ell,\bar{R}_\ell : \ell\in\{1,\ldots,N\}\}$. From (15), we observe that the order of terms in $R_\ell,\bar{R}_\ell$ is at least quadratic in the step size. As such, a crude estimate suffices to establish that the relevant terms in (16) grow slowly with N as shown in Lemmas 21 and 22 (postponed to the appendix):

$$\prod_{\ell=1}^N(1+\kappa\mathrm{e}^a\|R_\ell\|)^p \le \exp\left\{pC^{(0)}h^2\sum_{\ell=1}^N\alpha_{j_{\ell-1}+1}^2\right\} , \tag{17}$$

$$\mathbb{E}_{z_m}\Big[\prod_{\ell=1}^{N}(1+\kappa \mathrm{e}^{a}\|\bar{R}_\ell\|)^{2p}\Big] \leq \mathbb{E}_{z_m}\left[\exp\Big\{2pC^{(1)}2^h\sum_{\ell=1}^{N}\alpha_{j_{\ell-1}+1}^{1+\varepsilon}\sum_{k=j_{\ell-1}+1}^{j_\ell}W^\delta(Z_{k-m})\Big\}\right] , \quad (18)$$

where $C^{(0)}, C^{(1)}$ are defined in (81), (82), respectively. The exponents in (17), (18) are of the order $\mathcal{O}(\sum_{\ell=1}^{N}\alpha_{j_{\ell-1}+1}^{2})$, $\mathcal{O}(\sum_{\ell=1}^{N}\alpha_{j_{\ell-1}+1}^{1+\varepsilon})$, respectively, which are desirable for us.

However, similar crude estimates are not sufficient for controlling the last term of (16) which involves the linear term S_ℓ. We first apply the following useful bound (of independent interest):

Lemma 2 *(Lemma 20) Let $(\mathfrak{F}_\ell)_{\ell\geq 0}$ be some filtration and a sequence of non-negative random variables $(\xi_\ell)_{\ell\geq 0}$ which is $(\mathfrak{F}_\ell)_{\ell\geq 0}$-adapted. For any $P\in\mathbb{N}$, it holds*

$$\mathbb{E}\big[\prod_{\ell=1}^{P}\xi_\ell\big] \leq \big\{\mathbb{E}\big[\prod_{\ell=1}^{P}\mathbb{E}[\xi_\ell^2|\mathfrak{F}_{\ell-1}]\big]\big\}^{1/2} . \quad (19)$$

By the Markov property, the previous Lemma allow us to write:

$$\mathbb{E}_{z_m}^{1/(2p)}\Big[\prod_{\ell=1}^{N}\big(1+\kappa\mathrm{e}^{a}\|S_\ell\|\big)^{2p}\Big] \leq \mathbb{E}_{z_m}^{1/(4p)}\Big[\prod_{\ell=1}^{N}\mathbb{E}_{Z_{j_{\ell-1}}}[(1+\kappa\mathrm{e}^{a}\|S_\ell\|)^{4p}]\Big] . \quad (20)$$

Each of the conditional expectation on the r.h.s. can be controlled through studying the p-th moment of the linear statistics $\mathbb{E}_{Z_{j_{\ell-1}}}[\|S_\ell\|^{4p}]$. A tight bound can be obtained through applying the Rosenthal's inequalities derived in Appendix C. Formally, this is done by Corollary 24 in the appendix. Namely, for any $\ell=1,...,N$, it holds

$$\mathbb{E}_{Z_{j_{\ell-1}}}[(1+\kappa\mathrm{e}^{a}\|S_\ell\|)^{4p}] \leq \exp\big\{4pC_p^{(2)}h^{1/2}\alpha_{j_{\ell-1}+1}W^\delta(Z_{j_{\ell-1}})\big\} , \quad (21)$$

where $C_p^{(2)}$ is defined in (88). Note that the exponent on the r.h.s. has a sublinear growth rate with respect to the block size h. Combining (17)-(18)-(20)-(21) lead to the upper bound:

$$\mathbb{E}_{z_m}^{1/(2p)}\left[\prod_{\ell=1}^{N}\mathbb{E}_{Z_{j_{\ell-1}}}[(1+\kappa\mathrm{e}^{a}\|\overline{B}_\ell-B_\ell\|)^{2p}]\right] \leq \exp\Big\{C^{(0)}h^2\sum_{\ell=1}^{N}\alpha_{j_{\ell-1}+1}^{2}\Big\}\cdot T_1\cdot T_2 , \quad (22)$$

where T_1, T_2 are defined as

$$T_1 = \mathbb{E}_{z_m}^{1/(2p)}[\exp\{2pC^{(1)}2^h\textstyle\sum_{\ell=1}^{N}\alpha_{j_{\ell-1}+1}^{1+\varepsilon}\sum_{k=j_{\ell-1}+1}^{j_\ell}W^\delta(Z_{k-m})\}] ,$$
$$T_2 = \mathbb{E}_{z_m}^{1/(4p)}[\exp\{4pC_p^{(2)}h^{1/2}\textstyle\sum_{\ell=1}^{N}\alpha_{j_{\ell-1}+1}W^\delta(Z_{j_{\ell-1}-m})\}] .$$

Constructing an appropriately defined supermartingale (that we deduce from the super-Lyapunov drift condition) and assuming that $2^{h+1}pC^{(1)}\alpha_{\infty,p}^{1+\varepsilon}\leq c$, $4pC_p^{(2)}h^{1/2}\alpha_{\infty,p}\leq c$, in Lemmas 10 and 11 we show that T_1, T_2 can be bounded by

$$\begin{aligned} T_1 &\leq \exp\{C^{(1)}2^h(\alpha_{\infty,p}^{1+\varepsilon}W(z_m)+\tilde{\mathrm{b}}h\textstyle\sum_{\ell=1}^{N}\alpha_{j_{\ell-1}+1}^{1+\varepsilon})\} ,\\ T_2 &\leq \exp\{C_p^{(2)}h^{1/2}(\alpha_{\infty,p}W(z_m)+(\tilde{\mathrm{b}}-\log(1-\lambda))\textstyle\sum_{\ell=1}^{N}\alpha_{j_{\ell-1}+1})\} ,\end{aligned} \quad (23)$$

where $\tilde{\mathrm{b}} = \log\mathrm{b} + \sup_{r>0}\{cr^\delta - r\}$.

Step 3: Collecting Terms The proof is concluded by adjusting the block size and combining upper bounds on $\alpha_{\infty,p}$. The technical details are given in Appendix D.3.

3. Application to Linear Stochastic Approximation

This section illustrates how to apply Theorem 1 to analyze LSA schemes with Markovian noise. First, we state the assumptions on $\bar{b}(\cdot)$ and step sizes which can be either constant or diminishing. For $\mathsf{K} \in \mathbb{N}^*$, consider the following assumption:

A3 (K) *There exists* $\mathrm{C}_{b,\mathsf{K}} > 0$ *such that* $\max_{1 \leq \ell \leq d} \left\| \bar{b}_\ell \right\|_{V^{1/\mathsf{K}}} \leq \mathrm{C}_{b,\mathsf{K}}$*, where* $\bar{b}_\ell$ *is the* ℓ*-th component of* $\bar{b}$*.*

A4 *There exists a constant* $0 < \mathrm{c}_\alpha \leq a/16$ *such that for* $k \in \mathbb{N}$*,* $\alpha_k/\alpha_{k+1} \leq 1 + \alpha_{k+1}\,\mathrm{c}_\alpha$*.*

It is easy to check that A4 is satisfied by diminishing step sizes $\alpha_n = \mathrm{C}_a(n+n_0)^{-\mathrm{t}}$, $\mathrm{t} \in (0,1]$ and constant step sizes.

Theorem 3 *Let* $\mathsf{K} \geq 8$ *and* $\varepsilon \in (0,1)$*. Assume UE 1, A1(*ε*), A2 and A3(*K*). For any* $2 \leq p \leq \mathsf{K}/4$*, there exists* $\alpha^{(0)}_{\infty,p}$ *defined in (25) such that for any non-increasing sequence* $(\alpha_k)_{k\in\mathbb{N}^*}$ *satisfying* $\alpha_1 \in (0, \alpha^{(0)}_{\infty,p})$ *and A4,* $z \in \mathsf{Z}$*, and* $n \in \mathbb{N}$*, it holds*

$$\mathbb{E}_z^{1/p}[\|\tilde{\theta}_n\|^p] \leq \mathrm{M}_0\,\mathrm{C}_{\mathsf{st},2p}\,\mathrm{e}^{-(a/4)\sum_{\ell=1}^n \alpha_\ell} V^{1/(4p)}(z) + (\mathrm{C}^{(0)}_{\mathsf{J},p} + \mathrm{C}^{(0)}_{\mathsf{H},p})\sqrt{\alpha_n} V^{2/\mathsf{K}+1/(4p)}(z), \quad (24)$$

where $\mathrm{M}_0 = \mathbb{E}_z^{1/(2p)}[\|\tilde{\theta}_0\|^{2p}]$ *and* $\mathrm{C}^{(0)}_{\mathsf{J},p}$*,* $\mathrm{C}^{(0)}_{\mathsf{H},p}$ *are defined in (33), (36), respectively.*

Most often, the distribution of the initial value $\tilde{\theta}_0$ does not depend on the initial value of the Markov chain z. In this case $\mathbb{E}_z^{1/(2p)}[\|\tilde{\theta}_0\|^{2p}]$ is a constant. With a sufficiently small step size, Theorem 3 shows that the L_p norm of error vector converges under UE 1 for the Markov chain. Compared to (Srikant and Ying, 2019), we consider relaxed conditions on the Markov chain and allow for diminishing step sizes in the LSA.

Finite-time L_p error bound of LSA *[Proof of Theorem 3]* Define the following constraint on the step size

$$\alpha^{(0)}_{\infty,p} := \alpha_{\infty,2p} \wedge \rho \wedge \mathrm{e}^{-1}, \quad (25)$$

where $\alpha_{\infty,2p}$ and ρ are defined in (90) and (6) respectively. Below, we show that the finite-time L_p error bound can be derived through applying the stability of random matrix product (see Theorem 1). We recall that the error vector $\tilde{\theta}_{n+1} = \theta_{n+1} - \theta^\star$ may be expressed as

$$\tilde{\theta}_{n+1} = \Gamma_{1:n+1}\tilde{\theta}_0 + \textstyle\sum_{j=1}^{n+1} \alpha_j \Gamma_{j+1:n+1}\bar{\varepsilon}(Z_j) \equiv \tilde{\theta}^{(\mathsf{tr})}_{n+1} + \tilde{\theta}^{(\mathsf{fl})}_{n+1}\,. \quad (26)$$

Using the Hölder's inequality and Theorem 1, the transient term $\tilde{\theta}^{(\mathsf{tr})}_{n+1}$ can be bounded as follows

$$\mathbb{E}_z^{1/p}[\|\tilde{\theta}^{(\mathsf{tr})}_{n+1}\|^p] \leq \mathbb{E}_z^{1/(2p)}[\|\Gamma_{1:n+1}\|^{2p}]\mathbb{E}_z^{1/(2p)}[\|\tilde{\theta}_0\|^{2p}] \leq \mathrm{M}_0\,\mathrm{C}_{\mathsf{st},2p}\,\mathrm{e}^{-(a/4)\sum_{\ell=1}^{n+1}\alpha_\ell} V^{1/(4p)}(z). \quad (27)$$

As for the fluctuation term $\tilde{\theta}^{(\mathsf{fl})}_{n+1}$, it can be verified that $\tilde{\theta}^{(\mathsf{fl})}_{n+1} = J^{(0)}_{n+1} + H^{(0)}_{n+1}$, where the latter terms are defined by the following pair of recursions:

$$\begin{aligned} J^{(0)}_{n+1} &= (\mathrm{I}_d - \alpha_{n+1}A)\,J^{(0)}_n + \alpha_{n+1}\bar{\varepsilon}(Z_{n+1}), & J^{(0)}_0 &= 0, \\ H^{(0)}_{n+1} &= \left(\mathrm{I}_d - \alpha_{n+1}\bar{A}(Z_{n+1})\right)H^{(0)}_n - \alpha_{n+1}\widetilde{A}(Z_{n+1})J^{(0)}_n, & H^{(0)}_0 &= 0, \end{aligned} \quad (28)$$

and $\widetilde{A}(z) = \bar{A}(z) - A$. Furthermore, we observe that

$$J^{(0)}_{n+1} = \sum_{j=1}^{n+1} \alpha_j G_{j+1:n+1} \bar{\varepsilon}(Z_j), \quad H^{(0)}_{n+1} = -\sum_{j=1}^{n+1} \alpha_j \Gamma_{j+1:n+1} \widetilde{A}(Z_j) J^{(0)}_{j-1} . \tag{29}$$

From (29), we observe that $J^{(0)}_{n+1}$ is an additive functional of $\{\bar{\varepsilon}(Z_j)\}_{j=1}^{n+1}$ whose L_p norm can be bounded using a Rosenthal-type inequality for Markov chains (see Proposition 12). We obtain the following estimate for the function $\bar{\varepsilon}(\cdot)$ and the coefficients $\alpha_k G_{k+1:n+1}$. By A1, A3(K), we have

$$\max_{\ell \in \{1,\dots,d\}} \|\bar{\varepsilon}_\ell\|_{V^{1/\mathsf{K}}} \le \mathrm{C}_{\bar{\varepsilon}} := \sqrt{d}\, \mathrm{C}_{b,\mathsf{K}} + 2d(\beta \mathsf{K}/\mathrm{e})^\beta\, \mathrm{C}_A \|\theta^\star\|. \tag{30}$$

From A2, we recall that $\|G_{k+1:n+1}\| \le \kappa \prod_{\ell=k+1}^{n+1} \sqrt{1 - a\alpha_\ell}$ [cf. Lemma 18]. Together with A4, this implies that

$$\|\alpha_k G_{k+1:n+1} - \alpha_{k+1} G_{k+2:n+1}\| \le \kappa(\mathrm{c}_\alpha + 2\|A\|)\alpha_{k+1}^2 \prod_{\ell=k+1}^{n+1} \sqrt{1 - a\alpha_\ell} , \tag{31}$$

By A4, we also have $\alpha_1 \|G_{2:n+1}\| \le \kappa \alpha_{n+1} \prod_{j=2}^{n+1} (1 + \mathrm{c}_\alpha \alpha_j)(1 - a\alpha_j/2) \le \kappa \alpha_{n+1}$. We can now apply the Rosenthal inequality (see Proposition 12) to obtain the following estimate:

$$\begin{aligned}\mathbb{E}_z^{1/p}[\|J^{(0)}_{n+1}\|^p] \le d\, \mathrm{C}_{\bar{\varepsilon}}\, C_{\mathrm{Ros},p}^{1/p} V^{1/\mathsf{K}}(z) \Big\{ \big[\kappa + 1\big] \alpha_{n+1} \\ + \Big(\kappa^2 \sum_{k=1}^{n+1} \alpha_k^2 \prod_{\ell=k+1}^{n+1} (1 - \alpha_\ell a)\Big)^{1/2} + \kappa(\mathrm{c}_\alpha + 2\|A\|) \sum_{k=1}^{n+1} \alpha_{k+1}^2 \prod_{\ell=k+1}^{n+1} \sqrt{1 - a\alpha_\ell} \Big\} .\end{aligned}$$

Using the inequality $\sum_{k=1}^{n+1} \alpha_{k+1}^2 \prod_{\ell=k+1}^{n+1} \sqrt{1 - a\alpha_\ell} \le (4/a)\alpha_{n+1}$ [cf. Lemma 26] yields that

$$\mathbb{E}_z^{1/p}[\|J^{(0)}_{n+1}\|^p] \le \mathrm{C}^{(0)}_{\mathsf{J},p} \sqrt{\alpha_{n+1}} V^{1/\mathsf{K}}(z) , \tag{32}$$

where

$$\mathrm{C}^{(0)}_{\mathsf{J},p} = d\kappa\, \mathrm{C}_{\bar{\varepsilon}} (2 + 4(\mathrm{c}_\alpha + 2\|A\|)/a + 2/\sqrt{a}) C_{\mathrm{Ros},p}^{1/p} ; \tag{33}$$

Finally, to analyze $H^{(0)}_{n+1}$, from (29) we apply the Hölder's inequality twice to get

$$\mathbb{E}_z^{1/p}[\|H^{(0)}_{n+1}\|^p] \le \sum_{j=1}^{n+1} \alpha_j \mathbb{E}_z^{1/(2p)}[\|\Gamma_{j+1:n+1}\|^{2p}]\, \mathbb{E}_z^{1/(4p)}[\|\widetilde{A}(Z_j)\|^{4p}]\, \mathbb{E}_z^{1/(4p)}[\|J^{(0)}_{j-1}\|^{4p}]. \tag{34}$$

Notice that $\mathbb{E}_z^{1/(4p)}[\|\widetilde{A}(Z_j)\|^{4p}] \le \bar{\mathrm{C}}_A V^{1/\mathsf{K}}(z)$ where $\bar{\mathrm{C}}_A$ is defined in (78) [cf. Lemma 16]. Using Theorem 1 and (32), we obtain

$$\begin{aligned}\mathbb{E}_z^{1/p}[\|H^{(0)}_{n+1}\|^p] &\le \mathrm{C}_{\mathsf{st},2p}\, \mathrm{C}^{(0)}_{\mathsf{J},4p}\, \bar{\mathrm{C}}_A \sum_{j=1}^{n+1} \alpha_j \sqrt{\alpha_{j-1}} \mathrm{e}^{-(a/4)\sum_{\ell=j+1}^{n+1} \alpha_\ell} V^{2/\mathsf{K}+1/(4p)}(z) \\ &\overset{(a)}{\le} \sqrt{1 + \alpha^{(1)}_{\infty,p}\, \mathrm{c}_\alpha}\, \mathrm{C}_{\mathsf{st},2p}\, \mathrm{C}^{(0)}_{\mathsf{J},4p}\, \bar{\mathrm{C}}_A \sum_{j=1}^{n+1} \alpha_j^{3/2} \prod_{\ell=j+1}^{n+1} \big(1 - \alpha_\ell a/8\big) V^{2/\mathsf{K}+1/(4p)}(z) \\ &\overset{(b)}{\le} \mathrm{C}^{(0)}_{\mathsf{H},p} \sqrt{\alpha_{n+1}} V^{2/\mathsf{K}+1/(4p)}(z),\end{aligned} \tag{35}$$

where

$$\mathrm{C}^{(0)}_{\mathsf{H},p} = 16\sqrt{1 + \alpha^{(1)}_{\infty,p}\, \mathrm{c}_\alpha}\, \mathrm{C}_{\mathsf{st},2p}\, \mathrm{C}^{(0)}_{\mathsf{J},4p}\, \bar{\mathrm{C}}_A / a . \tag{36}$$

In the above, (a) is due to A4 and the inequality $\mathrm{e}^{-\alpha_j a/4} \le 1 - \alpha_j a/8$ since $\alpha_j a/4 \le 1$, (b) is due to the inequality $\sum_{j=1}^{n+1} \alpha_j^{3/2} \prod_{\ell=j+1}^{n+1} \big(1 - \alpha_\ell a/8\big) \le (16/a)\sqrt{\alpha_{n+1}}$ [cf. Lemma 26]. By observing that

$$\tilde{\theta}_{n+1} = \tilde{\theta}^{(\mathsf{tr})}_{n+1} + \tilde{\theta}^{(\mathsf{fl})}_{n+1} = \tilde{\theta}^{(\mathsf{tr})}_{n+1} + J^{(0)}_{n+1} + H^{(0)}_{n+1} , \tag{37}$$

applying Minkowski's inequality yields the bound in (24).

