ISSN 1063-7796, Physics of Particles and Nuclei, 2020, Vol. 51, No. 4, pp. 589–594. © Pleiades Publishing, Ltd., 2020.
Russian Text © The Author(s), 2020, published in Fizika Elementarnykh Chastits i Atomnogo Yadra, 2020, Vol. 51, No. 4.

On Some Equations Modeling the Yang–Mills Equations

N. G. Marchuk[a], * and D. S. Shirokov[b, c], **

[a]Steklov Mathematical Institute, Russian Academy of Sciences, Moscow, 119991 Russia
[b]National Research University Higher School of Economics, Moscow, 101000 Russia
[c]Institute for Information Transmission Problems, Russian Academy of Sciences, Moscow, 1127051 Russia
*e-mail: nmarchuk@mi-ras.ru
**e-mail: dm.shirokov@gmail.com

Received December 20, 2019; revised January 16, 2020; accepted January 29, 2020

Abstract—The paper considers plane-wave solutions of the Yang–Mills equations, which allow one to write out three systems of equations modeling the Yang–Mills system. An explicit form of all plane-wave solutions of the Yang–Mills equations with the SU(2) gauge symmetry and zero current in a (pseudo)Euclidean space of arbitrary finite dimension is presented.

DOI: 10.1134/S1063779620040498

INTRODUCTION

The Yang–Mills equations were introduced in 1954 and have long been considered fundamental equations of quantum physics. The Yang–Mills equations comprise a class of equations depending on a gauge Lie group and its real Lie algebra. In physics, as a rule, unitary gauge groups, or (more general) semisimple Lie groups, are used. In the Standard Model, the Yang–Mills equations are used to describe electroweak and strong interactions of elementary particles [1]. As noted by many authors, the study of the Yang–Mills equations and their solution involves difficulties arising from the nonlinearity of the equations. To simplify the situation, mathematicians introduce certain additional conditions restricting the class of solutions of the Yang–Mills equations. In particular, they consider self-dual (instanton) solutions, as well as solutions depending on a smaller number of independent variables. In this article, we present some plane wave solutions of the Yang–Mills equations, which enable one to write out three systems of equations modeling the Yang–Mills equations.

Particular classes of solutions of the Yang–Mills equations are presented in classical works [2–7], review [8], and other works. Particular classes of plane-wave solutions of the Yang–Mills equations were considered in [9–16]. In this paper, we present all plane-wave solutions of the Yang–Mills equations with the SU(2) gauge symmetry with zero current in a Euclidean or pseudo-Euclidean space of arbitrary finite dimension. These results generalize the results of one of the authors about all constant solutions in Euclidean and pseudo-Euclidean spaces of arbitrary finite dimension [17, 18]. Issues concerning constant and covariantly constant solutions of the Yang–Mills equations were also considered in [19–21].

1. THE YANG–MILLS EQUATIONS

Let p, and q be nonnegative integers and $n = p + q$ be a positive integer. We consider equations in a (pseudo)Euclidean space with Cartesian coordinates x^μ, $\mu = 1,\dots,n$. Partial derivatives are denoted by $\partial_\mu = \partial/\partial x^\mu$. We assume that all functions of $x \in \mathbb{R}^{p,q}$ considered below are sufficiently smooth (their smoothness is sufficient for the validity of the reasoning).

Let K be a semisimple Lie group (in particular, K may be a unitary Lie group) and L be a real Lie algebra of the Lie group K. We assume that the Lie group K and the Lie algebra L are represented by square matrices of some dimension N. In this case, the Lie bracket $[A, B]$ defining the multiplication in the Lie algebra L is implemented in the form of a matrix commutator $[A, B] = AB - BA$.

We denote by $L\mathrm{T}^r_s$ the set of tensor fields of the type (r, s) of the (pseudo)Euclidean space $\mathbb{R}^{p,q}$ with the values in L.

Let $x \in \mathbb{R}^{p,q}$, $A_\mu \in L\mathrm{T}_1$, $J^\nu \in L\mathrm{T}^1$, $F_{\mu\nu} \in L\mathrm{T}_2$, and $F_{\mu\nu} = -F_{\nu\mu}$. The equations

$$\begin{aligned}\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] &= F_{\mu\nu},\\ \partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] &= J^\nu\end{aligned} \tag{1}$$

are called the Yang–Mills equations (the Yang–Mills system).

