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GENERAL SOLUTIONS OF ONE CLASS OF FIELD EQUATIONS

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We find general solutions of some field equations (systems of equations) in pseudo-Euclidean spaces (the so-called primitive field equations). These equations are used in the study of the Dirac equation and Yang–Mills equations. These equations are invariant under orthogonal $\mathrm{O}(p,q)$ coordinate transformations and invariant under gauge transformations, which depend on some Lie groups. In this paper we use some new geometric objects—Clifford field vector and an algebra of h-forms which is a generalization of the algebra of differential forms and the Atiyah–Kähler algebra.

Keywords: Clifford algebra, gauge symmetry, primitive field equation, Dirac equation, Dirac gamma matrices, Yang–Mills equations.

1. Introduction

In physics field equations describe physical fields and (using quantization) elementary particles. The following equations are fundamental relativistic field equations: Maxwell's equations (1862), the Klein–Gordon–Fock equation (1926), the Dirac equation (1928), the Yang–Mills equations (1954). These equations are considered in Minkowski space $\mathbb{R}^{1,3}$, they are invariant under Lorentz coordinate transformations. They are also invariant under certain unitary gauge transformations.

In this paper we consider a class of the so-called *primitive field equations* (systems of equations) (27). These equations are considered in pseudo-Euclidean spaces $\mathbb{R}^{p,q}$ and have different Lie groups of gauge symmetry. We find general solutions of primitive field equations corresponding to a wide class of gauge Lie groups. A partial

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case of such equations was considered in 1930–1939 in the theory of Dirac equation on curved pseudo-Riemannian manifolds of signature $(1,3)$. Namely, it is a condition of generalized covariant constancy of γ-matrices, which is gauge invariant w.r.t. the spinor Lie group $\mathrm{Spin}(1,3)$ (see, for example, [12] formula (4.5.13)).

In the theory of model field equations developed by authors in a series of papers and in monograph [7] there arises necessity to solve primitive field equations in pseudo-Euclidean space $\mathbb{R}^{1,3}$ with various gauge Lie groups (see Section 2). Also, primitive field equations are used in the theory of Yang–Mills equations. Namely, we present a new class of gauge-invariant solutions of Yang–Mills equations, which correspond to solutions of primitive field equations (see [8]).

In Section 2 of this paper we consider the Dirac equation and make some important notes about the Lie group $\mathrm{SU}(2,2)$ in connection with the Dirac equation. As a result of these notes we get new system of equations (8) and (9). Eqs. (8) are a special case of primitive field equations and we study these equations in the next sections of the paper in pseudo-Euclidean spaces $\mathbb{R}^{p,q}$.

In Section 3 we discuss some known facts about Clifford algebras. We actively use tensor fields with values in Clifford algebra. Also we discuss some Lie algebras in Clifford algebra, especially Lie algebras $\mathrm{w}(\mathcal{C}\ell(p,q)))$. Results about Lie subalgebras of the Lie algebras $\mathrm{w}(\mathcal{C}\ell(p,q)))$ of pseudo-unitary Lie group are our original results published in [7, 11].

In Section 4 we present original results about projection operators and contractions in Clifford algebras. We use these results in Section 7.

In Section 5 of this paper we present some new geometric objects—Clifford field vector and an algebra of h-forms which is a generalization of the algebra of differential forms and the Atiyah-Kähler algebra [1, 5][1]. These objects are helpful for consideration of some problems related to field theory equations.

In Sections 6 and 7 we consider a primitive field equation and present original results about its gauge symmetry and about general solutions of this equation.

Note that all considerations of this paper are valid for a general pseudo-Euclidean metric of signature (p,q) and, in particular, for the Lorentzian metric $(+,-,-,-)$. Results of the paper can be understood either on the base of Dirac gamma matrices or on the base of Clifford algebras.

2. A new view on the Dirac equation and γ-matrices

Consider the Dirac equation for an electron in the Minkowski space $\mathbb{R}^{1,3}$ with coordinates x^μ, $\mu=0,1,2,3$ ($\partial_\mu=\partial/\partial x^\mu$—partial derivatives),

$$i\gamma^\mu(\partial_\mu\psi-ia_\mu\psi)-m\psi=0, \tag{1}$$

where γ^μ are 4 complex square matrices of order 4 satisfying conditions

[1]We combine the technique of the Dirac gamma matrices and the technique of differential forms, in particular, the Atiyah–Kähler algebra of differential forms.

$$\partial_\mu \gamma^\nu = 0, \tag{2}$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I, \tag{3}$$

where $\eta = \|\eta^{\mu\nu}\| = \mathrm{diag}(1, -1, -1, -1)$, I is the identity matrix of order 4, $a_\mu = a_\mu(x)$ is a covector potential of electromagnetic field, $\psi = \psi(x)$ is a Dirac spinor (column of four complex functions $\psi : \mathbb{R}^{1,3} \to \mathbb{C}^4$), i is the imaginary unit, m is a real number (mass of electron).

In the theory of the Dirac equation it is assumed that we have a fixed set of matrices γ^μ that satisfy conditions (2) and (3) and the condition[2] for Hermitian conjugated matrices

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0. \tag{4}$$

Matrices γ^μ satisfying conditions (2), (3) and (4) are defined up to a similarity transformation with a unitary matrix $U \in \mathrm{U}(4)$, i.e. matrices

$$\acute{\gamma}^\mu = U^{-1} \gamma^\mu U, \quad \text{where} \quad U^{-1} = U^\dagger, \tag{5}$$

satisfy the same conditions (2), (3) and (4).

In particular, matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ in the Dirac representation satisfy these conditions and the matrix γ^0 is diagonal $\gamma^0 = \mathrm{diag}(1, 1, -1, -1)$. This matrix γ^0 changes under unitary transformation (5).

Denote $\beta = \mathrm{diag}(1, 1, -1, -1)$ and consider Lie group $\mathrm{SU}(2, 2)$ of special pseudo-unitary matrices and its real Lie algebra $\mathfrak{su}(2, 2)$ (see [3])

$$\mathrm{SU}(2, 2) = \{S \in \mathrm{Mat}(4, \mathbb{C}) : S^\dagger \beta S = \beta,\ \det S = 1\},$$
$$\mathfrak{su}(2, 2) = \{s \in \mathrm{Mat}(4, \mathbb{C}) : \beta s^\dagger \beta = -s,\ \mathrm{tr}\, s = 0\},$$

where $\mathrm{Mat}(4, \mathbb{C})$ is the algebra of complex matrices of order 4. Dirac gamma matrices γ^μ satisfy (4) and $\mathrm{tr}\,\gamma^\mu = 0$, therefore

$$i\gamma^\mu \in \mathfrak{su}(2, 2). \tag{6}$$

We may consider conditions (2) and (3) together with condition (6) and allow a similarity transformation

$$i\gamma^\mu \to i\acute{\gamma}^\mu = S^{-1} i\gamma^\mu S \tag{7}$$

with a matrix $S \in \mathrm{SU}(2, 2)$, which preserves (2), (3) and (6).

If we consider conditions (2) and (3) as equations for matrices γ^μ with condition (6), then we can consider transformation (7) as a global symmetry (it does not depend on $x \in \mathbb{R}^{1,3}$) of this system of equations.

Now we change Eqs. (2) and obtain a system of equations with local (gauge) symmetry with respect to the pseudo-unitary group $\mathrm{SU}(2, 2)$.

Namely, consider the following system of equations [7]:

$$\partial_\mu \gamma^\nu - [C_\mu, \gamma^\nu] = 0, \tag{8}$$

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I, \tag{9}$$

[2] This condition is required when we consider bilinear covariants of the Dirac spinors.

where $i\gamma^\mu = i\gamma^\mu(x)$ and $C_\mu = C_\mu(x)$ are smooth functions of $x \in \mathbb{R}^{1,3}$ with values in the Lie algebra $\mathfrak{su}(2,2)$. The system of equations (8) and (9) is invariant under the local (gauge) transformation

$$i\gamma^\mu \to i\acute{\gamma}^\mu = S^{-1} i\gamma^\mu S, \tag{10}$$

$$C_\mu \to \acute{C}_\mu = S^{-1} C_\mu S - S^{-1}\partial_\mu S, \tag{11}$$

where the matrix $S = S(x)$ is a function of $x \in \mathbb{R}^{1,3}$ with values in the Lie group $\mathrm{SU}(2,2)$.

We consider the system of equations (8) and (9) as a new field equation (system of equations). We call this equation *a primitive field equation*. Let us analyze this equation in pseudo-Euclidean spaces. We use a formalism of Clifford algebras because, in our opinion, this formalism is the most convenient for this task.

3. Clifford algebras

Consider real $\mathcal{C}\ell^{\mathbb{R}}(p,q)$ or complexified $\mathcal{C}\ell(p,q) = \mathbb{C} \otimes \mathcal{C}\ell^{\mathbb{R}}(p,q)$ (see [6]) Clifford algebra with $p+q=n$, $n \geq 1$. Note that $\mathcal{C}\ell^{\mathbb{R}}(p,q) \subset \mathcal{C}\ell(p,q)$. When our argumentation is applicable to both cases, we write $\mathcal{C}\ell^{\mathbb{F}}(p,q)$, implying that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. The construction of Clifford algebra is discussed in details in [6, 10, 11].

Let e be the identity element and let e^a, $a = 1, \dots, n$, be generators[3] of the Clifford algebra $\mathcal{C}\ell^{\mathbb{R}}(p,q)$

$$e^a e^b + e^b e^a = 2\eta^{ab} e, \tag{12}$$

where $\eta = ||\eta^{ab}|| = ||\eta_{ab}||$ is the diagonal matrix with p pieces of $+1$ and q pieces of -1 on the diagonal. Elements

$$e^{a_1 \dots a_k} = e^{a_1} \cdots e^{a_k}, \qquad a_1 < \cdots < a_k,\ \ k = 1, \dots, n, \tag{13}$$

together with the identity element e form the basis of the Clifford algebra. The number of basis elements is equal to 2^n.