Refining the error bound $\mathbb{E}_z^{1/p}[\|\tilde{\theta}_n^{(\mathsf{fl})}\|^p]$ It is possible to obtain a bound on $\mathbb{E}_z^{1/p}[\|H_n^{(0)}\|^p]$ tighter than $\mathcal{O}(\sqrt{\alpha_n})$ obtained in (35). This establishes in particular that $J_n^{(0)}$ is the leading term in the decomposition of the fluctuation term $\tilde{\theta}_{n+1}^{(\mathsf{fl})} = J_{n+1}^{(0)} + H_{n+1}^{(0)}$. To this end, we rely on an extra decomposition step similar to (28). We may further decompose the error term $H_n^{(0)}$ as $H_n^{(0)} = J_n^{(1)} + H_n^{(1)}$ such that

$$\begin{aligned} J_{n+1}^{(1)} &= (\mathrm{I}_d - \alpha_{n+1} A) J_n^{(1)} - \alpha_{n+1} \widetilde{A}(Z_{n+1}) J_n^{(0)}, & J_0^{(1)} &= 0, \\ H_{n+1}^{(1)} &= (\mathrm{I}_d - \alpha_{n+1} \bar{A}(Z_{n+1})) H_n^{(1)} - \alpha_{n+1} \widetilde{A}(Z_{n+1}) J_n^{(1)}, & H_0^{(1)} &= 0, \end{aligned} \tag{38}$$

where $J_n^{(0)}$ is defined in (28). For diminishing step sizes, here we should strengthen the previous assumption A4 as:

A5 *We have $\mathcal{A}_0 < \infty$, where $\mathcal{A}_n = \sum_{\ell=n}^{\infty} \alpha_\ell^2$. There exists a constant $0 < \mathrm{c}_\alpha \leq a/32$ such that for $k \in \mathbb{N}$, $\alpha_k / \alpha_{k+1} \leq 1 + \alpha_{k+1} \mathrm{c}_\alpha$ and $\alpha_k / \mathcal{A}_{k+1} \leq (2/3) \mathrm{c}_\alpha$.*

It is easy to check that A5 is satisfied by diminishing step sizes $\alpha_n = \mathrm{C}_a (n + n_0)^{-\mathrm{t}}$, $\mathrm{t} \in (\frac{1}{2}, 1]$.

Using the decomposition in (38), we obtain the the following result:

Theorem 4 *(Theorem 31) Let $\mathsf{K} \geq 32$, $\varepsilon \in (0,1)$ and assume UE 1, A 1(ε), A 2, and A 3(K). For any $2 \leq p \leq \mathsf{K}/16$ and any non-increasing sequence $(\alpha_k)_{k \in \mathbb{N}}$ satisfying $\alpha_0 \in (0, \alpha_{\infty,p}^{(1)})$ such that $\alpha_k \equiv \alpha$ or A5 holds. For any $z \in \mathsf{Z}$, $n \in \mathbb{N}$, it holds*

$$\mathbb{E}_z^{1/p}[\|H_n^{(0)}\|^p] \leq V^{3/\mathsf{K} + 9/(16p)}(z) \begin{cases} \mathrm{C}_p^{(\mathsf{f})} \, \alpha \sqrt{\log(1/\alpha)}, & \textit{if } \alpha_n \equiv \alpha, \\ \mathrm{C}_p^{(\mathsf{d})} \, \sqrt{\alpha_n \mathcal{A}_n \log(1/\alpha_n)}, & \textit{if under A5}, \end{cases} \tag{39}$$

where $\alpha_{\infty,p}^{(1)}$, $\mathrm{C}_p^{(\mathsf{f})}$, $\mathrm{C}_p^{(\mathsf{d})}$ are given in (95), (97), respectively.

The theorem shows that the previous bound of $\mathbb{E}_z^{1/p}[\|H_n^{(0)}\|^p] = \mathcal{O}(\sqrt{\alpha_n})$ can be improved to $\mathcal{O}(\sqrt{\alpha_n \mathcal{A}_n \log(1/\alpha_n)})$. Take for example a diminishing step size as $\alpha_n = \mathrm{C}_a (n + n_0)^{-1}$, our result shows that the fluctuation term admits a *clear separation of scales* as

$$\tilde{\theta}_n^{(\mathsf{fl})} = J_n^{(0)} + H_n^{(0)} \text{ with } \mathbb{E}_z^{1/p}[\|J_n^{(0)}\|^p] = \mathcal{O}(n^{-1/2}), \; \mathbb{E}_z^{1/p}[\|H_n^{(0)}\|^p] = \mathcal{O}(n^{-1}\sqrt{\log n}).$$

Proof Sketch We study $J_{n+1}^{(1)}$ first. By (38) and the definition of $J_n^{(0)}$ in (28), we obtain

$$J_{n+1}^{(1)} = \textstyle\sum_{j=1}^{n} \alpha_j S_{j+1:n+1} \bar{\varepsilon}(Z_j), \; \text{ with } \; S_{j+1:n+1} = \sum_{k=j+1}^{n+1} \alpha_k G_{k+1:n} \widetilde{A}(Z_k) G_{j+1:k-1} \, . \tag{40}$$

For illustrative purpose, in this proof sketch we will only consider the case when $\{Z_i\}_{i \geq 1}$ are i.i.d.. Here, we have $\mathbb{E}[S_{j+1:n+1} \bar{\varepsilon}(Z_j) | Z_{j+1}, \ldots Z_{n+1}] = 0$ and therefore $J_{n+1}^{(1)}$ is a Martingale. It follows:

$$\mathbb{E}^{1/p}[\|J_{n+1}^{(1)}\|^p] \overset{(a)}{\lesssim} \sqrt{\sum_{j=1}^{n} \alpha_j^2 \mathbb{E}^{2/p}[\|S_{j+1:n+1}\|^p]} \overset{(b)}{\lesssim} \sqrt{\sum_{j=1}^{n+1} \alpha_j^2 \mathcal{A}_j \prod_{\ell=j+1}^{n+1} (1 - a\alpha_\ell)} \lesssim \sqrt{\alpha_{n+1} \mathcal{A}_{n+1}}, \tag{41}$$

where (a) applied the Burkholder inequality (Hall and Heyde, 1980, Theorem 2.10) for Martingales, and (b) can be obtained by applying the Rosenthal inequality for i.i.d. random variables to the expectation $\mathbb{E}^{1/p}[\|S_{j+1:n+1}\|^p]$ (Hall and Heyde, 1980, Theorem 2.12).

Furthermore, we observe that $H_{n+1}^{(1)} = \sum_{j=1}^{n+1} \alpha_j \Gamma_{j+1:n+1} \widetilde{A}(Z_j) J_{j-1}^{(1)}$. Similar to (34), we can apply (41) and the Hölder inequality to obtain $\mathbb{E}^{1/p}[\|H_{n+1}^{(1)}\|^p] = \mathcal{O}(\sqrt{\alpha_{n+1}\mathcal{A}_{n+1}})$. Combining both bounds yields the conclusion of the theorem.

Unfortunately, in the Markovian case we cannot apply the same arguments directly since $J_{n+1}^{(1)}$ is no longer a martingale. Instead, we first decouple the dependent random variables $\bar{\varepsilon}(Z_j)$ and $S_{j+1:n+1}$. This is done in Lemma 32 in the appendix by using the Berbee's coupling construction exploiting the fact that V-uniformly ergodic Markov chains are special cases of β-mixing processes (Rio, 2017). We leave the detailed derivations in the appendix for interested readers.

3.1. Temporal Difference Learning Algorithms

Following the notation from (Sutton and Barto, 2018, Chapter 12), we consider a discounted Markov Reward Process (MRP) denoted by the tuple $(\mathsf{X}, \mathrm{Q}, \mathrm{R}, \gamma)$, where Q is the state transition kernel defined on a general state space $(\mathsf{X}, \mathcal{X})$. We do not assume that X is finite and countable, the only requirement being that $\mathcal{X}$ is countably generated: we may assume for example that $\mathsf{X} = \mathbb{R}^d$. For any state $x, x' \in \mathsf{X}^2$, the scalar $\mathrm{R}(x, x')$ represents the instantaneous reward for going from state x to x'. The reward function is possibly unbounded. Finally, $\gamma \in (0, 1)$ is the discount factor. The value function $V^\star : \mathsf{X} \to \mathbb{R}$ is defined as the expected discounted reward $V^\star(x) = \mathbb{E}_x[\sum_{k=0}^{\infty} \gamma^k \mathrm{R}(X_k, X_{k+1})]$. The function V^* satisfies the Bellman equation $V^*(x) = \int \mathrm{Q}(x, \mathrm{d}x')\mathrm{R}(x, x') + \gamma \mathrm{Q} V^*(x)$.

We approximate $V^\star(x)$ using the linear value function estimation (LVE). Let $d \in \mathbb{N}^*$, we associate with every state $x \in \mathsf{X}$ a *feature vector* $\psi(x) \in \mathbb{R}^d$ and approximate $V^\star(x)$ by a linear combination $V_\theta(x) = \psi(x)^\top \theta$ (see Tsitsiklis and Van Roy (1997); Sutton and Barto (2018)). Temporal difference learning algorithms may be expressed as

$$\theta_{k+1} = \theta_k + \alpha_{k+1} \varphi_k \{\mathrm{R}(X_k, X_{k+1}) + \gamma \psi(X_{k+1})^\top \theta_k - \psi(X_k)^\top \theta_k\}, \tag{42}$$

where $\{\varphi_k\}_{k \in \mathbb{N}}$ is a sequence of eligibility vectors. For the TD(0) algorithm, $\varphi_k = \psi(X_k)$. For the TD(λ) algorithm, $\varphi_k = (\lambda\gamma)\varphi_{k-1} + \psi(X_k)$. Note that for TD($\lambda$), (42) corresponds to (1) with the extended Markov chain $Z_k = (X_k, X_{k+1}, \varphi_k)$ and $\bar{A}(Z_k) = -\varphi_k(\psi(X_k)^\top - \gamma\psi(X_{k+1})^\top)$, $b(Z_k) = \varphi_k \mathrm{R}(X_k, X_{k+1})$. Srikant and Ying (2019) were able to study TD(λ) while that $(Z_k)_{k \in \mathbb{N}^*}$ is not necessary uniformly ergodic. Indeed, a core argument in their application is the use of (Bertsekas and Tsitsiklis, 1996, Lemma 6.7) which implies that if Z is a finite state space and $(X_k)_{k \in \mathbb{N}}$ is uniformly ergodic, then $\|\mathbb{E}_z[\bar{A}(Z_k)] - A\| \leq C\rho^k$ and $\|\mathbb{E}_z[b(Z_k)] - b\| \leq C\rho^k$, for any $z \in \mathsf{Z}$, $k \in \mathbb{N}^*$ and for some $C \geq 0$, $\rho \in (0, 1)$. This is precisely the condition considered by Srikant and Ying (2019) to derive their bounds. (Bertsekas and Tsitsiklis, 1996, Lemma 6.7) does not extend to general (unbounded) state space.

As a replacement, to verify our assumption UE 1, we consider here a τ-truncated version of the eligibility trace

$$\varphi_k = \phi_\tau(X_{k-\tau+1:k}) \quad \text{where} \quad \phi_\tau(x_{0:\tau-1}) = \textstyle\sum_{s=0}^{\tau-1} (\lambda\gamma)^s \psi(x_{\tau-1-s}) \, . \tag{43}$$

TD(0) algorithm is a special case of (43) with $\tau = 1$ and we recover the TD(λ) algorithm by letting $\tau \to \infty$. The recursion (42) with eligibility vector defined in (43) is a special case of (1). To see this, we define $Z_k = [X_{k-\tau}, \ldots, X_k]^\top$ and observe that (42) can be obtained by using in (1) the following matrix/vector, for $z = [x_0, \ldots, x_\tau]^\top = x_{0:\tau} \in \mathsf{X}^{\tau+1}$,

$$\bar{A}(z) = \phi_\tau(x_{0:\tau-1})\{\psi(x_{\tau-1}) - \gamma\psi(x_\tau)\}^\top, \quad \bar{b}(z) = \phi_\tau(x_{0:\tau-1})\mathrm{R}(x_{\tau-1}, x_\tau) \, . \tag{44}$$

Note that compared to (Srikant and Ying, 2019), we do not consider TD(λ) but (43). Consider the following assumptions.

M1 *The Markov kernel* $\mathrm{Q} : \mathsf{X} \times \mathcal{X} \to \mathbb{R}_+$ *is irreducible and strongly aperiodic. There exist* $c_{\mathrm{Q}} > 0, \mathrm{b}_{\mathrm{Q}} > 0, \delta_{\mathrm{Q}} \in (1/2, 1]$, $R_{\mathrm{Q}} \geq 0$, *and* $\tilde{V} : \mathsf{X} \to [\mathrm{e}, \infty)$ *such that by setting* $\tilde{W} = \log \tilde{V}$, $\mathsf{C}_{\mathrm{Q}} = \{x : \tilde{W}(x) \leq R_{\mathrm{Q}}\}$, $\mathsf{C}_{\mathrm{Q}}^{\complement} = \{x : \tilde{W}(x) > R_{\mathrm{Q}}\}$, *we have*

$$\mathrm{Q}\tilde{V}(x) \leq \exp[-c_{\mathrm{Q}} \tilde{W}^{\delta_{\mathrm{Q}}}(x)] \tilde{V}(x) \mathbb{1}_{\mathsf{C}_{\mathrm{Q}}^{\complement}}(x) + \mathrm{b}_{\mathrm{Q}} \mathbb{1}_{\mathsf{C}_{\mathrm{Q}}}(x) \; . \tag{45}$$

In addition, for any $R \geq 1$, *the level sets* $\{x : \tilde{W}(x) \leq R\}$ *are* $(1, \varepsilon_{\mathrm{Q},R}\nu)$*-small for* Q, *with* $\varepsilon_{\mathrm{Q},R} \in (0, 1]$ *and* ν *being a probability measure on* $(\mathsf{X}, \mathcal{X})$.

It follows from (Douc et al., 2018, Theorem 15.2.4) that the Markov kernel Q admits a unique stationary distribution π_0. Set the state-space as $\mathsf{Z} = \mathsf{X}^{\tau+1}$ and the Markov kernel P is given by

$$\mathrm{P}(x_{0:\tau}; \mathrm{d}x'_{0:\tau}) = \textstyle\prod_{\ell=1}^{\tau} \delta_{x_\ell}(\mathrm{d}x'_{\ell-1}) \mathrm{Q}(x_\tau, \mathrm{d}x'_\tau) \; , \tag{46}$$

for any $z = x_{0:\tau} \in \mathsf{X}^{\tau+1}$, where δ_x denotes the Dirac measure at $x \in \mathsf{X}$. Define

$$V(x_{0:\tau}) = \exp\Big(\iota_0 \textstyle\sum_{i=0}^{\tau-1}(i+1)\tilde{W}^{\delta_{\mathrm{Q}}}(x_i) + \tilde{W}(x_\tau)\Big) \; , \tag{47}$$

where

$$\iota_0 = c_{\mathrm{Q}}^{\delta_{\mathrm{Q}}} / (1 + \tau c_{\mathrm{Q}}^{\delta_{\mathrm{Q}}}) \; . \tag{48}$$

Lemma 5 *Assume M 1. Then the Markov kernel* P *has a unique invariant distribution given by* $\pi(\mathrm{d}x_{0:\tau}) = \pi_0(\mathrm{d}x_0) \prod_{\ell=1}^{\tau} \mathrm{Q}(x_{\ell-1}, \mathrm{d}x_\ell)$. *In addition, UE 1 is satisfied with* V *given by* (47), *where the constants* c, b, *and* R *are defined in* (126).

Proof Follows from Lemma 36 and Lemma 37. ∎

Consider the following assumptions on ψ and R. Fix $\varepsilon \in (0, 1)$.

M2 $\pi_0(\psi\psi^\top)$ *is positive definite.*

M3 $(\varepsilon, \mathsf{K})$ *There exist* $\mathrm{C}_\psi, \mathrm{C}_{\mathrm{R},\mathsf{K}} > 0$ *such that* $\|\psi(x)\| \leq \mathrm{C}_\psi \, \tilde{W}^{\beta\delta_{\mathrm{Q}}/2}(x)$ *and*

$$|\mathrm{R}(x, x')| \leq \mathrm{C}_{\mathrm{R},\mathsf{K}} \, \mathrm{e}^{\iota_0(\tilde{W}^{\delta_{\mathrm{Q}}}(x) + \tilde{W}^{\delta_{\mathrm{Q}}}(x'))/(2\mathsf{K})} \; ,$$

where $\beta < \min(2\delta_{\mathrm{Q}} - 1, \delta_{\mathrm{Q}}/(1+\varepsilon))$ *and* δ_{Q} *is given in M 1.*

In the following, we show that under M 1–M 3, the TD(λ) algorithm with truncated eligibility trace (42) satisfies the assumptions in Section 3. In this case,

Theorem 6 (Finite-time bound for TD(λ) (42)–(43)) *Let* $\mathsf{K} \geq 8$ *and* $\varepsilon \in (0, 1)$. *Assume M 1– M 3*$(\varepsilon, \mathsf{K})$. *For any* $2 \leq p \leq \mathsf{K}/4$, *there exists* $\alpha_{\infty,p}^{(0)}$ *defined in* (25) *such that for any non-increasing sequence* $(\alpha_k)_{k \in \mathbb{N}^*}$ *satisfying* $\alpha_1 \in (0, \alpha_{\infty,p}^{(0)})$ *and A4,* $z = x_{0:\tau} \in \mathsf{X}^{\tau+1}$, *and* $n \in \mathbb{N}$, *it holds*

$$\mathbb{E}_z^{1/p}[\|\theta_n - \theta^\star\|^p] \leq \mathrm{M}_0 \, \mathrm{C}_{\mathsf{st},2p} \, \mathrm{e}^{-(a/4)\sum_{\ell=1}^{n} \alpha_\ell} V^{1/(4p)}(z) + (\mathrm{C}_{\mathsf{J},p}^{(0)} + \mathrm{C}_{\mathsf{H},p}^{(0)}) \sqrt{\alpha_n} V^{2/\mathsf{K} + 1/(4p)}(z),$$

where $\mathrm{M}_0 = \mathbb{E}_z^{1/(2p)}[\|\theta_0 - \theta^\star\|^{2p}]$, $\mathrm{C}_{\mathsf{J},p}^{(0)}, \mathrm{C}_{\mathsf{H},p}^{(0)}$ *are defined in* (33), (36), *respectively, with*

$$\mathrm{C}_A = (1+\gamma)\,\mathrm{C}_\psi^2 \big(\iota_0^{-\beta} \vee 1\big)(1 - \lambda\gamma)^{-1}, \quad \mathrm{C}_{b,\mathsf{K}} = \frac{1}{2(1-\lambda\gamma)} \left(\mathrm{C}_\psi^2 \left\{ \frac{\beta K}{\mathrm{e}\iota_0} \right\}^{\beta} + \mathrm{C}_{\mathrm{R},\mathsf{K}}^2 \right) . \tag{49}$$

Additionally, we remark that the bound can be tightened through applying Theorem 4 if we strengthen the stepsize condition A4 to A5.

Proof We apply Theorem 3. Lemma 5 shows that UE 1 is satisfied. Using definition (44) it is straightforward to check that A 1(ε) and A 3(K) hold with C_A and $\mathrm{C}_{b,\mathsf{K}}$ given in (49). Detailed calculations may be found in Lemma 38 and Lemma 39, respectively. Finally, A 2 follows from Lemma 35 in the appendix.

■

Conclusions We have established the (V, q)-exponential stability of the sequence of random matrices $\{\bar{A}(Z_k)\}_{k \in \mathbb{N}^*}$ under relaxed conditions on the Markov chain and the matrix functions. The results are applied to obtain finite-time p-th moment bounds of LSA error, and a family of TD learning algorithms.

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Appendix A. Formal statement and Proof for Example 1

The proof is by contradiction. Assume that $(Z_k)_{k\in\mathbb{N}}$ is not geometrically ergodic and let $\alpha > 0$. First for any $\varepsilon > 0$, $\int_{\mathsf{Z}} A_{\bar{\varepsilon}}(x)\mathrm{d}\pi(x) = -\varepsilon[1-\pi(\{1\})] + \pi(\{1\})$. Then for any $\varepsilon \in (0, \bar{\varepsilon})$, setting $\bar{\varepsilon} = \pi(\{1\})$, we get that $\int_{\mathsf{Z}} A_{\bar{\varepsilon}}(z)\mathrm{d}\pi(z) > 0$. In addition, we have by definition of $(\theta_n^{\bar{\varepsilon}})_{n\in\mathbb{N}}$,

$$\mathbb{E}\left[|\theta_n^{\bar{\varepsilon}}|\right] \geq \theta_0(1+\alpha\bar{\varepsilon})^n \mathbb{P}(\mathrm{Y}_1 > n+1) \; . \tag{50}$$

By (Douc et al., 2018, Theorem 15.1.5), $(Z_k)_{k\in\mathbb{N}}$ is not geometrically ergodic and for any $\eta > 0$, $\mathbb{E}\left[(1+\eta)^{\mathrm{Y}_1}\right] = +\infty$. Therefore, $\limsup_{n\to+\infty}[(1+\alpha\bar{\varepsilon})^n\mathbb{P}(\mathrm{Y}_1 \geq n)] = +\infty$, otherwise we would obtain that for any $\varepsilon \in (0, \alpha\bar{\varepsilon})$, $\mathbb{E}\left[(1+\varepsilon)^{\mathrm{Y}_1}\right] \leq \sup_{n\in\mathbb{N}}[(1+\alpha\bar{\varepsilon})^n\mathbb{P}(\mathrm{Y}_1 \geq n)]\sum_{k=1}^{+\infty}[(1+\varepsilon)/(1+\alpha\bar{\varepsilon})]^k < +\infty$, which is absurd. Applying this result to (50) completes the proof.

