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**Advances in Applied Clifford Algebras**

# Method of Averaging in Clifford Algebras

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**Abstract.** In this paper we consider different operators acting on Clifford algebras. We consider Reynolds operator of Salingaros' vee group. This operator "average" an action of Salingaros' vee group on Clifford algebra. We consider conjugate action on Clifford algebra. We present a relation between these operators and projection operators onto fixed subspaces of Clifford algebras. Using method of averaging we present solutions of system of commutator equations.

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## 1. Introduction

In this paper we consider the following operators acting on real Clifford algebras

$$F_S(U)=\frac{1}{|S|}\sum_{A\in S}(e^A)^{-1}Ue^A, \tag{1.1}$$

where

$$e^A=e^{a_1a_2\dots a_k}=e^{a_1}e^{a_2}\cdots e^{a_k},\quad A=a_1a_2\dots a_k,\quad a_1<a_2<\cdots<a_k,$$

are basis elements generated by an orthonormal basis in vector space $V$ over a field $\mathbb{R}$. Here $S\subseteq \mathrm{I}$ is a subset of the set of all ordered multi-indices $A$ of the length from 0 to $n$. We denote the number of elements in $S$ by $|S|$. Note that not for every subset $S\subseteq I$ in (1.1), the set $\{e^A|A\in S\}$ is a group.

We can consider Reynolds operator (see, for example, [6]) acting on a Clifford algebra element $U\in C\!\ell(p,q)$

$$R_G(U)=\frac{1}{|G|}\sum_{g\in G}g^{-1}Ug, \tag{1.2}$$

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where $|G|$ is the number of elements in a finite subgroup $G \subset C\ell(p,q)^\times$. We denote the group of all invertible Clifford algebra elements by $C\ell(p,q)^\times$. These operators "average" an action of group $G$ on Clifford algebra $C\ell(p,q)$.

We can take Salingaros' vee group $G = \{\pm e^A, A \in \mathrm{I}\}$ (see [1–3]), where $e^A$ are basis elements of Clifford algebra $C\ell(p,q)$. Note that Salingaros' vee group is a finite subroup of spin groups (groups $\mathrm{Pin(p,q)}$ and $\mathrm{Spin(p,q)}$, see [2,8,19,22]).

We can write in this case

$$\frac{1}{|G|}\sum_{g\in G} g^{-1}Ug = \frac{1}{2^{n+1}}\sum_{A\in\mathrm{I}}((e^A)^{-1}Ue^A + (-e^A)^{-1}U(-e^A))$$
$$= \frac{1}{2^n}\sum_{A\in\mathrm{I}}(e^A)^{-1}Ue^A.$$

We consider such operators further in this paper.

Note that operators (1.2) are often used in representation theory of finite groups (see [4,7,16]). We use these operators in Clifford algebras to obtain some new properties.

We present a relation between these operators and projection operators onto fixed subspaces of Clifford algebras.

Using method of averaging we present solutions $X$ of the system of commutator equations

$$e^A X + \epsilon X e^A = Q^A, \quad A \in S \subseteq \mathrm{I}, \quad \epsilon \in \mathbb{R}^\times$$

for some given elements[1] $Q^A \in C\ell(p,q)$. We use notation $\mathbb{R}^\times = \mathbb{R}\setminus 0$.

## 2. Clifford algebras, ranks, projection operators

Consider real Clifford algebra $C\ell(p,q)$ with $p+q=n$, $n\geq 1$. The construction of Clifford algebra is discussed in details in [9–11] or [12].

Let $e$ be the identity element and let $e^a$, $a=1,\ldots,n$ be generators[2] of the Clifford algebra $C\ell(p,q)$,

$$e^a e^b + e^b e^a = 2\eta^{ab}e,$$

where $\eta = ||\eta^{ab}|| = ||\eta_{ab}||$ is the diagonal matrix with $p$ pieces of $+1$ and $q$ pieces of $-1$ on the diagonal. Elements

$$e^{a_1\ldots a_k} = e^{a_1}\cdots e^{a_k}, \quad a_1<\cdots<a_k,\ k=1,\ldots,n,$$

together with the identity element $e$ form the basis of the Clifford algebra. The number of basis elements is equal to $2^n$.

Let us denote the set of ordered multi-indices of the length from 0 to $n$ by

$$\mathrm{I} = \{-,\ 1,\ \ldots,\ n,\ 12,\ 13,\ \ldots,\ 1\ldots n\}, \tag{2.1}$$

[1] Note that $A$ in $Q^A$ is a label (multi-index) that serves to pair off an arbitrary Clifford element $Q^A$ with basis element $e^A$.

[2] We use notation from [5] (see, also [13]). Note that there exists another notation instead of $e^a$—with lower indices. But we use upper indices because we take into account relation with differential forms. Note that $e^a$ is not exponent.

where "–" is an empty multi-index. So, we have the basis of Clifford algebra $\mathfrak{B} = \{e^A,\ A \in \mathrm{I}\}$, where $A$ is an arbitrary ordered multi-index[3]. Let us denote the length of multi-index $A$ by $|A|$. So, we use notation

$$e^A = e^{a_1 a_2 \dots a_k} = e^{a_1} e^{a_2} \cdots e^{a_k}, \quad A = a_1 a_2 \dots a_k, \quad a_1 < a_2 < \cdots < a_k.$$

Below we also consider different subsets $S \subseteq \mathrm{I}$:

$$\mathrm{I}_{\mathrm{Even}} = \{A \in \mathrm{I},\ |A| - \text{even}\}, \quad \mathrm{I}_{\mathrm{Odd}} = \{A \in \mathrm{I},\ |A| - \text{odd}\}.$$

We have $e_a = \eta_{ab} e^b$, $e^a = \eta^{ab} e_b$, where we use Einstein summation convection (there is a sum over index $b$). We denote[4] expressions

$$\eta_{a_1 b_1} \cdots \eta_{a_k b_k} e^{b_k} \cdots e^{b_1} = e_{a_k} \cdots e_{a_1} = (e^{a_1 \dots a_k})^{-1}, \quad a_1 < \cdots < a_k.$$

by $e_{a_1 \dots a_k}$ (not $e_{a_k \dots a_1}$). So, $e_A = (e^A)^{-1}\ \ \forall A \in \mathrm{I}$ in our notation.