It is usually assumed that A_m, and $F_{\mu\nu}$ are unknown and J^ν is a known vector with values in the Lie algebra L.

It is said that Eqs. (1) define a Yang–Mills field $(A_\mu, F_{\mu\nu})$, where A_μ is the potential and $F_{\mu\nu}$ is the Yang–Mills field strength. The vector J^ν is called a non-Abelian current (in the case of an Abelian group K, the vector J^ν is called a current).

We can verify that the system of equations (1) implies the relationship

$$\partial_\mu J^\mu - [A_\mu, J^\mu] = 0,$$

which is called the non-Abelian conservation law (in the case of an Abelian group K, we have $\partial_\nu J^\nu = 0$; i.e., the divergence of the vector J^ν is zero).

Let the tensor fields J^ν satisfy Yang–Mills equations (1). Take some element $U = U(x) \in K$ (a function $U: \mathbb{R}^{p,q} \to K$) and consider the transformed tensor fields

$$\acute{A}_\mu = U^{-1}A_\mu U - U^{-1}\partial_\mu U,$$
$$\acute{F}_{\nu\mu} = U^{-1}F_{\mu\nu}U, \tag{2}$$
$$\acute{J}^\nu = U^{-1}J^\nu U.$$

Then, these quantities satisfy the same Yang–Mills equations

$$\partial_\mu \acute{A}_\nu - \partial_\nu \acute{A}_\mu - [\acute{A}_\mu, \acute{A}_\nu] = \acute{F}_{\nu\mu},$$
$$\partial_\mu \acute{F}^{\nu\mu} - [\acute{A}_\mu, \acute{F}^{\nu\mu}]\acute{J}^\nu,$$

i.e., Eqs (1) are invariant with respect to transformations (2). Transformation (2) is called the gauge transformation (or gauge symmetry), and the group K is called the gauge group of Yang–Mills equations (1).

The components of the skew-symmetric tensor field $F_{\mu\nu}$ defined by the first equation (1) can be substituted into the second equation to obtain a second-order equation for the covector potential of the Yang–Mills field:

$$\begin{aligned}\partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu - [\partial_\mu A^\mu, A^\nu] - 2[A^\mu, \partial_\mu A^\nu]\\ + [A_\mu, \partial^\nu A^\mu] + [A_\mu, [A^\mu, A^\nu]] = J^\nu.\end{aligned} \tag{3}$$

1.1. Particular Solutions of the Yang–Mills Equations

The system of Yang–Mills equations (1) (or (3)) is considered as a system of nonlinear partial differential equations for unknown tensor fields $A_\mu(x), F_{\mu\nu}(x)$ with a known right-hand side $J^\nu(x)$. The knowledge of the theory of partial differential equations suggests that, for an in-depth analysis of the solutions of the Yang–Mills equations, it is necessary to consider boundary value problems in a certain region of space $R^{p,q}$. The formulation of boundary value problems (that is, which boundary conditions should be imposed, and where they should be imposed) depends on the signature (p, q) of the pseudo-Euclidean space. A correct formulation of boundary value problems for the Yang–Mills equations, first of all, can be obtained in the case of signatures $(1, n-1)$ and $(n-1, 1)$ hyperbolic cases, and in the case of signatures $(0, n)$ and $(n, 0)$: elliptic cases.

It is also of interest to search for particular solutions of the Yang–Mills equations in which the right-hand side $J^\nu(x)$ has one or another special form (e.g., dictated by the physical formulation of the problem).

If $J^\nu = 0$, we have a zero solution of the Yang–Mills equations: $A_\mu = 0$, $F_{\mu\nu} = 0$. If $U = U(x) \in K$, then, using gauge transformation (2), from the zero solution, we obtain another trivial (gauge equivalent to zero) solution of the Yang–Mills equations: $A_\mu = -U^{-1}\partial_\mu U$, $F_{\mu\nu} = 0$, $J^\nu = 0$.

2. PLANE-WAVE SOLUTION OF THE YANG–MILLS EQUATIONS

In the theory of linear partial differential equations in the (pseudo)Euclidean space $\mathbb{R}^{p,q}$, an important role is played by plane-wave solutions. In such solutions, the dependence on points x of the (pseudo)Euclidean space reduces to the dependence on a scalar (invariant) $\rho := \xi_\mu x^\mu$, which is defined by a given constant (independent of points x) covector field ξ_μ (real or complex).

