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Basis-free Solution to Sylvester Equation in Clifford Algebra of Arbitrary Dimension

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Abstract. The Sylvester equation and its particular case, the Lyapunov equation, are widely used in image processing, control theory, stability analysis, signal processing, model reduction, and many more. We present basis-free solution to the Sylvester equation in Clifford (geometric) algebra of arbitrary dimension. The basis-free solutions involve only the operations of Clifford (geometric) product, summation, and the operations of conjugation. To obtain the results, we use the concepts of characteristic polynomial, determinant, adjugate, and inverse in Clifford algebras. For the first time, we give alternative formulas for the basis-free solution to the Sylvester equation in the case $n = 4$, the proofs for the case $n = 5$ and the case of arbitrary dimension n. The results can be used in symbolic computation.

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Keywords. Clifford algebra, Geometric algebra, Sylvester equation, Lyapunov equation, characteristic polynomial, Basis-free solution.

1. Introduction

This paper is an extended version of the short note in Conference Proceedings [24]. We present for the first time the alternative formulas for the basis-free solution to the Sylvester equation in the case $n = 4$ (see the remarks after Theorem 3.1), the proofs of Theorems 4.1 and 5.1, and the simplification of the statement of Theorem 5.1 in the case of odd $n = p + q$ (see the remarks after Theorem 5.1).

The Sylvester equation [26] is a linear equation of the form $AX - XB = C$ for known A, B, C (quaternions, matrices, or multivectors depending on the formalism) and unknown X. The Sylvester equation and its particular

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case, the Lyapunov equation (with $B = -A^H$), are widely used in different applications – image processing, control theory, stability analysis, signal processing, model reduction, and many more. In this paper, we study the Sylvester equation in Clifford's geometric algebra $C\ell_{p,q}$ and present basis-free solution to this equation in the case of arbitrary $n = p + q$.

The Sylvester equation over quaternions corresponds to the Sylvester equation in geometric algebra of a vector space of dimension $n = 2$, because we have the isomorphism $C\ell_{0,2} \cong \mathbb{H}$. Thus the basis-free solution to the Sylvester equation in $C\ell_{p,q}$, $p + q = 2$, is constructed similarly to the basis-free solution to the Sylvester equation over quaternions. The same ideas as in the case $n = 2$ work in the case $n = 3$. The cases $n \leq 3$ are also discussed by Acus and Dargys [5].

In this paper, we present basis-free solutions in the cases $n = 4$ and $n = 5$, which are the most important cases for the applications. The geometric algebra $C\ell_{1,3}$ (the space-time algebra [9]) of a space of dimension 4 is widely used in physics. The conformal geometric algebra $C\ell_{4,1}$ of a space of dimension 5 is widely used in geometry, robotics, and computer vision (see [3,7,11,12,17]). Also we present recursive basis-free formulas to the Sylvester equation in $C\ell_{p,q}$ in the case of arbitrary $n = p + q$. They can be used in symbolic computation. We use our previous results on explicit and recursive formulas for the characteristic polynomial coefficients and inverse in Clifford algebras [25]. Note also the papers on the characteristic polynomial [8] and inverse [2,4,13,14,23].

An arbitrary linear quaternion equation with two terms

$$KXL + MXN = P \tag{1.1}$$

for known $K, L, M, N, P \in \mathbb{H}$ and unknown $X \in \mathbb{H}$ can be reduced to the Sylvester equation. Any nonzero quaternion $Q = a + bi + cj + dk \neq 0$, where $a, b, c, d \in \mathbb{R}$ are real numbers, and i, j, and k are the quaternion units, is invertible and the inverse is equal to

$$Q^{-1} = \frac{\overline{Q}}{Q\overline{Q}},$$

where $\overline{Q} := a - bi - cj - dk$ is the conjugate of Q. Multiplying both sides of (1.1) on the left by M^{-1} and on the right by L^{-1}, we obtain $M^{-1}KX + XNL^{-1} = M^{-1}PL^{-1}$. Denoting $A := M^{-1}K$, $B := -NL^{-1}$, $C := M^{-1}PL^{-1}$, we get the Sylvester equation

$$AX - XB = C \tag{1.2}$$

for known $A, B, C \in \mathbb{H}$ and unknown $X \in \mathbb{H}$ (see also [15,21]).