Any Clifford algebra element[5] $U \in C\ell(p,q)$ can be written in the form

$$U = ue + u_a e^a + \sum_{a_1 < a_2} u_{a_1 a_2} e^{a_1 a_2} + \cdots + u_{1\dots n} e^{1 \dots n} = u_A e^A, \tag{2.2}$$

where we have a sum[6] over ordered multi-index $A$ and

$$\{u_A\} = \{u, u_a, u_{a_1 a_2}, \dots, u_{1 \dots n}\}$$

are real numbers.

We denote by $C\ell_k(p,q)$, $k = 0, 1, \dots, n$ the vector spaces that span over the basis elements $e^{a_1 \cdots a_k}$. Elements of $C\ell_k(p,q)$ are said to be elements of rank[7] $k$. We have

$$C\ell(p,q) = \bigoplus_{k=0}^{n} C\ell_k(p,q). \tag{2.3}$$

We consider (linear) projection operators on the vector subspaces $C\ell_k(p,q)$

$$\pi_k : C\ell(p,q) \to C\ell_k(p,q), \quad \pi_k(U) = \sum_{a_1 < \cdots < a_k} u_{a_1 \dots a_k} e^{a_1 \cdots a_k}. \tag{2.4}$$

Clifford algebra $C\ell(p,q)$ is a superalgebra. It is represented as the direct sum of even and odd subspaces (of *even* and *odd* elements respectively)

$$C\ell(p,q) = C\ell_{\mathrm{Even}}(p,q) \oplus C\ell_{\mathrm{Odd}}(p,q),$$

$$C\ell_{\mathrm{Even}}(p,q) = \bigoplus_{k-even} C\ell_k(p,q), \quad C\ell_{\mathrm{Odd}}(p,q) = \bigoplus_{k-odd} C\ell_k(p,q).$$

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[3] We use notation $e^A$ from [5]. Note that $e^A$ is not exponent, $A$ is a multi-index.

[4] We want to deal only with ordered multi-indices. So, multi-index $a_1 \dots a_k$ is indivisible object in our consideration. It is not a set of indices $a_1, a_2, \dots a_k$. That's why $e_{a_1 \dots a_k} \neq e_{a_1} \cdots e_{a_k}$ in our notation.

[5] We denote Clifford algebra elements by capital letters (not small letters that is more traditional) to avoid confusion with numbers because sometimes Clifford algebra elements have indices or multi-indices in this paper too (see [13]).

[6] We use Einstein summation convection for multi-indices too.

[7] There is a difference in notation in literature. We use term "rank" and notation $C\ell_k(p,q)$ because we take into account relation with differential forms, see [13].

## 3. Reynolds operator of the Salingaros' vee group

We have the following well-known statement about center

$$\mathrm{Cen}(C\ell(p,q)) = \{U \in C\ell(p,q) \,|\, UV = VU \quad \forall V \in C\ell(p,q)\}$$

of Clifford algebra $C\ell(p,q)$.

**Theorem 3.1.** *We have*

$$\mathrm{Cen}(C\ell(p,q)) = \begin{cases} C\ell_0(p,q), & n \text{ is even;} \\ C\ell_0(p,q) \oplus C\ell_n(p,q), & n \text{ is odd.} \end{cases} \tag{3.1}$$

Let us consider the following operator (Reynolds operator of the Salingaros' vee group $\{e^A, A \in \mathrm{I}\}$, see above)

$$F(U) = \frac{1}{2^n} e_A U e^A,$$

where we have a sum over multi-index $A \in \mathrm{I}$.

**Theorem 3.2.** *We have*

$$F(U) = \frac{1}{2^n} e_A U e^A = \begin{cases} \pi_0(U), & \text{if } n \text{ is even;} \\ \pi_0(U) + \pi_n(U), & \text{if } n \text{ is odd,} \end{cases} \tag{3.2}$$

*where $\pi_0$ and $\pi_n$ are projection operators (see (2.4)) onto the subspaces of fixed ranks. Operator $F$ is a projector $F^2 = F$ (on the center of Clifford algebra).*

*Proof.* We have

$$(e^a)^{-1} F(U) e^a = \sum_A (e^A e^a)^{-1} F(U)(e^A e^a) = \sum_B (e^B)^{-1} F(U) e^B = F(U).$$

So, $F(U)$ is in the center of Clifford algebra (see Theorem 3.1). For elements $U$ of ranks $k = 1, \ldots, n-1$ (and $k = n$ in the case of even $n$) we have $F(U) = 0$. In other particular cases we have $e_A e^A = 2^n e$ and (in the case of odd $n$) $e_A e^{1\ldots n} e^A = 2^n e^{1\ldots n}$. It is also easy to verify that $F^2 = F$. $\square$

Note that $\mathrm{Cen}(C\ell(p,q))$ is the "ring of invariants" (in the language of [6]) of Salingaros' vee group.

**Theorem 3.3.** *Let an element $X \in C\ell(p,q)$ satisfy the system of $2^n$ equations with some given elements*[8] *$Q^A \in C\ell(p,q)$*

$$e^A X + \epsilon X e^A = Q^A \quad \forall A \in \mathrm{I}, \quad \epsilon \in \mathbb{R}^\times. \tag{3.3}$$

*If $\epsilon = -1$ (commutator case), then this system of equations either has no solution or it has a unique solution up to element of center:*

$$X = -\frac{1}{2^n} Q^A e_A + Z, \quad Z \in \mathrm{Cen}(C\ell(p,q)). \tag{3.4}$$

[8] Note that $A$ in $Q^A$ is a label (multi-index) that serves to pair off an arbitrary Clifford element $Q^A$ with basis element $e^A$.