Appendix B. Super-Lyapunov drift conditions UE 1

We gather the technical results needed for the proof of our main theorems. Define

$$\mathsf{C}_R = \{z \in \mathsf{Z} \, ,: \, W(z) \geq R\} \; , \text{ for any } R \geq 0 \; , \tag{51}$$

$$\varphi_\delta : z \mapsto cW^\delta(z) \; . \tag{52}$$

Lemma 7 *Assume UE 1. Then for any $n \in \mathbb{N}$, we have*

$$\mathrm{P}^n V(z) \leq \lambda^n V(z) + \mathrm{b}/(1-\lambda) \; , \quad \mathrm{P}^n V(z) \leq \mathrm{e}^{-\varphi_\delta(z)} V(z) + [\mathrm{b}/(1-\lambda)]\mathbb{1}_{\mathsf{C}_{R_1}}(z) \; , \tag{53}$$

where λ is defined in (5), C_R in (51) and

$$R_1 = \inf\{R \geq R_0 \, : \, \exp(R - cR^\delta) > [\mathrm{b}/(1-\lambda)^2]\} \; . \tag{54}$$

Proof We first show the left-hand side inequality in (53). First, UE 1 and (5) shows that $\mathrm{P}V \leq \lambda V + \mathrm{b}$ which implies by a straightforward induction that for any $n \in \mathbb{N}$,

$$\mathrm{P}^n V(z) \leq \lambda^n V(z) + \mathrm{b}\sum_{k=0}^{n-1}\lambda^k \; . \tag{55}$$

Using $\sum_{k=0}^{n-1}\lambda^k \leq (1-\lambda)^{-1}$ completes the proof.

We now show the right-hand side inequality of (53). (55) applied for $n-1 \in \mathbb{N}$ and (4) implies that

$$\begin{aligned}
\mathrm{P}^n V(z) &\leq \lambda^{n-1} \mathrm{P} V(z) + \mathrm{b} \sum_{k=0}^{n-2} \lambda^k \\
&\leq \mathrm{e}^{-\varphi_\delta(z)} \lambda^{n-1} V(z) + \mathrm{b} \sum_{k=0}^{n-1} \lambda^k \\
&\leq \mathrm{e}^{-\varphi_\delta(z)} V(z) - \mathrm{e}^{-\varphi_\delta(z)} (1-\lambda) V(z) + \mathrm{b}[1-\lambda]^{-1} .
\end{aligned}$$

Then, using by definition of R_1 that for any $z \in \mathsf{C}_{R_1}^{\complement}$, $\mathrm{e}^{-\varphi_\delta(z)} V(z) \geq \mathrm{b}[1-\lambda]^{-2}$ completes the proof. ■

Lemma 8 *Assume UE 1. Then, for any $\gamma > 0$,*

$$\mathrm{P} W^{\gamma+1-\delta}(z) \leq W^{\gamma+1-\delta}(z) - c_\gamma W^\gamma(z) + \mathrm{b}_\gamma \mathbb{1}_{\mathsf{C}_{R_\gamma}}(z) , \tag{56}$$

where the constants c_γ, R_γ and b_γ are given by: if $\gamma \leq \delta$,

$$R_\gamma = R_0 , \qquad c_\gamma = 1 \wedge [(\gamma+1-\delta)c] , \qquad \mathrm{b}_\gamma = \log^{\gamma+1-\delta} \mathrm{b} , \tag{57}$$

and if $\gamma > \delta$,

$$\begin{aligned}
&R_\gamma = R_0 \vee (2(\gamma+1-\delta)/c)^{1/\delta} \vee c^{1/(\delta-1)} , \quad \mathrm{b}_\gamma = \log^{\gamma+1-\delta} \left[(\mathrm{b} + \mathrm{e}^{\gamma-\delta}) \vee (\exp(R_\gamma + \mathrm{e}^{\gamma-\delta})) \right] \\
&c_\gamma = 1 \wedge [(\gamma+1-\delta)(1 - c R_\gamma^{\delta-1}/2)^{\gamma-\delta}(c/2)] .
\end{aligned} \tag{58}$$

Proof We consider separately the cases $\gamma \leq \delta$ and $\gamma > \delta$.

If $\gamma \leq \delta$, the function $z \mapsto \log^{\gamma+1-\delta} z$ is concave. Using Jensen's inequality and UE 1, we get that

$$\begin{aligned}
\mathrm{P} W^{\gamma+1-\delta}(z) &\leq (\mathrm{P} W(z))^{\gamma+1-\delta} \leq (W(z) - c W^\delta(z))^{\gamma+1-\delta} \mathbb{1}_{\mathsf{C}_0^\complement}(z) + \log^{\gamma+1-\delta}(\mathrm{b}) \mathbb{1}_{\mathsf{C}_0}(z) \\
&= W^{\gamma+1-\delta}(z)(1 - c W^{\delta-1}(z))^{\gamma+1-\delta} \mathbb{1}_{\mathsf{C}_0^\complement}(z) + \log^{\gamma+1-\delta}(\mathrm{b}) \mathbb{1}_{\mathsf{C}_0}(z) .
\end{aligned}$$

Note that UE 1 implies that for any $z \in \mathsf{C}_0^\complement$, $1 \leq \mathrm{P} V(z) \leq V(z) \mathrm{e}^{-c W^\delta(z)}$ and therefore, $c W^{\delta-1}(z) \leq 1$ since $\delta \leq 1$. Then, Using that $(1-x)^{\gamma+1-\delta} < 1 - (\gamma+1-\delta)x$ for all $x \in [0,1]$ since $\gamma+1-\delta \leq 1$ and $c W^{\delta-1}(z) \leq 1$ on $\mathsf{C}_0^\complement$, we get that

$$W^{\gamma+1-\delta}(z)(1 - c W^{\delta-1}(z))^{\gamma+1-\delta} \mathbb{1}_{\mathsf{C}_0^\complement}(z) \leq W^{\gamma+1-\delta}(z) - (\gamma+1-\delta) c W^\gamma(z) ,$$

which completes the proof for $\gamma \leq \delta$.

Consider now the case $\gamma > \delta$ and note that the function $z \mapsto \log^{\gamma+1-\delta} z$ is concave on $[\exp(\gamma-\delta), +\infty)$ and therefore $\psi_\gamma : z \mapsto \log^{\gamma+1-\delta}(z + \mathrm{e}^{\gamma-\delta})$ is concave on $\mathbb{R}_+$. Using Jensen's inequality, we obtain

$$\mathrm{P} W^{\gamma+1-\delta}(z) = \mathrm{P} \log^{\gamma+1-\delta}(V(z)) \leq \mathrm{P} \psi_\gamma \circ V(z) \leq \psi_\gamma[\mathrm{P} V(z)] . \tag{59}$$

Now by UE 1 and $a+b \leq a(b+1)$ for $a, b \geq 1$, $\mathrm{e}^c + 1 \leq \mathrm{e}^{c+1}$, we get

$$\begin{aligned}
\mathrm{P}V(z) + \mathrm{e}^{\gamma-\delta} &\leq (\exp(W(z) - cW^{\delta}(z)) + \mathrm{e}^{\gamma-\delta})\mathbb{1}_{\mathsf{C}_0^{\complement}}(z) + (\mathrm{b} + \mathrm{e}^{\gamma-\delta})\mathbb{1}_{\mathsf{C}_0}(z) \\
&\exp(W(z) - cW^{\delta}(z) + \gamma + 1 - \delta)\mathbb{1}_{\mathsf{C}_0^{\complement}}(z) + (\mathrm{b} + \mathrm{e}^{\gamma-\delta})\mathbb{1}_{\mathsf{C}_0}(z) \\
&\leq \exp(W(z) - (c/2)W^{\delta}(z))\mathbb{1}_{\mathsf{C}_{R_\gamma}^{\complement}}(z) + (\mathrm{b} + \mathrm{e}^{\gamma-\delta}) \vee (\exp(R_\gamma + \mathrm{e}^{\gamma-\delta}))\mathbb{1}_{\mathsf{C}_{R_\gamma}}(z) ,
\end{aligned}$$

where we used for the last inequality that for any $z \notin \mathsf{C}_{R_\gamma}$ and the definitions (51), (58), $W^{\delta}(z) \geq 2\mathrm{e}^{\gamma-\delta}/c$ and $R_\gamma \geq R_0$. Using the previous result in (59), we get that

$$\mathrm{P}W^{\gamma+1-\delta}(z) \leq \left(W(z) - (c/2)W^{\delta}(z)\right)^{\gamma+1-\delta} + \mathrm{b}_\gamma \mathbb{1}_{\mathsf{C}_{R_\gamma}}(z) , \tag{60}$$

where b_γ is given in (58). Note that $(1-x)^{\gamma+1-\delta} \leq 1 - (\gamma+1-\delta)(1 - cR_\gamma^{\delta-1}/2)^{\gamma-\delta}x$ for all $x \in [0, cR_\gamma^{\delta-1}/2]$ since $cR_\gamma^{\delta-1}/2 \leq 1/2$ by definition of R_γ (58). Therefore, using that on C_{R_γ}, we have $0 < cW^{\delta-1}(z)/2 \leq cR_\gamma^{\delta-1}/2$, we get $\left(W(z) - (c/2)W^{\delta}(z)\right)^{\gamma+1-\delta} = W^{\gamma+1-\delta}(z)\{1 - (c/2)W^{\delta-1}\}^{\gamma+1-\delta} \leq W^{\gamma+1-\delta}(z) - c_\gamma W^{\gamma}(z)$. Plugging this result in (60) concludes the proof of (56) for $\gamma > \delta$. ■

Corollary 9 *Assume UE 1. Then, for any $\gamma > 0$, it holds that $\pi(W^\gamma) \leq \mathrm{b}_\gamma / c_\gamma$ and $\pi(V) \leq b/(1-\lambda)$. where $\mathrm{b}_\gamma, c_\gamma$ are given in Theorem 8.*

Proof As mentioned previously (see (6)), P has a unique stationary distribution satisfying $\pi(V) < +\infty$. Therefore, since $\|W\|_V < +\infty$, we can take the integral in (56) and (53) with respect to π. Rearranging terms completes the proof. ■

Note that UE 1 implies for any $z \in \mathsf{Z}$,

$$\mathrm{P}V(z) \leq \exp\left(W(z) - cW^{\delta}(z) + \tilde{\mathrm{b}}\mathbb{1}_{\mathsf{C}_0}(z)\right) , \tag{61}$$

where $\tilde{\mathrm{b}} = \log \mathrm{b} + \sup_{r \geq \mathrm{e}}\{cr^{\delta} - r\}$. Similarly, (53) implies for any $h \in \mathbb{N}$ and $z \in \mathsf{Z}$,

$$\mathrm{P}^h V(z) \leq \exp\left(W(z) - cW^{\delta}(z) + \mathrm{b}' \mathbb{1}_{\mathsf{C}_{R_1}}(z)\right) , \tag{62}$$

where

$$\mathrm{b}' = \log\{\mathrm{b}/(1-\lambda)\} + \sup_{r>0}\{cr^{\delta} - r\} . \tag{63}$$

Lemma 10 *Assume UE 1. Let $(\alpha_i)_{i \in \mathbb{N}^*}$ be a non-increasing sequence, such that $0 < \alpha_i \leq 1$ for any $i \geq 1$. Then, for any $z \in \mathsf{Z}$ and $n \in \mathbb{N}$,*

$$\mathbb{E}_z[\exp\{c\textstyle\sum_{k=0}^{n-1}\alpha_k W^{\delta}(Z_k)\}] \leq \exp\{\tilde{\mathrm{b}}\textstyle\sum_{k=0}^{n-1}\alpha_k\}\exp\{\alpha_1 W(z)\} ,$$

where $\tilde{\mathrm{b}} = \log \mathrm{b} + \sup_{r \geq \mathrm{e}}\{cr^{\delta} - r\}$ and c in UE 1.

Proof Define, for $n \geq 0$,

$$M_n = \exp\{\alpha_n W(Z_n) + \textstyle\sum_{k=0}^{n-1} \alpha_k (cW^\delta(Z_k) - \tilde{\mathrm{b}}\mathbb{1}_{\mathsf{C}_0}(Z_k))\} \,, \tag{64}$$

with the convention $\sum_{k=0}^{-1} = 0$. Consider $(\mathcal{F}_n)_{n\in\mathbb{N}}$, the canonical filtration: $\mathcal{F}_n = \sigma(Z_0, \ldots, Z_n)$. Then, we have for $n \geq 1$,

$$\mathbb{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}\exp\{-\alpha_{n-1}W(Z_{n-1}) + \alpha_{n-1}(cW^\delta(Z_{n-1}) - \tilde{\mathrm{b}}\mathbb{1}_{\mathsf{C}_0}(Z_{n-1}))\}\mathbb{E}[\mathrm{e}^{\alpha_n W(Z_n)}|\mathcal{F}_{n-1}] \,.$$

Using the Markov property, $\alpha_n \leq \alpha_{n-1} \leq 1$, $V \geq 1$, (61) and Jensen's inequality, for $n \geq 1$,

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{\alpha_n W(Z_n)}|\mathcal{F}_{n-1}] &= \mathrm{P}V^{\alpha_n}(Z_{n-1}) \leq \mathrm{P}V^{\alpha_{n-1}}(Z_{n-1}) \leq (\mathrm{P}V(Z_{n-1}))^{\alpha_{n-1}} \\ &\leq \exp\{\alpha_{n-1}(W(Z_{n-1}) - cW^\delta(Z_{n-1}) + \tilde{\mathrm{b}}\mathbb{1}_{\mathsf{C}_0}(Z_{n-1}))\} \,.\end{aligned}$$

Therefore, $(M_n)_{n\geq 0}$ is $(\mathcal{F}_n)_{n\geq 0}$-supermartingale, and $\mathbb{E}_z[M_n] \leq \mathbb{E}_z[M_0] \leq \mathrm{e}^{\alpha_1 W(z)}$. We conclude the proof upon noting that $\mathbb{E}_z[\exp\{c\sum_{k=0}^{n-1}\alpha_k W^\delta(Z_k)\}] \leq \exp\{\tilde{\mathrm{b}}\sum_{k=0}^{n-1}\alpha_k\}\mathbb{E}_z[M_n]$. ■

Lemma 11 *Assume UE 1. Let $(\alpha_i)_{i\in\mathbb{N}^*}$ be a non-increasing sequence, such that $0 < \alpha_i \leq 1$ for any $i \geq 1$. Then, for any $z \in \mathsf{Z}$ and $n \in \mathbb{N}$, $h \in \mathbb{N}$,*

$$\mathbb{E}_z[\exp\{c\textstyle\sum_{k=0}^{n-1}\alpha_k W^\delta(Z_{hk})\}] \leq \exp\{\mathrm{b}'\textstyle\sum_{k=0}^{n-1}\alpha_k\}\exp\{\alpha_1 W(z)\} \,,$$

where b' is given in (63) and c in UE 1.

Proof The proof follows the same lines as Lemma 10, using (62) in place of (61). ■

Appendix C. Rosenthal inequality for Markov chains

In this section, we state a general weighted Rosenthal inequality for f-ergodic Markov chain. This result is a simple adaptation of (Fort and Moulines, 2003, Proposition 12). In addition, we apply this result to obtain bounds which will be useful in the proof of our main results.

In all this section, $(Z_k)_{k\in\mathbb{N}}$ is the canonical Markov chain corresponding to the Markov kernel P on the filtered canonical space $(\mathsf{Z}^{\mathbb{N}}, \mathcal{Z}^{\otimes\mathbb{N}}, (\mathcal{F}_n)_{n\in\mathbb{N}})$, where $\mathcal{F}_n = \sigma(Z_0, \ldots, Z_n)$ for $n \in \mathbb{N}$. We still denote by $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ the corresponding probability distribution and expectation with initial distribution μ. In the case $\mu = \delta_z$, $z \in \mathsf{Z}$, $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$ are denoted by $\mathbb{P}_z$ and $\mathbb{E}_z$.

Proposition 12 (Rosenthal's inequality) *Let $p \geq 2$ and $f, \mathcal{W}, \mathcal{V} : \mathsf{Z} \to [1, +\infty]$ such that $\|f\|_{\mathcal{W}} \leq 1$ and $\|\mathcal{W}^p\|_{\mathcal{V}} \leq 1$. Let $(\beta_k)_{k\in\mathbb{N}}$ be a real sequence. Assume that P has a unique stationary distribution π and satisfies for any $z \in \mathsf{Z}$,*

$$\sum_{n\in\mathbb{N}} \|\delta_z \mathrm{P}^n - \pi\|_f \leq C_f \mathcal{W}(z) \,, \quad \sum_{n\in\mathbb{N}} \|\delta_z \mathrm{P}^n - \pi\|_{\mathcal{W}^p} \leq C_{\mathcal{W}} \mathcal{V}(z) \,, \tag{65}$$

for some constants $C_f, C_{\mathcal{W}} < +\infty$. Then, for any $g \in \mathrm{L}_\infty^f$, it holds that for any $z \in \mathsf{Z}$,

$$\begin{aligned}&\mathbb{E}_z\left[\left|\textstyle\sum_{k=1}^n \beta_k\{g(Z_k) - \pi(g)\}\right|^p\right] \\ &\leq \|g\|_f^p C_{\mathrm{Ros},p}\left[\left\{\textstyle\sum_{k=1}^n \beta_k^2\right\}^{p/2} + \left\{\textstyle\sum_{k=1}^{n-1}|\beta_k - \beta_{k+1}|\right\}^p + \beta_1^p + \beta_n^p\right]\mathcal{V}(z) \,,\end{aligned} \tag{66}$$

where

$$C_{\mathrm{Ros},p} = 6^p C_f^p \{C_{\mathcal{W}} + \pi(\mathcal{W}^p)\}(p^p + 2) \ . \tag{67}$$

Proof Let $g \in \mathrm{L}_\infty^f$ and $z \in \mathsf{Z}$. Without loss of generality, we assume that $\|g\|_f \leq 1$. Denote by $S_n = \sum_{k=1}^n \beta_k \{g(Z_k) - \pi(g)\}$. By (65), the function $\hat{g}(x) = \sum_{n\in\mathbb{N}} \{\mathrm{P}^n g(x) - \pi(g)\}$ is well defined, $\hat{g} \in \mathrm{L}_\infty^{\mathcal{W}}$,

$$\|\hat{g}\|_{\mathcal{W}} \leq C_f \tag{68}$$

and is a solution of the Poisson equation $\hat{g} - \mathrm{P}\hat{g} = g - \pi(g)$. Then, we have

$$\begin{aligned} S_n &= M_n + R_{1,n} + R_{2,n} \ , \\ M_n &= \sum_{k=0}^{n-1} \beta_{k+1}\{\hat{g}(Z_{k+1}) - \mathrm{P}\hat{g}(Z_k)\} \\ R_{1,n} &= \sum_{k=1}^{n-1} (\beta_{k+1} - \beta_k)\mathrm{P}\hat{g}(Z_k) \ , \quad R_{2,n} = \beta_1 \mathrm{P}\hat{g}(Z_0) - \beta_n \mathrm{P}\hat{g}(Z_n) \ . \end{aligned}$$

Therefore, by Young inequality, we get that

$$\mathbb{E}_z\left[|S_n|^p\right] \leq 3^{p-1}\{\mathbb{E}_z\left[|M_n|^p\right] + \mathbb{E}_z\left[|R_{1,n}|^p\right] + \mathbb{E}_z\left[|R_{1,n}|^p\right]\} \ . \tag{69}$$

We now bound each term on the right-hand side.