Multiplying both sides of (1.2) on the right by $-\overline{B}$, we get

$$-AX\overline{B} + XB\overline{B} = -C\overline{B}. \tag{1.3}$$

Multiplying both sides of (1.2) on the left by A, we get

$$A^2X - AXB = AC. \tag{1.4}$$

Summing (1.3) and (1.4) and using $B+\overline{B}\in\mathbb{R}$, $B\overline{B}\in\mathbb{R}$, we obtain

$$A^2X-(B+\overline{B})AX+B\overline{B}X=AC-C\overline{B}.$$

If

$$D:=A^2-BA-\overline{B}A+B\overline{B}\neq 0,$$

then D is invertible and we get the basis-free solution to (1.2):

$$X=D^{-1}(AC-C\overline{B})=\frac{\overline{D}(AC-C\overline{B})}{D\overline{D}}.$$

2. The Cases $n\leq 3$

Let us consider the Clifford's geometric algebra $C\ell_{p,q}$, $p+q=n$, [6,10,16,18,19,22] with the identity element e and the generators e_a, $a=1,\ldots,n$, satisfying

$$e_ae_b+e_be_a=2\eta_{ab}e,\qquad a,b=1,\ldots,n,$$

where $\eta=(\eta_{ab})=\operatorname{diag}(1,\ldots,1,-1,\ldots,-1)$ is the diagonal matrix with its first p entries equal to 1 and the last q entries equal to -1 on the diagonal. We call the subspace of $C\ell_{p,q}$ of geometric algebra elements, which are linear combinations of basis elements with multi-indices of length k, the subspace of grade k and denote it by $C\ell^k_{p,q}$, $k=0,1,\ldots,n$. We identify elements of the subspace of grade 0 with scalars: $C\ell^0_{p,q}\equiv\mathbb{R}$, $e\equiv 1$. Denote the operation of projection onto the subspace $C\ell^k_{p,q}$ by $\langle\quad\rangle_k$. The center of $C\ell_{p,q}$ is $\operatorname{cen}(C\ell_{p,q})=C\ell^0_{p,q}$ in the case of even n and $\operatorname{cen}(C\ell_{p,q})=C\ell^0_{p,q}\oplus C\ell^n_{p,q}$ in the case of odd n.

We use the following two standard operations of conjugation in $C\ell_{p,q}$: the grade involution $\widehat{\ }$ and the reversion (an anti-involution) $\widetilde{\ }$

$$\widehat{U}=\sum_{k=0}^{n}(-1)^k\langle U\rangle_k,\qquad \widehat{UV}=\widehat{U}\widehat{V},\qquad \forall U,V\in C\ell_{p,q},\tag{2.1}$$

$$\widetilde{U}=\sum_{k=0}^{n}(-1)^{\frac{k(k-1)}{2}}\langle U\rangle_k,\qquad \widetilde{UV}=\widetilde{V}\widetilde{U},\qquad \forall U,V\in C\ell_{p,q}.\tag{2.2}$$

Let us consider the Sylvester equation in geometric algebra

$$AX-XB=C\tag{2.3}$$

for known $A,B,C\in C\ell_{p,q}$ and unknown $X\in C\ell_{p,q}$.

In the case $n=1$, the geometric algebra $C\ell_{p,q}$ is commutative and we get $(A-B)X=C$. Denoting $D:=A-B$ and using[1]

$$\operatorname{Adj}(D)=\widehat{D},\qquad \operatorname{Det}(D)=D\widehat{D}\in C\ell^0_{p,q}\equiv\mathbb{R},\qquad D^{-1}=\frac{\operatorname{Adj}(D)}{\operatorname{Det}(D)},$$

we conclude that if

$$Q:=D\widehat{D}\neq 0,\tag{2.4}$$

[1]The definitions of adjugate $\operatorname{Adj}(D)$, determinant $\operatorname{Det}(D)$, and inverse D^{-1} in $C\ell_{p,q}$ for an arbitrary n are given in [25].

then

$$X = \frac{\widehat{D}C}{Q}.$$

In the case $n = 2$, we can do the same as for the Sylvester equation over quaternions (see Introduction). Multiplying both sides of (2.3) on the right by $-\widehat{\widetilde{B}}$ and on the left by A, we get

$$-AX\widehat{\widetilde{B}} + XB\widehat{\widetilde{B}} = -C\widehat{\widetilde{B}}, \qquad A^2X - AXB = AC.$$

Summing and using $\mathrm{Det}(B) = B\widehat{\widetilde{B}} \in C\ell^0_{p,q} \equiv \mathbb{R}$, $B + \widehat{\widetilde{B}} \in C\ell^0_{p,q} \equiv \mathbb{R}$, we get

$$(A^2 - (B + \widehat{\widetilde{B}})A + B\widehat{\widetilde{B}})X = AC - C\widehat{\widetilde{B}}. \tag{2.5}$$

Using

$$\mathrm{Adj}(D) = \widehat{\widetilde{D}}, \qquad \mathrm{Det}(D) = D\widehat{\widetilde{D}} \in C\ell^0_{p,q} \equiv \mathbb{R}, \qquad D^{-1} = \frac{\mathrm{Adj}(D)}{\mathrm{Det}(D)},$$

we conclude that if

$$Q := D\widehat{\widetilde{D}} \neq 0, \tag{2.6}$$

then for $D := A^2 - (B + \widehat{\widetilde{B}})A + B\widehat{\widetilde{B}}$, we get

