On the Asymptotic Capacity of Slotted Multiple Access Channel

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***Abstract*—We consider a special class of multiple access system over vector-disjunctive channel when users transmit some vector of bits of length L. We estimate the asymptotic capacity of this channel and derive the lower bound on this value in the case of different number of active users and channel slots. It was shown that the channel capacity achieves maximum when the number of active users is strictly higher than the number of orthogonal slots.**

I. INTRODUCTION

At present one of the most important features of the wireless networks evolution process is a sharp increase of the number of active users in the system due to the development of IoT (Internet of Things) and Smart House concepts. In this case the level of the mutual interference becomes extremely high. At the same time, the requirements on the data rate and the probability of successful transmission become more strict. Thus new coding methods are needed which allow a large number of users to work simultaneously in the system at high rate (per user) and in presence of interference. In this setting, a very large number of users in a wireless network operate in an uncoordinated fashion. Out of the total number of users, there is some subset of U users which are active at any time; and each of them wishes to communicate a relatively short message to a central base station. The base station is interested only in recovering the list of messages without regard to the identity of the user who transmitted a particular message. The uncoordinated nature of the problem and the small block lengths represent a substantial departure from the traditional multiple access channel and, consequently, has important implications both on the fundamental limits as well as the design of pragmatic low-complexity coding schemes. Due to small block lengths, information rates do not provide reasonable benchmarks and finite block length bounds are more meaningful.

Almost all well-known low-complexity coding solutions for the traditional MAC channel such as code-division multiple access (CDMA), rate-splitting [1], and interleave-division multiple access [2], implicitly assume some form of coordination between the users and that some parameters of the coding scheme such as the spreading sequence, code rates, time sharing parameters, Tanner graph of the code, etc., are user dependent.

In this paper we focuse on a special class of uncoordinated MAC, namely vector-disjunctive channel (logical OR of Z-channel) that was introduced by Cohen, Heller and Viterbi in [3]. We assume that channel consists of $Q > 2$ undependent and distinct subchannels. Moreover,we consider $U > 1$ active users that are trying to transmit information over channel mentioned above. In this paper we consider a vector-disjunctive channel that was introduced by Cohen, Heller and Viterbi in [3]. The channel model is a generalization of the A-channel from [4]– [8]. We estimate the capacity of this channel in the case of equiprobable choice of each binary vector of length L. Thus we derive the lower bound on the capacity. We also present some simulation results for maximal achievable relative sum-rate of the channel.

II. VECTOR-DISJUNCTIVE CHANNEL MODEL

A. Channel Description

We consider vector-disjunctive channel that consists of $Q > 1$ independent and distinct sub-channels. Also we assume that there are $U > 0$ active users which use channel for transmission. Let us assume that $Q = LS$ and our MAC channel is divided into S independent channels that consist of L elementary sub-channels. We will call these "grouped" channels as L-channels or slots. At each time moment τ each i-th user choose an arbitrary j-th L-channel ($1 \le j \le S$) and transmit L bits in this slot. For each slot receiver obtains a disjunction of all transmitted data (binary vectors of length L) in it.

For instance, let us consider a transmission of two users: i and k who choose one common slot for transmission in j-th time moment.

Let us assume that users i and k transmit L binary symbols $\mathbf{X}_{ij} = (x_{ij}^{(1)}, x_{ij}^{(2)}, \dots, x_{ij}^{(L)})$ and $\mathbf{X}_{kj} = (x_{kj}^{(1)}, x_{kj}^{(2)}, \dots, x_{kj}^{(L)})$

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respectively in L-channel. Since we suppose that our channel is vector-disjunctive then in common L-channel we obtain a disjunction of transmitted binary vectors: $\mathbf{X}_{ij} \vee \mathbf{X}_{kj}$.

For instance, let us assume that $L = 3$ and two users choose some common slot for transmission. If first user transmit vector $(0, 1, 0)$ and second one – $(0, 1, 1)$ then receiver obtains $(0, 1, 0) \vee (0, 1, 1) = (0, 1, 1)$.

It is obvious that the maximal number of users that can transmit information orthogonal is S.

B. Collision model

Let us consider transmission model described above. In this scheme collisions may take place when two or more users choose common L-channel for transmission at the same time moment.