Any element U of the Clifford algebra $\mathcal{C}\ell^{\mathbb{F}}(p,q)$ can be expanded in the basis

$$U = ue + u_a e^a + \sum_{a_1 < a_2} u_{a_1 a_2} e^{a_1 a_2} + \cdots + u_{1\dots n} e^{1\dots n}, \tag{14}$$

where $u, u_a, u_{a_1 a_2}, \dots, u_{1\dots n}$ are real or complex numbers (in the respective cases $\mathcal{C}\ell^{\mathbb{R}}(p,q)$ or $\mathcal{C}\ell(p,q)$).

Vector (real or complex) subspaces spanned on basis elements $e^{a_1 \dots a_k}$ labeled by ordered multi-indices of length k are denoted by $\mathcal{C}\ell^{\mathbb{F}}_k(p,q)$, $k = 0, \dots, n$. Elements of the subspace $\mathcal{C}\ell^{\mathbb{F}}_k(p,q)$ are called *elements of rank*[4] k. We have $\mathcal{C}\ell^{\mathbb{F}}(p,q) = \mathcal{C}\ell^{\mathbb{F}}_0(p,q) \oplus \cdots \oplus \mathcal{C}\ell^{\mathbb{F}}_n(p,q)$.

[3] We use notation from [2] (see, also [7]). Note that there exists another notation instead of e^a - with lower indices. But we use upper indices because we take into account relation with differential forms. Note that e^a is not exponent.

[4] There is a difference in notation in literature. We use term "rank" and notation $\mathcal{C}\ell_k(p,q)$ because we take into account a relation with differential forms, see [7].

The Clifford algebra $\mathcal{C}\ell^{\mathbb{F}}(p,q)$ is a superalgebra. It is represented as the direct sum of even and odd subspaces (of *even* and *odd* elements respectively)

$$\mathcal{C}\ell^{\mathbb{F}}(p,q) = \mathcal{C}\ell^{\mathbb{F}}_{\mathrm{Even}}(p,q) \oplus \mathcal{C}\ell^{\mathbb{F}}_{\mathrm{Odd}}(p,q),$$
$$\mathcal{C}\ell^{\mathbb{F}}_{\mathrm{Even}}(p,q) = \bigoplus_{k-even} \mathcal{C}\ell^{\mathbb{F}}_k(p,q), \qquad \mathcal{C}\ell^{\mathbb{F}}_{\mathrm{Odd}}(p,q) = \bigoplus_{k-odd} \mathcal{C}\ell^{\mathbb{F}}_k(p,q).$$

We introduce the operations of projection onto subspaces of rank-k elements ($k = 0, 1, \ldots, n$):

$$\pi_k : \mathcal{C}\ell^{\mathbb{F}}(p,q) \to \mathcal{C}\ell^{\mathbb{F}}_k(p,q), \qquad \pi_k(U) = \sum_{a_1<\cdots<a_k} u_{a_1\ldots a_k} e^{a_1\ldots a_k}. \tag{15}$$

The Clifford algebra $\mathcal{C}\ell^{\mathbb{F}}(p,q)$, $n = p+q$, has the center

$$\mathrm{Cen}(\mathcal{C}\ell^{\mathbb{F}}(p,q)) = \begin{cases} \mathcal{C}\ell^{\mathbb{F}}_0(p,q), & \text{if } n \text{ is even;} \\ \mathcal{C}\ell^{\mathbb{F}}_0(p,q) \oplus \mathcal{C}\ell^{\mathbb{F}}_n(p,q) & \text{if } n \text{ is odd.} \end{cases}$$

Pseudo-Euclidean space $\mathbb{R}^{p,q}$ and changes of coordinates. Let p, q be nonnegative integers and $n = p+q \geq 1$. We denote an n-dimensional pseudo-Euclidean space of signature (p,q) with Cartesian coordinates x^μ, $\mu = 1, \ldots, n$, by $\mathbb{R}^{p,q}$. Tensor indices corresponding to the coordinates are denoted by small Greek letters. The metric tensor of pseudo-Euclidean space $\mathbb{R}^{p,q}$ is given by the diagonal matrix of order n,

$$\eta = \|\eta_{\mu\nu}\| = \|\eta^{\mu\nu}\| = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1) \tag{16}$$

with p copies of 1 and q copies of -1 on the diagonal.

In $\mathbb{R}^{p,q}$ we deal with linear coordinate transformations[5]

$$x^\mu \to \acute{x}^\mu = p^\mu_\nu x^\nu, \tag{17}$$

preserving the metric tensor. So, the real numbers p^μ_ν satisfy relations $p^\mu_\alpha p^\nu_\beta \eta^{\alpha\beta} = \eta^{\mu\nu}$, $p^\mu_\alpha p^\nu_\beta \eta_{\mu\nu} = \eta_{\alpha\beta}$. In matrix formalism we can write $P^T \eta P = \eta$, $P \eta P^T = \eta$, where T is the matrix transposition and the matrix $P = \|p^\mu_\nu\|$ is from the pseudo-orthogonal group $O(p,q) = \{P \in \mathrm{Mat}(n, \mathbb{R}) : P^T \eta P = \eta\}$.

We denote the set of (r,s) tensor fields (of rank $r+s$) of pseudo-Euclidean space $\mathbb{R}^{p,q}$ by T^r_s. Real or complex tensor field $u \in \mathrm{T}^r_s$ has components $u^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s}$ in coordinates x^μ. These components are smooth functions $\mathbb{R}^{p,q} \to \mathbb{F}$, where $\mathbb{F}$ is the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. In all considerations of this work it is sufficient that all functions of $x \in \mathbb{R}^{p,q}$ have continuous partial derivatives up to the second order.

Functions with values in Clifford algebra. Further we consider functions $\mathbb{R}^{p,q} \to \mathcal{C}\ell(p,q)$ with values in Clifford algebra. We assume that the basis elements (13)

[5] We use the Einstein summation convention. For example $p^\mu_\nu x^\nu = \sum_{\nu=1}^n p^\mu_\nu x^\nu$.

do not depend on the points $x \in \mathbb{R}^{p,q}$, i.e.

$$\partial_\mu e^a = 0, \qquad \forall \mu, a = 1, \dots, n,$$

where $\partial_\mu = \partial/\partial x^\mu$ are partial derivatives. The coefficients in the basis expansion of the Clifford algebra element $u_{a_1 \dots a_k} = u_{a_1 \dots a_k}(x)$ may depend on $x \in \mathbb{R}^{p,q}$. In the present paper we also consider the functions with values in Lie algebras generated by the Clifford algebra (see p. 312).

Tensor fields with values in Clifford algebra. A tensor at the point $x \in \mathbb{R}^{p,q}$ with values in Clifford algebra is a mathematical object that belongs to the tensor product of the tensor algebra and Clifford algebra.

If a tensor field of rank (r, s) in $\mathbb{R}^{p,q}$ has components $u^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} = u^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}(x)$ in Cartesian coordinates x^μ, then these components are considered as functions $\mathbb{R}^{p,q} \to \mathbb{F}$. These functions transform by the standard tensor transformation rule.

Components $U^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}$ of tensor fields with values in Clifford algebra $\mathcal{C}\ell^{\mathbb{F}}(p, q)$ are considered as functions $\mathbb{R}^{p,q} \to \mathcal{C}\ell^{\mathbb{F}}(p, q)$ that transform under changes of coordinates by the standard tensor transformation rule.

We use the following notation for tensor fields with values in Clifford algebra: $U^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \in \mathcal{C}\ell(p, q)\mathrm{T}^r_s$ or $U \in \mathcal{C}\ell(p, q)\mathrm{T}^r_s$. In this notation the letter T means that this object is a tensor field. In particular, for scalar functions $U : \mathbb{R}^{p,q} \to \mathcal{C}\ell^{\mathbb{F}}(p, q)$ we use the notation $U \in \mathcal{C}\ell^{\mathbb{F}}(p, q)\mathrm{T}$.

For example, if we consider a tensor field $U^\mu_\nu \in \mathcal{C}\ell(p, q)\mathrm{T}^1_1$ with values in Clifford algebra, then we can write

$$U^\mu_\nu = u^\mu_\nu e + u^\mu_{\nu a} e^a + \sum_{a_1 < a_2} u^\mu_{\nu a_1 a_2} e^{a_1 a_2} + \dots + u^\mu_{\nu 1 \dots n} e^{1 \dots n},$$

where $u^\mu_\nu, u^\mu_{\nu a}, u^\mu_{\nu a_1 a_2}, \dots, u^\mu_{\nu 1 \dots n}$ are real (in the case of $\mathcal{C}\ell^{\mathbb{R}}(p, q)$) or complex (in the case of $\mathcal{C}\ell(p, q)$) tensor fields from T^1_1.

In the present paper we also consider tensor fields with values in Lie algebras and scalar fields with values in Lie groups (see p. 312).

Lie algebras in Clifford algebras. Let us consider the commutator (Lie bracket) $[U, V] = UV - VU$ of Clifford algebra elements $U, V \in \mathcal{C}\ell(p, q)$. This operation satisfies the Jacobi identity

$$[[U, V], W] + [[V, W], U] + [[W, U], V] = 0, \qquad \forall U, V, W \in \mathcal{C}\ell(p, q).$$

Therefore, Clifford algebra $\mathcal{C}\ell(p, q)$ can be considered as a Lie algebra with respect to the commutator. We can consider vector subspaces $L \subset \mathcal{C}\ell(p, q)$ of Clifford algebra that are closed under commutator, i.e. with the condition: if $U, V \in L$ then $[U, V] \in L$. These subspaces are Lie algebras (generated by Clifford algebra), see also [4]. Primarily we are interested in Lie algebras that are direct sums (as vector spaces) of subspaces of Clifford algebra elements of fixed ranks [13].