*If $\epsilon \neq -1$, then this system of equations either has no solution or it has a unique solution*

$$X = \begin{cases} \frac{1}{2^n \epsilon}\left(Q^A e_A - \frac{1}{(\epsilon+1)}\pi_0(Q^A e_A)\right), & \text{if } n \text{ is even,} \\ \\ \frac{1}{2^n \epsilon}\left(Q^A e_A - \frac{1}{(\epsilon+1)}(\pi_0(Q^A e_A) + \pi_n(Q^A e_A))\right), & \text{if } n \text{ is odd.} \end{cases} \tag{3.5}$$

*Proof.* Let us multiply each equation by $e_A$ on the right and add them (see Theorem 3.2):

$$e^A X e_A + \epsilon X e^A e_A = Q^A e_A \quad \Rightarrow \quad 2^n \pi_{center}(X) + \epsilon X 2^n = Q^A e_A,$$

where $\pi_{center}$ is the projection on the center of Clifford algebra. Using $X = \sum_{k=0}^{n} \pi_k(X)$ and Theorem 3.1, we obtain statement of the theorem. □

Note that we have a solution or have no solution of the system of commutator equations. It depends on elements $Q^A$ (it suffices to substitute solution in equation and check the equality).

As suggested by one of the referees, in the case $\epsilon \neq -1$ we can take solution of the "first" equation (3.3) $X = \frac{1}{1+\epsilon}Q$ (here $Q$ has empty multi-index) and substitute it in the other equations. We obtain the condition

$$Q^A = \frac{1}{1+\epsilon}(e^A Q + \epsilon Q e^A), \tag{3.6}$$

that we can also rewrite in the form[9] $(e^A)^{-1} Q^A = \pi_{[A]}(Q) + \frac{1-\epsilon}{1+\epsilon}\pi_{\{A\}}(Q)$.

So, we can say, that system of Eq. (3.3) has solution in the case $\epsilon \neq -1$ if and only if given elements $Q^A$ have the following connection (3.6) with the first of them ($Q$).

In the case $\epsilon = -1$ it is not difficult to understand that $\pi_{center}(Q^A (e^A)^{-1}) = 0$ is the necessary condition for the system (3.3) to have a solution. Sufficient condition is

$$-e^A \left(\frac{1}{2^n} Q^B e_B\right) + \left(\frac{1}{2^n} Q^B e_B\right) e^A = Q^A, \quad \forall A \in \mathrm{I}$$

with summation over multi-index $B$. It can be rewritten in the form

$$(e^A)^{-1} Q^A = \frac{1}{2^n}\pi_{\{A\}}(e_B Q^B), \quad \forall A \in \mathrm{I}.$$

## 4. Adjoint sets of multi-indices

We call ordered multi-indices $a_1 \dots a_k$ and $b_1 \dots b_l$ *adjoint multi-indices* if they have no common indices and they form multi-index $1 \dots n$ of the length $n$. We write $b_1 \dots b_l = \widetilde{a_1 \dots a_k}$ and $a_1 \dots a_k = \widetilde{b_1 \dots b_l}$. We call corresponding basis elements $e^{a_1 \dots a_m}$, $e^{b_1 \dots b_l}$ *adjoint* and write $e^{b_1 \dots b_l} = e^{\widetilde{a_1 \dots a_m}}$, $e^{a_1 \dots a_m} = e^{\widetilde{b_1 \dots b_l}}$. We can also write that $e^{a_1 \dots a_m} e^{b_1 \dots b_l} = \pm e^{1 \dots n}$ and $\star e^{a_1 \dots a_m} = \pm e^{b_1 \dots b_l}$,

[9] We use operators $\pi_{[A]}$ and $\pi_{\{A\}}$ here. See about them below.

where $\star$ is Hodge operator[10]. We denote the sets of corresponding $2^{n-1}$ multi-indices by $\mathrm{I}_{\mathrm{Adj}}$ and $\widetilde{\mathrm{I}_{\mathrm{Adj}}} = \mathrm{I} \setminus \mathrm{I}_{\mathrm{Adj}}$. So, for each multi-index in $\mathrm{I}_{\mathrm{Adj}}$ there exists adjoint multi-index in $\widetilde{\mathrm{I}_{\mathrm{Adj}}}$. We have

$$\mathfrak{B} = \{e^A \mid A \in \mathrm{I}\} = \{e^A \mid A \in \mathrm{I}_{\mathrm{Adj}}\} \cup \{e^A \mid A \in \widetilde{\mathrm{I}_{\mathrm{Adj}}}\}. \tag{4.1}$$

For Clifford algebra $C\ell(p,q)$ of dimension $n = p+q$ we have $2^{2^{n-1}-1}$ different partitions[11] of the form (4.1). For example,

$$\mathrm{I}_{\mathrm{Adj}} = \mathrm{I}_{\mathrm{First}}, \quad \widetilde{\mathrm{I}_{\mathrm{Adj}}} = \mathrm{I} \setminus \mathrm{I}_{\mathrm{First}} = \mathrm{I}_{\mathrm{Last}},$$

where $\mathrm{I}_{\mathrm{First}}$ consists of the first (in the order) $2^{n-1}$ multi-indices of the set I (2.1). In the case of odd $n$ we can write

$$\mathrm{I}_{\mathrm{First}} = \left\{A \in \mathrm{I}, \quad |A| \leq \frac{n-1}{2}\right\}, \quad \mathrm{I}_{\mathrm{Last}} = \left\{A \in \mathrm{I}, \quad |A| \geq \frac{n+1}{2}\right\}.$$