Let us denote by κ the order of collision. This value might be considered as a number of additional active users that choose adjusted j-th L-channel for transmission which was selected by user we consider. It is obvious that $0 \le \kappa \le U - 1$. It is easy to calculate the probability that $\kappa = t$:

$$P(\kappa = t) = \binom{U-1}{t} \left(\frac{1}{S}\right)^t \left(1 - \frac{1}{S}\right)^{U-t-1}.$$

In this paper we consider noiseless channel and uncoordinated transmission scenario. Thus for a given user we can assume transmissions of all other $U - 1$ users as a noise (cause of collisions).

In some cases the transmission of other users does not result in incorrect transmission for a given user. For instance if the transmitted information in slot for considering user is $(1, 1, \ldots, 1)$ then it will be received correctly regardless of what other users transmit in the same slot.

In the most general case there is no collision for i-th user if the binary vector which this user transmit covers all vectors transmitted by all other users who have chosen the same slot for transmission.

C. Channel analysis

The previous reasoning can be generalized for arbitrary order of collision. For the better clarity of our proposed technique let us consider the channel presented in Fig. 1

In fig. 1 we represent all possible relations between inputs and outputs of the channel. As we can see for each vector of length L and weight w there are 2^{L-w} possible transitions. Thus the number of transmissions and transition probabilities does not depend on vector structure (support). Moreover, we can notice that in the case of error the weight of output vector weight increases, otherwise the vector weight is preserved. So, we can represent our initial channel model with 2^L inputs (which are equiprobable) and 2^L outputs as a channel with $L + 1$ inputs and $L + 1$ outputs, which correspond to input and output vectors weights respectively (see Fig. 2).

We would like to point out that in this case the inputs of the channel are not equiprobable. In particular, the input distribution are as follows:

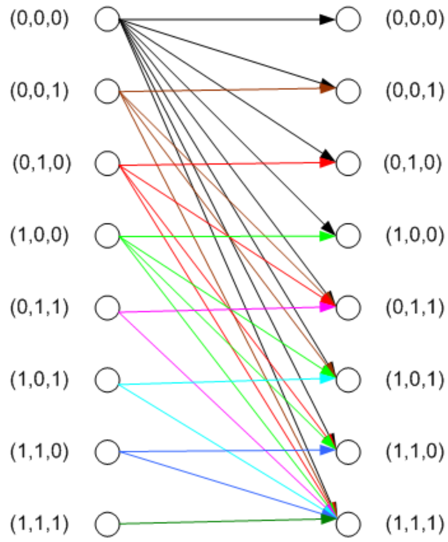

Fig. 1. Possible transitions for $L = 3$

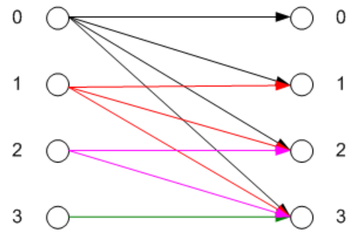

Fig. 2. Possible weights transitions for $L = 3$

$$P_{in}(w) = \binom{L}{w} 2^{-L}.$$

It is obvious that input distribution does not depend on collisions order κ. In the same time output distribution depends on it. Let us assume that $\kappa = t$. If the input vector $\mathbf{u}$ has weight $w < L$ then the output vector $\mathbf{v}$ has weight $w' > w$ if at least one of t other users transmit at least one 1 in any positions from $[L] \setminus \{i : \ u_i = 1\}$, where $[L] = \{1, 2, \ldots, L\}$. The probability $p(t)$ that at least one of t users transmit 1 in any fixed position $k \in [L]$ is as follows:

$$p(t) = 1 - (1 - p)^t,$$

where p is the probability of transmitting 1. Since we assume that $p = \frac{1}{2}$ then

$$p(t) = 1 - 2^{-t}.$$

Taking this fact into account it is easy to calculate the conditional probabilities $P(w'|w, t)$ to obtain output vector with weight w' for a given input vector with weight w in the case of collision order t:

$$P(w'|w,t) = \begin{cases} 0, & w' < w \\ \binom{L-w}{w'-w} p(t)^{w'-w} (1 - p(t))^{L-w'}, & w' \ge w \end{cases}.$$

Finally, the output distribution of channel is as follows:

$$P_{out}(w'|t) = \sum_{i=0}^{w'} P(w'|i,t)P_{in}(i).$$

D. Channel capacity estimation

Now let us calculate the capacity of channel described above. First we obtain the capacity for a given order of collision $\kappa = t$ and then we obtain the mathematical expectation over all possible collision orders of a given quantity which is the lower bound on capacity of MAC channel we described in the paper.