First, since $\hat{g} \in \mathrm{L}_\infty^{\mathcal{W}}$ and (65), note that $(M_k)_{k\in\mathbb{N}}$ is a $(\mathcal{F}_n)_{n\in\mathbb{N}}$-martingale with martingale increment $(\Delta M_k = \beta_{k+1}\{\hat{g}(Z_{k+1}) - \mathrm{P}\hat{g}(Z_k)\})_{k\in\mathbb{N}}$. Therefore, using (Osekowski, 2012, Theorem 8.6) and Jensen inequality, we have

$$\begin{aligned} \mathbb{E}_z[|M_n|^p] &\leq p^p \mathbb{E}_z[|\textstyle\sum_{k=0}^{n-1} \Delta M_k^2|^{p/2}] \\ &\leq p^p \{\textstyle\sum_{k=0}^{n-1} \beta_{k+1}^2\}^{p/2-1} \textstyle\sum_{k=0}^{n-1} \beta_{k+1}^2 \mathbb{E}_z[|\hat{g}(Z_{k+1}) - \mathrm{P}\hat{g}(Z_k)|^p] \ . \end{aligned}$$

Using (65) and Jensen inequality, we get

$$\begin{aligned} \mathbb{E}_z[|M_n|^p] &\leq 2^{p-1} p^p \{\textstyle\sum_{k=0}^{n-1} \beta_{k+1}^2\}^{p/2-1} \textstyle\sum_{k=0}^{n-1} \beta_{k+1}^2 \mathbb{E}_z[|\hat{g}(Z_{k+1})|^p + |\mathrm{P}\hat{g}(Z_k)|^p] \\ &\leq 2^p p^p \{\textstyle\sum_{k=0}^{n-1} \beta_{k+1}^2\}^{p/2-1} \|\hat{g}\|_{\mathcal{W}}^p \textstyle\sum_{k=0}^{n-1} \beta_{k+1}^2 [|\mathbb{E}_z[\mathcal{W}(Z_{k+1})^p] - \pi(\mathcal{W}^p)| + \pi(\mathcal{W}^p)] \\ &\leq 2^p p^p \|\hat{g}\|_{\mathcal{W}}^p \{\textstyle\sum_{k=0}^{n-1} \beta_{k+1}^2\}^{p/2} \{C_{\mathcal{W}} \mathcal{V}(z) + \pi(\mathcal{W}^p)\} \ . \end{aligned} \tag{70}$$

Using (65) and Jensen inequality, we get that

$$\begin{aligned} \mathbb{E}_z[|R_{1,n}|^p] &\leq \{\textstyle\sum_{k=1}^{n-1} |\beta_k - \beta_{k+1}|\}^{p-1} \textstyle\sum_{k=1}^{n-1} |\beta_k - \beta_{k+1}| \mathbb{E}_z\left[|\mathrm{P}\hat{g}(Z_k)|^p\right] \\ &\leq \{\textstyle\sum_{k=1}^{n-1} |\beta_k - \beta_{k+1}|\}^p \|\hat{g}\|_{\mathcal{W}}^p \{C_{\mathcal{W}} \mathcal{V}(z) + \pi(\mathcal{W}^p)\} \ . \end{aligned} \tag{71}$$

Finally, by (65) and Young inequality, we get

$$\begin{aligned} \mathbb{E}_z[|R_{2,n}|^p] &\leq 2^{p-1} \beta_1^p |\mathrm{P}\hat{g}(z)|^p + 2^{p-1} \beta_n^p \mathbb{E}_z\left[|\mathrm{P}\hat{g}(Z_n)|^p\right] \\ &\leq 2^{p-1}\{\beta_1^p + \beta_n^p\} \|\hat{g}\|_{\mathcal{W}}^p \{C_{\mathcal{W}} \mathcal{V}(z) + \pi(\mathcal{W}^p)\} \ . \end{aligned} \tag{72}$$

Combining (68), (70), (71) and (72) in (69) completes the proof.

■

Proposition 13 (Proposition 13, Fort and Moulines (2003)) *Assume that* P *is irreducible and aperiodic and satisfies for* $\mathbf{W}, \mathbf{f} : \mathsf{Z} \to [1, +\infty)$, $\|\mathbf{f}\|_{\mathbf{W}} \leq 1$, $b \in \mathbb{R}_+$ *and* $\mathsf{C} \in \mathcal{Z}$,

$$\mathrm{P}\mathbf{W} \leq \mathbf{W} - \mathbf{f} + b\mathbb{1}_{\mathsf{C}} .$$

Assume in addition that $\mathsf{C} \cup \{\mathbf{f} \leq 2b\} \subset \mathsf{D}$, *where* D *is a* (m, ϵ)*-small set and* $\sup_{\mathsf{D}} \mathbf{W} < +\infty$. *Then, for any distribution* λ, μ *on* Z, $\lambda(\mathbf{f}), \mu(\mathbf{f}) < +\infty$, *we have*

$$\textstyle\sum_{n\in\mathbb{N}} \|\lambda \mathrm{P}^n - \mu \mathrm{P}^n\|_{\mathbf{f}} \leq 8\epsilon^{-1}\{bm + \sup_{\mathsf{D}} \mathbf{W}\} + 2\{\lambda(\mathbf{W}) + \mu(\mathbf{W})\} ,$$

Proposition 14 *Assume UE 1.*

a) For any $\gamma > 0$, *the inequality* (65) *holds with* $f \leftarrow W^{\gamma}$, $\mathcal{W} \leftarrow W^{\gamma+1-\delta}/c_{\gamma}$ *and* $\mathcal{V} \leftarrow W^{p(\gamma+1-\delta)+1-\delta}/[c_{\gamma}^{p} c_{p(\gamma+1-\delta)}]$ *and*

$$C_f(\gamma) = \psi(\gamma) , \qquad C_{\mathcal{W}}(\gamma) = \psi(p(\gamma + 1 - \delta)) \tag{73}$$

where for any $\tilde{\gamma} > 0$, $\psi(\tilde{\gamma}) = 8\varepsilon_{\tilde{R}_{\tilde{\gamma}}}\{\mathrm{b}_{\tilde{\gamma}}/c_{\tilde{\gamma}} m_{\tilde{R}_{\tilde{\gamma}}} + \tilde{R}_{\tilde{\gamma}}^{\gamma+1-\delta}\} + 2[\mathrm{b}_{\tilde{\gamma}+1-\delta}/c_{\tilde{\gamma}+1-\delta} + 1]$, $\tilde{R}_{\tilde{\gamma}} = \{2\,\mathrm{b}_{\tilde{\gamma}}/c_{\tilde{\gamma}}\}^{1/\tilde{\gamma}} \vee R_{\tilde{\gamma}}$, *and* $R_{\tilde{\gamma}}, \mathrm{b}_{\tilde{\gamma}}, c_{\tilde{\gamma}}$ *are given in Theorem 8.*

b) For any $\gamma > 0$ *and* $p \geq 1$, (66) *holds with* $f \leftarrow W^{\gamma}$, $\mathcal{V} \leftarrow W^{p(\gamma+1-\delta)+1-\delta}$ *and*

$$C_{\mathrm{Ros},p} = 6^p C_f^p(\gamma)\{C_{\mathcal{W}}(\gamma) + \mathrm{b}_{p(\gamma+1-\delta)}/[c_{\gamma}^p c_{p(\gamma+1-\delta)}]\}(p^p + 2)/[c_{\gamma} c_{p(\gamma+1-\delta)}] . \tag{74}$$

where $C_f(\gamma), C_{\mathcal{W}}(\gamma)$ *are defined in* (73).

Proof First note that b) is an easy consequence of a), Proposition 12 and Theorem 9.
We now show a). Let $\tilde{\gamma} > 0$. Theorem 8 shows that

$$c_{\tilde{\gamma}}^{-1} \mathrm{P} W^{\tilde{\gamma}+1-\delta} \leq c_{\tilde{\gamma}}^{-1} W^{\tilde{\gamma}+1-\delta} - W^{\tilde{\gamma}} + \mathrm{b}_{\tilde{\gamma}}/c_{\tilde{\gamma}} \mathbb{1}_{\mathsf{C}_{R_{\tilde{\gamma}}}} .$$

Then, using that for any $R \geq 0$, $\{W \leq R\}$ is an (ϵ_R, m_R)-small set for P under UE 1, $\mathsf{C}_{R_{\tilde{\gamma}}} \cap \{W^{\tilde{\gamma}} \leq 2\,\mathrm{b}_{\tilde{\gamma}}/c_{\tilde{\gamma}}\} \subset \mathsf{C}_{R_{\tilde{\gamma}}}$, Proposition 13 and Theorem 9, we get that for any $z \in \mathsf{Z}$,

$$\sum_{n\in\mathbb{N}} \|\delta_z \mathrm{P}^n - \pi\|_{W^{\tilde{\gamma}}} \leq \psi(\gamma) W^{\tilde{\gamma}+1-\delta}(z)/c_{\tilde{\gamma}} ,$$

where ψ is defined by (73). Applying this result for $\tilde{\gamma} \leftarrow \gamma$ and $\tilde{\gamma} \leftarrow p(\gamma + 1 - \delta)$ completes the proof. ■

Proposition 15 *Assume UE 1.*

a) For any $\tau \geq 1$, *the inequality* (65) *holds with* $f \leftarrow V^{1/\tau}$, $\mathcal{W} \leftarrow V^{1/\tau}/(1 - \lambda^{1/\tau})$ *and* $\mathcal{V} \leftarrow V/[(1 - \lambda)(1 - \lambda^{1/\tau})]$ *and*

$$C_f(\tau) = \phi(\tau) , \qquad C_{\mathcal{W}}(\tau) = \phi(1) \tag{75}$$

with for any $\tilde{\tau} > 0$, $\phi(\tilde{\tau}) = 8\varepsilon_{R_{\tilde{\tau}}}\{\mathrm{b}^{1/\tilde{\tau}}/(1-\lambda^{1/\tilde{\tau}}) m_{R_{\tilde{\tau}}} + 2\,\mathrm{b}^{1/\tilde{\tau}}/(1-\lambda^{1/\tilde{\tau}})\} + 2[\mathrm{b}/(1-\lambda) + 1]$, $R_{\tilde{\tau}} = \log(R_0) \vee \log[2^{\tilde{\tau}}\,\mathrm{b}/(1 - \lambda^{1/\tilde{\tau}})^{\tilde{\tau}}]$ *and* λ *is defined by* (5).

b) For any $p \geq 1$, (66) holds with $f \leftarrow V^{1/p}$, $\mathcal{V} \leftarrow V$ and

$$D_{\mathrm{Ros},p} = 6^p C_f^p(p)\{C_{\mathcal{W}}(p) + b/(1-\lambda)\}(p^p+2)/[(1-\lambda)(1-\lambda^{1/p})] \,, \tag{76}$$

where $C_f(p), C_{\mathcal{W}}(p)$ are defined in (75).

Proof First note that b) is an easy consequence of a), Proposition 12 and Theorem 9.

Let $\tilde{\tau} \geq 1$. First, Jensen inequality, the fact that $t \mapsto t^{1/\tilde{\tau}}$ is sub-additive on $\mathbb{R}_+$ and UE 1 and the definition of λ in (5) imply that $\mathrm{P}V^{1/\tilde{\tau}} \leq \lambda^{1/\tilde{\tau}} V^{1/\tilde{\tau}} + \mathrm{b}^{1/\tilde{\tau}} \mathbb{1}_{\mathsf{C}_0} = V^{1/\tilde{\tau}} - (1-\lambda^{1/\tilde{\tau}})V^{1/\tilde{\tau}} + \mathrm{b}^{1/\tilde{\tau}} \mathbb{1}_{\mathsf{C}_0}$. Therefore, since $\lambda \in [0,1)$, $\mathsf{C}_0 \cup \{V^{1/\tilde{\tau}} \leq 2\,\mathrm{b}^{1/\tilde{\tau}}/(1-\lambda^{1/\tilde{\tau}})\} \subset \mathsf{C}_{R_{\tilde{\tau}}}$, using Proposition 13 and by Theorem 9, $\pi(V^{1/\tilde{\tau}}) \leq \pi(V) \leq b/(1-\lambda)$ we obtain that $\sum_{n\in\mathbb{N}} \|\delta_z \mathrm{P}^n - \pi\|_{V^{1/\tilde{\tau}}} \leq \phi(\tau) V^{1/\tilde{\tau}}(z)/(1-\lambda^{1/\tau})$ for any $z \in \mathsf{Z}$. Applying this result twice for $\tilde{\tau} \leftarrow \tau$ and $\tilde{\tau} \leftarrow 1$ completes the proof. ∎

Lemma 16 *Under assumptions of Theorem 3 for any $1 \leq q \leq \mathsf{K}$, $z \in \mathsf{Z}$ and $j \in \mathbb{N}$*

$$\begin{aligned} \mathbb{E}_z^{1/q}[\|\widetilde{A}(Z_j)\|^q] &\leq \bar{\mathrm{C}}_A V^{1/\mathsf{K}}(z), \\ \mathbb{E}_z^{1/q}[\|\bar{b}(Z_j) - b\|^q] &\leq \bar{\mathrm{C}}_b V^{1/\mathsf{K}}(z) \,, \end{aligned} \tag{77}$$

where

$$\bar{\mathrm{C}}_A := (\|A\| + d\,\mathrm{C}_A(\beta\mathsf{K}/\mathrm{e})^\beta\{1 + b/(1-\lambda)\}^{1/\mathsf{K}}) \tag{78}$$

and

$$\bar{\mathrm{C}}_b := (\|b\| + d\,\mathrm{C}_b\{1 + b/(1-\lambda)\}^{1/\mathsf{K}}). \tag{79}$$

Proof We first note that using

$$\begin{aligned} \mathbb{E}_z^{1/q}[\|\widetilde{A}(Z_j)\|^q] &\leq \|A\| + \mathbb{E}_z^{1/q}[\|\bar{A}(Z_j)\|^q] \leq \|A\| + d\,\mathrm{C}_A \sup_{x\geq 1} \frac{\log^\beta x}{x^{1/\mathsf{K}}}\{\mathrm{P}^j V^{q/\mathsf{K}}(z)\}^{1/q} \\ &\leq (\|A\| + d\,\mathrm{C}_A(\beta\mathsf{K})^\beta \mathrm{e}^{-\beta}\{1 + b/(1-\lambda)\}^{1/\mathsf{K}}) V^{1/\mathsf{K}}(z). \end{aligned}$$

Similarly, one may prove the second statement of the lemma. ∎

Appendix D. Proofs for Theorem 1

In this section, we provide the core lemmas that are employed for the proof of Theorem 1 in Section 2.1.

D.1. Technical and preliminary results

Lemma 17 (Lyapunov Lemma) *A matrix A is Hurwitz if and only if for any positive symmetric matrix $P = P^\top \succ 0$ there is $Q = Q^\top \succ 0$ that satisfies the Lyapunov equation*

$$A^\top Q + QA = -P \,.$$

In addition, Q is unique.

Proof See (Poznyak, 2008, Lemma 9.1, p. 140). ■

Lemma 18 *Assume that $-A$ is a Hurwitz matrix. Let Q be the unique solution of the Lyapunov equation $A^\top Q + QA = \mathrm{I}$. Then, for any $\alpha \in [0, (1/2)\|A\|_Q^{-2}\|Q\|^{-1}]$, we get $\|\mathrm{I} - \alpha A\|_Q^2 \leq (1 - a\alpha)$ with $a = (1/2)\|Q\|^{-1}$. In particular, for any $\alpha \in [0, (1/2)\|A\|_Q^{-2}\|Q\|^{-1}]$, $\|\mathrm{I} - \alpha A\| \leq \sqrt{\kappa_{\mathsf{Q}}}(1 - a\alpha/2)$, where $\kappa_{\mathsf{Q}} = \lambda_{\mathsf{min}}^{-1}(Q)\lambda_{\mathsf{max}}(Q)$. If in addition $\alpha \leq \|Q\|^2$ then $1 - a\alpha \geq 1/2$.*

Proof For any $x \in \mathbb{R}^d \setminus \{0\}$, we get

$$\frac{x^\top(\mathrm{I} - \alpha A)^\top Q(\mathrm{I} - \alpha A)x}{x^\top Q x} = 1 - \alpha\frac{\|x\|^2}{x^\top Q x} + \alpha^2\frac{x^\top A^\top Q A x}{x^\top Q x}$$

Hence, we get that for all $\alpha \in [0, (1/2)\|A\|_Q^{-2}\|Q\|^{-1}]$,

$$1 - \alpha\frac{\|x\|^2}{x^\top Q x} + \alpha^2\frac{x^\top A^\top Q A x}{x^\top Q x} \leq 1 - \alpha\|Q\|^{-1} + \alpha^2\|A\|_Q^2 \leq 1 - (1/2)\|Q\|^{-1}\alpha\,.$$

The proof is completed using that for any $t \in [0, 1]$, $(1 - t)^{1/2} \leq 1 - t/2$ and that for any matrix $A \in \mathbb{R}^{d \times d}$, $\|A\|_Q \leq \kappa_{\mathsf{Q}}^{1/2}\|A\|$. ■

Lemma 19 *Let $\alpha > 0$ and $(u_i)_{i \geq 1}$ be a sequence of non-negative numbers. Then, for any $n \in \mathbb{N}$, $n \geq 2$, and any $\epsilon \in (0, 1)$,*

$$1 + \alpha^n \textstyle\prod_{i=1}^n u_i \leq \exp\left\{\alpha^{1+\epsilon}(1 + \epsilon)^{-1}\sum_{i=1}^n u_i^{1+\epsilon}\right\}\,.$$

Proof First note that for any $\beta \geq 1$ and $t \geq 0$, $1 \leq t^{1/\beta}(t^{-1} + 1)$ which implies that $1 + t \leq \exp\left(\beta t^{1/\beta}\right)$. Using this inequality for $\beta = n/(1 + \epsilon)$ and the inequality of arithmetic and geometric means, we get

$$\begin{aligned} 1 + \alpha^n \textstyle\prod_{i=1}^n u_i &\leq \exp\left(\tfrac{n}{1+\epsilon}\alpha^{1+\epsilon}(\textstyle\prod_{i=1}^n u_i)^{(1+\epsilon)/n}\}\right) \leq \exp\left(\tfrac{n}{1+\epsilon}\alpha^{1+\epsilon}(\tfrac{1}{n}\textstyle\sum_{i=1}^n u_i)^{1+\epsilon}\right) \\ &\leq \exp\left(\tfrac{\alpha^{1+\epsilon}}{1+\epsilon}\textstyle\sum_{i=1}^n u_i^{1+\epsilon}\right)\,, \end{aligned}$$

where the last inequality follows from Jensen's inequality. ■

Lemma 20 *Let $(\mathfrak{F}_\ell)_{\ell \geq 0}$ be some filtration and a sequence of non-negative random variables $(\xi_\ell)_{\ell \geq 0}$ is adopted to this filtration. Then, for any $N \in \mathbb{N}$, $p \in \mathbb{N}$, it holds*

$$\mathbb{E}\left[\prod_{\ell=1}^N \xi_\ell^p\right] \leq \left\{\mathbb{E}\left[\prod_{\ell=1}^N \mathbb{E}[\xi_\ell^{2p}|\mathfrak{F}_{\ell-1}]\right]\right\}^{1/2}\,. \tag{80}$$

Proof Denote for any $k \in \mathbb{N}$, $p \in \mathbb{N}$,

$$B_{k,p} = (\prod_{\ell=1}^k \mathbb{E}[\xi_\ell^p|\mathfrak{F}_{\ell-1}])^{-1},\; B_{0,p} := 1,\; y_{k,p} = \xi_k^p y_{k-1},\; y_{0,p} := 1\,.$$

It is straightforward to check that for any $p \in \mathbb{N}$,

$$\mathbb{E}[B_{k+1,p} y_{k+1,p}] = \mathbb{E}[B_{k+1,p} \mathbb{E}[\xi_{k+1}^p | \mathfrak{F}_k] y_{k,p}] = \mathbb{E}[B_{k,p} y_{k,p}] = \cdots = \mathbb{E}[B_{0,p} y_{0,p}] = 1 .$$