In the case of odd $n$ we can consider the following adjoint sets

$$\mathrm{I}_{\mathrm{Adj}} = \mathrm{I}_{\mathrm{Even}}, \quad \widetilde{\mathrm{I}_{\mathrm{Adj}}} = \mathrm{I}_{\mathrm{Odd}}.$$

## 5. Commutative properties of basis elements

**Theorem 5.1.** *Consider real Clifford algebra $C\ell(p,q)$, $p+q=n$ and the set of basis elements $\mathfrak{B} = \{e^A, A \in \mathrm{I}\}$.*

*Then each element of this set (if it is neither $e$ nor $e^{1\ldots n}$) commutes with $2^{n-2}$ even elements of the set $\mathfrak{B}$, commutes with $2^{n-2}$ odd elements of the set $\mathfrak{B}$, anticommutes with $2^{n-2}$ even elements of the set $\mathfrak{B}$ and anticommutes with $2^{n-2}$ elements of the set $\mathfrak{B}$. Element $e$ commutes with all elements of the set $\mathfrak{B}$.*

1. *if $n$—even, then $e^{1\ldots n}$ commutes with all $2^{n-1}$ even elements of the set $\mathfrak{B}$ and anticommutes with all $2^{n-1}$ odd elements of the set $\mathfrak{B}$;*
2. *if $n$—odd, then $e^{1\ldots n}$ commutes with all $2^n$ elements of the set $\mathfrak{B}$.*

*Proof.* The cases $k=0$ and $k=n$ are trivial (see Theorem 3.1).

Let us fix one multi-index $A$ of the length $k$. Then there are $C_k^i C_{n-k}^{m-i}$ different multi-indices of the fix length $m$ that have fixed number $i$ coincident indices with multi-index $A$. Here $C_n^k = \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$ is binomial coefficient (we have $C_n^k = 0$ for $k > n$). Note, that $\sum_{i=0}^n C_k^i C_{n-k}^{m-i} = C_n^m$ (Vandermonde's convolution)—the full number of ordered multi-indices of the length $m$. When we swap basis element with multi-index $A$ of the length $k$ with another basis element with multi-index of the length $m$, then we obtain coefficient $(-1)^{km-i}$, where $i$ is the number of coincident indices in these 2 multi-indices, i.e. $e^{a_1\ldots a_k} e^{b_1\ldots b_m} = (-1)^{km-i} e^{b_1\ldots b_m} e^{a_1\ldots a_k}$.

[10] It is the analogue of Hodge operator in Clifford algebra $\star U = \tilde{U} e^{1\ldots n}$, where $\tilde{}$ is the reversion anti-automorphism in the Clifford algebra $C\ell(p,q)$ [11].

[11] For example, in the case $n=2$ we have 2 partitions $\{e, e^1, e^2, e^{12}\} = \{e, e^1\} \cup \{e^{12}, e^2\} = \{e, e^2\} \cup \{e^{12}, e^2\}$.

If $k$ is even and does not equal to 0 and $n$, then the number of even and odd elements $e^{b_1\dots b_m}$ that commute (in this case coefficient $km-i$ must be even, and so $i$ is even) with fixed $e^{a_1\dots a_k}$ respectively equals

$$\sum_{m-even}\sum_{i-even} C_k^i C_{n-k}^{m-i} = 2^{n-2},\qquad \sum_{m-odd}\sum_{i-even} C_k^i C_{n-k}^{m-i} = 2^{n-2}.$$

If $k$ is odd and does not equal to $n$, then the number of even and odd elements $e^{b_1\dots b_m}$ that anticommute (in this case $km-i$ is odd, and so $m-i$ is even) with $e^{a_1\dots a_k}$ respectively equals

$$\sum_{m-even}\sum_{i-even} C_k^i C_{n-k}^{m-i} = 2^{n-2},\qquad \sum_{m-odd}\sum_{i-odd} C_k^i C_{n-k}^{m-i} = 2^{n-2}.$$

We can prove the last 4 identities if we regroup summands. For example, we have

$$\sum_{m-even}\sum_{i-even} C_k^i C_{n-k}^{m-i} = (\sum_{j=even} C_k^j)(\sum_{l-even} C_{n-k}^l) = 2^{k-1}2^{n-k-1} = 2^{n-2}.$$

□

Also we have the following theorem about adjoint sets of multi-indices.

**Theorem 5.2.** *Consider real Clifford algebra $C\ell(p,q)$, $p+q=n$ and the set of basis elements $\mathfrak{B}=\{e^A, A\in \mathrm{I}\}$. Suppose we have a partition $\mathrm{I}=\mathrm{I}_{\mathrm{Adj}}\cup\widetilde{\mathrm{I}_{\mathrm{Adj}}}$.*

*If $n$ is even then any even (not odd!) basis element (if it is not $e$) commutes with $2^{n-2}$ basis elements from $\{e^A \,|\, A\in \mathrm{I}_{\mathrm{Adj}}\}$, anticommutes with $2^{n-2}$ basis elements from $\{e^A \,|\, A\in \mathrm{I}_{\mathrm{Adj}}\}$, commutes with $2^{n-2}$ basis elements from $\{e^A \,|\, A\in \widetilde{\mathrm{I}_{\mathrm{Adj}}}\}$ and anticommutes with $2^{n-2}$ basis elements from $\{e^A \,|\, A\in \widetilde{\mathrm{I}_{\mathrm{Adj}}}\}$.*

*If $n$ is odd then any basis element (if it is neither $e$ nor $e^{1\dots n}$) commutes with $2^{n-2}$ basis elements from $\{e^A \,|\, A\in \mathrm{I}_{\mathrm{Adj}}\}$, anticommutes with $2^{n-2}$ basis elements from $\{e^A \,|\, A\in \mathrm{I}_{\mathrm{Adj}}\}$, commutes with $2^{n-2}$ basis elements from $\{e^A \,|\, A\in \widetilde{\mathrm{I}_{\mathrm{Adj}}}\}$ and anticommutes with $2^{n-2}$ basis elements from $\{e^A \,|\, A\in \widetilde{\mathrm{I}_{\mathrm{Adj}}}\}$.*

Note that in the case of odd $n$ we can take $\mathrm{I}_{\mathrm{Adj}}=\mathrm{I}_{\mathrm{Even}}$, $\widetilde{\mathrm{I}_{\mathrm{Adj}}}=\mathrm{I}_{\mathrm{Odd}}$ and obtain the statement from the Theorem 5.1.