With the help of the operator $\pi_0 : \mathcal{C}\ell^{\mathbb{F}}(p,q) \to \mathcal{C}\ell^{\mathbb{F}}_0(p,q)$ we define operation of *Clifford algebra trace* $\mathrm{Tr} : \mathcal{C}\ell^{\mathbb{F}} \to \mathbb{F}$,

$$\mathrm{Tr}(U) = \pi_0(U)|_{e\to 1}, \qquad \forall U \in \mathcal{C}\ell^{\mathbb{F}}(p,q).$$

THEOREM 1. *In a Clifford algebra $\mathcal{C}\ell^{\mathbb{F}}(p,q)$ of arbitrary dimension $n = p+q$ we have* $\mathrm{Tr}([U,V]) = 0$ *for all $U, V \in \mathcal{C}\ell^{\mathbb{F}}(p,q)$. In a Clifford algebra $\mathcal{C}\ell^{\mathbb{F}}(p,q)$ of odd dimension $n = p+q$ we have $\pi_n([U,V]) = 0$ for all $U, V \in \mathcal{C}\ell^{\mathbb{F}}(p,q)$.*

Proof: Using (12) for 2 arbitrary basis elements we obtain

$$[e^{a_1\dots a_k}, e^{b_1\dots b_l}] = (1-(-1)^{kl-s})e^{a_1\dots a_k}e^{b_1\dots b_l} \in \mathcal{C}\ell^{\mathbb{F}}_{k+l-2s}(p,q), \tag{18}$$

where s is the number of coincident indices in the ordered multi-indices $a_1\dots a_k$ and $b_1\dots b_l$. If $k = l = s$, then $1-(-1)^{kl-s}$ equals 0. If $k+l = n$, $s = 0$ and n is odd, then it equals 0 again. For more details see Theorem 1 in [13]. □

Consider the set of Clifford algebra elements with zero projection onto the Clifford algebra center

$$\mathcal{C}\ell_{\circledS}(p,q) = \mathcal{C}\ell(p,q) \setminus \mathrm{Cen}(\mathcal{C}\ell(p,q)).$$

THEOREM 2. *The set $\mathcal{C}\ell_{\circledS}(p,q)$ is a Lie algebra with respect to the commutator $[A,B] = AB - BA$.*

Proof: See the previous theorem. □

THEOREM 3. *Let $F = F(x)$ be a function with values in the Lie algebra $\mathcal{C}\ell_{\circledS}(p,q)$. Then the partial derivatives $\partial_\mu F$ are functions (components of a covariant vector field) with values in the same Lie algebra $\mathcal{C}\ell_{\circledS}(p,q)$.*

Proof: If n is even, then the function $F = F(x)$ can be written as basis expansion (13)

$$F = f_a e^a + \sum_{a_1<a_2} f_{a_1a_2}e^{a_1a_2} + \dots + f_{1\dots n}e^{1\dots n}. \tag{19}$$

Since $\mathrm{Tr}\,F = 0$, then the first term fe is absent. We assume that the Clifford algebra generators e^a do not depend on $x \in \mathbb{R}^{p,q}$. So $\partial_\mu e^a = 0$ for all $\mu, a = 1,\dots,n$, and

$$\partial_\mu F = (\partial_\mu f_a)e^a + \sum_{a_1<a_2}(\partial_\mu f_{a_1a_2})e^{a_1a_2} + \dots + (\partial_\mu f_{1\dots n})e^{1\dots n}.$$

We obtain $\mathrm{Tr}\,F = 0$ and $\mathrm{Tr}(\partial_\mu F) = 0$, i.e. $\partial_\mu F \in \mathcal{C}\ell_{\circledS}(p,q)$.

If n is odd, then the function $F = F(x) \in \mathcal{C}\ell_{\circledS}(p,q)$ can be written as basis expansion (19) without the first term fe and without the last term $f_{1\dots n}e^{1\dots n}$. We obtain $\partial_\mu F \in \mathcal{C}\ell_{\circledS}(p,q)$ again. □

The following subspaces of Clifford algebra are Lie algebras with respect to the commutator: $\mathcal{C}\ell_2(p,q)$, $\mathcal{C}\ell_1(p,q) \oplus \mathcal{C}\ell_2(p,q)$, $\mathcal{C}\ell_2(p,q) \oplus \mathcal{C}\ell_3(p,q)$, $\mathcal{C}\ell_0(p,q)$, $\mathrm{Cen}(\mathcal{C}\ell(p,q))$, $\mathcal{C}\ell_{\circledS}(p,q)$.

In Section 2 we have considered gamma matrices in the Dirac representation (which are used in the Dirac equation for an electron) and found that $i\gamma^\mu \in \mathfrak{su}(2,2)$. In the Clifford algebra $\mathcal{C}\ell(p,q)$ the following Lie algebra is an analogue[6] of the Lie algebra $\mathfrak{su}(2,2)$,

$$\mathrm{w}(\mathcal{C}\ell(p,q)) = \bigoplus_{k=1}^{\acute{n}} i^{\frac{k(k-1)}{2}+1}\mathcal{C}\ell^{\mathbb{R}}_k(p,q)$$

where $\acute{n} = n$ in the case of even n and $\acute{n} = n-1$ in the case of odd n.

We are interested in Lie subalgebras of this Lie algebra. As we will see, the Lie algebra $\mathcal{C}\ell^{\mathbb{R}}_2(p,q)$ plays an important role in field theory equations. Other important Lie algebras contain Lie subalgebra $i\mathcal{C}\ell^{\mathbb{R}}_1(p,q) \oplus \mathcal{C}\ell^{\mathbb{R}}_2(p,q)$:

- For $n \geq 2$: $i\mathcal{C}\ell^{\mathbb{R}}_1(p,q) \oplus \mathcal{C}\ell^{\mathbb{R}}_2(p,q)$.
- For $n \geq 6$: $i\mathcal{C}\ell^{\mathbb{R}}_1(p,q) \oplus \mathcal{C}\ell^{\mathbb{R}}_2(p,q) \oplus a_{\acute{n}-1}\mathcal{C}\ell^{\mathbb{R}}_{\acute{n}-1}(p,q) \oplus a_{\acute{n}}\mathcal{C}\ell^{\mathbb{R}}_{\acute{n}}(p,q)$, where $\acute{n} = n$ for even n and $\acute{n} = n-1$ for odd n.
- For $n \geq 8$: $i\mathcal{C}\ell^{\mathbb{R}}_1(p,q) \oplus \mathcal{C}\ell^{\mathbb{R}}_2(p,q) \oplus i\mathcal{C}\ell^{\mathbb{R}}_5(p,q) \oplus \mathcal{C}\ell^{\mathbb{R}}_6(p,q) \oplus i\mathcal{C}\ell^{\mathbb{R}}_9(p,q) \oplus \mathcal{C}\ell^{\mathbb{R}}_{10}(p,q) \oplus \cdots \oplus a_r\mathcal{C}\ell^{\mathbb{R}}_r(p,q)$, where $r = n-2$ if $n = 0 \mod 4$, $r = n-3$ if $n = 1 \mod 4$, $r = n$ if $n = 2 \mod 4$, $r = n-1$ if $n = 3 \mod 4$.

We consider pinor groups as the following sets of Clifford algebra elements,

$$\mathrm{Pin}(p,q) = \{S \in \mathcal{C}\ell^{\mathbb{R}}_{\mathrm{Even}}(p,q) \,\text{or}\, S \in \mathcal{C}\ell^{\mathbb{R}}_{\mathrm{Odd}}(p,q) : S^{\sim}S = \pm e,\ S^{-1}e^a S \in \mathcal{C}\ell^{\mathbb{R}}_1(p,q)\},$$

where the linear operation $\sim$: $\mathcal{C}\ell_k(p,q) \to \mathcal{C}\ell_k(p,q)$, $k = 0, 1, \ldots, n$, is called *reversion*. This operation reverses the order of generators in products: $(e^{a_1}\cdots e^{a_k})^{\sim} = e^{a_k}\cdots e^{a_1}$.

Note that the set of rank 2 Clifford algebra elements $\mathcal{C}\ell^{\mathbb{F}}_2(p,q)$ is closed w.r.t. commutator and hence generates a Lie algebra. The Lie algebra $\mathcal{C}\ell^{\mathbb{R}}_2(p,q) \subset \mathrm{w}(\mathcal{C}\ell(p,q))$ is a real Lie algebra of the Lie group $\mathrm{Pin}(p,q)$ (see [14]).

4. Relation between projection operators and contractions in Clifford algebras

Consider operations of projection (15) onto subspaces $\mathcal{C}\ell_k(p,q)$ of Clifford algebra elements of rank k.

The following sum is called a *generator contraction* of an arbitrary Clifford algebra element $U \in \mathcal{C}\ell(p,q)$,

$$F(U) = e^a U e_a, \tag{20}$$

where $e_a = \eta_{ab}e^b$. We use the notation $F^0(U) = U$, $F^1(U) = F(U)$, $F^2(U) = F(F(U))$, etc. Note that $F^l : \mathcal{C}\ell_k(p,q) \to \mathcal{C}\ell_k(p,q)$ for all $k, l = 0, 1, \ldots, n$.

According to the theorem on generator contraction [10] we have

$$F(U) = \sum_{k=0}^{n} \lambda_k \pi_k(U), \qquad \text{where} \quad \lambda_k = (-1)^k(n-2k). \tag{21}$$

[6]See [7, 11].