We will seek solutions of the Yang–Mills equations (3) in the following form

$$A_\mu = a_\mu e^\rho, \text{ where } \rho = \xi_\mu x^\mu \tag{4}$$

and a_μ are the components of a constant covector field with values in the Lie algebra L. Partial derivatives act on the exponential e^ρ according to the standard rule

$$\partial_\nu e^\rho = (\partial_\nu \rho)e^\rho = \xi_\nu e^\rho.$$

Substituting A_μ from (4) into the left-hand side of (3), we have

$$\begin{aligned}\partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu - [\partial_\mu A^\mu, A^\nu] - 2[A^\mu, \partial_\mu A^\nu]\\ + [A_\mu, \partial^\nu A^\mu] + [A_\mu, [A^\mu, A^\nu]]\\ = (\xi_\mu\xi^\mu a^\nu - \xi^\nu\xi_\mu a^\mu)e^\rho\\ - 3\xi_\mu[a^\mu, a^\nu]e^{2\rho} + [a_\mu, [a^\mu, a^\nu]]e^{3\rho}.\end{aligned} \tag{5}$$

Now assume that the right-hand side of Eq. (3) has the form

$$J^{\nu} = j^{\nu}_{(1)}e^{\rho} + j^{\nu}_{(2)}e^{2\rho} + j^{\nu}_{(3)}e^{3\rho}, \tag{6}$$

where $j^{\nu}_{(k)}$, $k = 1,2,3$, independent of the x components of three vectors (vector fields) with values in the Lie algebra L. Equating the right-hand sides of equalities (5) and (6), we obtain the system of equations

$$\xi_{\mu}\xi^{\mu}a^{\nu} - \xi^{\nu}\xi_{\mu}a^{\mu} = j^{\nu}_{(1)}, \tag{7}$$

$$-3\xi_{\mu}[a^{\mu}, a^{\nu}] = j^{\nu}_{(2)}, \tag{8}$$

$$[a_{\mu},[a^{\mu}, a^{\nu}]] = j^{\nu}_{(3)}, \tag{9}$$

which can be considered a system of algebraic equations for finding a complex vector ξ^{μ} and a vector a_{μ} with values in the Lie algebra L from the known right-hand side $j^{\nu}_{(k)}$, $k = 1,2,3$.

3. ANALYSIS OF SYSTEM (7)–(9) AND THE SOLUTION IN THE FORM OF A SUM OF WAVES

Let us discuss in more detail the systems of algebraic equations (7)–(9), proposed in the previous section. System (9) is a system of equations for finding constant solutions of the Yang–Mills equations (a similar system can be obtained from (3) if we assume that the solutions are independent of the point x of the (pseudo)Euclidean space under consideration). The solution of such a system of equations for an arbitrary current in the case of the Lie group SU(2) is discussed in [17, 18]. Under certain conditions on the current, solutions may not exist. For example, in the case of the Lie group SU(2) in the Euclidean space $\mathbb{R}^n$, the condition for the existence of solutions is the following restriction rank $(j^{\nu}_{(3)k}) \neq 1$ on the rank of the matrix of coefficients of the expansion of the current $j^{\nu}_{(3)} = j^{\nu}_{(3)k}\tau^k$ via the basis τ^k, $k = 1,2,3$, of the Lie algebra $su(2)$ (see [17]). In the cases in which constant solutions a^{μ} exist, we can find them from system (9) and substitute them into (7) and (8). Note that system (8) is a system of linear equations for unknowns ξ_{μ} and the matrix of this system is the skew-symmetric matrix of the corresponding constant Yang–Mills field strength:

$$\xi_{\mu}f^{\mu\nu} = \frac{1}{3}j^{\nu}_{(2)}, \quad f^{\mu\nu} := -[a^{\mu}, a^{\nu}].$$

As is known, the determinant of a skew-symmetric matrix of an odd order $n = 2k+1$ is equal to zero and the determinant of a skew-symmetric matrix of an even order $n = 2k$ is the square of a homogeneous polynomial of degree k, which is called the Pfaffian. System (7)–(9) under consideration has solutions under certain constraints on the currents $j^{\nu}_{(1)}$, $j^{\nu}_{(2)}$, and $j^{\nu}_{(3)}$. The existence of the solution and its explicit form depend on the signature and dimension of the space, as well as on the Lie group under consideration.