First let us introduce a definition of channel capacity. Let $\mathcal{X}$ and $\mathcal{Y}$ are random variables representing the input and output of the channel, respectively. The channel capacity $\mathcal{C}$ is defined as follows:

$$\mathcal{C} = \sup_{p_{in}(\mathcal{X})} I(\mathcal{X};\mathcal{Y}),$$

where $I(\mathcal{X};\mathcal{Y})$ is a mutual information between random variables $\mathcal{X}$ and $\mathcal{Y}$. Supremum takes over all possible distributions $p_{in}(\mathcal{X})$ of input $\mathcal{X}$.

If one fixes an input distribution $P_{in}(w)$ and collision order $\kappa = t$ then the capacity $\mathcal{C}(t)$ of MAC channel with fixed collision order t can be estimated as follows:

$$\begin{aligned}\mathcal{C}(t) \geq \sum_{w=0}^{L} P_{in}(w) \sum_{w'=w}^{L} P(w'|w,t)\log_2 P(w'|w,t) - \\ - \sum_{w''=0}^{L} P_{out}(w''|t)\log_2 P_{out}(w''|t).\end{aligned} \quad (1)$$

Thus one can estimate the capacity of vector-disjunctive channel by the mathematical expectation of $\mathcal{C}(t)$ over collision order t:

$$\mathcal{C} \geq C^{\star} = \sum_{t=0}^{U-1} P(\kappa = t)\mathcal{C}(t). \quad (2)$$

Since $C^{\star}$ can be considered as fraction of information that is guaranteed to be transmitted correctly then it has sense to consider "throughput" $T = C^{\star}L$ because each user can transmit up to L bits per channel use.

In fig. 3 the dependence between the lower bound on "throughput" T and number of active users U for fixed $Q = 1024$ and different values of the number of orthogonal slots $S = Q/L$ is represented. Now we can notice that for some U-ranges $[0; U_{th}]$ the smaller S the larger T. Thus to increase the single user throughput for relatively small number of users it has sense to increase L (the number of bits per channel use) and with the increasing of number of users it has sense to increase the number of orthogonal slots S.

It can be shown that for large enough L the value $\mathcal{C}(t)$ for $t \geq 1$ can be approximated as follows: Let us consider the first part of (1):

$$Z_1 = \sum_{w=0}^{L} P_{in}(w) \sum_{w'=w}^{L} P(w'|w,t)\log_2 P(w'|w,t). \quad (3)$$

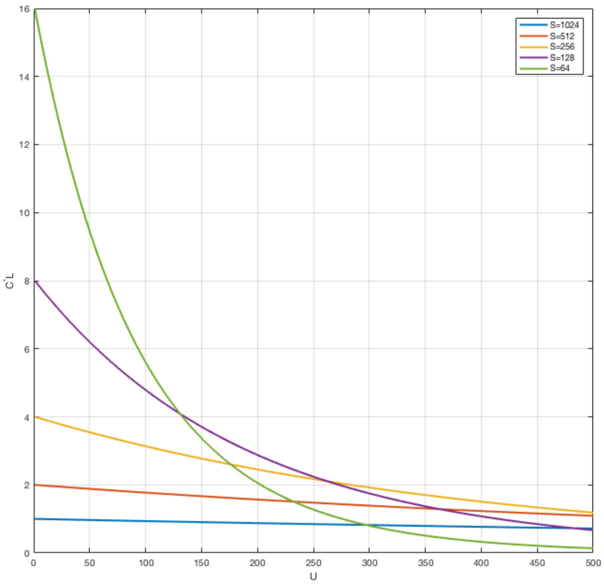

Fig. 3. Lower bound on "throughput" $T = C^{\star}L$ for $Q = 1024$ and different values of $S = Q/L$

More precisely:

$$\begin{aligned}Z_1 = 2^{-L}\sum_{w=0}^{L}\binom{L}{w}\sum_{w'=w}^{L}\binom{L-w}{w'-w}\left(1-2^{-t}\right)^{w'-w} \times \\ \times \left(2^{-t}\right)^{L-w'} \log_2\left(\binom{L-w}{w'-w}\left(1-2^{-t}\right)^{w'-w}\left(2^{-t}\right)^{L-w'}\right).\end{aligned} \quad (4)$$