This fact implies that

$$\begin{aligned}\mathbb{E}\left[\prod_{\ell=1}^N \xi_\ell\right] &= \mathbb{E}[y_N] = \mathbb{E}[y_{N,p} B_{N,2p}^{1/2} B_{N,2p}^{-1/2}] \\ &\leq \{\mathbb{E}[y_{N,2p} B_{N,2p}]\}^{1/2} \{\mathbb{E}[B_{N,2p}^{-1}]\}^{1/2} = \left\{\mathbb{E}\left[\prod_{\ell=1}^N \mathbb{E}[\xi_\ell^{2p} | \mathfrak{F}_{\ell-1}]\right]\right\}^{1/2} .\end{aligned}$$

Hence, (80) is proved. ∎

D.2. Core Lemmas

Lemma 21 *Assume that the conditions of Theorem 1 holds. Then, for any $\ell \in \{1, \ldots, N\}$, and $p \geq 1$,*

$$\left(1 + \kappa_{\mathsf{Q}}^{1/2} \mathrm{e}^a \|R_\ell\|\right)^p \leq \exp\{p C^{(0)} h^2 \alpha_{j_{\ell-1}+1}^2\} ,$$

where R_ℓ is given in (15) and

$$C^{(0)} = (1/2) \kappa_{\mathsf{Q}}^{1/2} \|A\|^2 \exp(\|A\| + a) . \tag{81}$$

Proof Let $\ell \in \{1, \ldots, N\}$, and $p \geq 1$. Using the definition of R_ℓ and since $(\alpha_i)_{i \in \mathbb{N}}$ is non-increasing, we get

$$\begin{aligned}\|R_\ell\| &\leq \sum_{r=2}^h \binom{h}{r} \alpha_{j_{\ell-1}+1}^r \|A\|^r \leq \alpha_{j_{\ell-1}+1}^2 \|A\|^2 \sum_{r=0}^{h-2} \binom{h}{r+2} \alpha_{j_{\ell-1}+1}^r \|A\|^r \\ &\leq 2^{-1} \alpha_{j_{\ell-1}+1}^2 h^2 \|A\|^2 (1 + \alpha_{j_{\ell-1}+1} \|A\|)^{h-2} \leq 2^{-1} \alpha_{j_{\ell-1}+1}^2 h^2 \|A\|^2 \exp\{\alpha_{j_{\ell-1}+1} h \|A\|\} ,\end{aligned}$$

where we have used for the last two inequalities, the upper bounds $\binom{h}{r+2} \leq \binom{h-2}{r}(h^2/2)$ for any $r \in \{0, \ldots, h-2\}$ and $(1+t) \leq \mathrm{e}^t$ for any $t \geq 0$. It yields using that $\alpha_{\infty,p} h \leq 1$ that $\|R_\ell\| \leq 2^{-1} \alpha_{j_{\ell-1}+1}^2 h^2 \|A\|^2 \mathrm{e}^{\|A\|}$. The proof is then completed using the bound $(1+t) \leq \mathrm{e}^t$ for any $t \geq 0$ again. ∎

Lemma 22 *Assume that the conditions of Theorem 1 holds. Then, for any $\ell \in \{1, \ldots, N\}$, and $p \geq 1$, almost surely it holds*

$$\left(1 + \kappa_{\mathsf{Q}}^{1/2} \mathrm{e}^a \|\bar{R}_\ell\|\right)^{2p} \leq \exp\left\{2^{h+1} p C^{(1)} \alpha_{j_{\ell-1}+1}^{1+\varepsilon} \textstyle\sum_{k=j_{\ell-1}+1}^{j_\ell} W^\delta(Z_{k-m})\right\} ,$$

where $\bar{R}_\ell$ is defined by (15) and

$$C^{(1)} = (\kappa_{\mathsf{Q}}^{1/2} d \mathrm{e}^a \, \mathrm{C}_A)^{1+\varepsilon} / (1+\varepsilon) . \tag{82}$$

Proof Let $\ell \in \{1, \dots, N\}$, and $p \geq 1$. Using the definition of $\bar{R}_\ell$, we consider the following decomposition

$$\bar{R}_\ell = \sum_{r=2}^h \bar{R}_\ell^{(r)}, \quad \bar{R}_\ell^{(r)} = (-1)^r \sum_{(i_1,\dots,i_r)\in \mathsf{I}_r^\ell} (\prod_{u=1}^r \alpha_{i_u}) A^r \;.$$

Then, using that $(1+a+b) \leq (1+a)(1+b)$ for $a, b \geq 0$ and $(\alpha_i)_{i\in\mathbb{N}}$ is non-increasing yields

$$\begin{aligned}(1+\kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a\|\bar{R}_\ell\|)^{2p} &\leq \prod_{r=2}^h (1+\kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a\|\bar{R}_\ell^{(r)}\|)^{2p} \\ &\leq \prod_{r=2}^h \prod_{(i_1,\dots,i_r)\in \mathsf{I}_r^\ell} (1+\kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a \alpha_{j_{\ell-1}+1}^r \prod_{k=1}^r \|\bar{A}(Z_{i_k - m})\|)^{2p}\end{aligned}$$

Using Theorem 19, $r \geq 2$ and $\varepsilon \in (0,1)$ in A1,

$$\begin{aligned}(1+\kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a\|\bar{R}_\ell\|)^{2p} &\leq \prod_{r=2}^h \prod_{(i_1,\dots,i_r)\in \mathsf{I}_r^\ell} \exp\left(\frac{2p(\kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a)^{1+\varepsilon}\alpha_{j_{\ell-1}+1}^{1+\varepsilon}}{1+\varepsilon} \sum_{u=1}^r \|\bar{A}(Z_{i_u-m})\|^{1+\varepsilon}\right) \\ &\leq \exp\left(\frac{2p(\kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a)^{1+\varepsilon}\alpha_{j_{\ell-1}+1}^{1+\varepsilon}}{1+\varepsilon} \sum_{k=j_{\ell-1}+1}^{j_\ell} \|\bar{A}(Z_{k-m})\|^{1+\varepsilon} \sum_{r=2}^h \sum_{(i_1,\dots,i_r)\in \mathsf{I}_r^\ell}\right) \\ &\leq \exp\left(\frac{2^{h+1}p(\kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a)^{1+\varepsilon}\alpha_{j_{\ell-1}+1}^{1+\varepsilon}}{1+\varepsilon} \sum_{k=j_{\ell-1}+1}^{j_\ell} \|\bar{A}(Z_{k-m})\|^{1+\varepsilon}\right) \;.\end{aligned}$$

The proof follows from A1 which implies that $\|\bar{A}(z)\|^{1+\varepsilon} \leq \left\|\bar{A}(z)\right\|_{\mathrm{F}}^{1+\varepsilon} \leq d^{1+\varepsilon}\,\mathrm{C}_A^{1+\varepsilon}\,W^\delta(z)$, for any $z \in \mathsf{Z}$. ■

Under A1, define for $n \in \mathbb{N}$, $n \geq 1$, $(\alpha_i)_{i\in\mathbb{N}}$ a non-increasing positive sequence,

$$r_A = \min\{s \geq 0 \,:\, \beta \leq 2\delta - 1 - (1-\delta)/s\} \;, \qquad \tilde{S}_n = \sum_{k=1}^n \alpha_k(\bar{A}(Z_k) - A) \;. \tag{83}$$

Lemma 23 *Assume that the conditions of Theorem 1 holds. For any $n \in \mathbb{N}^*$, $p \geq 1 \vee (r_A/4)$,*

$$\mathbb{E}_z^{1/4p}[(1+\kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a\|\tilde{S}_n\|)^{4p}] \leq \exp\{C_p^{(2)}\alpha_1 n^{1/2} W^\delta(z)\} \;,$$

where $r_A, \tilde{S}_n$ are defined in (83), and

$$C_p^{(2)} = \kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a d\,\mathrm{C}_A(4C_{\mathrm{Ros},4p})^{1/4p} \;, \tag{84}$$

with $C_{\mathrm{Ros},4p}$ given in (74).

Proof First by Minkowski's inequality, we get

$$\mathbb{E}_z^{1/4p}[(1+\kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a\|\tilde{S}_n\|)^{4p}] \leq 1 + \kappa_{\mathsf{Q}}^{1/2}\mathrm{e}^a\mathbb{E}_z^{1/4p}[\|\tilde{S}_n\|^{4p}] \;. \tag{85}$$

In addition, note that denoting by $[\tilde{S}_n]_{i,j}$, the (i,j)-th component of $\tilde{S}_n$, using the Jensen inequality, we get

$$\begin{aligned}\mathbb{E}_z[\|\tilde{S}_n\|^{4p}] &\leq \mathbb{E}_z[(\sum_{i_1,i_2=1}^d [\tilde{S}_n]_{i_1,i_2}^2)^{4p/2}] \leq d^{4p-2}\mathbb{E}_z[\sum_{i_1,i_2=1}^d |[\tilde{S}_n]_{i_1,i_2}|^{4p}] \\ &\leq d^{4p}\max_{i_1,i_2\in\{1,\cdots,d\}} \mathbb{E}_z[|[\tilde{S}_n]_{i_1,i_2}|^{4p}] \;.\end{aligned} \tag{86}$$

Using UE 1 and applying Proposition 14-b) with $\gamma \leftarrow \beta$ and using that $(\alpha_i)_{i\in\mathbb{N}}$ is non-increasing, we obtain that for any $i_1, i_2 \in \{1, \ldots, N\}$,

$$\mathbb{E}_z[|[\tilde{S}_n]_{i_1,i_2}|^{4p}] \leq C_{\mathrm{Ros},4p} C_A^{4p} (\alpha_1^{4p} n^{4p/2} + 3\alpha_1^{4p}) W^{4p(\beta+1-\delta)+1-\delta} \leq 4C_{\mathrm{Ros},4p} C_A^{4p} \alpha_1^{4p} n^{2p} W^{4p\delta}(z) , \tag{87}$$

using for the last inequality that $W(z) \geq 1$ and $4p(\beta+1-\delta)+1-\delta \geq 4p\delta$ since $4p \geq r_A$, $\beta \leq 2\delta - 1 - (1-\delta)/r_A$ by (83). Combining (85)-(86)-(87), we get

$$\mathbb{E}_z^{1/4p}[(1 + \kappa_{\mathsf{Q}}^{1/2} \mathrm{e}^a \|\tilde{S}_n\|)^{4p}] \leq 1 + \kappa_{\mathsf{Q}}^{1/2} \mathrm{e}^a [4C_{\mathrm{Ros},4p}]^{1/4p} d C_A \alpha_1 n^{1/2} W^{\delta}(z) .$$

Using that $1 + t \leq \mathrm{e}^t$ completes the proof. ■

Corollary 24 *Assume that the conditions of Theorem 1 holds. For any $n \in \mathbb{N}^*$, $p \geq 1$,*

$$\mathbb{E}_z^{1/4p}[(1 + \kappa_{\mathsf{Q}}^{1/2} \mathrm{e}^a \|\tilde{S}_n\|)^{4p}] \leq \exp\{C_p^{(2)} \alpha_1 n^{1/2} W^{\delta}(z)\} ,$$

where $r_A, \tilde{S}_n$ are defined in (83), and

$$C_p^{(2)} = \kappa_{\mathsf{Q}}^{1/2} \mathrm{e}^a d\, C_A (4C_{\mathrm{Ros},4\tilde{p}})^{1/4\tilde{p}} , \quad \tilde{p} = \max(p, r_A/4) , \tag{88}$$

with $C_{\mathrm{Ros},4\tilde{p}}$ given in (74).

Proof The proof is a simple consequence of Lemma 23 and Jensen's inequality. ■

D.3. Proof of Theorem 1

Details on the Step 3. We have all the elements to conclude the proof of the theorem. It is essentially a question of adjusting the constants and combining the different bounds obtained above. To simplify notations, we first introduce the auxiliary quantities:

$$D_p^{(1)} = C^{(1)} 2^h \alpha_{\infty,p}^{1+\varepsilon} + C_p^{(2)} h^{1/2} \alpha_{\infty,p}$$
$$D_p^{(2)} = C^{(0)} h \alpha_{\infty,p} + C^{(1)} 2^h \alpha_{\infty,p}^{\varepsilon} \tilde{\mathrm{b}} + C_p^{(2)} h^{-1/2} (\tilde{\mathrm{b}} - \log(1-\lambda)).$$

Substituting (22), (23) into (13) and using that $\sup_{i\in\mathbb{N}^*} \alpha_i \leq \alpha_{\infty,p}$, we get

$$\mathbb{E}_{z_m}^{1/p}[\|\Gamma_{m+1:n}(Z_{1:n-m})\|^p]$$
$$\leq \kappa \exp(a\alpha_{\infty,p} h) \exp\big\{ -(a/2) \textstyle\sum_{i=m+1}^n \alpha_i + D_p^{(2)} h \sum_{\ell=1}^N \alpha_{j_{\ell-1}+1} \big\} \exp[D_p^{(1)} W(z_m)] .$$

We set the block size and the upper bound to the step sizes as

$$h = \big\lceil \big(12 C_p^{(2)} (\tilde{\mathrm{b}} - \log(1-\lambda))/a\big)^2 \big\rceil , \tag{89}$$

$$\alpha_{\infty,p} = \min\left[\frac{1}{a}, \frac{1}{h}, \frac{1}{2\|A\|_{Q}^2 \|Q\|}, \frac{a}{12hC^{(0)}}, \left(\frac{a}{12C^{(1)}2^h}\right)^{\frac{1}{\varepsilon}}, \left(\frac{c \wedge 1/2}{2pC^{(1)}2^h}\right)^{\frac{1}{1+\varepsilon}}, \frac{c \wedge 1}{4pC_p^{(2)} h^{1/2}}\right] . \tag{90}$$

This yields $D_p^{(1)} \leq 1/(2p)$, $D_p^{(2)} \leq a/4$. Together with $h \sum_{\ell=1}^N \alpha_{j_{\ell-1}+1} \leq \alpha_{\infty,p} + \sum_{i=m+1}^n \alpha_i$, we get

$$\mathbb{E}_{z_m}^{1/p}[\|\Gamma_{m+1:n}(Z_{1:n-m})\|^p] \leq \kappa \mathrm{e}^{5a\alpha_{\infty,p} h/4} \exp\{-(a/4) \textstyle\sum_{i=m+1}^n \alpha_i\} V^{1/(2p)}(z_m) . \tag{91}$$

Combining (91), (10), and Jensen's inequality yields the statement of the theorem with the constant

$$\mathrm{C}_{\mathsf{st},p} = \kappa_{\mathsf{Q}}^{1/2} \exp(5a\alpha_{\infty,p} h/4) \left(\lambda^{m/(2p)} + [\mathrm{b}/(1-\lambda)]^{1/(2p)}\right) . \tag{92}$$

Appendix E. Proofs of Section 3

This section provides the missing lemmas and proofs that were required in Section 3.

E.1. Technical lemmas

Lemma 25 *Let $a > 0$ and $(\alpha_k)_{k\geq 0}$ be a non-increasing sequence such that $\alpha_0 < 1/a$. Then*

$$\sum_{j=0}^{n+1} \alpha_j \prod_{l=j+1}^{n+1} (1 - \alpha_l a) = \frac{1}{a} \left\{ 1 - \prod_{l=1}^{n+1} (1 - \alpha_l a) \right\}$$

Proof Let us denote $u_{j:n+1} = \prod_{l=j}^{n+1}(1 - \alpha_l a)$. Then, for $j \in \{1, \dots, n+1\}$, $u_{j+1:n+1} - u_{j:n+1} = a\alpha_j u_{j+1:n+1}$. Hence,

$$\sum_{j=0}^{n+1} \alpha_j \prod_{l=j+1}^{n+1} (1 - \alpha_l a) = \frac{1}{a} \sum_{j=1}^{n+1} (u_{j+1:n+1} - u_{j:n+1}) = a^{-1}(1 - u_{1:n+1}) \,.$$

■

Lemma 26 *Let $b > 0$ and $(\alpha_k)_{k\geq 0}$ be a non-increasing sequence such that $\alpha_0 < 1/(2b)$.*

- *Assume $\alpha_k - \alpha_{k+1} \leq \mathrm{c}_\alpha \, \alpha_{k+1}^2$ with $\mathrm{c}_\alpha \leq b/2$. Then for $p \in (1, 2]$,*

$$\sum_{k=1}^{n+1} \alpha_k^p \prod_{j=k+1}^{n+1} (1 - b\alpha_j) \leq (2/b)\alpha_{n+1}^{p-1}.$$

- *Assume $\alpha_k - \alpha_{k+1} \leq \mathrm{c}_\alpha \, \alpha_{k+1}^2$, $\alpha_k/\mathcal{A}_{k+1} \leq (2/3)\,\mathrm{c}_\alpha$ with $\mathrm{c}_\alpha \leq b/4$. We additionally assume that $\alpha_0 \leq (2\,\mathrm{c}_\alpha)^{-1}$. Then for any $p \in (1, 2], q \in [0, 1]$*

$$\sum_{k=1}^{n+1} \alpha_k^p \mathcal{A}_k^q \prod_{j=k+1}^{n+1} (1 - b)\alpha_j) \leq (2/b)\alpha_{n+1}^{p-1}\mathcal{A}_{n+1}^q.$$

Proof For the first part

$$\begin{aligned}
\sum_{k=1}^{n+1} \alpha_k^p \prod_{j=k+1}^{n+1} (1 - b\alpha_j) &= \alpha_{n+1}^{p-1} \sum_{k=1}^{n+1} \alpha_k \prod_{j=k+1}^{n+1} \left(\frac{\alpha_{j-1}}{\alpha_j} \right)^{p-1} (1 - b\alpha_j) \\
&\leq \sqrt{\alpha_{n+1}} \sum_{k=1}^{n+1} \alpha_k \prod_{j=k+1}^{n+1} (1 + \mathrm{c}_\alpha \, \alpha_j)(1 - b\alpha_j) \\
&\leq \alpha_{n+1}^{p-1} \sum_{k=1}^{n+1} \alpha_k \prod_{j=k+1}^{n+1} (1 - (b/2)\alpha_j) \leq (2/b)\alpha_{n+1}^{p-1},
\end{aligned}$$

where on the last step we used Lemma 25. For the second part, we first note that

$$\frac{\mathcal{A}_{j-1}}{\mathcal{A}_j} \leq 1 + \frac{\alpha_{j-1}^2}{\mathcal{A}_j} \leq 1 + (2/3)\,\mathrm{c}_\alpha\,\alpha_{j-1} \leq 1 + (2/3)\,\mathrm{c}_\alpha\,\alpha_j + (2/3)\,\mathrm{c}_\alpha^2\,\alpha_j^2 \leq 1 + \mathrm{c}_\alpha\,\alpha_j .$$

Similarly to the first part,

$$\begin{aligned}\sum_{k=1}^{n+1} \alpha_k^p \mathcal{A}_k^q \prod_{j=k+1}^{n+1} (1 - b\alpha_j) &= \alpha_{n+1}^{p-1}\mathcal{A}_{n+1}^q \sum_{k=1}^{n+1} \alpha_k \prod_{j=k+1}^{n+1} \left(\frac{\alpha_{j-1}}{\alpha_j}\right)^{p-1} \left(\frac{\mathcal{A}_{j-1}}{\mathcal{A}_j}\right)^q (1 - b\alpha_j) \\ &\leq \alpha_{n+1}^{p-1}\mathcal{A}_{n+1}^q \sum_{k=1}^{n+1} \alpha_k \prod_{j=k+1}^{n+1} (1 + \mathrm{c}_\alpha\,\alpha_j)^2 (1 - b\alpha_j) \\ &\leq \alpha_{n+1}^{p-1}\mathcal{A}_{n+1}^q \sum_{k=1}^{n+1} \alpha_k \prod_{j=k+1}^{n+1} (1 - (b/2)\alpha_j) \leq (2/b)\alpha_{n+1}^{p-1}\mathcal{A}_{n+1}^q ,\end{aligned}$$

where we also used Lemma 25. ■

To estimate moments of $\|S_{j+1:n+1}\|^p$ that was defined in (40), we first derive an alternative expression for the term. For this aim we prove the following lemma. Define

$$D_{j:k} := \sum_{\ell=j}^{k} \alpha_\ell \widetilde{A}(Z_\ell).$$

Here we also assume that $D_{j:k} = 0$ if $j > k$. Recall that $S_{j:k} = 0$ if $j > k$ and $G_{j:k} = 0$ if $j > k + 1$.