In the case of zero current $j^{\nu}_{(1)} = j^{\nu}_{(2)} = j^{\nu}_{(3)} = 0$, we obtain the system

$$\xi_{\mu}\xi^{\mu}a^{\nu} - \xi^{\nu}\xi_{\mu}a^{\mu} = 0, \tag{10}$$

$$-3\xi_{\mu}[a^{\mu}, a^{\nu}] = 0, \tag{11}$$

$$[a_{\mu},[a^{\mu}, a^{\nu}]] = 0. \tag{12}$$

If the Lie algebra L is compact, then, in the case of a space $R^{p,q}$ of the Lorentz signature, $(p,q) = (1, n-1)$ or $(n-1, 1)$, or Euclidean signature $(n,0)$ or $(0,n)$, for all solutions a_{μ} of system (12), we have $f^{\mu\nu} = -[a^{\mu}, a^{\nu}] = 0$ (this fact was proved in [22, 23]); i.e., system (11) is satisfied automatically for any ξ_{μ} and expressions ξ_{μ} should be sought from system (10). This is not true in the case of other signatures; a counterexample for the case of $\mathbb{R}^{p,q}$ with $p \geq 2$ and $q \geq 2$ is given below.

We restrict ourselves to the case of the Lie group SU(2) and a (pseudo)Euclidean space $\mathbb{R}^{p,q}$, $p+q = n$, of arbitrary finite dimension n. To solve system (12), we use the previously obtained results [17, 18] on all constant solutions SU(2) of the Yang–Mills equations. Denote by $\tau^k = \dfrac{\sigma^k}{2i}$, $k = 1,2,3$, the standard basis of the Lie algebra $su(2)$, constructed using the Pauli matrices σ^k, $k = 1,2,3$. Let us list the explicit form of all solutions $\{a^{\mu}, \xi_{\nu}\}$ of system (10)–(12). The solutions are written out with an appropriate choice[1] of the coordinate system and gauge:

[1] In other words, any solution of the system under study can be reduced using an orthogonal change of coordinates and fixing the gauge to the solutions listed below (for more details, see [17, 18]). When listing the solutions, we write out only nonzero components a^{μ}, ξ_{ν}. The zero solution $a^1 = \ldots = a^n = 0$ with arbitrary ξ_{ν} is not considered.

$$p \geq 1: \quad a^1 = a\tau^1, \quad a \in \mathbb{R}\setminus\{0\}, \; \xi_1 \in \mathbb{C};$$

$$p \geq 1: \quad a^1 = a\tau^1, \quad a \in \mathbb{R}\setminus\{0\}, \quad \xi_\mu\xi^\mu - \xi_1\xi^1 = 0;$$

$$q \geq 1: \quad a^{p+1} = a\tau^1, \quad a \in \mathbb{R}\setminus\{0\}, \quad \xi_{p+1} \in \mathbb{C};$$

$$q \geq 1: \quad a^{p+1} = a\tau^1, \quad a \in \mathbb{R}\setminus\{0\}, \quad \xi_\mu\xi^\mu - \xi_{p+1}\xi^{p+1} = 0;$$

$$p \geq 1, q \geq 1: \quad a^1 = a^{p+1} = \tau^1, \quad \forall\xi_1 \in \mathbb{C};$$

$$p \geq 1, q \geq 1: \quad a^1 = a^{p+1} = \tau^1, \quad \forall\xi_{p+1} \in \mathbb{C};$$

$$p \geq 1, q \geq 1: \quad a^1 = a^{p+1} = \tau^1, \quad \forall\xi_{p+1} = -\xi_1 \in \mathbb{C}, \quad \xi_\mu\xi^\mu = 0;$$

$$p \geq 2, q \geq 2: \quad a^1 = a^{p+1} = \tau^1, \quad a^2 = a^{p+2} = \tau^2, \quad \forall\xi_{p+1} = -\xi_1 \in \mathbb{C},$$
$$\forall\xi_{p+2} = -\xi_2 \in \mathbb{C}, \quad \xi_\mu\xi^\mu = 0;$$

$$p \geq 3, q \geq 3: \quad a^1 = a^{p+1} = \tau^1, \quad a^2 = a^{p+2} = \tau^2, \quad a^3 = a^{p+3} = \tau^3,$$
$$\forall\xi_{p+1} = -\xi_1 \in \mathbb{C}, \quad \forall\xi_{p+2} = -\xi_2 \in \mathbb{C}, \quad \forall\xi_{p+3} = -\xi_3 \in \mathbb{C}, \quad \xi_\mu\xi^\mu = 0.$$