Let us make a substitution $z = w' - w$ and consider inner sum of (4):

$$\begin{aligned}\sum_{z=0}^{L}\binom{L-w}{z}\left(1-2^{-t}\right)^{z}\left(2^{-t}\right)^{L-w-z} \times \\ \times \log_2\left(\binom{L-w}{z}\left(1-2^{-t}\right)^{z}\left(2^{-t}\right)^{L-w-z}\right).\end{aligned} \quad (5)$$

If we apply an expression for the entropy of binomial distribution

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\log_2\left(\binom{n}{k}p^k(1-p)^{n-k}\right) \approx \\ \approx \frac{1}{2}\log_2\left(2\pi enp(1-p)\right) + O\left(\frac{1}{n}\right)\end{aligned} \quad (6)$$

for (5), then (omitting $O(\cdot)$) Z_1 can represented as follows:

$$\begin{aligned}Z_1 \approx 2^{-L}\sum_{w=0}^{L}\frac{1}{2}\binom{L}{w}\log_2\left(2\pi e\left(L-w\right)\left(2^{-t}-2^{-2t}\right)\right) = \\ = \frac{1}{2}\log_2\left(2\pi e\left(\frac{2^t-1}{4^t}\right)\right) + \frac{1}{2^{L+1}}\sum_{w=0}^{L-1}\binom{L}{w}\log_2(L-w).\end{aligned} \quad (7)$$

Now let us consider the second part of (1):

$$Z_2 = \sum_{w''=0}^{L} P_{out}(w''|t)\log_2 P_{out}(w''|t). \quad (8)$$

Or more precisely:

$$Z_2 = \sum_{w=0}^{L} \sum_{i=0}^{w} f(i, w, L) \log_2 \left(\sum_{i=0}^{w} f(i, w, L) \right),$$

where

$$f(i, w, L) = 2^{-L} \binom{L-i}{w-i} \binom{L}{i} \left(1 - 2^{-t}\right)^{w-i} \left(2^{-t}\right)^{L-w}.$$

Since

$$\binom{L-i}{w-i} \binom{L}{i} = \binom{L}{L-w} \binom{w}{i}$$

and

$$2^{-L} \binom{L}{L-w} \left(2^{-t}\right)^{L-w} \sum_{i=0}^{w} \binom{w}{i} \left(1 - 2^{-t}\right)^{w-i} =$$

$$= \binom{L}{L-w} \left(2^{-(t+1)}\right)^{L-w} \left(1 - 2^{-(t+1)}\right)^{w}$$

then applying an expression for the entropy of binomial distribution we can evaluate Z_2 as follows:

$$Z_2 \approx \frac{1}{2} \log_2 \left(2\pi e L 2^{-(t+1)} (1 - 2^{-(t+1)})\right) + O\left(\frac{1}{L}\right) \quad (9)$$

Thus omitting $O\left(\frac{1}{L}\right)$:

$$\mathcal{C}(t) \approx Z_1 - Z_2,$$

Thus, using (7) and (9):

$$\mathcal{C}(t) \approx \frac{1}{2} \log_2 \left(\frac{L(2^{t+1} - 1)}{4(2^t - 1)} \right) -$$

$$- \frac{1}{2^{L+1}} \sum_{w=0}^{L-1} \binom{L}{w} \log_2 (L - w)$$

Since $\mathcal{C}(t) \to 0$ when $t \to \infty$ then

$$\frac{1}{2^{L+1}} \sum_{w=0}^{L-1} \binom{L}{w} \log_2 (L - w) \approx$$

$$\approx \lim_{t \to \infty} \frac{1}{2} \log_2 \left(\frac{L(2^{t+1} - 1)}{4(2^t - 1)} \right) = \frac{1}{2} \log_2 \left(\frac{L}{2} \right).$$

Thus

$$\mathcal{C}(t) \approx \frac{1}{2} \log_2 \left(\frac{2^{t+1} - 1}{2^{t+1} - 2} \right)$$

or

$$\mathcal{C}(t) \approx \frac{1}{2} \log_2 \left(1 + \frac{1}{2^{t+1} - 2} \right). \quad (10)$$

For the case when $S \to \infty$, $U \to \infty$ and $\psi = \frac{U}{S}$ we can calculate $\lim_{S \to \infty} P(\kappa = t)$ as follows:

$$\lim_{S \to \infty} \binom{\psi S - 1}{t} \left(\frac{1}{S}\right)^t \left(1 - \frac{1}{S}\right)^{\frac{-S(t+1-\psi S)}{S}} =$$

$$= e^{-\psi} \lim_{S \to \infty} \binom{\psi S - 1}{t} \left(\frac{1}{S}\right)^t = \frac{e^{-\psi} \psi^t}{t!}. \quad (11)$$