THEOREM 4. *Consider an arbitrary Clifford algebra element* $U \in \mathcal{C}\ell(p,q)$, $n = p+q$. *Then we have*

$$\begin{aligned} \pi_k(U) &= \sum_{l=0}^{n} b_{kl} F^l(U) \quad \text{if } n \text{ is even;} \\ \pi_{k,n-k}(U) &= \sum_{l=0}^{\frac{n-1}{2}} g_{kl} F^l(U) \ \text{ if } n \text{ is odd,} \end{aligned} \tag{22}$$

where $B = ||b_{kl}||$ *is inverse of matrix* $A_{(n+1)\times(n+1)} = ||a_{kl}||$, $a_{kl} = (\lambda_{l-1})^{k-1}$, $G = ||g_{kl}||$ *is inverse of matrix* $D_{\frac{n+1}{2}\times\frac{n+1}{2}} = ||d_{kl}||$, $d_{kl} = (\lambda_{l-1})^{k-1}$ *and* $\lambda_k = (-1)^k(n-2k)$ *and* $\pi_{k,n-k} = \pi_k + \pi_{n-k}$ *is operation of projection onto subspace* $\mathcal{C}\ell_k(p,q) \oplus \mathcal{C}\ell_{n-k}(p,q)$.

Proof: We have $F^l(U) = \sum_{k=0}^{n} (\lambda_k)^l \pi_k(U)$, then

$$\begin{pmatrix} F^0(U) \\ F^1(U) \\ \dots \\ F^n(U) \end{pmatrix} = A \begin{pmatrix} \pi_0(U) \\ \pi_1(U) \\ \dots \\ \pi_n(U) \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_0 & \lambda_1 & \dots & \lambda_n \\ \dots & \dots & \dots & \dots \\ (\lambda_0)^n & (\lambda_1)^n & \dots & (\lambda_n)^n \end{pmatrix}.$$

The matrix A is a Vandermonde matrix. Its determinant equals

$$\det A = \prod_{0 \le i < j \le n} (\lambda_j - \lambda_i).$$

In the case of even n we have $\lambda_k = -\lambda_{n-k}$, because $\lambda_{n-k} = (-1)^{n-k}(n-2(n-k)) = (-1)^k(2k-n) = -\lambda_k$. In particular, $\lambda_{\frac{n}{2}} = 0$. It is easy to see that all λ_k are different in the case of even n, and the Vandermonde matrix is invertible. Denote the inverse matrix by $B = ||b_{ij}||$:

$$\begin{pmatrix} \pi_0(U) \\ \pi_1(U) \\ \dots \\ \pi_n(U) \end{pmatrix} = \begin{pmatrix} b_{00} & b_{01} & \dots & b_{0n} \\ b_{10} & b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots \\ b_{n0} & b_{n1} & \dots & b_{nn} \end{pmatrix} \begin{pmatrix} F^0(U) \\ F^1(U) \\ \dots \\ F^n(U) \end{pmatrix}.$$

There exists an explicit formula for inverse of the Vandermonde matrix but we do not use it.

In the case of odd n we have $\lambda_k = \lambda_{n-k}$, and hence the Vandermonde matrix is singular and projection operations are not expressed through contractions. In this case we use projections $\pi_{k,n-k}$,

$$\begin{pmatrix} F^0(U) \\ F^1(U) \\ \dots \\ F^{\frac{n-1}{2}}(U) \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_0 & \lambda_1 & \dots & \lambda_{\frac{n-1}{2}} \\ \dots & \dots & \dots & \dots \\ (\lambda_0)^{\frac{n-1}{2}} & (\lambda_1)^{\frac{n-1}{2}} & \dots & (\lambda_{\frac{n-1}{2}})^{\frac{n-1}{2}} \end{pmatrix} \begin{pmatrix} \pi_{0,n}(U) \\ \pi_{1,n-1}(U) \\ \dots \\ \pi_{\frac{n-1}{2},\frac{n+1}{2}}(U) \end{pmatrix}.$$

We denote the invertible matrix from the last formula by D and inverse of D by $G = ||g_{ij}||$.

We obtain the relation between projection operations and contractions in the following form:

$$\begin{pmatrix} \pi_{0,n}(U) \\ \pi_{1,n-1}(U) \\ \dots \\ \pi_{\frac{n-1}{2},\frac{n+1}{2}}(U) \end{pmatrix} = \begin{pmatrix} g_{00} & g_{01} & \cdots & g_{0\frac{n-1}{2}} \\ g_{10} & g_{11} & \cdots & g_{1\frac{n-1}{2}} \\ \dots & \dots & \dots & \dots \\ g_{\frac{n-1}{2}0} & g_{\frac{n-1}{2}1} & \cdots & g_{\frac{n-1}{2}\frac{n-1}{2}} \end{pmatrix} \begin{pmatrix} F^0(U) \\ F^1(U) \\ \dots \\ F^{\frac{n-1}{2}}(U) \end{pmatrix}. \qquad \square$$

So, in the case of even n the operations of projection of Clifford algebra elements $U \in \mathcal{C}\ell(p,q)$ are uniquely expressed through contractions (of order not more than n) of element U. Note that we can use these formulae as the definition of operations of projection onto subspaces of fixed ranks.

Let us give some examples. In the case of $n = 2$ we have

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 4 & 0 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{8} \\ 1 & 0 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{8} \end{pmatrix},$$

$$F^0(U) = U, \qquad F^1(U) = 2\pi_0(U) - 2\pi_2(U), \qquad F^2(U) = 4\pi_0(U) + 4\pi_2(U),$$

$$\pi_0(U) = \frac{1}{4}e^a U e_a + \frac{1}{8}e^a e^b U e_b e_a, \qquad \pi_1(U) = U - \frac{1}{4}e^a e^b U e_a e_b,$$

$$\pi_2(U) = -\frac{1}{4}e^a U e_a + \frac{1}{8}e^a e^b U e_b e_a.$$

In the case of $n = 4$ we have

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 16 & 4 & 0 & 4 & 16 \\ 64 & -8 & 0 & 8 & -64 \\ 256 & 16 & 0 & 16 & 256 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -\frac{1}{24} & -\frac{1}{96} & \frac{1}{96} & \frac{1}{384} \\ 0 & -\frac{1}{3} & \frac{1}{6} & \frac{1}{48} & -\frac{1}{96} \\ 1 & 0 & -\frac{5}{16} & 0 & \frac{1}{64} \\ 0 & \frac{1}{3} & \frac{1}{6} & -\frac{1}{48} & -\frac{1}{96} \\ 0 & \frac{1}{24} & -\frac{1}{96} & -\frac{1}{96} & \frac{1}{384} \end{pmatrix}.$$

In the case of odd dimension $n = 3$ we have

$$
\begin{aligned}
F^0(U) &= U = \pi_0(U) + \pi_1(U) + \pi_2(U) + \pi_3(U),\\
F^1(U) &= 3\pi_0(U) - \pi_1(U) - \pi_2(U) + 3\pi_3(U),\\
F^2(U) &= 9\pi_0(U) + \pi_1(U) + \pi_2(U) + 9\pi_3(U),\\
F^3(U) &= 27\pi_0(U) - \pi_1(U) - \pi_2(U) + 27\pi_3(U).
\end{aligned}
$$

The matrix of this system of equations is singular. But we can consider expressions $\pi_{03}(U) = \pi_0(U) + \pi_3(U)$, $\pi_{12}(U) = \pi_1(U) + \pi_2(U)$ and obtain

$$
F^0(U) = U = \pi_{03}(U) + \pi_{12}(U), \qquad F^1(U) = 3\pi_{03}(U) - \pi_{12}(U),
$$

$$
\pi_{03}(U) = \frac{1}{4}F^0(U) + \frac{1}{4}F^1(U) = \frac{1}{4}U + \frac{1}{4}e^a U e_a,
$$

$$
\pi_{12}(U) = \frac{3}{4}F^0(U) - \frac{1}{4}F^1(U) = \frac{3}{4}U - \frac{1}{4}e^a U e_a,
$$

$$
D = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}, \qquad G = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & -\frac{1}{4} \end{pmatrix}.
$$

5. Clifford field vectors and an algebra of h-forms

In this section we introduce new geometric objects—Clifford field vector and an algebra of h-forms which is a generalization of the algebra of differential forms and the Atiyah-Kähler algebra [1, 5]. We combine the technique of the Dirac gamma matrices and the technique of differential forms, in particular, the Atiyah-Kähler algebra of differential forms. From our point of view, these objects are helpful for consideration of some problems related to field theory equations.

Frame field y^μ_a. A set of n real vector fields $y^\mu_a = y^\mu_a(x) \in \mathrm{T}^1$ of pseudo-Euclidean space $\mathbb{R}^{p,q}$ enumerated by the Latin index $(a = 1, \dots, n)$ and satisfying

$$
y^\mu_a y^\nu_b \eta^{ab} = \eta^{\mu\nu}, \qquad \forall x \in \mathbb{R}^{p,q},
$$

is called a *frame field*. Using a local (that depends on x) pseudo-orthogonal transformation, we can get another frame field from the frame field y^μ_a,

$$
y^\mu_a \to \hat{y}^\mu_a = q^b_a y^\mu_b,
$$

where $q^b_a = q^b_a(x)$ are smooth functions of $x \in \mathbb{R}^{p,q}$ and the matrix $Q = Q(x) = \|q^b_a\|$ is such that $Q \in O(p,q)$ for any x. It is easy to see that

$$
\hat{y}^\mu_a \hat{y}^\nu_b \eta^{ab} = \eta^{\mu\nu}, \qquad \forall x \in \mathbb{R}^{p,q},
$$

i.e. the set of n vector fields $\hat{y}^\mu_a$ is also a frame field.

Coframe field y^b_ν. A set of n real covector fields $y^b_\nu = y^b_\nu(x) \in \mathrm{T}_1$ of pseudo-Euclidean space $\mathbb{R}^{p,q}$ enumerated by the Latin index $(b = 1, \dots, n)$ and satisfying

$$
y^a_\mu y^b_\nu \eta_{ab} = \eta_{\mu\nu}, \qquad \forall x \in \mathbb{R}^{p,q},
$$

is called a *coframe field*.