Note that, in the last two cases, the expression for the Yang–Mills field strength is nonzero, $f^{\mu\nu} \neq 0$, and the calculations included Eqs. (11).

We also consider a more general formulation of the problem than that proposed in (4) and (6). Namely, we consider the solutions of Eqs. (3) in the form of the sum of waves (Fourier series[2]). Let the current have the form

$$J^\nu = \sum_{k=-\infty}^{\infty} j^\nu_{(k)} e^{k\rho}, \quad \rho = \xi_\mu x^\mu. \tag{13}$$

We will seek solutions of system (3) in the form

$$A^\nu = \sum_{k=-\infty}^{\infty} a^\nu_{(k)} e^{k\rho}. \tag{14}$$

We obtain the system

$$\begin{aligned}\sum_k (a^\nu_{(k)}\xi_\mu\xi^\mu - a^\mu_{(k)}\xi_\mu\xi^\nu)k^2 e^{k\rho} - \sum_k\sum_l \xi_\mu[a^\mu_{(k)}, a^\nu_{(l)}]k e^{(k+l)\rho} - 2\sum_k\sum_l \xi_\mu[a^\mu_{(k)}, a^\nu_{(l)}]l e^{(k+l)\rho} \\ + \sum_k\sum_l \xi^\nu[a_{\mu(k)}, a^\mu_{(l)}]l e^{(k+l)\rho} + \sum_k\sum_l\sum_m [a_{\mu(k)}, [a^\mu_{(l)}, a^\nu_{(m)}]] e^{(k+l+m)\rho} = \sum_k j^\nu_{(k)} e^{k\rho}.\end{aligned} \tag{15}$$

This system is split into a system of equations of the form

$$F_k(a^\mu_{(l)}, \xi^\nu) = j^\nu_{(k)}, \quad k = -\infty, \dots, -1, 0, 1, \dots, \infty,$$

where F_k are given expressions that are polynomials of degree not higher than 2 in the unknowns ξ^ν and polynomials of degree not higher than 3 in the unknowns $a^\mu_{(l)}$. We can also consider a more particular formulation of the problem, when the summation in (13) and (14) is carried out from 0 to ∞ (i.e., $j^\nu_{(k)} = a^\nu_{(k)} = 0$ for all $k = -1, -2, \dots$). The system obtained in this case is split into a system of equations with respect to the unknowns $a^\nu_{(k)}$, $k = 0, 1, \dots, \infty$, and ξ^μ. The first three equations of this system have the form:

$$j^\nu_{(0)} = [a_{\mu(0)}, [a^\mu_{(0)}, a^\nu_{(0)}]], \tag{16}$$

$$\begin{aligned} j^\nu_{(1)} &= \xi_\mu\xi^\mu a^\nu_{(1)} - \xi_\mu\xi^\nu a^\mu_{(1)} - \xi_\mu[a^\mu_{(1)}, a^\nu_{(0)}] \\ &- 2\xi_\mu[a^\mu_{(0)}, a^\nu_{(1)}] + \xi^\nu[a_{\mu(0)}, a^\mu_{(1)}] + [a_{\mu(1)}, [a^\mu_{(0)}, a^\nu_{(0)}]] \\ &+ [a_{\mu(0)}, [a^\mu_{(1)}, a^\nu_{(0)}]] + [a_{\mu(0)}, [a^\mu_{(0)}, a^\nu_{(1)}]], \end{aligned} \tag{17}$$