*Proof.* If $n$ is odd then $e^{1\dots n}$ is in the center of Clifford algebra. So if basis element commutes with some basis element, then it commutes with adjoint basis element. But we know from Theorem 5.1 that basis elements (except $e$ and $e^{1\dots n}$) commutes with $2^{n-1}$ basis elements and anticommutes with $2^{n-1}$ basis elements. So we obtain the statement of theorem for the case of odd $n$.

If $n$ is even then even (not odd) basis element commutes with $e^{1\dots n}$. So if even basis element commutes with some basis element, then it commutes with adjoint basis element. □

Let us represent the commutative property of basis elements in the following tables. At the intersection of two basis elements is a sign "+" if they commute and the sign "–" if they anticommute. For small dimensions we have the following tables:

| $n=1$ | $e$ | $e^1$ |
|---|---|---|
| $e$ | $+$ | $+$ |
| $e^1$ | $+$ | $+$ |

| $n=2$ | $e$ | $e^1$ | $e^2$ | $e^{12}$ |
|---|---|---|---|---|
| $e$ | $+$ | $+$ | $+$ | $+$ |
| $e^1$ | $+$ | $+$ | $-$ | $-$ |
| $e^2$ | $+$ | $-$ | $+$ | $-$ |
| $e^{12}$ | $+$ | $-$ | $-$ | $+$ |

| $n=3$ | $e$ | $e^1$ | $e^2$ | $e^3$ | $e^{12}$ | $e^{13}$ | $e^{23}$ | $e^{123}$ |
|---|---|---|---|---|---|---|---|---|
| $e$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ |
| $e^1$ | $+$ | $+$ | $-$ | $-$ | $-$ | $-$ | $+$ | $+$ |
| $e^2$ | $+$ | $-$ | $+$ | $-$ | $-$ | $+$ | $-$ | $+$ |
| $e^3$ | $+$ | $-$ | $-$ | $+$ | $+$ | $-$ | $-$ | $+$ |
| $e^{12}$ | $+$ | $-$ | $-$ | $+$ | $+$ | $-$ | $-$ | $+$ |
| $e^{13}$ | $+$ | $-$ | $+$ | $-$ | $-$ | $+$ | $-$ | $+$ |
| $e^{23}$ | $+$ | $+$ | $-$ | $-$ | $-$ | $-$ | $+$ | $+$ |
| $e^{123}$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ | $+$ |

Consider the following operator

$$F_{\mathrm{Adj}}(U) = \frac{1}{2^{n-1}} \sum_{A\in \mathrm{I}_{\mathrm{Adj}}} e_A U e^A.$$

**Theorem 5.3.** *Consider an arbitrary Clifford algebra element $U$. Suppose we have a partition $\mathrm{I} = \mathrm{I}_{\mathrm{Adj}} \cup \widetilde{\mathrm{I}_{\mathrm{Adj}}}$. In the case of arbitrary $n$ we have*

$$F_{\mathrm{Adj}}(U) = F(U).$$

*Proof.* If $n$ is odd, then $e^{1\ldots n}$ is in the center of Clifford algebra,

$$\begin{aligned}(e^{a_1\ldots a_m})^{-1} U e^{a_1\ldots a_m} &= e^{1\ldots n}(e^{1\ldots n})^{-1}(e^{a_1\ldots a_m})^{-1} U e^{a_1\ldots a_m}\\ &= (e^{\widetilde{a_1\ldots a_m}})^{-1} U e^{\widetilde{a_1\ldots a_m}},\end{aligned}$$

and

$$e_A U e^A = 2 \sum_{A\in \mathrm{I}_{\mathrm{Adj}}} e_A U e^A. \tag{5.1}$$

If $n$ is even, then $e^{1\ldots n}$ anticommutes with all odd basis elements and commutes with all even basis elements (see Theorem 5.1). So if $U = U_0 + U_1$, $U_0 \in C\ell_{\mathrm{Even}}(p,q)$, $U_1 \in C\ell_{\mathrm{Odd}}(p,q)$, then for $k = 0, 1$ we have

$$\begin{aligned}(e^{a_1\ldots a_m})^{-1} U_k e^{a_1\ldots a_m} &= e^{1\ldots n}(e^{1\ldots n})^{-1}(e^{a_1\ldots a_m})^{-1} U_k e^{a_1\ldots a_m}\\ &= (-1)^{2m+k}(e^{a_1\ldots a_m})^{-1}(e^{1\ldots n})^{-1} U_k e^{1\ldots n} e^{a_1\ldots a_m}\\ &= (-1)^k (e^{\widetilde{a_1\ldots a_m}})^{-1} U_k e^{\widetilde{a_1\ldots a_m}}, \end{aligned} \tag{5.2}$$

and we obtain (5.1) again. □

So we can use operator $F_{\mathrm{Adj}}$ (with $2^{n-1}$ summands) instead of operator $F(U)$ (with $2^n$ summands) in all calculations.