Thus

$$P(\kappa = t) \longrightarrow \frac{e^{-\psi} \psi^t}{t!}, \quad (12)$$

then combining together (10) and (12), knowing that $\mathcal{C}(0) = 1$, we can estimate $\mathcal{C}$ as follows:

$$\mathcal{C} \geq e^{-\psi} + \frac{1}{2} \sum_{t=1}^{U-1} \frac{e^{-\psi} \psi^t}{t!} \log_2 \left(1 + \frac{1}{2^{t+1} - 2} \right). \quad (13)$$

Since it is well known that $\ln(1 + x) \geq \frac{x}{x+1}$ for all $x > -1$ then

$$\frac{1}{2} \sum_{t=1}^{U-1} \frac{e^{-\psi} \psi^t}{t!} \log_2 \left(1 + \frac{1}{2^{t+1} - 2} \right) \geq$$

$$\geq \frac{1}{2 \ln 2} \sum_{t=1}^{U-1} e^{-\psi} \frac{\psi^t}{t!} \frac{1}{2^{t+1} - 1} > \frac{e^{-\psi}}{4 \ln 2} \sum_{t=1}^{U-1} \frac{\left(\frac{\psi}{2}\right)^t}{t!}.$$

In the case when $U \to \infty$ the latest expression can be represented as follows:

$$\frac{e^{-\psi}}{4 \ln 2} \sum_{t=1}^{U-1} \frac{\left(\frac{\psi}{2}\right)^t}{t!} \to \frac{e^{-\psi}}{4 \ln 2} \left(e^{\frac{\psi}{2}} - 1 \right).$$

Finally, applying well-known logarithmic unequality $\ln(1 + x) \geq \frac{x}{x+1}$ for all $x > -1$, for (13) we obtain:

$$\mathcal{C} \geq e^{-\psi} \left(1 - \frac{1}{4 \ln 2} \right) + \frac{1}{4 \ln 2} e^{-\frac{\psi}{2}}. \quad (14)$$

In Fig. 4 present a dependencies between $C^\star$ and U for $Q = 2^{15}$, $L = 32$, $S = 1024$ which were calculated with (2), (13) and (14):

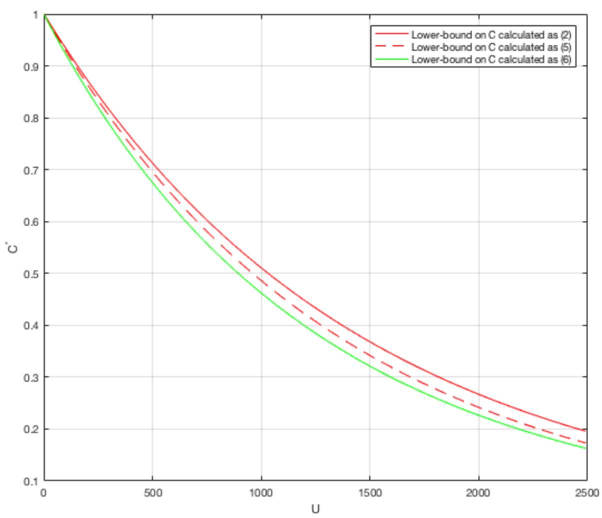

Fig. 4. Lower bounds on $\mathcal{C}$ obtained by (2), (13) and (14)

It is easy to see that expressions (13) and (14) are rather good approximations of exact formula (2). We apply expression (14) for further analysis since it has a very simple form.

In order to evaluate the efficiency of proposed scheme of transmission let us consider the sum-rate of transmission over vector-disjunctive channel:

$$R_\sigma = \frac{ULC^\star}{Q} = \psi C^\star(\psi).$$

This value has the following meaning: it represents the relation between the overall throughput of total number of users (UT) and the maximal achievable throughput (Q) for the system we consider.