$$\begin{aligned} j^\nu_{(2)} &= 4(\xi_\mu\xi^\mu a^\nu_{(2)} - \xi_\mu\xi^\nu a^\mu_{(2)}) - \xi_\mu[a^\mu_{(1)}, a^\nu_{(1)}] \\ &- 2\xi_\mu[a^\mu_{(2)}, a^\nu_{(0)}] - 2\xi_\mu[a^\mu_{(1)}, a^\nu_{(1)}] - 4\xi_\mu[a^\mu_{(0)}, a^\nu_{(2)}] \\ &+ \xi^\nu[a_{\mu(1)}, a^\mu_{(1)}] + 2\xi^\nu[a_{\mu(0)}, a^\mu_{(2)}] \\ &+ [a_{\mu(2)}, [a^\mu_{(0)}, a^\nu_{(0)}]] + [a_{\mu(0)}, [a^\mu_{(2)}, a^\nu_{(0)}]] \\ &+ [a_{\mu(0)}, [a^\mu_{(0)}, a^\nu_{(2)}]] + [a_{\mu(1)}, [a^\mu_{(1)}, a^\nu_{(0)}]] \\ &+ [a_{\mu(1)}, [a^\mu_{(0)}, a^\nu_{(1)}]] + [a_{\mu(0)}, [a^\mu_{(1)}, a^\nu_{(1)}]]. \end{aligned} \tag{18}$$

[2] In this paper, we do not consider the convergence of series. All series are considered only if they converge.

The first system (16) is a system for finding all constant solutions $a^{\mu}_{(0)}$ of the Yang–Mills equations with a constant current $j^{\nu}_{(0)}$. Finding all the constant solutions $a^{\mu}_{(0)}$ (in the case of the Lie group SU(2), such a problem was solved in [17, 18]), we can substitute them into the second system (17), which is linear in the unknowns $a^{\mu}_{(1)}$ and quadratic in the unknowns ξ^{μ}. System (18), following it, is linear in $a^{\mu}_{(2)}$ and quadratic in ξ^{μ}, etc. The existence and the explicit form of the solutions depend on the considered Lie group ξ^{μ}, the dimension and signature of the space $\mathbb{R}^{p,q}$, and the currents $j^{\nu}_{(k)}$, $k = 0,1,\ldots,\infty$.

4. ON SYSTEMS OF EQUATIONS MODELING THE YANG–MILLS EQUATIONS

The reasoning in the previous section, which led us to algebraic system of equations (7)–(9), leads us to the idea that we can postulate three systems of equations, each of which models certain aspects of the Yang–Mills equations. Namely, it is proposed to consider the following systems of equations:

$$\partial_{\mu}\partial^{\mu}A^{\nu} - \partial^{\nu}\partial_{\mu}A^{\mu} = J^{\nu}_{(1)}, \tag{19}$$

$$-[\partial_{\mu}A^{\mu}, A^{\nu}] - 2[A^{\mu}, \partial_{\mu}A^{\nu}] + [A_{\mu}, \partial^{\nu}A^{\mu}] = J^{\nu}_{(2)}, \tag{20}$$

$$[A_{\mu}, [A^{\mu}, A^{\nu}]] = J^{\nu}_{(3)}, \tag{21}$$

where $J^{\nu}_{(k)}$, $k = 1,2,3$, depending on the x components of three vector fields with values in the Lie algebra L.

The system of equations (19) is just several instances of Maxwell's equations, and this number of instances is equal to the dimension of the Lie algebra L, considered as a vector space.

The system of equations (21) is called an algebraic approximation of the Yang–Mills equations. Note that this system of equations contains, as a particular class of solutions, all constant solutions of the Yang–Mills equations. This system also contains other classes of (nonconstant) solutions of the Yang–Mills equations, which are both solutions of systems (19) and (20).

In [17, 18], explicit formulas were presented for all solutions A_{μ} of system (21) in the case in which all the expressions A_{μ} and $J^{\nu}_{(3)}$ in this system do not depend on the point x of the (pseudo)Euclidean space $\mathbb{R}^{p,q}$ in the case of the Lie group SU(2). These results can be reformulated locally for system (21) in the general case, since it does not include any differential operators.

It seems interesting to further study the system of equations (7)–(9) in the general formulation, with an arbitrary current, as well as for other semisimple Lie groups, in particular, the Lie group SU(3). Systems of equations (19), (20), and (21) are of interest for further studies. It would be interesting to find particular classes of solutions to system (20). In the future, it is planned to generalize the results of this article to the system of Yang–Mills–Dirac equations in Minkowski space.