## 6. Conjugate action on Clifford algebras

We denote the corresponding square symmetric matrices of size $2^n$ from the previous section (with elements 1 and $-1$, see tables) by $M_n = ||m_{AB}||$. For arbitrary element of these matrices we have[12] $m_{AB} = e^A e^B (e^A)^{-1}(e^B)^{-1}$, $A, B \in \mathrm{I}$, $e \equiv 1$. We have[13]

$$m_{AB} = m_{BA} = \begin{cases} 1, & \text{if } [e^A, e^B] = 0; \\ -1, & \text{if } \{e^A, e^B\} = 0. \end{cases} \tag{6.1}$$

In the case of odd $n$ we also consider symmetric matrix $L$ of size $2^{n-1}$ $L_n = ||l_{AB}||$, $l_{AB} = m_{AB}$, $A, B \in \mathrm{I}_{\mathrm{First}} = \{A \in \mathrm{I}, \quad |A| \leq \frac{n-1}{2}\}$.

**Theorem 6.1.** *Matrix $M_n$ is invertible in the case of even $n$ and $M_n^{-1} = \frac{1}{2^n} M_n$. Matrix $M_n$ is not invertible in the case of odd $n$.*

*Matrix $L_n$ is invertible in the case of odd $n$ and $L_n^{-1} = \frac{1}{2^{n-1}} L_n$.*

*Proof.* Matrices are symmetric $M_n^T = M_n$, $L_n^T = L_n$ by definition. Let us multiply matrix $M_n$ by itself. For two arbitrary rows we have

$$\begin{aligned}\sum_B m_{AB} m_{BC} &= \sum_B e^A e^B (e^A)^{-1} (e^B)^{-1} e^B e^C (e^B)^{-1} (e^C)^{-1} \\ &= e^A \left( \sum_B e^B (e^A)^{-1} e^C (e^B)^{-1} \right) (e^C)^{-1}.\end{aligned}$$

In the last expression sum is equal to zero if $A \neq C$ and (in the case of odd $n$) $A, C$ are not adjoint multi-indices, because $e_B U e^B$ is projection onto the center of Clifford algebra (see Theorem 3.2). In other cases the last expression equals to $2^n$. In the case of odd $n$ we must use matrix $L_n$ because we do not have adjoint multi-indices in this matrix. $\square$

Let us consider the following operators (for different $A \in \mathrm{I}$)

$$F_{e^A} : C\ell(p,q) \to C\ell(p,q), \quad U \to (e^A)^{-1} U e^A, \quad U \in C\ell(p,q).$$

Note that $F_{e^A}(U) = (e^A)^{-1} U e^A$ is a *conjugation* of Clifford algebra element $U \in C\ell(p,q)$ by element $e^A$ of Salingaros' vee group.

**Theorem 6.2.** *For operator $F_{e^A}(U) = (e^A)^{-1} U e^A$ we have*

$$F_{e^A}(U) = \sum_B m_{AB} \pi_{e^B}(U), \tag{6.2}$$

*where $\pi_{e^B}$ is a projection*[14] *onto subspace spanned over element $e^B$. We have $F_{e^A}(F_{e^A}(U)) = U$.*

*Proof.* The statement follows from the definition of matrix $M_n = ||m_{AB}||$ and definition of conjugation. $\square$

---

[12] Note that $e^A e^B (e^A)^{-1} (e^B)^{-1}$ is the group commutator of $e^A$ and $e^B$ in Salingaros' vee group.

[13] We can say that commutator subgroup of Salingaros' vee group is $\{1, -1\}$.

[14] We do not use notation $\pi_B$ instead of $\pi_{e^B}$ because it will conflict with notation $\pi_k$ for projection onto subspace $C\ell_k(p,q)$.

Fixed multi-index $A$ divides the set $\mathrm{I}$ into 2 sets $\mathrm{I} = \mathrm{I}_{[A]} \cup \mathrm{I}_{\{A\}}$, where $e^B$, $B \in \mathrm{I}_{[A]}$ commute with $e^A$, and $e^B$, $B \in \mathrm{I}_{\{A\}}$ anticommute with $e^A$. Denote the corresponding subspaces of Clifford algebra by $\mathcal{C}\ell_{[A]}(p,q)$ and $\mathcal{C}\ell_{\{A\}}(p,q)$ and corresponding projection operators by $\pi_{[A]}$ and $\pi_{\{A\}}$. We have $\mathcal{C}\ell(p,q) = \mathcal{C}\ell_{[A]}(p,q) \oplus \mathcal{C}\ell_{\{A\}}(p,q)$ and

$$F_{e^A}(U) = (e^A)^{-1} U e^A = \pi_{[A]}(U) - \pi_{\{A\}}(U), \quad \forall A.$$

**Theorem 6.3.** *For arbitrary Clifford algebra element $U$ we have*

$$\pi_{[A]}(U) = \frac{1}{2}(U + (e^A)^{-1} U e^A), \quad \pi_{\{A\}}(U) = \frac{1}{2}(U - (e^A)^{-1} U e^A).$$

*Proof.* Using

$$(e^A)^{-1} U e^A = \pi_{[A]}(U) - \pi_{\{A\}}(U), \quad U = \pi_{[A]}(U) + \pi_{\{A\}}(U)$$

we obtain the statement of theorem. □

For empty multi-index $A = -$ we have $m_{-,B} = 1$ for all $B$, $\mathrm{I} = \mathrm{I}_{[A]}$, $\mathrm{I}_{\{A\}} = \emptyset$. For multi-index $A = 1 \ldots n$ we have $m_{1\ldots n,B} = 1$ for all $B$ in the case of odd $n$ and

$$m_{1\ldots n,B} = \begin{cases} 1, & \text{if } B \text{ is even;} \\ -1, & \text{if } B \text{ is odd,} \end{cases} \tag{6.3}$$

and $e_{1\ldots n} U e^{1\ldots n} = \pi_{\mathrm{Even}}(U) - \pi_{\mathrm{Odd}}(U)$ in the case of even $n$, where $\pi_{\mathrm{Even}}$ and $\pi_{\mathrm{Odd}}$ are projection operations onto the even and odd subspaces of Clifford algebra. In other cases (when $A$ is not empty and in not $1\ldots n$) we have $2^{n-1}$ elements in each of the sets $\mathrm{I}_{[A]}$, $\mathrm{I}_{\{A\}}$ (see Theorem 5.1) i.e. we have $\dim \mathcal{C}\ell_{[A]}(p,q) = \dim \mathcal{C}\ell_{\{A\}}(p,q) = 2^{n-1}$ in these cases.