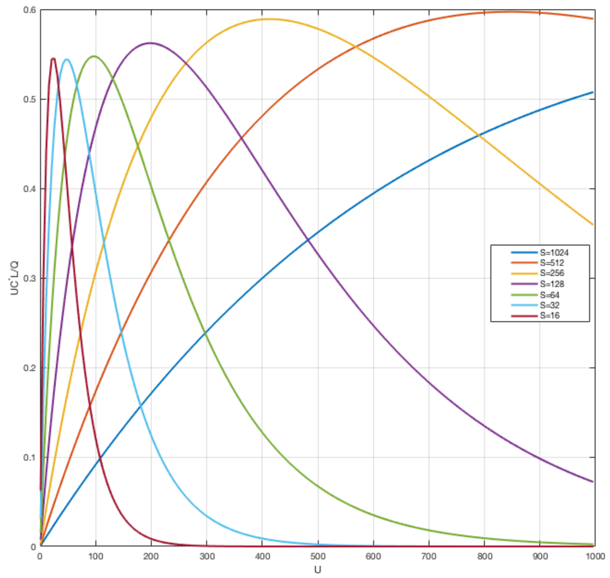

Fig. 5. Lower bound on sum-rate $R_\sigma = UC^\star L/Q$ for $Q = 1024$ and different values of $S = Q/L$

In fig. 5 the dependency between sum-rate R_σ and number of users U for fixed value $Q = 1024$ and different values of $S \in \{16, 32, 64, 128, 256, 512, 1024\}$ is represented. It is obvious that the smaller S the smaller number of users where R_σ achieves maximum. Moreover, maximal achievable values of R_σ increase with growth of S. But there are some areas (when the number of users is relatively small) where sum-rate for smaller S is higher than the one for higher value of S.

Now let us prove that R_σ achieves maximum when a number of users is greater than a number of orthogonal slots S.

In order to do it let us consider a derivative of R_σ:

$$\frac{dR_\sigma}{d\psi} = C_1 e^{-\psi} + C_2 e^{-\frac{\psi}{2}} + \psi\left(-C_1 e^{-\psi} - \frac{1}{2} C_2 e^{-\frac{\psi}{2}}\right),$$

where $C_1 = 1 - \frac{1}{4\ln 2} > 0$ and $C_2 = \frac{1}{4\ln 2} > 0$. It is easy to show that $\frac{dR_\sigma}{d\psi}$ decreases monotonically with growth of ψ. Thus R_σ achieves maximum when $\frac{dR_\sigma}{d\psi} = 0$:

$$C_1(1-\psi) + C_2\left(1 - \frac{\psi}{2}\right) e^{\frac{\psi}{2}} = 0. \quad (15)$$

If $\psi \leq 1$ then $1 - \psi \geq 0$ and $1 - \frac{\psi}{2} > 0$ thus $\frac{dR_\sigma}{d\psi} > 0$. Thereby $\frac{dR_\sigma}{d\psi} = 0$ only when $\frac{U}{S} = \psi > 1$.

Since it is rather difficult to find an exact solution of (15) we apply numerical methods to find solution $\psi \approx 1.357$. For this ψ we have $R_\sigma \approx 0.4717$. So, when the number of users are more than the number of orthogonal slots on 35%, then the maximum sum-rate is 0.4717.

Now let us consider this "optimal" number of users $U_{opt} = \underset{U}{\operatorname{argmax}}\, R_\sigma(U, Q, S)$ for different Q and L, applying exact formula (2).

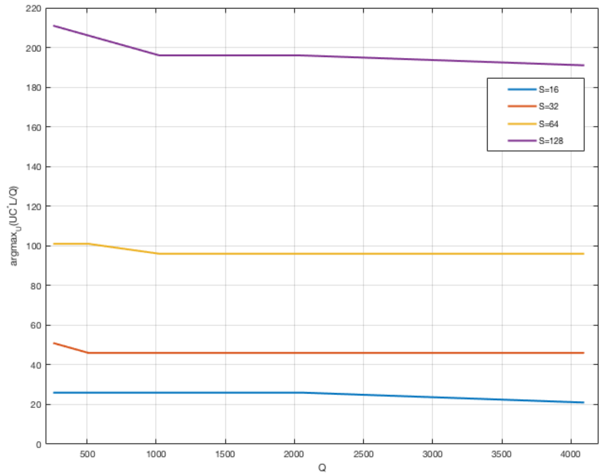

Fig. 6. Number of users when maximum of R_σ is achieved: for different Q and different of $S = Q/L$.

In fig. 6 the dependency between U_{opt} and different values of Q and S is represented. As it was proved U_{opt} always greater than the S – number of users that can transmit information orthogonal.

III. Conclusion

In this paper we consider a vector-disjunctive channel where users transmit some vectors of bits. We estimate the capacity of this channel deriving the lower bound on this value for a different number of active users and various channel parameters.

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