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Geometric structures on flat affine manifolds

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a) P. Osipov "Selfsimilar Hessian manifolds", Journal of Geometry and Physics, 2022, Vol. 175.

b) P. Osipov "Self-similar Hessian and conformally Kähler manifolds", Annals of Global Analysis and Geometry, 2022, Vol. 62, No. 3.

c) P. Osipov "Statistical Lie algebras of constant curvature and locally conformally Kähler Lie algebras", Bulletin Mathematique de la Société des Sciences Mathématiques de Roumanie, 2022, vol. 65 (113), No. 3, pp. 341–358.

Contents

1 Self-similar manifolds

In [O1], we study Riemannian manifolds endowed with a homothetic vector field. A **self-similar manifold** is a Riemannian manifold (M,g) endowed with a vector field ξ satisfying $\mathcal{L}_\xi g = 2g$. Moreover, if the vector field ξ is complete then we say that (M,g,ξ) is a **globally self-similar manifold**.

Example 1 *Let $(C = M \times \mathbb{R}^{>0}, g = s^2 g_M + ds^2)$ be a **Riemannian cone** and $\xi = s\frac{\partial}{\partial s}$. Then (C,g,ξ) is a globally self-similar manifold.*

Riemannian cones have important applications in supegravity ([ACDM], [ACM], [CDM], [CDMV]).

Example 2 ([O1]) *Let φ and s are coordinates on S^1 and $\mathbb{R}^{>0}$ then the collection $\left(C = S^1 \times \mathbb{R}^{>0}, g = s^2 d\varphi^2 + s ds\cdot d\varphi + ds^2, s\frac{\partial}{\partial s}\right)$ is a global self-similar manifold but (C,g) is not isometric to a Riemannian cone.*

We describe global self-similar manifolds.

Theorem 1 ([O1]) *Any global self-similar manifold (C,g,ξ) is isomorphic to one of the following:*

(i) $\left(\mathbb{R}^n, \sum_{i=1}^n \left(dx^i\right)^2, \rho + \eta\right)$, *where $a \in \mathbb{R}$, $\rho = \sum_{i=1}^n x^i \frac{\partial}{\partial x^i}$ is a radiant vector field and $\eta \in \mathfrak{so}(n)$ is a Killing vector field.*

(ii) $\left(\hat{M} = M \times \mathbb{R}^{>0}, \hat{g} = s^2 g_M + s ds\cdot\alpha + ds^2, s\frac{\partial}{\partial s}\right)$, *where s is a coordinate on $\mathbb{R}^{>0}$, g_M is a Riemannian metric on M, α a 1-form on M, and $g_M(X,X) + 2\alpha(X) + 1 > 0$, for any $X \in \Gamma(TM)$.*

Any self-similar manifold is locally isomorphic to a global self-similar manifold.

We say that (C,g,ξ) is a **self-similar manifold with a potential homothetic vector field** if (M,g,ξ) is a selfimilar manifold and ξ is locally defined as a gradient of a function. If $\xi = \operatorname{grad} f$ on a domain U then $\iota_\xi g|_U = df$. Moreover, a form is closed if and only if it is locally exact. Therefore, the vector field ξ is potential if and only if $d\iota_\xi g = 0$.

Theorem 2 ([O1]) *Let* (M, g, ξ) *is a global self-similar manifold with a potential homothetic vector field.*

(i) If ξ *vanishes at a point then* (M, g, ξ) *is Euclidean space with a radiant vector field* $\left(\mathbb{R}^n, \sum_{i=1}^n (dx^i)^2, \sum x^i \frac{\partial}{\partial x^i}\right)$.

(ii) If ξ *does not vanishes at any point then* (M, g, ξ) *is a Riemannian cone* $\left(\hat{M}, \hat{g}, \hat{\xi}\right)$.

2 Selfimilar Hessian manifolds

A **flat affine manifold** is a differentiable manifold equipped with a flat torsion-free connection. Equivalently, it is a manifold equipped with an atlas such that all transition functions between charts are affine transformations (see [FGH] or [Sh]). A **Hessian manifold** is a flat affine manifold with a Riemannian metric wich is locally equivalent to a Hessian of a function.

Hessian manifolds have many different application: in supersymmetry ([CMMS], [CM], [AC]), in convex programming ([N], [NN]), in the Monge-Ampère Equation ([F1], [F2], [Gu]), in the WDVV equations ([To]).

A **self-similar Hessian manifold** (C, ∇, g, ξ) is a Hessian manifold (C, ∇, g) endowed with a vector field ξ such that (C, g, ξ) is a self-similar manifold and the flow along ξ preserves ∇. If ξ is complete then (C, g, ξ) is called a **global self-similar Hessian manifold**.

A **radiant manifold** (C, ∇, ρ) is a flat affine manifold (C, ∇) endowed with a **radiant vector field** i.e. a field ρ satisfying

$$\nabla \rho = \mathrm{Id}.$$

We call a self-similar Hessian manifold (C, ∇, g, ξ) a **radiant Hessian manifold** if and only if there exists a radiant vector field ρ on C and a constant $\lambda \in \mathbb{R}$ such that $\xi = \lambda \rho$.

Theorem 3 *[O1] Let* (C, ∇, ξ) *be a self-similar Hessian manifold. Then* ξ

is potential if and only if (C, ∇, ξ) *is locally isomorphic to a direct product of radiant Hessian manifolds.*

3 Self-similar Hessian and conformally Kähler manifolds

A Kähler structure (I, g^{r}) on TM can be constructed by a Hessian structure (∇, g) on M (see [Sh]). The correspondence

$$\mathrm{r} : \{\text{Hessian manifolds}\} \to \{\text{Kähler manifolds}\}$$

$$(M, \nabla, g) \quad \to \quad (TM, I, g^{\mathrm{r}})$$

is called the **(affine) r-map**. In particular, this map associates some special Kähler manifolds to special real manifolds (see [AC]). In this case, r-map describes a correspondence between the scalar geometries for supersymmetric theories in dimension 5 and 4. See [CMMS] for details on the r-map and supersymmetry.