ACKNOWLEDGMENTS

We are grateful to the organizers of the international conference "Problems of Mathematical and Theoretical Physics", dedicated to the 110th birthday of N.N. Bogolyubov. We are grateful to the participants of this conference for useful discussions.

FUNDING

The work of D.S. Shirokov (Section 3) was supported by the Russian Science Foundation (project no. 18-71-00010).

REFERENCES

1. L. D. Faddeev and A. A. Slavnov, *Gauge Fields: An Introduction to Quantum Theory*, 2nd ed. (CRC Press, Boca Raton, 2018).
2. T. T. Wu and C. N. Yang, "Some solutions of the classical isotopic gauge field equations," in *Properties of Matter Under Unusual Conditions*, Ed. by H. Mark and S. Fernbach (Interscience, New York, 1968).
3. G. 't Hooft, "Magnetic monopoles in unified gauge theories," Nucl. Phys. B **79**, 276–284 (1974).
4. A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, "Pseudoparticle solutions of the Yang-Mills equations," Phys. Lett. B **59**, 85 (1975).
5. V. de Alfaro, S. Fubini, and G. Furlan, "A new classical solution of the Yang–Mills field equations," Phys. Lett. B **65**, 163 (1976).
6. E. Witten, "Some exact multipseudoparticle solutions of classical Yang–Mills theory," Phys. Rev. Lett. **38**, 121 (1977).
7. M. Atiyah, V. Drinfeld, N. Hitchin, and Yu. Manin, "Construction of instantons," Phys. Lett. A **65**, 185–187 (1978).
8. A. Actor, "Classical solutions of SU(2) Yang–Mills theories," Rev. Mod. Phys. **51**, 461–525 (1979).
9. S. Coleman, "Non-Abelian plane waves," Phys. Lett. B **70**, 59–60 (1977).
10. F. Melia and S. Lo, "Linear plane waves solutions of the Yang–Mills theory," Phys. Lett. B **77,** 71–72 (1978).
11. G. Z. Baseyan, S. G. Matinyan, and G. K. Savvidi, "Nonlinear plane waves in the massless Yang–Mills theory," JETP Lett. **29**, 587 (1979).
12. W. B. Campbell and T. A. Morgan, "Non-abelian plane-fronted waves," Phys. Lett. B **84**, 87–88 (1979).
13. C. H. Oh and R. Teh, "Periodic solutions of the Yang–Mills field equations," Phys. Lett. B **87**, 83–86 (1979).
14. C. H. Oh and R. Teh, "Nonabelian progressive waves," J. Math. Phys. **26**, 841–844 (1985).

15. A. Tsapalisa, E. P. Politisa, X. N. Maintasa, and F. K. Diakonosa, "Gauss' law and non-linear plane waves for Yang–Mills theory," Phys. Rev. D **93**, 085003 (2016).

16. W. Li, "Wave solutions to the Yang-Mills equation," https://www.physics.nus.edu.sg/student/Honours Projects Repository 2016-17/Li weijun.pdf.

17. D. S. Shirokov, "On constant solutions of SU(2) Yang–Mills equations with arbitrary current in Euclidean space R^n," J. Nonlinear Math. Phys. **27**, 199–218 (2020); arXiv:1804.04620.

18. D. S. Shirokov, "On constant solutions of SU(2) Yang-Mills equations with arbitrary current in pseudo-Euclidean space $R^{p,q}$," arXiv:1912.04996.

19. N. G. Marchuk, "On a field equation generating a new class of particular solutions to the Yang–Mills equations." Proc. Steklov Inst. Math. **285**, 197–210 (2014).

20. N. G. Marchuk and D. S. Shirokov, "Constant solutions of Yang–Mills equations and generalized Proca equations." J. Geom. Symmetry Phys. **42**, 53–72 (2016).

21. D. S. Shirokov, "Covariantly constant solutions of the Yang–Mills equations," Advances in Applied Clifford Algebras **28**, 53 (2018).

22. R. Schimming, "On constant solutions of the Yang–Mills equations," Arch. Math. **24**, 65–73 (1988).

23. R. Schimming and E. Mundt, "Constant potential solutions of the Yang–Mills equation," J. Math. Phys. **33**, 4250 (1992).

Translated by E. Chernokozhin