If we have a frame field y_a^μ, then we can get the coframe field using the Minkowski matrix,

$$y_\nu^b = \eta^{ab}\eta_{\mu\nu}y_a^\mu.$$

Clifford field vector h^μ**.** If $h^\mu = h^\mu(x)$ are components of a vector field with values in $\mathcal{C}\ell(p,q)$ that satisfy the following relations

$$h^\mu h^\nu + h^\nu h^\mu = 2\eta^{\mu\nu}e, \qquad \mu,\nu = 1,\dots,n, \tag{23}$$

for any $\forall x \in \mathbb{R}^{p,q}$ and the condition

$$\mathrm{Tr}(h^1\cdots h^n) = 0, \tag{24}$$

then the vector $h^\mu \in \mathcal{C}\ell(p,q)\mathrm{T}^1$ is called a *Clifford field vector*.

Note that condition (24) holds automatically in the case of even n, i.e. this condition is necessary for the case of odd n.

Denote the set of invertible Clifford algebra elements by $\mathcal{C}\ell^\times(p,q)$. Note that $\mathcal{C}\ell^\times(p,q)$ is a Lie group with respect to the Clifford multiplication.

If h^μ is a Clifford field vector and $S \in \mathcal{C}\ell^\times(p,q)\mathrm{T}$ is a continuous function, then we can get the pair of new Clifford field vectors using similarity transformation[7] $\hat{h}^\mu = \pm S^{-1}h^\mu S$.

For example, let us consider a frame field $y_a^\mu = y_a^\mu(x)$ and a smooth function $S \in \mathcal{C}\ell^\times(p,q)\mathrm{T}$ with values in the set of invertible Clifford algebra elements. With the help of generators e^a we get the vector field

$$h^\mu = h^\mu(x) = y_a^\mu S^{-1}e^a S \in \mathcal{C}\ell(p,q)\mathrm{T}^1.$$

It is easy to see that components of this vector field satisfy relations (23) and (24), i.e. h^μ is a Clifford field vector.

Components of field vector transform under (orthogonal) changes of coordinates (17) using the standard tensor transformation law

$$h^\mu \to \acute{h}^\mu = p_\nu^\mu h^\nu, \qquad P = \|p_\nu^\mu\| \in \mathrm{O}(p,q). \tag{25}$$

With the help of the metric tensor we can raise and lower indices:

$$h_\nu = \eta_{\mu\nu}h^\mu, \quad h^\mu = \eta^{\mu\nu}h_\nu.$$

THEOREM 5. *If $n = p+q \geq 2$ and h^μ is a Clifford field vector, then $h^\mu \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}^1$.*

Proof: Let us consider a coframe field y_μ^a. We define n elements $h^a = y_\mu^a h^\mu \in \mathcal{C}\ell(p,q)$, satisfying $h^a h^b + h^b h^a = 2\eta^{ab}e$ for all $a,b = 1,\dots,n$.

Let $n = p+q$ be even. We prove that for any $x \in \mathbb{R}^{p,q}$ we have $\mathrm{Tr}\, h^\mu = 0$. By the generalized Pauli's theorem [15] there exists an invertible element $S \in \mathcal{C}\ell(p,q)$ (at any $x \in \mathbb{R}^{p,q}$) such that $h^a = S^{-1}e^a S$, $a = 1,\dots n$. So

$$\mathrm{Tr}\, h^a = \mathrm{Tr}(S^{-1}e^a S) = \mathrm{Tr}\, e^a = 0, \quad \mathrm{Tr}\, h^\mu = \mathrm{Tr}(y_a^\mu h^a) = 0.$$

It proves the theorem for the case of even n.

[7]In the case of even n it is sufficiently to consider only the relation $\hat{h}^\mu = S^{-1}h^\mu S$ (see [11]).

Let $n = p + q \geq 3$ be odd. We prove that for any $x \in \mathbb{R}^{p,q}$ we have $\operatorname{Tr} h^\mu = 0$ and $\operatorname{Tr}(e^{1\ldots n} h^\mu) = 0$. By the generalized Pauli theorem [15] there exists an invertible element $S \in \mathcal{C}\ell(p,q)$ (at any $x \in \mathbb{R}^{p,q}$) such that two sets of n elements $\{e^a\}$ and $\{h^a\}$ are related by one of two following formulae: $h^a = \epsilon\, S^{-1} e^a S,\ a = 1,\ldots n$, $\epsilon = \pm 1$. Then $e^{1\ldots n} h^a = \epsilon\, e^{1\ldots n} S^{-1} e^a S,\ a = 1,\ldots,n$.

Note that the element $e^{1\ldots n} e^a$ is an element of rank $n-1$. Therefore $\operatorname{Tr}(e^{1\ldots n} e^a) = 0$ for $n > 1$. Since the element $e^{1\ldots n}$ (for n odd) is from the center of the Clifford algebra $\mathcal{C}\ell(p,q)$, then for $n \geq 3$

$$\operatorname{Tr} h^a = \epsilon \operatorname{Tr} e^a = 0, \qquad \operatorname{Tr}(e^{1\ldots n} h^a) = \epsilon \operatorname{Tr}(e^{1\ldots n} e^a) = 0, \qquad a = 1,\ldots,n.$$

Consequently, for odd $n \geq 3$ and for any $x \in \mathbb{R}^{p,q}$ we have

$$\operatorname{Tr} h^\mu = 0, \qquad \operatorname{Tr}(e^{1\ldots n} h^\mu) = 0, \qquad a = 1,\ldots,n.$$

It means that $h^\mu \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}^1$. □

***h*-forms.** Let us consider a covariant skew-symmetric tensor field $u_{\mu_1\ldots\mu_k} \in \mathrm{T}_{[k]}$ of rank k and a Clifford field vector $h^\mu \in \mathcal{C}\ell(p,q)\mathrm{T}^1$. We say that the expression

$$\frac{1}{k!} u_{\mu_1\ldots\mu_k} h^{\mu_1} \cdots h^{\mu_k} = \sum_{\nu_1 < \cdots < \nu_k} u_{\nu_1\ldots\nu_k} h^{\nu_1} \cdots h^{\nu_k}$$

is an *h-form of rank k*.

If we have a scalar function $u = u(x)$ and n covariant skew-symmetric tensor fields $u_{\mu_1\ldots\mu_k} \in \mathrm{T}_{[k]}$ of ranks $k = 1, 2, \ldots, n$, then we say that

$$U = ue + \sum_{k=1}^{n} \frac{1}{k!} u_{\mu_1\ldots\mu_k} h^{\mu_1} \cdots h^{\mu_k} = ue + \sum_{k=1}^{n} \sum_{\nu_1 < \cdots < \nu_k} u_{\nu_1\ldots\nu_k} h^{\nu_1} \cdots h^{\nu_k} \tag{26}$$

is an *h-form* or a *heterogeneous h-form.*

An h-form is invariant under orthogonal changes of coordinates (17). Components $u_{\mu_1\ldots\mu_k}$ of an h-form are components of covariant skew-symmetric tensor fields of ranks $k = 0,\ldots,n$.

If we do not pay attention to the difference between the tensor (Greek) and nontensor (Latin) indices[8], then, by relations (23), we can consider components of the field vector h^μ as generators of Clifford algebra. A set of h-forms over the field $\mathbb{F}$ is called[9] the *algebra of h-forms* $\mathcal{C}\ell[h]^{\mathbb{F}}(p,q)$. We denote the set of h-forms of rank k by $\mathcal{C}\ell[h]^{\mathbb{F}}_k(p,q)$. If U is an h-form (26) then we denote projections of U onto $\mathcal{C}\ell[h]^{\mathbb{F}}_k(p,q)$ by $\pi[h]_k(U)$, $k = 0, 1, \ldots, n$. To calculate projections $\pi[h]_k(U)$ we can use the method of contractions by components of Clifford field vector using the Vandermonde matrix (as in Section 4). These structure of the algebra of h-forms is considered as a geometrization of the structure of Clifford algebra.

[8]The difference between the tensor and nontensor indices appears only when we consider coordinate transformations of pseudo-Euclidean space $\mathbb{R}^{p,q}$.

[9]In notation $\mathcal{C}\ell[h]^{\mathbb{F}}(p,q)$ the symbol h means that the basis is generated by the Clifford field vector h^μ.

Lie algebras generated by Clifford algebra were considered on page 312. We will use the following Lie algebras generated by the algebra of h-forms,

$$\mathcal{C}\ell[h]_2(p,q),\ \ \mathcal{C}\ell[h]_1(p,q)\oplus\mathcal{C}\ell[h]_2(p,q),\ \ \mathcal{C}\ell[h]_2(p,q)\oplus\mathcal{C}\ell[h]_3(p,q),$$
$$\mathrm{Cen}(\mathcal{C}\ell[h](p,q)),\ \ \mathcal{C}\ell[h]_{\circledS}(p,q),$$

where $\mathrm{Cen}(\mathcal{C}\ell[h](p,q))$ is the center of algebra of h-forms, $\mathcal{C}\ell[h]_{\circledS}(p,q)=\mathcal{C}\ell[h](p,q)\setminus\mathrm{Cen}(\mathcal{C}\ell[h](p,q))$ is the set of h-forms with zero projection onto the center of algebra of h-forms.

Note that $\mathcal{C}\ell[h]_{\circledS}(p,q)\simeq\mathcal{C}\ell_{\circledS}(p,q)$, because $\mathcal{C}\ell[h]_0(p,q)\simeq\mathcal{C}\ell_0(p,q)$ for any natural $n=p+q$ and $\mathcal{C}\ell[h]_n(p,q)\simeq\mathcal{C}\ell_n(p,q)$ for any odd n.