An open cone $V \subset \mathbb{R}^n$ is called **regular** if it does not contain full straight lines. Any convex regular cone admits a function φ called **characteristic function** such that $g_{can} = \mathrm{Hess}\,(\ln\varphi)$ is a Hessian metric which is invariant with respect to all automorphisms of the cone. ([V]). The r-map constructs an invariant Kähler structure $(I, g^{\mathrm{r}}_{can})$ on $TV \simeq V \oplus \sqrt{-1}\mathbb{R}^n$. Thus, any homogeneous Siegel domain of the first kind admits an invariant Kähler structure. The Kähler potential of g^{r}_{can} equals $4\pi^*(\ln\varphi)$.

The construction of the invariant Kähler structure on $V \oplus \sqrt{-1}\mathbb{R}^n$ is well known (see [VGP] or [C]). We modify this construction. A **(globally) conformally Kähler manifold** (M, I, ω) is a complex manifold endowed with Riemannian metric g which is (globally) conformally equivalent to a Kahler one. We consider the metric $g_{con} = \mathrm{Hess}\,\varphi$ on a regular homogeneous cone V. This metric is invariant under $\mathrm{Aut}(V) \cap \mathrm{SL}(\mathbb{R}^n)$ and coincides with g_{can} on the hypersurface $\{\varphi(x) = 1\}$. The dilation $x \mapsto qx$ acts on g_{con} by

$\lambda_q^* g_{con} = q^{-n} g_{con}$. The Kähler metric $g^{\mathrm{r}}{}_{con}$ on $V \oplus \sqrt{-1}\mathbb{R}^n$ constructed by the r-map is not invariant but it is conformally equivalent to the invariant Riemannian metric $r^{-2} g^{\mathrm{r}}{}_{con}$ on the homogeneous domain $V \times \sqrt{-1}\mathbb{R}^n$. Thus, Siegel domains of the first kind admit two different invariant structures: Kähler and conformally Kähler.

In [O2], we generalize this construction to self-similar Hessian manifolds. The main result of [O2] is the following.

Theorem 4 *Let (M, ∇, g, ξ) be a simply connected self-similar Hessian manifold such that ξ is complete and G be the group of affine isometries of (M, ∇, g) preserving ξ. Suppose that G acts simply transitively on the level set $\{g(\xi, \xi) = 1\}$. Then TM admits a homogeneous conformally Kähler structure.*

4 Statistical manifolds of constant curvature

A **statistical manifold** (C, D, g) is a manifold M endowed with a torsion-free connection D and a Riemannian metric g such that the tensor Dg is totally symmetric. The term "statistical manifolds" arose in information geometry (see [AN]). In this sense, statistical manifolds is a space of probability distributions endowed with the Fisher information metric. For example, the statistical manifold corresponding to the family of normal distributions is isometric to hyperbolic plane.

A statistical manifold (C, D, g) is said to be **of constant curvature** c if the curvature tensor Θ_D satisfies

$$\Theta_D(X, Y)Z = c\left(g(Y, Z)X - g(X, Z)Y\right),$$

for any $X, Y, Z \in TM$. For example, a Riemannian manifold of constant section curvature is statistical manifold of constant curvature. The definition of statistical manifolds of constant curvature arose in the context of geometry of affine hypersurfaces ([Ku]). Note that Hessian manifolds are statistical manifolds of curvature 0. We assume that the curvature of a statistical manifold is not equal to 0.

Convex projective geometry provides a wide class of statistical manifolds. A domain $U \subset \mathbb{RP}^n$ is called properly convex if the closure of U is a compact convex set in some affine chart. If Γ is a discrete subgroup of the group of projective automorphisms of a properly convex domain $U \subset \mathbb{RP}^n$ such that $M = U/\Gamma$ is a manifold then M is called a **properly convex** $\mathbb{RP}^n$-manifold. For examples of compact properly convex $\mathbb{RP}^n$-manifolds see [B].

Theorem 5 ([KO]) *Any properly convex $\mathbb{RP}^n$-manifold admits a statistical structure of negative constant curvature. Any compact statistical manifold of negative constant curvature admits a properly convex $\mathbb{RP}^n$ structure.*

In [O3], we construct the correspondence between radiant Hessian manifolds and statistical manifolds of constant curvature. Precisely, we show that a Riemannian cone $(M \times \mathbb{R}^{>0}, s^2 g_M + ds^2)$ over a statistical manifold (M, g, D) of constant curvature admits a structure of a radiant Hessian manifold. Conversely, level sets of a Hessian potential on a radiant Hessian manifold are statistical manifolds of constant curvature.

By dd^c Lemma, Any Kähler form can be locally represented as a complex Hessian $dd^c\varphi$. Hence, we can consider Hessian manifold are a real analogue of Kähler manifolds. A **Sasakian manifold** is a Riemmanian manifold (M, g) such that the cone metric $g = s^2 g_M + ds^2$ on $M \times \mathbb{R}^{>0}$ is Kähler with respect to a dilatation-invariant complex structure I (see [OV]). Thus, we can consider statistical manifolds of constant curvature as an analogue of Sasakian manifolds.

The r-map construct a Kähler structure on TM by a Hessian structure on M. The following theorem provides an analogue of the r-map for statistical manifolds of constant curvature.

Theorem 6 ([O3]) *Let (M, g, ∇) be a statistical manifold of a constant curvature. Then $TM \times \mathbb{R}$ admits a structure of a Sasakian manifold.*

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