Lemma 27 *For any* $0 \leq k \leq n$

$$S_{k+1:n+1} = -\sum_{\ell=k+1}^{n+1} \alpha_\ell G_{\ell+1:n+1} A D_{\ell:n+1} G_{k+1:\ell-2} + \sum_{\ell=k+1}^{n+1} \alpha_{\ell-1} G_{\ell+1:n+1} D_{\ell:n+1} A G_{k+1:\ell-2}.$$

Proof By definition of $D_{k:n+1}$

$$S_{k+1:n+1} = -\sum_{\ell=k+1}^{n+1} G_{\ell+1:n+1}(D_{\ell:n+1} - D_{\ell+1:n+1}) G_{k+1:\ell-1}.$$

Simple algebraic manipulations lead to

$$\begin{aligned}S_{k+1:n+1} &= -\sum_{\ell=k+1}^{n+1} G_{\ell+1:n+1} D_{\ell:n+1} G_{k+1:\ell-1} + \sum_{\ell=k+1}^{n+1} G_{\ell:n+1} D_{\ell:n+1} G_{k+1:\ell-2} \\ &= -\sum_{\ell=k+1}^{n+1} (G_{\ell+1:n+1} - G_{\ell:n+1}) D_{\ell:n+1} G_{k+1:\ell-2} \\ &\quad - \sum_{\ell=k+1}^{n+1} G_{\ell:n+1} D_{\ell:n+1} (G_{k+1:\ell-1} - G_{k+1:\ell-2}).\end{aligned}$$

Calculating the difference in the brackets we obtain the statement of this lemma. ■

Lemma 28 *Under assumptions of Theorem 3 for any $2 \le p \le \mathsf{K}$ and $z \in \mathsf{Z}$,*

$$\mathbb{E}_z^{1/p}[\|D_{\ell:n+1}\|^p] \le 4dC_{\mathrm{Ros},p}^{1/p}(\mathrm{C}_A + \|A\|)V^{1/\mathsf{K}}(z)\mathcal{A}_{\ell:n+1}^{1/2}.$$

Proof Proof follows from Proposition 12 with $f = \mathcal{W} = V^{1/\mathsf{K}}$, $\mathcal{V} = V^{p/\mathsf{K}}$. ■

Lemma 29 *Under assumptions of Theorem 3 for any $2 \le p \le \mathsf{K}$ and $z \in \mathsf{Z}$,*

$$\mathbb{E}_z^{1/p}[\|S_{k+1:n+1}\|^p] \le \mathrm{C}_{\mathsf{S},p} \sum_{\ell=k+1}^{n+1} \alpha_\ell \mathcal{A}_{\ell:n+1}^{1/2} \prod_{j=k+1}^{n+1} (1 - a\alpha_j)^{1/2} V^{1/\mathsf{K}}(z),$$

where

$$\mathrm{C}_{\mathsf{S},p} := 24\kappa_{\mathsf{Q}} dC_{\mathrm{Ros},p}^{1/p}(\mathrm{C}_A + \|A\|)\|A\|. \tag{93}$$

Proof Recall that

$$S_{k+1:n+1} = -\sum_{\ell=k+1}^{n+1} \alpha_\ell G_{\ell+1:n+1} A D_{\ell:n+1} G_{k+1:\ell-2} + \sum_{\ell=k+1}^{n+1} \alpha_{\ell-1} G_{\ell+1:n+1} D_{\ell:n+1} A G_{k+1:\ell-2}.$$

Applying Minkowski's inequality and Lemma 28 we get

$$\mathbb{E}_z^{1/p}[\|S_{k+1:n+1}\|^p] \le \mathrm{C}_{\mathsf{S},p} \sum_{\ell=k+1}^{n+1} \alpha_\ell \prod_{j=\ell+1}^{n+1} \sqrt{1 - a\alpha_j} \prod_{j=k+1}^{\ell-1} \sqrt{1 - a\alpha_j} \mathcal{A}_{\ell:n+1}^{1/2} V^{1/\mathsf{K}}(z).$$

■

Lemma 30 *Denote $\mathfrak{F}_k := \sigma\{Z_s, s \ge k\}, k \ge 0$. Let A_k be a sequence of $d \times d$ random matrices such that A_k is $\mathfrak{F}_k$-measurable. Assume that Z_k^* is independent of $\mathfrak{F}_k$. Then*

$$\mathbb{E}_z^{\frac{1}{p}}\big[\big\|\sum_{k=1}^{n} A_k \bar{\varepsilon}(Z_k^*)\big\|_2^p\big] \le \mathrm{C}_{\mathsf{B},p}\big(\sum_{k=1}^{n} \mathbb{E}^{\frac{2}{p}}[\|A_k\|^p]\big)^{1/2} V^{1/\mathsf{K}}(z),$$

where

$$\mathrm{C}_{\mathsf{B},p} := d^{3/2}\left\{\frac{2\mathrm{B}_V \, \mathrm{C}_{\bar{\varepsilon}}}{\sqrt{1-\rho}} + 2\bar{\mathrm{C}}_{\bar{\varepsilon}}(18\sqrt{2}p)\right\}. \tag{94}$$

Proof We first reduce the problem to univariate one. Applying Minkowski's inequality we get

$$\begin{aligned}
\mathbb{E}_z^{1/p}\big[\big\|\sum_{k=1}^{n} A_k \bar{\varepsilon}(Z_k^*)\big\|_2^p\big] &= \mathbb{E}_z^{1/p}\big[\big|\sum_{\ell_1=1}^{d}\big\{\sum_{\ell_2=1}^{d}\sum_{k=1}^{n}[A_k]_{\ell_1\ell_2}[\bar{\varepsilon}(Z_k^*)]_{\ell_2}\big\}^2\big|^{p/2}\big] \\
&\le \left\{\sum_{\ell_1=1}^{d} \mathbb{E}_z^{2/p}\big[\big|\sum_{\ell_2=1}^{d}\sum_{k=1}^{n}[A_k]_{\ell_1\ell_2}[\bar{\varepsilon}(Z_k^*)]_{\ell_2}\big|^p\big]\right\}^{1/2} \\
&\le \left\{\sum_{\ell_1=1}^{d}\left\{\sum_{\ell_2=1}^{d} \mathbb{E}_z^{1/p}\big[\big|\sum_{k=1}^{n}[A_k]_{\ell_1\ell_2}[\bar{\varepsilon}(Z_k^*)]_{\ell_2}\big|^p\big]\right\}^2\right\}^{1/2}.
\end{aligned}$$

Consider

$$I := \mathbb{E}_z^{1/p}\big[\big|\sum_{k=1}^{n}[A_k]_{\ell_1\ell_2}[\bar{\varepsilon}(Z_k^*)]_{\ell_2}\big|^p\big].$$

We decompose it into two parts, $I \leq I_1 + I_2$,

$$I_1 := \mathbb{E}_z^{\frac{1}{p}}\big[\big|\sum_{k=1}^{n}[A_k]_{\ell_1\ell_2}([\bar{\varepsilon}(Z_k^*)]_{\ell_2} - \mathbb{E}_z[[\bar{\varepsilon}(Z_k^*)]_{\ell_2}])\big|^p\big],$$

$$I_2 := \mathbb{E}_z^{\frac{1}{p}}\big[\big|\sum_{k=1}^{n}[A_k]_{\ell_1\ell_2}\mathbb{E}_z[[\bar{\varepsilon}(Z_k^*)]_{\ell_2}]\big|^p\big]$$

The term I_2 may be estimated as follows

$$\begin{aligned}|I_2| &\leq 2\mathrm{B}_V\,\mathrm{C}_{\bar{\varepsilon}}\,\mathbb{E}_z^{\frac{1}{p}}\big[\big|\sum_{k=1}^{n}[A_k]_{\ell_1\ell_2}\rho^k\big|^p\big]V^{1/\mathsf{K}}(z)\\ &\leq \frac{2\mathrm{B}_V\,\mathrm{C}_{\bar{\varepsilon}}}{\sqrt{1-\rho}}\mathbb{E}_z^{\frac{1}{p}}\big[\big|\sum_{k=1}^{n}[A_k]_{\ell_1\ell_2}^2\big|^{p/2}\big]V^{1/\mathsf{K}}(z)\\ &\leq \frac{2\mathrm{B}_V\,\mathrm{C}_{\bar{\varepsilon}}}{\sqrt{1-\rho}}\bigg\{\sum_{k=1}^{n}\mathbb{E}_z^{\frac{2}{p}}[|[A_k]_{\ell_1\ell_2}|^p]\bigg\}^{1/2}V^{1/\mathsf{K}}(z)\end{aligned}$$

Applying Burkholder's inequality, see (Hall and Heyde, 1980, Theorem 2.10), Minkowski's inequality and lemma 16 we obtain

$$\begin{aligned}\mathbb{E}^{\frac{1}{p}}\big[\big|\sum_{k=1}^{n}[A_k]_{\ell_1\ell_2}[\xi_k]_{\ell_2}\big|^p\big] &\leq (18\sqrt{2}p)\mathbb{E}^{\frac{1}{p}}\Bigg[\bigg\{\sum_{k=1}^{n}|[A_k]_{\ell_1\ell_2}|^2([\bar{\varepsilon}(Z_k^*)]_{\ell_2} - \mathbb{E}_z[[\bar{\varepsilon}(Z_k^*)]_{\ell_2}])^2\bigg\}^{p/2}\Bigg]\\ &\leq 2\bar{\mathrm{C}}_{\bar{\varepsilon}}(18\sqrt{2}p)\bigg\{\sum_{k=1}^{n}\mathbb{E}^{\frac{2}{p}}[|[A_k]_{\ell_1\ell_2}|^p]\bigg\}^{1/2}V^{1/\mathsf{K}}(z)\end{aligned}$$

Finally,

$$\mathbb{E}_z^{\frac{1}{p}}\big[\big\|\sum_{k=1}^{n}A_k\bar{\varepsilon}(Z_k^*)\big\|_2^p\big] \leq \mathrm{C}_{\mathsf{B},p}\big(\sum_{k=1}^{n}\mathbb{E}^{\frac{2}{p}}[\|A_k\|^p]\big)^{1/2}V^{1/\mathsf{K}}(z).$$

■

E.2. Proof of Theorem 4

Define the following constraint on the step size

$$\alpha_{\infty,p}^{(1)} := \alpha_{\infty,2p} \wedge \rho \wedge \mathrm{e}^{-1} \wedge (2\,\mathrm{c}_\alpha)^{-1}, \tag{95}$$

where $\alpha_{\infty,2p}$ and ρ are defined in (90) and (6) respectively, and c_α is from A 5. Let us re-state Theorem 4 as follows.

Theorem 31 *Let $\mathsf{K} \geq 32$ and assume UE 1, A 1, A 2, and A 3. For any $2 \leq p \leq \mathsf{K}/16$, any non-increasing sequence $(\alpha_k)_{k\in\mathbb{N}}$ satisfying $\alpha_0 \in (0, \alpha^{(1)}_{\infty,p})$ and such that $\alpha_k \equiv \alpha$ or A 5 holds, $z \in \mathsf{Z}$, $n \in \mathbb{N}$, it holds*

$$\mathbb{E}_z^{1/p}[\|H_n^{(0)}\|^p] \leq V^{3/\mathsf{K}+9/(16p)}(z) \begin{cases} \mathrm{C}_p^{(\mathsf{f})}\, \alpha\sqrt{\log(1/\alpha)} & \alpha_n \equiv \alpha, \\ \mathrm{C}_p^{(\mathsf{d})}\, \sqrt{\alpha_n \mathcal{A}_n \log(1/\alpha_n)} & \text{under A 5}, \end{cases} \tag{96}$$

where the constants $\mathrm{C}_p^{(\mathsf{f})}, \mathrm{C}_p^{(\mathsf{d})}$ are defined as

$$\mathrm{C}_p^{(\mathsf{f})} := \mathrm{C}_{\mathsf{H},p}^{(\mathsf{f})} + \mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{f})}, \quad \mathrm{C}_p^{(\mathsf{d})} := \mathrm{C}_{\mathsf{H},p}^{(\mathsf{d})} + \mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{d})}. \tag{97}$$

Lemma 32 *Under conditions of Theorem 31:*

1. *If the step sizes are constant $\alpha_k \equiv \alpha$, then*

$$\mathbb{E}_z^{\frac{1}{p}}[\|J_n^{(1)}\|^p] \leq \mathrm{C}_{\mathsf{J},\mathsf{p}}^{(1,\mathsf{f})}\, \alpha\sqrt{\log(1/\alpha)} V^{\frac{2}{\mathsf{K}}+\frac{1}{4p}}(z), \tag{98}$$

   where $\mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{f})}$ is defined in (111).

2. *If the step sizes $\alpha_k, k \in \mathbb{N}$, satisfy A 5, then*

$$\mathbb{E}_z^{\frac{1}{p}}[\|J_n^{(1)}\|^p] \leq \mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{d})}\, \sqrt{\alpha_n \mathcal{A}_n}\sqrt{\log(1/\alpha_n)} V^{\frac{2}{\mathsf{K}}+\frac{1}{4p}}(z), \tag{99}$$

   where $\mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{d})}$ is defined in (113).

Lemma 33 *Under conditions of Theorem 31:*

1. *If the step sizes are constant $\alpha_k \equiv \alpha$, then*

$$\mathbb{E}_z^{\frac{1}{p}}[\|H_n^{(1)}\|^p] \leq \mathrm{C}_{\mathsf{H},p}^{(\mathsf{f})}\, \alpha\sqrt{\log(1/\alpha)} V^{\frac{3}{\mathsf{K}}+\frac{9}{16p}}(z), \tag{100}$$

   where $\mathrm{C}_{\mathsf{H},p}^{(\mathsf{f})}$ is defined in (115).

2. *If the step sizes $\alpha_k, k \in \mathbb{N}$, satisfy A 5, then*

$$\mathbb{E}_z^{\frac{1}{p}}[\|H_n^{(1)}\|^p] \leq \mathrm{C}_{\mathsf{H},p}^{(\mathsf{d})}\, \sqrt{\alpha_n \mathcal{A}_n}\sqrt{\log(1/\alpha_n)} V^{\frac{3}{\mathsf{K}}+\frac{9}{16p}}(z), \tag{101}$$

   where $\mathrm{C}_{\mathsf{H},p}^{(\mathsf{d})}$ is defined in (117).

Proof [Proof of Lemma 32] For the second term $J_n^{(1)}$, solving the recursion in (38) yields the double summation:

$$J_{n+1}^{(1)} = -\sum_{k=1}^{n+1} \alpha_k G_{k+1:n+1} \widetilde{A}(Z_k) J_{k-1}^{(0)} = -\sum_{k=1}^{n+1} \alpha_k \sum_{j=1}^{k-1} \alpha_j G_{k+1:n+1} \widetilde{A}(Z_k) G_{j+1:k-1} \bar{\varepsilon}(Z_j).$$

Changing the order of summation gives

$$J_{n+1}^{(1)} = -\sum_{j=1}^{n} \alpha_j \Bigg\{ \sum_{k=j+1}^{n+1} \alpha_k G_{k+1:n+1} \widetilde{A}(Z_k) G_{j+1:k-1} \Bigg\} \bar{\varepsilon}(Z_j) = \sum_{j=1}^{n} \alpha_j S_{j+1:n+1} \bar{\varepsilon}(Z_j), \tag{102}$$

where for $j \leq n$ we have defined

$$S_{j:n} := -\sum_{k=j}^{n} \alpha_k G_{k+1:n} \widetilde{A}(Z_k) G_{j:k-1}.$$

Fix a constant $m \geq 1$ (to be determined later), we can further rewrite $S_{j+1:n+1}$ as

$$\begin{aligned} S_{j+1:n+1} &= -\sum_{k=j+1}^{j+m} \alpha_k G_{k+1:n+1} \widetilde{A}(Z_k) G_{j+1:k-1} - \sum_{k=j+m+1}^{n+1} \alpha_k G_{k+1:n+1} \widetilde{A}(Z_k) G_{j+1:k-1} \\ &= G_{j+m+1:n+1} S_{j+1:j+m} + S_{j+m+1:n+1} G_{j+1:j+m}. \end{aligned}$$

Let $N := \lfloor n/m \rfloor$. In these notations, we can express $J_{n+1}^{(1)}$ as the sum of three terms:

$$\begin{aligned} J_{n+1}^{(1)} = & \underbrace{\sum_{j=1}^{(m-1)N} \alpha_j G_{j+m+1:n+1} S_{j+1:j+m} \bar{\varepsilon}(Z_j)}_{=T_1} + \underbrace{\sum_{j=1}^{(m-1)N} \alpha_j S_{j+m+1:n+1} G_{j+1:j+m} \bar{\varepsilon}(Z_j)}_{=T_2} \\ & + \underbrace{\sum_{j=(m-1)N+1}^{n} \alpha_j S_{j+1:n+1} \bar{\varepsilon}(Z_j)}_{=T_3}. \end{aligned}$$

Denote $\bar{\mathrm{C}}_{\bar{\varepsilon}} := \bar{\mathrm{C}}_A \|\theta^\star\| + \bar{\mathrm{C}}_b$. By Lemma 16 for any $1 \leq q \leq \mathsf{K}$,

$$\mathbb{E}_z^{\frac{1}{q}}[\|\bar{\varepsilon}(Z_j)\|^q] \leq \bar{\mathrm{C}}_{\bar{\varepsilon}} V^{1/\mathsf{K}}(z). \tag{103}$$

Let us consider the first term T_1. By the Minkowski inequality, Lemma 16 and Lemma 29 (see the definition for $\mathrm{C}_{\mathsf{S},\mathsf{p}}$ in (93))

$$\begin{aligned} \mathbb{E}_z^{1/p}[\|T_1\|^p] &\leq \sqrt{\kappa_{\mathsf{Q}}} \bar{\mathrm{C}}_{\bar{\varepsilon}} \, \mathrm{C}_{\mathsf{S},p} \sum_{k=1}^{(m-1)N} \alpha_k \sum_{\ell=k+1}^{k+m} \alpha_\ell \mathcal{A}_{\ell:k+m}^{1/2} \prod_{j=k+1}^{n+1} \sqrt{1 - a\alpha_j} V^{2/\mathsf{K}}(z) \\ &\leq \sqrt{\kappa_{\mathsf{Q}} m} \bar{\mathrm{C}}_{\bar{\varepsilon}} \, \mathrm{C}_{\mathsf{S},p} \sum_{k=1}^{(m-1)N} \alpha_k \sum_{\ell=k+1}^{k+m} \alpha_\ell^2 \prod_{j=k+1}^{n+1} \sqrt{1 - a\alpha_j} V^{2/\mathsf{K}}(z) \\ &\leq \sqrt{\kappa_{\mathsf{Q}} m} \bar{\mathrm{C}}_{\bar{\varepsilon}} \, \mathrm{C}_{\mathsf{S},p} \sum_{\ell=1}^{n+1} \alpha_\ell^2 \prod_{j=\ell+1}^{n+1} \sqrt{1 - a\alpha_j} \sum_{k=1}^{\ell} \alpha_k \prod_{j=k+1}^{\ell} \sqrt{1 - a\alpha_j} V^{2/\mathsf{K}}(z) \\ &\leq \mathrm{C}_1 \sqrt{m} \alpha_{n+1} V^{2/\mathsf{K}}(z), \end{aligned} \tag{104}$$

where we have defined

$$\mathrm{C}_1 := 8a^{-2} \bar{\mathrm{C}}_{\bar{\varepsilon}} \, \mathrm{C}_{\mathsf{S},p} \sqrt{\kappa_{\mathsf{Q}}}.$$

Similar bound holds for T_3,

$$\mathbb{E}_z^{\frac{1}{p}}[\|T_3\|^p] \leq \mathrm{C}_1 \sqrt{m} \alpha_{n+1} V^{2/\mathsf{K}}(z). \tag{105}$$

The second term T_2 may be rewritten as $T_{21} + T_{22}$, where

$$T_{21} := \sum_{k=0}^{N-1} \sum_{i=1}^{m} \alpha_{km+i} S^{(1)}_{(k+1)m+i+1:n+1} G_{km+i+1:(k+1)m+i} \bar{\varepsilon}(Z^*_{km+i}),$$

$$T_{22} := \sum_{k=0}^{N-1} \sum_{i=1}^{m} \alpha_{km+i} S^{(1)}_{(k+1)m+i+1:n+1} G_{km+i+1:(k+1)m+i} (\bar{\varepsilon}(Z_{km+i}) - \bar{\varepsilon}(Z^*_{km+i})).$$

In the above, the set of r.v. Z^*_{km+i} is constructed for each $i \in [1, m]$, with $\{Z^*_{km+i}\}_{k=0}^{N-1}$ and the following properties

$$\begin{aligned} &1.\ Z^*_{km+i} \text{ is independent of } \mathfrak{F}^{n+1}_{(k+1)m+i} := \sigma\{Z_{(k+1)m+i}, \dots, Z_{n+1}\}; \\ &2.\ \mathbb{P}_z(Z^*_{km+i} \neq Z_{km+i}) \leq 2\mathrm{B}_V \rho^m V(z); \\ &3.\ Z^*_{km+i} \text{ and } Z_{km+i} \text{ have the same distribution,} \end{aligned} \tag{106}$$

where B_V, ρ are defined in (6). The existence of the r.v.s Z^*_{km+i} is guaranteed by Berbee's lemma, see e.g (Rio, 2017, Lemma 5.1). We also exploit the fact the V-uniformly ergodic Markov chains are a special instance of β-mixing processes. We control β-mixing coefficient via total variation distance; see (Douc et al., 2018, Theorem F.3.3).