Tensor fields with values in h-forms. Tensor field $U^{\nu_1\ldots\nu_k}_{\rho_1\ldots\rho_r}$ with values in h-forms (at a point $x\in\mathbb{R}^{p,q}$) belongs to the tensor product of tensor algebra and the algebra of h-forms. We write $U^{\nu_1\ldots\nu_k}_{\rho_1\ldots\rho_r}\in\mathcal{C}\ell[h](p,q)\mathrm{T}^k_r$. For example, a tensor field $U^\nu_\rho\in\mathcal{C}\ell[h](p,q)\mathrm{T}^1_1$ can be represented as

$$U^\nu_\rho=u^\nu_\rho e+\sum_{k=1}^n\frac{1}{k!}u^\nu_{\rho\mu_1\ldots\mu_k}h^{\mu_1}\cdots h^{\mu_k},$$

where $u^\nu_{\rho\mu_1\ldots\mu_k}=u^\nu_{\rho[\mu_1\ldots\mu_k]}$ are components of $(1,k+1)$ tensor field which are skew-symmetric w.r.t. k covariant indices (antisymmetrization is denoted by square brackets).

Note that we can consider Clifford field vector h^μ as vectors with values in h-forms of rank 1. Actually, $h^\mu=\delta^\mu_\nu h^\nu\in\mathcal{C}\ell[h]_1(p,q)\mathrm{T}^1$, where δ^μ_ν is Kronecker tensor ($\delta^k_r=0$ if $k\neq r$ and $\delta^k_r=1$ if $k=r$). Also we have $h_\mu=\eta_{\mu\nu}h^\nu\in\mathcal{C}\ell[h]_1(p,q)\mathrm{T}_1$, where $\eta_{\mu\nu}$ are components of the metric tensor of pseudo-Euclidean space $\mathbb{R}^{p,q}$.

Note that we also consider tensor fields with values in Lie algebras generated by algebra of h-form in this paper (see p. 318).

6. Primitive field equation and its gauge symmetry

Consider the equation (system of equations)

$$\partial_\mu h_\rho-[C_\mu,h_\rho]=0,\quad \mu,\rho=1,\ldots,n, \tag{27}$$

where $h^\rho\in\mathcal{C}\ell(p,q)\mathrm{T}^1$ is an arbitrary Clifford field vector and $C_\mu=C_\mu(x)$ ($x\in\mathbb{R}^{p,q}$) is a covector field with values in $\mathcal{C}\ell(p,q)$.

We consider the system of equations (27) as a new field equation. This equation is called *a primitive field equation.*

Note that if we have a solution $C_\mu=C_\mu(x)\in\mathcal{C}\ell(p,q)\mathrm{T}_1$ of the system of equations (27) and $\alpha_\mu=\alpha_\mu(x)$ are arbitrary continuous components of covector field with values in center of Clifford algebra, then components $C_\mu+\alpha_\mu\in\mathcal{C}\ell(p,q)\mathrm{T}_1$ also satisfy Eq. (27).

Therefore it is reasonable to assume that $C_\mu\in\mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$.

THEOREM 6. *Let* $h^\nu \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}^1$ *be a Clifford field vector and* $C_\mu \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$ *satisfy the primitive field equation*

$$\partial_\mu h_\rho - [C_\mu, h_\rho] = 0, \qquad \forall \mu, \rho = 1, \dots, n.$$

Let $S : \mathbb{R}^{p,q} \to \mathcal{C}\ell^\times(p,q)$ *be a function with values in* $\mathcal{C}\ell^\times(p,q)$ *such that* $S^{-1}\partial_\mu S \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$. *Then, the following components of covectors*

$$\acute{h}_\rho = S^{-1}h_\rho S \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1, \qquad \acute{C}_\mu = S^{-1}C_\mu S - S^{-1}\partial_\mu S \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$$

also satisfy the equation $\partial_\mu \acute{h}_\rho - [\acute{C}_\mu, \acute{h}_\rho] = 0, \quad \forall \mu, \rho = 1, \dots, n.$

Proof: The condition $\acute{h}_\rho \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$ holds automatically for every $S \in \mathcal{C}\ell^\times(p,q)T$ because $\mathrm{Tr}(S^{-1}h_\rho S) = \mathrm{Tr}(h_\rho)$ in the case of natural n and $\pi[h]_n(S^{-1}h_\rho S) = \pi[h]_n(h_\rho)$ in the case of odd n (see Theorems 1 and 5).

To satisfy the condition $\acute{C}_\mu \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$ we need functions S from the class $\mathbb{S}$

$$\mathbb{S} = \{S \in \mathcal{C}\ell^\times(p,q)T : S^{-1}\partial_\mu S \in \mathcal{C}\ell_{\circledS}(p,q)T_1\}.$$

Then

$$\begin{aligned}
\partial_\mu \acute{h}_\rho - [\acute{C}_\mu, \acute{h}_\rho] &= \partial_\mu(S^{-1}h_\rho S) - (S^{-1}C_\mu S - S^{-1}\partial_\mu S)S^{-1}h_\rho S \\
&\quad + S^{-1}h_\rho S(S^{-1}C_\mu S - S^{-1}\partial_\mu S) \\
&= \partial_\mu S^{-1}h_\rho S + S^{-1}\partial_\mu h_\rho S + S^{-1}h_\rho \partial_\mu S \\
&\quad - S^{-1}C_\mu h_\rho S + S^{-1}\partial_\mu S S^{-1}h_\rho S + S^{-1}h_\rho C_\mu S - S^{-1}h_\rho \partial_\mu S \\
&= S^{-1}(\partial_\mu h_\rho - [C_\mu, h_\rho])S + S^{-1}(S\partial_\mu S^{-1} + \partial_\mu S S^{-1})h_\rho S = 0. \quad \square
\end{aligned}$$

REMARK. Professor G. A. Alekseev called our attention to the following fact. If we consider elements $S = S(x)$ as matrices then we can use the well-known formula

$$\mathrm{Tr}(S^{-1}\partial_\mu S) = \partial_\mu(\ln(\det S)).$$

By this formula, from the condition $S^{-1}\partial_\mu S \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$ it follows that $\det S$ does not depend on $x \in \mathbb{R}^{p,q}$. So we may normalize S and take $\det S = 1$ or $\det S = -1$.

THEOREM 7. *Let* $h^\mu \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}^1$ *be a Clifford field vector and* $C_\mu \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$ *be a covector field. If* h^μ *and* C_ν *are related by the equation*

$$\partial_\mu h^\nu - [C_\mu, h^\nu] = 0, \qquad \forall \mu, \nu = 1, \dots, n,$$

then components of the covector field C_μ *satisfy the conditions*

$$\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] = 0, \qquad \forall \mu, \nu = 1, \dots, n. \tag{28}$$

Conditions (28) *are invariant under the gauge transformation*

$$C_\mu \to \acute{C}_\mu = S^{-1}C_\mu S - S^{-1}\partial_\mu S,$$

where $S = S(x)$ *is a function from* $\mathbb{S}$, *i.e.* $S \in \mathcal{C}\ell^\times(p,q)T$ *and* $S^{-1}\partial_\mu S \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$.

Proof: Let us differentiate conditions $\partial_\mu h^\lambda = [C_\mu, h^\lambda]$, and obtain

$$\partial_\nu\partial_\mu h^\lambda = [\partial_\nu C_\mu, h^\lambda] + [C_\mu, \partial_\nu h^\lambda] = [\partial_\nu C_\mu, h^\lambda] + [C_\mu, [C_\nu, h^\lambda]],$$

$$0 = (\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)h^\lambda = [\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu], h^\lambda]. \tag{29}$$

If an element of a Clifford algebra commutes with all generators (with h^μ, $\mu = 1, \dots, n$ in this case), then this element belongs to the center of this Clifford algebra. Therefore, (29) implies

$$\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] = c_{\mu\nu} e, \qquad \text{if } n = p + q \text{ is even},$$

$$\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] = c_{\mu\nu} e + d_{\mu\nu} e^1 \cdots e^n, \qquad \text{if } n = p + q \text{ is odd},$$

where $c_{\mu\nu}$, $d_{\mu\nu}$ are components of tensors of rank 2. Since $C_\mu \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$, then (by Theorem 3) $\partial_\mu C_\nu \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_2$. So

$$\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu] \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_2$$

and, hence, $c_{\mu\nu} = 0, d_{\mu\nu} = 0$. Eq. (28) is proved. Gauge invariance of Eq. (28) is proved by the formula

$$\partial_\mu \acute{C}_\nu - \partial_\nu \acute{C}_\mu - [\acute{C}_\mu, \acute{C}_\nu] = S^{-1}(\partial_\mu C_\nu - \partial_\nu C_\mu - [C_\mu, C_\nu])S. \qquad \square$$

7. General solution of the primitive field equation

In this section we find a general solution (up to elements of the center of Clifford algebra) of the primitive field equation (27).

THEOREM 8. *Suppose that n is a natural number and $C_\mu \in \mathcal{C}\ell_{\circledS}(p,q)\mathrm{T}_1$. Then the following two systems of equations are equivalent:*

$$\partial_\mu h_\rho - [C_\mu, h_\rho] = 0 \quad \Leftrightarrow \quad C_\mu = \sum_{k=1}^{\acute{n}} \mu_k \pi[h]_k((\partial_\mu h^\rho) h_\rho), \tag{30}$$

where $\acute{n} = n$ for even n, $\acute{n} = n - 1$ for odd n and

$$\mu_k = \frac{1}{n - (-1)^k (n - 2k)} = \frac{1}{n - \lambda_k}.$$

REMARK. Using formulae (22), we can rewrite the general solution (30) of the primitive field equation in the following form (we use contractions and do not use projection operators):

$$C_\mu = \sum_{k=1}^{n} \mu_k \sum_{l=0}^{n} b_{kl} F^l((\partial_\mu h^\rho) h_\rho) = \sum_{l=0}^{n} r_l F^l((\partial_\mu h^\rho) h_\rho), \quad r_l = \sum_{k=1}^{n} \mu_k b_{kl} \tag{31}$$

in the case of even n and

$$C_\mu = \sum_{k=1}^{n-1} \mu_k \sum_{l=0}^{\frac{n-1}{2}} g_{kl} F^l((\partial_\mu h^\rho) h_\rho) = \sum_{l=0}^{\frac{n-1}{2}} s_l F^l((\partial_\mu h^\rho) h_\rho), \quad s_l = \sum_{k=1}^{n-1} \mu_k g_{kl} \tag{32}$$

in the case of odd n.