In particular case we obtain the following identities (for $A = 1\ldots n$): in the case of even $n$ we have

$$\pi_{\mathrm{Even}}(U) = \frac{1}{2}(U + e_{1\ldots n} U e^{1\ldots n}), \quad \pi_{\mathrm{Odd}}(U) = \frac{1}{2}(U - e_{1\ldots n} U e^{1\ldots n}).$$

We have the following theorem.

**Theorem 6.4.** *Let an element $X \in \mathcal{C}\ell(p,q)$ satisfy the following equation with some given element*[15] $Q^A \in \mathcal{C}\ell(p,q)$

$$e^A X + \epsilon X e^A = Q^A, \quad \epsilon \in \mathbb{R}^\times. \tag{6.4}$$

*If $\epsilon \neq \pm 1$, then we have a unique solution*

$$X = \sum_B \frac{1}{1 + \epsilon m_{AB}} \pi_{e^B}((e^A)^{-1} Q^A).$$

*If $\epsilon = -1$ (commutator case), then:*

- *if $\pi_{[A]}((e^A)^{-1} Q^A) \neq 0$ (i.e. $\{e^A, Q^A\} \neq 0$), then there is no solution;*

[15] Note that $A$ in $Q^A$ is a label (multi-index) that serves to pair off an arbitrary Clifford element $Q^A$ with basis element $e^A$.

- *if* $\pi_{[A]}((e^A)^{-1}Q^A) = 0$ (*i.e.* $\{e^A, Q^A\} = 0$), *then the solution is*
$$\frac{1}{2}\pi_{\{A\}}((e^A)^{-1}Q^A) + \pi_{[A]}(U),$$
*where $U$ is an arbitrary Clifford algebra element.*

*If* $\epsilon = 1$ (*anticommutator case*), *then:*

- *if* $\pi_{\{A\}}((e^A)^{-1}Q^A) \neq 0$ (*i.e.* $[e^A, Q^A] \neq 0$), *then there is no solution;*
- *if* $\pi_{\{A\}}((e^A)^{-1}Q^A) = 0$ (*i.e.* $[e^A, Q^A] = 0$), *then the solution is*
$$\frac{1}{2}\pi_{[A]}((e^A)^{-1}Q^A) + \pi_{\{A\}}(U),$$
*where $U$ is an arbitrary Clifford algebra element.*

*Proof.* Multiply equation on the left by $(e^A)^{-1} = e_A$ and use Theorem 6.2:
$$X + \epsilon(e^A)^{-1}Xe^A = (e^A)^{-1}Q^A \quad \Rightarrow \quad X + \epsilon\sum_B m_{AB}\pi_{e^B}(X) = (e^A)^{-1}Q^A.$$

Using $X = \sum_B \pi_{e^B}(X)$ we obtain
$$\sum_B (1 + \epsilon m_{AB})\pi_{e^B}(X) = \sum_B \pi_{e^B}((e^A)^{-1}Q^A).$$
In the case $\epsilon \neq \pm 1$ we obtain the statement of the theorem.

Let $\epsilon = -1$. Then
$$\begin{aligned} &X - (e^A)^{-1}Xe^A = (e^A)^{-1}Q^A \\ &\quad \Rightarrow 2\pi_{\{A\}}(X) = \pi_{\{A\}}((e^A)^{-1}Q^A) + \pi_{[A]}((e^A)^{-1}Q^A) \end{aligned}$$
and we obtain the statement of the theorem for this case. Note that
$$\begin{aligned} \pi_{[A]}((e^A)^{-1}Q^A) &= \frac{1}{2}((e^A)^{-1}Q^A + (e^A)^{-1}(e^A)^{-1}Q^Ae^A) \\ &= \frac{1}{2}((e^A)^{-1}Q^A + e^A(e^A)^{-1}Q^A(e^A)^{-1}) \\ &= \frac{1}{2}\{(e^A)^{-1}, Q^A\} = \pm\frac{1}{2}\{e^A, Q^A\}, \end{aligned}$$
because $(e^A)^{-1} = \pm e^A$.

In the case $\epsilon = 1$ proof is similar. Analogously we have
$$\pi_{\{A\}}((e^A)^{-1}Q^A) = \pm\frac{1}{2}[e^A, Q^A].$$
□

**Theorem 6.5.** *In the case of even $n$ we have*
$$\pi_{e^A}(U) = \frac{1}{2^n}\sum_B m_{AB}(e^B)^{-1}Ue^B. \tag{6.5}$$
*In the case of odd $n$ we have*
$$\pi_{e^A,\widetilde{e^A}}(U) = \pi_{e^A}(U) + \pi_{\widetilde{e^A}}(U) = \frac{1}{2^{n-1}}\sum_{B\in \mathrm{I}_{\mathrm{First}}} l_{AB}(e^B)^{-1}Ue^B. \tag{6.6}$$
*Note that we can use instead of* $\mathrm{I}_{\mathrm{First}}$ *any adjoint set* $\mathrm{I}_{\mathrm{Adj}}$.