To analyze T_{21} we use Lemma 30

$$\begin{aligned} \mathbb{E}_z^{1/p}[\|T_{21}\|^p] &\leq \sum_{i=1}^{m} \mathbb{E}_z^{\frac{1}{p}}\left[\left\| \sum_{k=0}^{N-1} \alpha_{km+i} S_{(k+1)m+i+1:n+1} G_{km+i+1:(k+1)m+i} \bar{\varepsilon}(Z^*_{km+i}) \right\|^p\right] \\ &\leq \mathrm{C}_{\mathsf{B},p} \sqrt{\kappa_{\mathsf{Q}}} \sum_{i=1}^{m} \left(\sum_{k=0}^{N-1} \alpha^2_{km+i} \mathbb{E}^{\frac{2}{p}}[\|S_{(k+1)m+i+1:n+1}\|^p] \prod_{\ell=km+i+1}^{(k+1)m+i} (1 - a\alpha_\ell) \right)^{1/2} V^{1/\mathsf{K}}(z) \\ &\leq \mathrm{C}_{\mathsf{B},p} \sqrt{\kappa_{\mathsf{Q}}} m \left(\sum_{k=1}^{n+1} \alpha^2_k \mathbb{E}^{\frac{2}{p}}[\|S_{k+m+1:n+1}\|^p] \prod_{\ell=k+1}^{k+m} (1 - a\alpha_\ell) \right)^{1/2} V^{1/\mathsf{K}}(z), \end{aligned}$$

where C_{B} is defined in (94). Applying Lemma 29 we may estimate the term in the brackets by

$$\begin{aligned} &\frac{4(\mathrm{C}_{\mathsf{S},p})^2}{a^2} \sum_{k=1}^{n+1} \alpha^2_k \sum_{\ell=k+1}^{n+1} \alpha_\ell \mathcal{A}_{\ell:n+1} \prod_{j=\ell+1}^{n+1} \sqrt{1 - a\alpha_j} \prod_{j=k+1}^{\ell} (1 - a\alpha_j) V^{2/\mathsf{K}}(z) \\ &\quad \leq \frac{4(\mathrm{C}_{\mathsf{S},p})^2}{a^2} \sum_{\ell=1}^{n+1} \alpha_\ell \mathcal{A}_{\ell:n+1} \prod_{j=\ell+1}^{n+1} \sqrt{1 - a\alpha_j} \sum_{k=1}^{\ell} \alpha^2_k \prod_{j=\ell+1}^{n+1} (1 - a\alpha_j) V^{2/\mathsf{K}}(z) \\ &\quad \leq \frac{16(\mathrm{C}_{\mathsf{S},p})^2}{a^2} \sum_{\ell=1}^{n+1} \alpha^2_\ell \mathcal{A}_{\ell:n+1} \prod_{j=\ell+1}^{n+1} \sqrt{1 - a\alpha_\ell} V^{2/\mathsf{K}}(z). \end{aligned}$$

Finally

$$\mathbb{E}_z^{1/p}[\|T_{21}\|^p] \leq \mathrm{C}_2 \sqrt{m} \left\{ \sum_{k=1}^{n+1} \alpha^2_k \mathcal{A}_{k:n+1} \prod_{\ell=k+1}^{\ell} \sqrt{1 - a\alpha_\ell} \right\}^{1/2} V^{2/\mathsf{K}}(z), \tag{107}$$

where

$$\mathrm{C}_2 := 4a^{-1}\,\mathrm{C}_{\mathsf{B},p}\,\mathrm{C}_{\mathsf{S},p}\sqrt{\kappa_{\mathsf{Q}}}.$$

For the term T_{22} we use Minkowski's inequality

$$\mathbb{E}_z^{1/p}[\|T_{22}\|^p] \le \sqrt{\kappa_{\mathsf{Q}}}\sum_{k=0}^{N-1}\sum_{i=1}^{m}\alpha_{km+i}\mathbb{E}^{\frac{1}{2p}}[\|S^{(1)}_{(k+1)m+i+1:n+1}\|^{2p}] \\ \times \prod_{\ell=km+i+1}^{(k+1)m+i}\sqrt{1-a\alpha_\ell}\,\mathbb{E}_z^{\frac{1}{2p}}[\|\bar{\varepsilon}(Z_{km+i})-\bar{\varepsilon}(Z^*_{km+i}))\|^{2p}].$$

Using definition of Z^*_{km+i} and and the Cauchy-Schwartz inequality

$$\begin{aligned}\mathbb{E}_z^{1/(2p)}&[\|\bar{\varepsilon}(Z_{km+i})-\bar{\varepsilon}(Z^*_{km+i}))\|^{2p}] \\ &= \mathbb{E}_z^{1/(2p)}[\|\bar{\varepsilon}(Z_{km+i})-\bar{\varepsilon}(Z^*_{km+i}))\mathbb{1}\{Z^*_{km+i}\neq Z_{km+i}\}\|^{2p}] \\ &\le 2\mathbb{E}_z^{1/(4p)}[\|\bar{\varepsilon}(Z_{km+i})\|^{4p}\mathbb{P}_z^{1/(4p)}(Z^*_{km+i}\neq Z_{km+i}) \\ &\le 4\bar{\mathrm{C}}_{\bar{\varepsilon}}\mathrm{B}_V^{1/(4p)}\rho^{m/(4p)}V^{1/(4p)+1/\mathsf{K}}(z),\end{aligned} \tag{108}$$

where we used (106). The last two inequalities, Lemma 29 and Lemma 25 imply

$$\begin{aligned}\mathbb{E}_z^{1/p}[\|T_{22}\|^p] &\le 4\bar{\mathrm{C}}_{\bar{\varepsilon}}\mathrm{B}_V^{1/(4p)}\bar{\rho}^m\sum_{k=1}^{n+1}\alpha_k\sum_{\ell=k+1}^{n+1}\alpha_\ell\mathcal{A}^{1/2}_{\ell:n+1}\prod_{\ell=k+1}^{n+1}\sqrt{1-a\alpha_\ell}V^{1/(4p)+2/\mathsf{K}}(z) \\ &\le \mathrm{C}_3\,\bar{\rho}^m\sum_{\ell=1}^{n+1}\alpha_\ell\mathcal{A}^{1/2}_{\ell:n+1}\prod_{j=\ell+1}^{n+1}\sqrt{1-a\alpha_j}V^{1/(4p)+2/\mathsf{K}}(z),\end{aligned} \tag{109}$$

where

$$\mathrm{C}_3 := 8a^{-1}\kappa_{\mathsf{Q}}^{1/2}\,\mathrm{C}_{\mathsf{S},p}\,\bar{\mathrm{C}}_{\bar{\varepsilon}}\mathrm{B}_V^{1/(4p)}, \quad \bar{\rho} := \rho^{1/(4p)}.$$

Bounds (104), (105), (107), (109) together imply

$$\mathbb{E}_z^{1/p}[\|J^{(1)}_{n+1}\|^p] \le 3\,\mathrm{C}_1\sqrt{m}\alpha_{n+1}V^{2/\mathsf{K}}(z) + \mathrm{C}_2\sqrt{m}\left\{\sum_{k=0}^{n+1}\alpha_k^2\mathcal{A}_{k:n+1}\prod_{\ell=k+1}^{n+1}\sqrt{1-a\alpha_\ell}\right\}^{1/2}V^{2/\mathsf{K}}(z) \\ + \mathrm{C}_3\,\bar{\rho}^m\sum_{k=1}^{n+1}\alpha_k\mathcal{A}^{1/2}_{k:n+1}\prod_{\ell=k+1}^{n+1}\sqrt{1-a\alpha_\ell}V^{1/(4p)+2/\mathsf{K}}(z).$$

We distinguish two cases:

1. $\alpha_k \equiv \alpha$ for any $k \in \mathbb{N}$. Then

$$\begin{aligned}\mathbb{E}_z^{\frac{1}{p}}[\|J^{(1)}_{n+1}\|^p] &\le 2\,\mathrm{C}_1\sqrt{m}\alpha V^{2/\mathsf{K}}(z) + \mathrm{C}_2\sqrt{m}\alpha^2\left\{\sum_{k=0}^{n+1}(n-k+2)(1-a\alpha)^{(n-k+1)/2}\right\}^{1/2}V^{2/\mathsf{K}}(z) \\ &+ \mathrm{C}_3\,\alpha^2\bar{\rho}^m\sum_{k=1}^{n+1}\sqrt{n-k+2}(1-a\alpha)^{(n-k+1)/2}V^{1/(4p)+2/\mathsf{K}}(z) \\ &\le \mathrm{C}_4\sqrt{m}\alpha V^{1/(4p)+2/\mathsf{K}}(z),\end{aligned}$$

where

$$\mathrm{C}_4 := 2\,\mathrm{C}_1 + 2\sqrt{\mathrm{e}}\,\mathrm{C}_2 / a + \sqrt{2\pi}\mathrm{e}\,\mathrm{C}_3 / a^{3/2}$$

and we took m such that

$$\bar{\rho}^m \leq \sqrt{\alpha}, \text{ i.e. } m = \left\lceil \frac{1}{2} \frac{\log(1/\alpha)}{\log(1/\bar{\rho})} \right\rceil.$$

We obtain

$$\mathbb{E}_z^{\frac{1}{p}}[\|J_{n+1}^{(1)}\|^p] \leq \mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{f})}\, \alpha \log^{1/2}(1/\alpha) V^{1/(4p)+2/\mathsf{K}}(z), \tag{110}$$

where

$$\mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{f})} := 2\sqrt{p}\,\mathrm{C}_4 \log^{-1/2}(1/\rho). \tag{111}$$

2. Assume that A5 is satisfied. Then we apply Lemma 26 and obtain

$$\mathbb{E}_z^{\frac{1}{p}}[\|J_{n+1}^{(1)}\|^p] \leq \mathrm{C}_5 \sqrt{m}\sqrt{\alpha_{n+1}\mathcal{A}_{n+1}} V^{1/(4p)+2/\mathsf{K}}(z),$$

where

$$\mathrm{C}_5 := (3\,\mathrm{C}_1 + 2\,\mathrm{C}_2 / \sqrt{a} + 4\,\mathrm{C}_3 / a)(\sqrt{\mathrm{c}_\alpha} + 1)$$

and

$$m = \left\lceil \frac{1}{2} \frac{\log(1/\alpha_{n+1})}{\log(1/\bar{\rho})} \right\rceil.$$

In both cases, we have

$$\mathbb{E}_z^{\frac{1}{p}}[\|J_{n+1}^{(1)}\|^p] \leq \mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{d})} \sqrt{\alpha_{n+1}\mathcal{A}_{n+1}} \sqrt{\log(1/\alpha_{n+1})} V^{1/(4p)+2/\mathsf{K}}(z), \tag{112}$$

where

$$\mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{f})} := 2\sqrt{p}\,\mathrm{C}_5 \log^{-1}(1/\rho). \tag{113}$$

■

Proof [Proof of Lemma 33] To estimate $H_n^{(1)}$ we rewrite it as follows

$$H_{n+1}^{(1)} = -\sum_{\ell=1}^{n+1} \alpha_\ell \Gamma_{\ell+1:n+1} \widetilde{A}(Z_\ell) J_{\ell-1}^{(1)}.$$

Using Minkowski's and Cauchy-Schwarz inequality,

$$\mathbb{E}_z^{1/p}[\|H_{n+1}^{(1)}\|^p] \leq \sum_{\ell=1}^{n+1} \alpha_\ell \mathbb{E}_z^{1/(2p)}[\|\Gamma_{\ell+1:n+1}\|^{2p}] \mathbb{E}_z^{1/(4p)}[\|\widetilde{A}(Z_\ell)\|^{4p}] \mathbb{E}_z^{1/4p}[\|J_{\ell-1}^{(1)}\|^{4p}].$$

We apply Theorem 1 to estimate $\mathbb{E}_z^{1/(2p)}[\|\Gamma_{\ell+1:n+1}\|^{2p}]$ and Lemma 16 to estimate $\mathbb{E}_z^{1/(4p)}[\|\widetilde{A}(Z_\ell)\|^{4p}]$. These bounds lead

$$\mathbb{E}_z^{1/p}[\|H_{n+1}^{(1)}\|^p] \leq \bar{\mathrm{C}}_A\, \mathrm{C}_{\mathsf{st},2p} \sum_{\ell=1}^{n+1} \alpha_\ell \mathrm{e}^{-(a/4)\sum_{k=\ell+1}^{n+1} \alpha_k} \mathbb{E}_z^{1/(4p)}[\|J_{k-1}^{(1)}\|^{4p}] V^{1/\mathsf{K}+1/(4p)}(z).$$

We again consider two cases:

1. $\alpha_k \equiv \alpha$ for any $k \in \mathbb{N}$. Then applying (110) we get

$$\mathbb{E}_z^{\frac{1}{p}}[\|H_{n+1}^{(1)}\|^p] \le \mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{f})}\, \bar{\mathrm{C}}_A\, \mathrm{C}_{\mathsf{st},2p}\, \alpha^2 \log^{1/2}(1/\alpha) \sum_{k=1}^{n+1} \mathrm{e}^{-\alpha a(n-k+1)/4} V^{3/\mathsf{K}+9/(16p)}(z).$$

This expression may be simplified. We come to the inequality

$$\mathbb{E}_z^{\frac{1}{p}}[\|H_{n+1}^{(1)}\|^p] \le \mathrm{C}_{\mathsf{H},p}^{(\mathsf{f})}\, \alpha \sqrt{\log(1/\alpha)} V^{3/\mathsf{K}+9/(16p)}(z), \tag{114}$$

where

$$\mathrm{C}_{\mathsf{H},p}^{(\mathsf{f})} := 8\, \mathrm{C}_{\mathsf{J}}^{(1,\mathsf{f})}\, \bar{\mathrm{C}}_A\, \mathrm{C}_{\mathsf{st},2p}\, /a. \tag{115}$$

2. Assume A5, then we use (112) and inequality $\mathrm{e}^{-x} \le 1 - x/2$ valid for $0 \le x \le 1$,

$$\begin{aligned}\mathbb{E}_z^{\frac{1}{p}}[\|H_{n+1}^{(1)}\|^p] &\le \mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{d})}\, \bar{\mathrm{C}}_A\, \mathrm{C}_{\mathsf{st},2p}(1 + \mathrm{c}_\alpha\, \alpha_{\infty,p}^{(2)}) \sqrt{\log(1/\alpha_{n+1})} \\ &\qquad \times \sum_{k=1}^{n+1} \alpha_k \mathrm{e}^{-(a/4)\sum_{\ell=k+1}^{n+1} \alpha_\ell} \sqrt{\alpha_k \mathcal{A}_k} V^{3/\mathsf{K}+9/(16p)}(z) \\ &\le \mathrm{C}_{\mathsf{J},p}^{(1,\mathsf{d})}\, \bar{\mathrm{C}}_A\, \mathrm{C}_{\mathsf{st},2p}(1 + \mathrm{c}_\alpha\, \alpha_{\infty,p}^{(2)}) \sqrt{\log(1/\alpha_{n+1})} \\ &\qquad \times \sum_{k=1}^{n+1} \alpha_k \sqrt{\alpha_k \mathcal{A}_k} \prod_{\ell=k+1}^{n} (1 - (a/8)\alpha_\ell) V^{3/\mathsf{K}+9/(16p)}(z).\end{aligned}$$

Applying Lemma 26 we get

$$\mathbb{E}_z^{\frac{1}{p}}[\|H_{n+1}^{(1)}\|^p] \le \mathrm{C}_{\mathsf{H},p}^{(\mathsf{d})} \sqrt{\alpha_{n+1}\mathcal{A}_{n+1}} \sqrt{\log(1/\alpha_{n+1})} V^{3/\mathsf{K}+9/(16p)}(z), \tag{116}$$

where

$$\mathrm{C}_{\mathsf{H},p}^{(\mathsf{d})} := 16\, \mathrm{C}_{\mathsf{J}}^{(1,\mathsf{d})}\, \bar{\mathrm{C}}_A\, \mathrm{C}_{\mathsf{st},2p}(1 + \mathrm{c}_\alpha\, \alpha_{\infty,p}^{(2)})/a. \tag{117}$$

■

Appendix F. Temporal-Difference Learning

We preface the proof by a a well-known elementary sufficient condition for a matrix $-A$ to be Hurwitz. We give the proof for completeness.

Lemma 34 *Let A be a $d \times d$ matrix. Assume that for all $x \in \mathbb{R}^d$, $x^\top A x > 0$, then for any $\ell \in \{1, \ldots, d\}$, $\operatorname{Re} \lambda_\ell(A) > 0$, where $\lambda_\ell(A)$, $\ell \in \{1, \ldots, d\}$ are the eigenvalues of A.*

Proof Fix $\ell = 1, \ldots, d$ and let $\lambda = \lambda_\ell(A) = \mu + \mathrm{i}\nu$ and $z = x + \mathrm{i}y$ be the eigenvector of A corresponding to λ. Then

$$(A - \lambda \mathrm{I})(x + \mathrm{i}y) = (A - \mu\mathrm{I})x + \nu y + \mathrm{i}(-\nu x + (A - \mu\mathrm{I})y) = 0.$$

This implies that

$$\begin{cases} x^\top (A - \mu\mathrm{I})x = -\nu x^\top y \\ y^\top (A - \mu\mathrm{I})y = \nu y^\top x. \end{cases}$$

Taking the sum of these equations we get $x^\top(A-\mu \mathrm{I})x + y^\top(A-\mu\mathrm{I})y = 0$, or

$$\mu = \frac{x^\top Ax + y^\top Ay}{x^\top x + y^\top y} > 0.$$

■

Recall that,

$$A = \sum_{\ell=0}^{\tau-1} \mathbb{E}_{\pi_0}[\psi(X_{\tau-1-\ell})\{\psi(X_{\tau-1}) - \gamma\psi(X_\tau)\}^\top] \;, \tag{118}$$

for $\tau \in \mathbb{N}^*$,

Lemma 35 *Assume M 2. Then for any $\ell = 1, \dots, d$*

$$\operatorname{Re}\lambda_\ell(A) > 0. \tag{119}$$

Proof We show that $x^\top Ax > 0$ for any $x \in \mathbb{R}^d$ and then apply Lemma 34. Fix $x \in \mathbb{R}^d$ and denote

$$\rho(\ell) = \mathbb{E}_{\pi_0}[x^\top\psi(X_0)\psi^\top(X_\ell)x]\{\mathbb{E}_{\pi_0}[x^\top\psi(X_0)\psi(X_0)^\top x]\}^{-1}.$$

Then

$$x^\top Ax = \mathbb{E}_{\pi_0}[x^\top\psi(X_0)\psi(X_0)^\top x]\bigg\{\sum_{\ell=0}^{\tau-1}(\lambda\gamma)^\ell(\rho(\ell) - \gamma\rho(\ell+1))\bigg\}.$$

The sum in the brackets could be rewritten as

$$\sum_{\ell=0}^{\tau-1}(\lambda\gamma)^\ell(\rho(\ell) - \gamma\rho(\ell+1)) = 1 - \gamma\big\{(1-\lambda)\sum_{\ell=1}^{\tau-1}(\lambda\gamma)^{\ell-1}\rho(\ell) + (\lambda\gamma)^{\tau-1}\rho(\tau)\big\}.$$

Since by the Cauchy-Schwartz inequality $|\rho(\ell)| \le 1$, we obtain

$$\begin{aligned}\sum_{\ell=0}^{\tau-1}(\lambda\gamma)^\ell(\rho(\ell) - \gamma\rho(\ell+1)) &\ge 1 - \gamma\big\{(1-\lambda)\sum_{\ell=1}^{\tau-1}(\lambda\gamma)^{\ell-1} + (\lambda\gamma)^{\tau-1}\big\} \\ &= 1 - \gamma\big\{(1-\lambda)(1-(\lambda\gamma)^{\tau-1})(1-\lambda\gamma)^{-1} + (\lambda\gamma)^{\tau-1}\big\} \\ &= \frac{1-\gamma}{1-\lambda\gamma}\{1-(\lambda\gamma)^\tau\}.\end{aligned}$$

Finally,

$$x^\top Ax \ge \frac{1-\gamma}{1-\lambda\gamma}\{1-(\lambda\gamma)^\tau\}\mathbb{E}_{\pi_0}[x^\top\psi(X_0)\psi(X_0)^\top x] > 0,$$

where we applied M 2. ■

Lemma 36 *Assume M 1. Then the Markov kernel* P *defined in (46), is irreducible and aperiodic.*

Proof Recall that the Markov kernel P is irreducible if it admits an accessible small set. We are going to construct such set.