On page 325 we write explicit formulae for solution of the primitive field equation in the cases of small dimensions $n = 2, 3, 4$.

Proof: Consider the decomposition of solution C_μ of system of equations (27),

$$C_\mu = \sum_{k=0}^{n} \pi[h]_k(C_\mu), \tag{33}$$

where $\pi[h]_k(C_\mu) \in \mathcal{C}\ell[h]_k(p,q)\mathrm{T}_1$. Multiply the left-hand side of Eq. (27) by h^ρ and consider the corresponding contraction (summation over index ρ): $h^\rho \partial_\mu h_\rho - h^\rho C_\mu h_\rho + h^\rho h_\rho C_\mu = 0$. Using formula (33) and formulae

$$h^\rho h_\rho = ne, \qquad h^\rho C_\mu h_\rho = \sum_{k=0}^{n} h^\rho \pi[h]_k(C_\mu) h_\rho = \sum_{k=0}^{n} (-1)^k (n-2k)\pi[h]_k(C_\mu),$$

we obtain

$$\sum_{k=0}^{n} (n - (-1)^k(n-2k))\pi[h]_k(C_\mu) = -h^\rho \partial_\mu h_\rho = (\partial_\mu h^\rho) h_\rho. \tag{34}$$

It is easy to see that $n - (-1)^k(n-2k) = 0$ holds for $k = 0$, $\forall n$ and for $k = n$, odd n. From (34) we obtain the required formula (27) for C_μ.

Now we shall prove that this expression for C_μ satisfies the primitive field equation.

Consider the following contractions $M^{a,s}_{(-1)^t}(U)$:

$$M^{a,s}_{1}(U, h_\nu) = h^{\mu_1} \cdots h^{\mu_s} h^{\rho_1} \cdots h^{\rho_a} U h_{\rho_a} \cdots h_{\rho_1} h_\nu h_{\mu_s} \cdots h_{\mu_1},$$
$$M^{a,s}_{-1}(U, h_\nu) = h^{\mu_1} \cdots h^{\mu_s} h_\nu h^{\rho_1} \cdots h^{\rho_a} U h_{\rho_a} \cdots h_{\rho_1} h_{\mu_s} \cdots h_{\mu_1}.$$

We contract an arbitrary element $U \in \mathcal{C}\ell(p,q)$ over $a + s$ indices. An element h_ν is on the right if $t = 0$ and on the left if $t = 1$. The number s is a distance between h_ν and the boundary of expression, the number a is a distance between h_ν and the center of expression.

LEMMA 1. *We have* $M^{a,s}_{(-1)^t}(U, h_\nu) = -M^{a-1,s+1}_{(-1)^t}(U, h_\nu) + 2M^{a-1,s}_{(-1)^{t+1}}(U, h_\nu)$.

Proof: In the case $t = 0$ we permute the neighbouring elements h_ν and h_{ρ_1} using $h_{\rho_1} h_\nu = -h_\nu h_{\rho_1} + 2\eta_{\nu\rho_1} e$ and obtain 2 another contractions from the statement. In the case $t = 1$ we use $h_\nu h^{\rho_1} = -h^{\rho_1} h_\nu + 2\eta^{\rho_1}_{\nu} e$. □

LEMMA 2. *We have* $M^{a,s}_{(-1)^t}(U, h_\nu) = \sum_{i=0}^{a} (-1)^i 2^{a-i} C^{a-i}_a M^{0,i+s}_{(-1)^{a-i+t}}(U, h_\nu)$.

Proof: We use the method of mathematical induction (over index a). For $a = 0$ we have $M^{0,s}_{(-1)^t}(U, h_\nu) = M^{0,s}_{(-1)^t}(U, h_\nu)$. Suppose that this formula is valid for some a. Let us prove the validity of this formula for $a + 1$. We have

$$
\begin{aligned}
M^{a+1,s}_{(-1)^t} &= -M^{a,s+1}_{(-1)^t} + 2M^{a,s}_{(-1)^{t+1}}\\
&= -\sum_{i=0}^{a}(-1)^i 2^{a-i} C_a^{a-i} M^{0,i+s+1}_{(-1)^{a-i+t}} + 2\sum_{i=0}^{a}(-1)^i 2^{a-i} C_a^{a-i} M^{0,i+s}_{(-1)^{a-i+t+1}}\\
&= \sum_{j=1}^{a+1}(-1)^j 2^{a-j+1} C_a^{a-j+1} M^{0,j+s}_{(-1)^{a-j+1+t}} + \sum_{i=0}^{a}(-1)^i 2^{a-i+1} C_a^{a-i} M^{0,i+s}_{(-1)^{a-i+t+1}}\\
&= \sum_{i=1}^{a}(-1)^i 2^{a+1-i}(C_a^{a-i+1} + C_a^{a-i}) M^{0,i+s}_{(-1)^{a-i+t+1}}\\
&\quad + (-1)^{a+1} M^{0,a+1-s}_{(-1)^t} + 2^{a+1} M^{0,s}_{(-1)^{a+t+1}}\\
&= \sum_{i=0}^{a+1}(-1)^i 2^{a+1-i} C_{a+1}^{a+1-i} M^{0,i+s}_{(-1)^{a+1-i+t}},
\end{aligned}
$$

where we use $C_n^{k+1}+C_n^k=C_{n+1}^{k+1}$ and we use the notation $M^{a,s}_{(-1)^t}(U,h_\nu)=M^{a,s}_{(-1)^t}$. $\square$

We continue the proof of the theorem in the case of even n. Let us substitute formulae (31) for C_μ in the primitive field equation,

$$
\partial_\mu h_\nu = \sum_{l=0}^{n} r_l F^l((\partial_\mu h^\rho)h_\rho)h_\nu - \sum_{l=0}^{n} r_l h_\nu F^l((\partial_\mu h^\rho)h_\rho).
$$

Using lemmata, we obtain

$$
\begin{aligned}
\partial_\mu h_\nu &= \sum_{l=0}^{n} r_l F^l((\partial_\mu h^\rho)h_\rho)h_\nu - \sum_{l=0}^{n} r_l h_\nu F^l((\partial_\mu h^\rho)h_\rho)\\
&= \sum_{l=0}^{n} r_l (M_1^{l,0}((\partial_\mu h^\rho)h_\rho, h_\nu) - M_{-1}^{l,0}((\partial_\mu h^\rho)h_\rho, h_\nu))\\
&= \sum_{l=0}^{n} r_l \sum_{i=0}^{l}(-1)^i 2^{l-i} C_l^{l-i}(M^{0,i}_{(-1)^{l-i}}((\partial_\mu h^\rho)h_\rho, h_\nu) - M^{0,i}_{(-1)^{l-i+1}}((\partial_\mu h^\rho)h_\rho, h_\nu)).
\end{aligned}
$$

We have

$$
\begin{aligned}
&M^{0,i}_{(-1)^{l-i}}((\partial_\mu h^\rho)h_\rho, h_\nu) - M^{0,i}_{(-1)^{l-i+1}}((\partial_\mu h^\rho)h_\rho, h_\nu)\\
&\qquad = (-1)^{l-i} h^{b_1}\cdots h^{b_i}((\partial_\mu h^\rho)h_\rho h_\nu - h_\nu(\partial_\mu h^\rho)h_\rho)h_{b_i}\cdots h_{b_1}\\
&\qquad = (-1)^{l-i} F^i((\partial_\mu h^\rho)h_\rho h_\nu - h_\nu(\partial_\mu h^\rho)h_\rho)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\partial_\mu h^\rho)h_\rho h_\nu - h_\nu(\partial_\mu h^\rho)h_\rho = (\partial_\mu h^\rho)(-h_\nu h_\rho + 2\eta_{\nu\rho}e) + h_\nu h_\rho(\partial_\mu h^\rho)\\
&\quad = -(\partial_\mu h^\rho)h_\nu h_\rho + 2\partial_\mu h_\nu + (-h_\rho h_\nu + 2\eta_{\rho\nu}e)(\partial_\mu h^\rho)\\
&\quad = 4\partial_\mu h_\nu - ((\partial_\mu h_\rho)h_\nu h^\rho + h_\rho h_\nu(\partial_\mu h^\rho)) = 4\partial_\mu h_\nu - (\partial_\mu(h_\rho h_\nu h^\rho) - h_\rho \partial_\mu(h_\nu)h^\rho)\\
&\quad = 4\partial_\mu h_\nu - ((2-n)\partial_\mu h_\nu - h_\rho\partial_\mu(h_\nu)h^\rho) = (2+n)\partial_\mu h_\nu + h_\rho(\partial_\mu h_\nu)h^\rho.
\end{aligned}
$$

Then

$$M^{0,i}_{(-1)^{l-i}}((\partial_\mu h^\rho)h_\rho, h_\nu) - M^{0,i}_{(-1)^{l-i+1}}((\partial_\mu h^\rho)h_\rho, h_\nu)$$
$$= (-1)^{l-i} F^i((2+n)\partial_\mu h_\nu + h_\rho(\partial_\mu h_\nu)h^\rho)$$
$$= (-1)^{l-i} \sum_{m=0}^{n} \lambda_m^i (2+n+\lambda_m)\pi[h]_m(\partial_\mu h_\nu),$$

where $\lambda_m = (-1)^m(n-2m)$. So

$$\partial_\mu h_\nu = \sum_{l=0}^{n} r_l \sum_{i=0}^{l} (-1)^l 2^{l-i} C_l^{l-i} \sum_{m=0}^{n} \lambda_m^i (2+n+\lambda_m)\pi[h]_m(\partial_\mu h_\nu),$$

where

$$r_l = \sum_{k=1}^{n} \mu_k b_{kl} = \sum_{k=1}^{n} \frac{1}{n-\lambda_k} b_{kl}$$

and $B = ||b_{kl}||$ is inverse of the Vandermonde matrix.