*Proof.* From (6.2) we obtain

$$\begin{pmatrix} F_e(U) \\ F_{e^a}(U) \\ \dots \\ F_{e^{1\dots n}}(U) \end{pmatrix} = M_n \begin{pmatrix} \pi_e(U) \\ \pi_{e^1}(U) \\ \dots \\ \pi_{e^{1\dots n}}(U) \end{pmatrix}.$$

Using Theorem 6.1 in the case of even $n$ we obtain

$$\begin{pmatrix} \pi_e(U) \\ \pi_{e^1}(U) \\ \dots \\ \pi_{e^{1\dots n}}(U) \end{pmatrix} = \frac{1}{2^n} M_n \begin{pmatrix} F_e(U) \\ F_{e^1}(U) \\ \dots \\ F_{e^{1\dots n}}(U) \end{pmatrix}.$$

In the case of odd $n$ we have

$$F_{e^A}(U) = (e^A)^{-1} U e^A = \sum_B m_{AB} \pi_{e^B}(U) = \sum_{B \in \mathrm{I}_{\mathrm{First}}} l_{AB} \pi_{e^B, \widetilde{e^B}}(U). \quad (6.7)$$

and use Theorem 6.1. $\square$

Let us give some examples.

In the case $n = 1$ we have

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 1 \end{pmatrix}, \quad F_e(U) = F_{e^1}(U) = U, \quad \pi_{e,e^1}(U) = U.$$

In the case $n = 2$ we have $M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$,

$$\begin{aligned} F_e(U) &= U, \\ F_{e^1}(U) &= \pi_e(U) + \pi_{e^1}(U) - \pi_{e^2}(U) - \pi_{e^{12}}(U), \\ F_{e^2}(U) &= \pi_e(U) - \pi_{e^1}(U) + \pi_{e^2}(U) - \pi_{e^{12}}(U), \\ F_{e^{12}}(U) &= \pi_e(U) - \pi_{e^1}(U) - \pi_{e^2}(U) + \pi_{e^{12}}(U). \\ \pi_e(U) &= \frac{1}{4}(e_A U e^A), \\ \pi_{e^1}(U) &= \frac{1}{4}(eUe + (e^1)^{-1}Ue^1 - (e^2)^{-1}Ue^2 - (e^{12})^{-1}Ue^{12}), \\ \pi_{e^2}(U) &= \frac{1}{4}(eUe - (e^1)^{-1}Ue^1 + (e^2)^{-1}Ue^2 - (e^{12})^{-1}Ue^{12}), \\ \pi_{e^{12}}(U) &= \frac{1}{4}(eUe - (e^1)^{-1}Ue^1 - (e^2)^{-1}Ue^2 + (e^{12})^{-1}Ue^{12}). \end{aligned}$$

In the case $n=3$ we have $L_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$,

$$\pi_{e,e^{123}}(U) = \frac{1}{4}(eUe + (e^1)^{-1}Ue^1 + (e^2)^{-1}Ue^2 + (e^3)^{-1}Ue^{12}),$$
$$\pi_{e^1,e^{23}}(U) = \frac{1}{4}(eUe + (e^1)^{-1}Ue^1 - (e^2)^{-1}Ue^2 - (e^3)^{-1}Ue^3),$$
$$\pi_{e^2,e^{13}}(U) = \frac{1}{4}(eUe - (e^1)^{-1}Ue^1 + (e^2)^{-1}Ue^2 - (e^3)^{-1}Ue^3),$$
$$\pi_{e^3,e^{12}}(U) = \frac{1}{4}(eUe - (e^1)^{-1}Ue^1 - (e^2)^{-1}Ue^2 + (e^3)^{-1}Ue^3).$$

## 7. Conclusion

In the present paper we consider different operators

$$F_S(U) = \frac{1}{|S|}\sum_{A\in S}(e^A)^{-1}Ue^A,$$

acting on real Clifford algebra $C\ell(p,q)$. Note that all theorems of this paper can be reformulated without changes for complex Clifford algebra.

We can also consider operators with other subsets $S \subseteq \mathrm{I}$. Note that not for every subset $S \subseteq \mathrm{I}$, the set $\{e^A | A \in S\}$ is a group.

In [12] we consider the following operator in Clifford algebra $C\ell(p,q)$

$$F_1(U) = e_a U e^a$$

and prove that $e_a U e^a = \sum_{k=0}^{n}(-1)^k(n-2k)\pi_k(U)$.

In [14] we present the relation between this operator and projections onto subspaces of fixed ranks. We use this relation to present new class of gauge invariant solutions of Yang–Mills equations.

We can consider operators (1.1) with the following subsets $S \subseteq \mathrm{I}$:

$$\mathrm{I}_{\mathrm{Even}} = \{A \in \mathrm{I},\, |A| - \text{even}\}, \quad \mathrm{I}_{\mathrm{Odd}} = \{A \in \mathrm{I},\, |A| - \text{odd}\},$$
$$\mathrm{I}_k = \{A \in \mathrm{I}, \quad |A| = k\}, \quad k = 0,1,\ldots,n,$$
$$\mathrm{I}_{\overline{k}} = \{A \in \mathrm{I}, \quad |A| = m \mod 4\}, \quad m = 0,1,2,3.$$

In the last case we use the concept of so-called *quaternion type m* [18] of Clifford algebra element, $m = 0,1,2,3$. There is a relation between these operators and projective operators onto fixed subspaces of Clifford algebras. This is a subject for further research.

In [21] we consider operators $\sum_A \gamma^A U \beta_A$ with 2 different sets $\gamma^a$, $\beta^a$, $a = 1,\ldots,n$ of Clifford algebra elements that satisfy

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}e, \quad \beta^a\beta^b + \beta^b\beta^a = 2\eta^{ab}e.$$

We use these operators to prove generalized Pauli's theorem and some other problems about spin groups (see [15,17,19,20,22]).

The results of this article (especially about the relation between projection operators and averaging operators; solving commutator equations) may be used in computer calculations.

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