Since the Markov kernel Q is strongly aperiodic, it admits an accessible $(1, \varepsilon\nu)$-small set C with $\nu(C) > 0$ (see (Douc et al., 2018, Definition 9.3.5)). Let us take $\tilde{C} = \mathsf{X} \times \cdots \times \mathsf{X} \times C$ and check that it is accessible and small for P. Note that, for $k \geq \tau$,

$$\mathbb{P}_{(x_{-\tau},\ldots,x_0)}(Z_k \in \tilde{C}) = \mathbb{P}_{x_0}(X_{k-\tau} \in C) = \mathrm{Q}^{k-\tau}(x_0, C) \ .$$

Since C is accessible for Q, for any $x_0 \in \mathsf{X}$ we can choose k, such that $\mathrm{Q}^{k-\tau}(x_0, C) > 0$, showing that $\tilde{C}$ is accessible for P. To check that $\tilde{C}$ is small, note that for any $D_1, \ldots, D_{\tau+1} \in \mathcal{X}$, and $(x_{-\tau}, \ldots, x_0) \in \tilde{C}$,

$$\mathrm{P}^{\tau+1}((x_{-\tau}, \ldots, x_0), D_1 \times \cdots \times D_{\tau+1}) = \int \prod_{k=0}^{\tau} \mathrm{Q}(x_k, \mathrm{d}x_{k+1}) \prod_{k=1}^{\tau+1} \mathbb{1}_{D_k}(x_k)$$

$$\geq \int \prod_{k=0}^{\tau} \mathrm{Q}(x_k, \mathrm{d}x_{k+1}) \prod_{k=1}^{\tau+1} \mathbb{1}_{D_k \cap C}(x_k) \overset{(a)}{\geq} \varepsilon^{\tau+1} \prod_{k=1}^{\tau+1} \nu(D_k \cap C) \geq \big(\varepsilon\nu(C)\big)^{\tau+1} \nu_C(D_1 \times \cdots \times D_{\tau+1}) \ ,$$

where (a) follows from $(\tau + 1)$ applications of the fact that C is $(1, \varepsilon\nu)$ small for Q and

$$\nu_C(D_1 \times \cdots \times D_{\tau+1}) = \prod_{k=1}^{\tau+1} \nu(D_k \cap C)/\nu(C) \ . \tag{120}$$

Hence, $\tilde{C}$ is $\big(\tau + 1, (\varepsilon\nu(C))^{\tau+1}\nu_C\big)$-small and accessible. This implies that the Markov kernel P is irreducible. To check that P is aperiodic, we first note that, due to (Douc et al., 2018, Lemma 9.3.3)), there exists such $n_0 \in \mathbb{N}$, that for any $k \geq n_0$, set C is $(k, \varepsilon_k\nu)$-small for Q with $\varepsilon_k > 0$. Hence, for any $k \geq n_0 + \tau$,

$$\inf_{x_{-\tau:0} \in \tilde{C}} \mathrm{P}^k(x_{-\tau:0}, \tilde{C}) = \inf_{x_0 \in C} \mathbb{P}_{x_0}(X_{k-\tau} \in C) \geq \varepsilon_{k-\tau}\nu(C) > 0 \ ,$$

yielding that the Markov kernel P is aperiodic. ■

Lemma 37 *Assume M 1. Then the Markov kernel* P *(see (46)) satisfies UE 1 with function* $V(x_{0:\tau})$ *defined in (47) and the constants* c, b, *and* R *given in (126). Moreover, for any* $R \geq 1$ *the sublevel sets* $\{x_{0:\tau} : W(x_{0:\tau}) \leq R\}$ *are* $\big(\tau + 1, (\varepsilon_R\nu(C_R))^{\tau+1}\nu_{C_R}\big)$*-small with respect to measure* ν_{C_R} *defined in (127).*

Proof Let us introduce the function $V_\iota(x_0, \ldots, x_\tau) = \mathrm{e}^{\iota c_{\mathrm{Q}} \sum_{i=0}^{\tau-1}(i+1)\tilde{W}^{\delta_{\mathrm{Q}}}(x_i) + \tilde{W}(x_\tau)}$ where c_{Q} is defined in M 1 and $\iota \in (0, 1/\tau)$ is a parameter to be chosen later. Then

$$\begin{aligned}
\mathrm{P}V_\iota(x_{0:\tau}) &= \int \cdots \int \mathrm{e}^{\iota c_{\mathrm{Q}} \sum_{i=0}^{\tau-1}(i+1)\tilde{W}^{\delta_{\mathrm{Q}}}(x'_i)} \mathrm{e}^{\tilde{W}(x'_\tau)} \Big\{\prod_{i=0}^{\tau-1} \delta_{x_{i+1}}(\mathrm{d}x'_i)\Big\} \mathrm{Q}(x_\tau, \mathrm{d}x'_\tau) \\
&\overset{(a)}{\leq} \int \cdots \int \mathrm{e}^{\iota c_{\mathrm{Q}} \sum_{i=0}^{\tau-1}(i+1)\tilde{W}^{\delta_{\mathrm{Q}}}(x'_i)} \Big(\mathrm{e}^{-c_{\mathrm{Q}}\tilde{W}^{\delta_{\mathrm{Q}}}(x_\tau)} \tilde{V}(x_\tau) + \mathrm{b}_{\mathrm{Q}}\Big) \Big\{\prod_{i=0}^{\tau-1} \delta_{x_{i+1}}(\mathrm{d}x'_i)\Big\} \\
&= \mathrm{e}^{\iota c_{\mathrm{Q}} \sum_{i=1}^{\tau-1} i\tilde{W}^{\delta_{\mathrm{Q}}}(x_i)} \mathrm{e}^{-(1-\iota\tau)c_{\mathrm{Q}}\tilde{W}^{\delta_{\mathrm{Q}}}(x_\tau)} \tilde{V}(x_\tau) + \mathrm{b}_{\mathrm{Q}}\, \mathrm{e}^{\iota c_{\mathrm{Q}} \sum_{i=1}^{\tau} i\tilde{W}^{\delta_{\mathrm{Q}}}(x_i)} \\
&= \mathrm{e}^{-\iota c_{\mathrm{Q}} \sum_{i=0}^{\tau-1} \tilde{W}^{\delta_{\mathrm{Q}}}(x_i) - (1-\iota\tau)c_{\mathrm{Q}}\tilde{W}^{\delta_{\mathrm{Q}}}(x_\tau)} V_\iota(x_{0:\tau}) + \mathrm{b}_{\mathrm{Q}}\, \mathrm{e}^{\iota c_{\mathrm{Q}} \sum_{i=1}^{\tau} i\tilde{W}^{\delta_{\mathrm{Q}}}(x_i)} \ ,
\end{aligned}$$

where (a) follows from UE 1. Now we select ι in order to ensure that

$$\mathrm{e}^{-\iota c_{\mathrm{Q}} \sum_{i=0}^{\tau-1} \tilde{W}^{\delta_{\mathrm{Q}}}(x_i) - (1-\iota\tau) c_{\mathrm{Q}} \tilde{W}^{\delta_{\mathrm{Q}}}(x_\tau)} \leq \mathrm{e}^{-\tilde{c} W_\iota^{\delta_{\mathrm{Q}}}(x_{0:\tau})}$$

for some constant $\tilde{c} > 0$ and $W_\iota(x_{0:\tau}) = \log V_\iota(x_{0:\tau})$. For this purpose, we first notice that

$$\begin{aligned} W_\iota^{\delta_{\mathrm{Q}}}(x_{0:\tau}) &\leq \sum_{i=0}^{\tau-1} ((i+1)\iota c_{\mathrm{Q}})^{\delta_{\mathrm{Q}}} \tilde{W}^{\delta_{\mathrm{Q}}}(x_i) + \tilde{W}^{\delta_{\mathrm{Q}}}(x_\tau) \\ &= \frac{\sum_{i=0}^{\tau-1} (1-\iota\tau) c_{\mathrm{Q}} ((i+1)\iota c_{\mathrm{Q}})^{\delta_{\mathrm{Q}}} \tilde{W}^{\delta_{\mathrm{Q}}}(x_i) + (1-\iota\tau) c_{\mathrm{Q}} \tilde{W}^{\delta_{\mathrm{Q}}}(x_\tau)}{(1-\iota\tau) c_{\mathrm{Q}}} . \end{aligned}$$

Now we select ι satisfying

$$(1-\iota\tau)(\tau\iota c_{\mathrm{Q}})^{\delta_{\mathrm{Q}}} \leq \iota . \tag{121}$$

Since $\tau\iota < 1$, it is enough to choose $\iota = \iota_0$, where ι_0 satisfies equation $(1-\iota_0\tau) c_{\mathrm{Q}}^{\delta_{\mathrm{Q}}} = \iota_0$, that is,

$$\iota_0 = \frac{c_{\mathrm{Q}}^{\delta_{\mathrm{Q}}}}{1 + \tau c_{\mathrm{Q}}^{\delta_{\mathrm{Q}}}} . \tag{122}$$

Then, setting $\tilde{c} = (1-\tau\iota_0) c_{\mathrm{Q}} = c_{\mathrm{Q}} / (1 + \tau c_{\mathrm{Q}}^{\delta_{\mathrm{Q}}})$, we get

$$\tilde{c} W_{\iota_0}^{\delta_{\mathrm{Q}}}(x_{0:\tau}) \leq \iota_0 c_{\mathrm{Q}} \sum_{i=0}^{\tau-1} \tilde{W}^{\delta_{\mathrm{Q}}}(x_i) + \tilde{c} \tilde{W}^{\delta_{\mathrm{Q}}}(x_\tau) .$$

From now on we fix $\iota = \iota_0$ defined in (122) and for ease of notations we write $V(x_{0:\tau}) = V_{\iota_0}(x_{0:\tau})$, $W(x_{0:\tau}) = W_{\iota_0}(x_{0:\tau})$. Then it holds

$$\mathrm{P} V(x_{0:\tau}) \leq \mathrm{e}^{-\tilde{c} W^{\delta_{\mathrm{Q}}}(x_{0:\tau})} V(x_{0:\tau}) + \mathrm{b}_{\mathrm{Q}} \, \mathrm{e}^{\iota_0 c_{\mathrm{Q}} \sum_{i=1}^{\tau} i \tilde{W}^{\delta_{\mathrm{Q}}}(x_i)} . \tag{123}$$

Moreover,

$$\sum_{i=1}^{\tau} i \tilde{W}^{\delta_{\mathrm{Q}}}(x_i) \leq W(x_{0:\tau}) - \tilde{c} W^{\delta_{\mathrm{Q}}}(x_{0:\tau}) - \tilde{W}(x_\tau) + c_{\mathrm{Q}} \tilde{W}^{\delta_{\mathrm{Q}}}(x_\tau) ,$$

which implies

$$\begin{aligned} \mathrm{b}_{\mathrm{Q}} \, \mathrm{e}^{\iota_0 c_{\mathrm{Q}} \sum_{i=1}^{\tau} i \tilde{W}^{\delta_{\mathrm{Q}}}(x_i)} &\leq \big\{ \mathrm{b}_{\mathrm{Q}} \sup_{r>0} \mathrm{e}^{c_{\mathrm{Q}} r^{\delta_{\mathrm{Q}}} - r} \big\} \mathrm{e}^{-\tilde{c} W^{\delta_{\mathrm{Q}}}(x_{0:\tau})} V(x_{0:\tau}) \\ &\leq c_0 \mathrm{e}^{-\tilde{c} W^{\delta_{\mathrm{Q}}}(x_{0:\tau})} V(x_{0:\tau}) , \end{aligned}$$

where we have defined

$$c_0 = \mathrm{b}_{\mathrm{Q}} \exp\{ c_{\mathrm{Q}} (c_{\mathrm{Q}} \delta_{\mathrm{Q}})^{\delta_{\mathrm{Q}}/(1-\delta_{\mathrm{Q}})} - (c_{\mathrm{Q}} \delta_{\mathrm{Q}})^{1/(1-\delta_{\mathrm{Q}})} \} . \tag{124}$$

Equation (123) now implies

$$\mathrm{P} V(x_{0:\tau}) \leq (c_0 + 1) \mathrm{e}^{-\tilde{c} W^{\delta_{\mathrm{Q}}}(x_{0:\tau})} V(x_{0:\tau}) . \tag{125}$$

We fix R such that for $(x_{0:\tau}) \in \{W(x_{0:\tau}) \geq R\}$, it holds

$$\mathrm{e}^{(\tilde{c}/2)W^{\delta_{\mathrm{Q}}}(x_{0:\tau})} \geq c_0 + 1 \;.$$

Now (125) implies UE 1 with constants

$$c = \frac{c_{\mathrm{Q}}}{2(1+\tau c_{\mathrm{Q}}^{\delta_{\mathrm{Q}}})}, \quad R = (2\log(1+c_0)/\tilde{c})^{1/\delta_{\mathrm{Q}}}, \quad \mathrm{b} = (c_0+1)\big(1 \vee \mathrm{e}^{R-2cR^{\delta_{\mathrm{Q}}}}\big) \;. \tag{126}$$

Now let us define, for $R \geq 1$, the sublevel sets $C_R = \{x \in \mathsf{X} : \tilde{W}(x) \leq R\}$, $\tilde{C}_R = \{x_{0:\tau} \in \mathsf{X}^{\tau+1} : W(x_{0:\tau}) \leq R\}$. To check that $\tilde{C}_R$ is small, we proceed similarly to Lemma 36. For any $D_1, \ldots, D_{\tau+1} \in \mathcal{X}$, and $x_{0:\tau} \in \tilde{C}_R$,

$$\begin{aligned}
\mathrm{P}^{\tau+1}(x_{0:\tau}, D_1 \times \cdots \times D_{\tau+1}) &= \int \prod_{k=0}^{\tau} \mathrm{Q}(x_{\tau+k}, \mathrm{d}x_{\tau+k+1}) \prod_{k=1}^{\tau+1} \mathbb{1}_{D_k}(x_{\tau+k}) \\
&\geq \int \prod_{k=0}^{\tau} \mathrm{Q}(x_{\tau+k}, \mathrm{d}x_{\tau+k+1}) \prod_{k=1}^{\tau+1} \mathbb{1}_{D_k \cap C_R}(x_{\tau+k}) \overset{(a)}{\geq} \varepsilon_R^{\tau+1} \prod_{k=1}^{\tau+1} \nu(D_k \cap C_R) \\
&\geq \big(\varepsilon_R \nu(C_R)\big)^{\tau+1} \nu_{C_R}(D_1 \times \cdots \times D_{\tau+1}) \;,
\end{aligned}$$

where (a) follows from $(\tau+1)$ applications of the fact that C_R is $(1, \varepsilon_R \nu)$-small for Q and

$$\gamma_{C_R}(D_1 \times \cdots \times D_{\tau+1}) = \prod_{k=1}^{\tau+1} \nu(D_k \cap C_R)/\nu(C_R) \;. \tag{127}$$

Hence, $\tilde{C}_R$ is $\big(\tau+1, (\varepsilon_R \nu(C_R))^{\tau+1} \nu_{C_R}\big)$-small for the Markov kernel P. ■

Lemma 38 *Under the assumptions of Theorem 6, Assumption A 1(ε) holds with*

$$\mathrm{C}_A = (1+\gamma)\,\mathrm{C}_\psi^2\big(\iota_0^{-\beta} \vee 1\big)(1-\lambda\gamma)^{-1} \;.$$

Proof Using (43), (44), and $ab \leq a^2/2 + b^2/2$, we get

$$\|\bar{A}(x_{0:\tau})\| \leq (1/2)\sum_{s=0}^{\tau-1} (\lambda\gamma)^s \big((1+\gamma)\|\psi(x_{\tau-1-s})\|^2 + \|\psi(x_{\tau-1})\|^2 + \gamma\|\psi(x_\tau)\|^2\big) \;.$$

Using (47) and M 3(ε, K), for any $s \in \{0, \ldots, \tau-1\}$,

$$\|\psi(x_{\tau-1-s})\|^2 \leq \mathrm{C}_\psi^2\, \tilde{W}^{\beta\delta_{\mathrm{Q}}}(x_{\tau-1-s}) \leq \mathrm{C}_\psi^2\, \iota_0^{-\beta} W^\beta(x_{0:\tau}) \;, \tag{128}$$

and, similarly,

$$\|\psi(x_\tau)\|^2 \leq \mathrm{C}_\psi^2\, W^\beta(x_{0:\tau}) \;.$$

Combining the above inequalities, we get

$$\|\bar{A}(x_{0:\tau})\| \leq (1+\gamma)\,\mathrm{C}_\psi^2\big(\iota_0^{-\beta} \vee 1\big)(1-\lambda\gamma)^{-1} W^\beta(x_{0:\tau}) \;.$$

■

Lemma 39 *Under the assumptions of Theorem 6, Assumption A3(K) holds with*

$$\mathrm{C}_{b,\mathsf{K}} = \frac{1}{2(1-\lambda\gamma)}\left(\mathrm{C}_{\psi}^{2}\left\{\frac{\beta K}{\mathrm{e}\iota_0}\right\}^{\beta} + \mathrm{C}_{\mathrm{R},\mathsf{K}}^{2}\right) .$$

Proof Using (43), (44), and $ab \leq a^2/2 + b^2/2$, we get

$$\|\bar{b}(x_{0:\tau})\| \leq (1/2)\sum_{s=0}^{\tau-1}(\lambda\gamma)^{s}\left(\|\psi(x_{\tau-1-s})\|^{2} + |\mathrm{R}(x_{\tau-1}, x_{\tau})|^{2}\right) . \tag{129}$$

Using (128) and M3(ε, K),

$$\|\|\psi(x_{\tau-1-s})\|^{2}\|_{V^{1/K}} \leq \mathrm{C}_{\psi}^{2}\,\iota_0^{-\beta}\sup_{y>0}\left\{\frac{y^{\beta}}{\mathrm{e}^{y/K}}\right\} = \mathrm{C}_{\psi}^{2}\left\{\frac{\beta K}{\mathrm{e}\iota_0}\right\}^{\beta} .$$

Moreover, using $\iota_0\tau < 1$, for any values $x_0, \ldots, x_{\tau-2}$ it holds

$$\begin{aligned}|\mathrm{R}(x_{\tau-1}, x_{\tau})|^{2} &\leq \mathrm{C}_{\mathrm{R},\mathsf{K}}^{2}\exp\bigl\{\frac{\iota_0}{K}\bigl(\tilde{W}^{\delta_{\mathrm{Q}}}(x_{\tau-1}) + \tilde{W}(x_{\tau})\bigr)\bigr\} \\ &\leq \mathrm{C}_{\mathrm{R},\mathsf{K}}^{2}\exp\bigl\{\frac{\iota_0}{K\iota_0\tau}\bigl(\iota_0\tau\tilde{W}^{\delta_{\mathrm{Q}}}(x_{\tau-1}) + \tilde{W}(x_{\tau})\bigr)\bigr\} \\ &\leq \mathrm{C}_{\mathrm{R},\mathsf{K}}^{2}\,V^{1/K}(x_{0:\tau}) .\end{aligned}$$

Combining the previous bounds with (129) yields the statement. ■