We change the index $j = l - i$ and change the order of summation

$$\partial_\mu h_\nu = \sum_{m=0}^{n} (2+n+\lambda_m)\pi[h]_m(\partial_\mu h_\nu) \sum_{k=1}^{n} \frac{1}{n-\lambda_k} \sum_{l=0}^{n} b_{kl}(-1)^l \sum_{j=0}^{l} 2^j C_l^j \lambda_m^{l-j}$$
$$= \sum_{m=0}^{n} (2+n+\lambda_m)\pi[h]_m(\partial_\mu h_\nu) \sum_{k=1}^{n} \frac{1}{n-\lambda_k} \sum_{l=0}^{n} b_{kl}(-1)^l (2+\lambda_m)^l.$$

Further we consider the sum over m starting with $m = 1$ because $\pi[h]_0(\partial_\mu h_\nu) = 0$.

We have $-2 - \lambda_m = \lambda_{m+(-1)^{m+1}}$, $1 \le m \le n$. Indeed, in the cases of even and odd m we have, respectively,

$$-2 - \lambda_m = -2 - (n-2m) = -2 - n + 2m = -(n - 2(m-1)) = \lambda_{m-1} = \lambda_{m+(-1)^{m+1}},$$
$$-2 - \lambda_m = -2 + (n-2m) = -2 + n - 2m = n - 2(m+1) = \lambda_{m+1} = \lambda_{m+(-1)^{m+1}}.$$

Using $\sum_{l=0}^{n} b_{kl}(\lambda_a)^l = \delta_{k,a}$, we obtain

$$\partial_\mu h_\nu = \sum_{m=1}^{n} (2+n+\lambda_m)\pi[h]_m(\partial_\mu h_\nu) \sum_{k=1}^{n} \frac{1}{n-\lambda_k} \sum_{l=0}^{n} b_{kl}(\lambda_{m+(-1)^{m+1}})^l$$
$$= \sum_{m=1}^{n} (2+n+\lambda_m)\pi[h]_m(\partial_\mu h_\nu) \sum_{k=1}^{n} \frac{\delta_{k,m+(-1)^{m+1}}}{n-\lambda_k}$$
$$= \sum_{m=1}^{n} \frac{(2+n+\lambda_m)\pi[h]_m(\partial_\mu h_\nu)}{n-\lambda_{m+(-1)^{m+1}}} = \sum_{m=1}^{n} \pi[h]_m(\partial_\mu h_\nu).$$

This completes the proof of theorem for the case of even n.

Let us prove the theorem in the case of odd n. In this case we have $\lambda_k = \lambda_{n-k}$, hence $\mu_k = \mu_{n-k}$.

We have

$$C_\mu = \sum_{k=1}^{\frac{n-1}{2}} \mu_k \pi[h]_{k,n-k}((\partial_\mu h^\rho)h_\rho), \qquad \pi[h]_{k,n-k}(U) = \pi[h]_k(U) + \pi[h]_{n-k}(U),$$

$$C_\mu = \sum_{k=1}^{\frac{n-1}{2}} \mu_k \sum_{l=0}^{\frac{n-1}{2}} g_{kl} F^l((\partial_\mu h^\rho)h_\rho) = \sum_{l=0}^{\frac{n-1}{2}} s_l F^l((\partial_\mu h^\rho)h_\rho), \qquad s_l = \sum_{k=1}^{\frac{n-1}{2}} \mu_k g_{kl}.$$

Substitute this expression for C_μ in the primitive field equation and obtain

$$\partial_\mu h_\nu = \sum_{l=0}^{\frac{n-1}{2}} s_l F^l((\partial_\mu h^\rho)h_\rho)h_\nu - \sum_{l=0}^{\frac{n-1}{2}} s_l h_\nu F^l((\partial_\mu h^\rho)h_\rho).$$

Similarly to the case of even n we get

$$\partial_\mu h_\nu = \sum_{l=0}^{\frac{n-1}{2}} s_l \sum_{i=0}^{l} (-1)^l 2^{l-i} C_l^{l-i} \sum_{m=0}^{\frac{n-1}{2}} \lambda_m^i (2+n+\lambda_m)\pi[h]_{m,n-m}(\partial_\mu h_\nu),$$

where

$$s_l = \sum_{k=1}^{\frac{n-1}{2}} \mu_k g_{kl} = \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{n-\lambda_k} g_{kl}$$

and $G = ||g_{kl}||$ is inverse of the Vandermonde matrix.

Further we consider the sum over m starting with $m = 1$ because $\pi[h]_{0,n}(\partial_\mu h_\nu) = 0$. We change the index $j = l - i$ and change the order of summation,

$$\begin{aligned}\partial_\mu h_\nu &= \sum_{m=1}^{\frac{n-1}{2}} (2+n+\lambda_m)\pi[h]_{m,n-m}(\partial_\mu h_\nu) \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{n-\lambda_k} \sum_{l=0}^{\frac{n-1}{2}} (-1)^l g_{kl} \sum_{j=0}^{l} 2^j C_l^j \lambda_m^{l-j} \\ &= \sum_{m=1}^{\frac{n-1}{2}} (2+n+\lambda_m)\pi[h]_{m,n-m}(\partial_\mu h_\nu) \sum_{k=1}^{\frac{n-1}{2}} \frac{1}{n-\lambda_k} \sum_{l=0}^{\frac{n-1}{2}} (-1)^l g_{kl}(2+\lambda_m)^l.\end{aligned}$$

Using $-2-\lambda_m = \lambda_{m+(-1)^{m+1}}$, $1 \le m \le n$ and $\sum_{l=0}^{\frac{n-1}{2}} g_{kl}(\lambda_a)^l = \delta_{k,a}$, we get

$$\begin{aligned}\partial_\mu h_\nu &= \sum_{m=1}^{\frac{n-1}{2}} (2+n+\lambda_m)\pi[h]_{m,n-m}(\partial_\mu h_\nu) \sum_{k=1}^{\frac{n-1}{2}} \frac{\delta_{k,m+(-1)^{m+1}}}{n-\lambda_k} \\ &= \sum_{m=1}^{\frac{n-1}{2}} \frac{(2+n+\lambda_m)\pi[h]_{m,n-m}(\partial_\mu h_\nu)}{n-\lambda_{m+(-1)^{m+1}}} = \sum_{m=1}^{\frac{n-1}{2}} \pi[h]_{m,n-m}(\partial_\mu h_\nu).\end{aligned}$$

So, in the case of odd n the theorem is also proved. □

In the case of small dimensions $n = 2, 3, 4$ the expressions C_μ from the formula (30) have the following explicit form.

In the case $n = 2$,

$$
\begin{aligned}
C_\mu &= \sum_{k=1}^{2} \mu_k \pi[h]_k((\partial_\mu h^\rho)h_\rho) = \frac{1}{2}\pi[h]_1((\partial_\mu h^\rho)h_\rho) + \frac{1}{4}\pi[h]_2((\partial_\mu h^\rho)h_\rho) \\
&= \frac{1}{2}(\partial_\mu h^\rho)h_\rho - \frac{1}{16}h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha - \frac{3}{32}h^\beta h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha h_\beta.
\end{aligned}
$$

In the case $n = 3$,

$$
\begin{aligned}
C_\mu &= \sum_{k=1}^{2} \mu_k \pi[h]_k((\partial_\mu h^\rho)h_\rho) = \frac{1}{4}\pi[h]_1((\partial_\mu h^\rho)h_\rho) + \frac{1}{4}\pi[h]_2((\partial_\mu h^\rho)h_\rho) \\
&= \frac{1}{4}\pi[h]_{12}((\partial_\mu h^\rho)h_\rho) = \frac{3}{16}(\partial_\mu h^\rho)h_\rho - \frac{1}{16}h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha.
\end{aligned}
$$

In the case $n = 4$,

$$
\begin{aligned}
C_\mu &= \sum_{k=1}^{4} \mu_k \pi[h]_k((\partial_\mu h^\rho)h_\rho) = \frac{1}{6}\pi[h]_1((\partial_\mu h^\rho)h_\rho) \\
&\quad + \frac{1}{4}\pi[h]_2((\partial_\mu h^\rho)h_\rho) + \frac{1}{2}\pi[h]_3((\partial_\mu h^\rho)h_\rho) + \frac{1}{8}\pi[h]_4((\partial_\mu h^\rho)h_\rho) \\
&= \frac{1}{4}(\partial_\mu h^\rho)h_\rho + \frac{67}{576}h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha + \frac{73}{2304}h^\beta h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha h_\beta \\
&\quad - \frac{19}{2304}h^\gamma h^\beta h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha h_\beta h_\gamma - \frac{25}{9216}h^\delta h^\gamma h^\beta h^\alpha(\partial_\mu h^\rho)h_\rho h_\alpha h_\beta h_\gamma h_\delta.
\end{aligned}
$$

8. Conclusion

We have invented a class of *primitive field equations* (27), which depend on the real Lie algebra of a Lie group G. Eq. (27) has gauge symmetry w.r.t. the Lie group G. Also we have discussed related mathematical structures— the Lie groups and Lie algebras in the Clifford algebra, tensor fields with values in the Clifford algebra, Clifford field vectors, an algebra of h-forms and so on. Also we have developed techniques needed to solve primitive fields equations—theory of h-forms, a method of calculation of projection operators onto vector subspaces of h-forms of different ranks using the inverse of Vandermonde matrices, etc. We gave general solutions of primitive field equations. In particular, we gave explicit formulae for solutions in cases of small dimensions $n = 2, 3, 4$.

Primitive field equations are models of Yang–Mills equations with a Clifford field vector as a current in the right-hand part [8]. A necessity to solve the primitive field equations arises, in particular, in model equations of field theory [7] and in the theory of Dirac equation on curved pseudo-Riemannian manifolds [12]. Presented in this article results on primitive field equations show us a direction for further investigation—to find new classes of solutions of Yang–Mills equations.

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