$$X = \frac{\widehat{\widetilde{D}}(AC - C\widehat{\widetilde{B}})}{Q}.$$

In the case $n = 3$, we have $B\widehat{\widetilde{B}} \in C\ell^0_{p,q} \oplus C\ell^3_{p,q} = \mathrm{cen}(C\ell_{p,q})$ and $B + \widehat{\widetilde{B}} \in C\ell^0_{p,q} \oplus C\ell^3_{p,q} = \mathrm{cen}(C\ell_{p,q})$ and obtain again (2.5). Using

$$\mathrm{Adj}(D) = \widehat{D}\widetilde{D}\widehat{\widetilde{D}}, \qquad \mathrm{Det}(D) = D\widehat{D}\widetilde{D}\widehat{\widetilde{D}} \in C\ell^0_{p,q} \equiv \mathbb{R}, \qquad D^{-1} = \frac{\mathrm{Adj}(D)}{\mathrm{Det}(D)}$$

for $D := A^2 - (B + \widehat{\widetilde{B}})A + B\widehat{\widetilde{B}}$, we conclude that if

$$Q := D\widehat{D}\widetilde{D}\widehat{\widetilde{D}} \neq 0, \tag{2.7}$$

then

$$X = \frac{\widehat{D}\widetilde{D}\widehat{\widetilde{D}}(AC - C\widehat{\widetilde{B}})}{Q}. \tag{2.8}$$

3. The Case $n = 4$

Let us consider one additional operation of conjugation $\triangle$ (compare with the grade involution (2.1) and the reversion (2.2), see also [25])

$$\begin{aligned} U^{\triangle} &= \sum_{k=0}^{n} (-1)^{\frac{k(k-1)(k-2)(k-3)}{4!}} \langle U \rangle_k \\ &= \sum_{k=0,1,2,3 \bmod 8} \langle U \rangle_k - \sum_{k=4,5,6,7 \bmod 8} \langle U \rangle_k, \qquad \forall U \in C\ell_{p,q}. \end{aligned} \tag{3.1}$$

In the general case, we have $(UV)^\triangle \neq U^\triangle V^\triangle$ and $(UV)^\triangle \neq V^\triangle U^\triangle$.

Theorem 3.1. *Let us consider the Sylvester equation in* $C\ell_{p,q}$, $p+q=4$

$$AX - XB = C, \tag{3.2}$$

for known $A, B, C \in C\ell_{p,q}$ *and unknown* $X \in C\ell_{p,q}$.

If

$$Q := D\widehat{\widetilde{D}}(\widehat{D}\widetilde{D})^\triangle \neq 0, \tag{3.3}$$

then

$$X = \frac{\widehat{\widetilde{D}}(\widehat{D}\widetilde{D})^\triangle F}{Q}, \tag{3.4}$$

where

$$\begin{aligned} D &:= A^4 - A^3(B + \widehat{\widetilde{B}} + \widehat{B}^\triangle + \widetilde{B}^\triangle) \\ &\quad + A^2(B\widehat{\widetilde{B}} + B\widehat{B}^\triangle + B\widetilde{B}^\triangle + \widehat{\widetilde{B}}\widehat{B}^\triangle + \widehat{\widetilde{B}}\widetilde{B}^\triangle + (\widehat{B}\widetilde{B})^\triangle) \\ &\quad - A(B\widehat{\widetilde{B}}\widehat{B}^\triangle + B\widehat{\widetilde{B}}\widetilde{B}^\triangle + B(\widehat{B}\widetilde{B})^\triangle + \widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^\triangle) + B\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^\triangle, \end{aligned} \tag{3.5}$$

$$\begin{aligned} F &:= A^3C - A^2C(\widehat{\widetilde{B}} + \widehat{B}^\triangle + \widetilde{B}^\triangle) \\ &\quad + AC(\widehat{\widetilde{B}}\widehat{B}^\triangle + \widehat{\widetilde{B}}\widetilde{B}^\triangle + (\widehat{B}\widetilde{B})^\triangle) - C\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^\triangle. \end{aligned} \tag{3.6}$$

As one of the anonymous reviewers of this paper noted, the formulas (3.3), (3.4), (3.5), and (3.6) can be rewritten using the new operation

$$B^\natural := (\widehat{B}\widetilde{B})^\triangle \tag{3.7}$$

in the following form

$$\begin{aligned} &Q = D\widehat{\widetilde{D}}D^\natural, \qquad X = \frac{\widehat{\widetilde{D}}D^\natural F}{Q}, \\ &D = A^4 - A^3(B + \widehat{\widetilde{B}} + \widehat{B}^\triangle + \widetilde{B}^\triangle) + A^2(B\widehat{\widetilde{B}} + B\widehat{B}^\triangle + B\widetilde{B}^\triangle + \widehat{\widetilde{B}}\widehat{B}^\triangle \\ &\quad + \widehat{\widetilde{B}}\widetilde{B}^\triangle + B^\natural) - A(B\widehat{\widetilde{B}}\widehat{B}^\triangle + B\widehat{\widetilde{B}}\widetilde{B}^\triangle + BB^\natural + \widehat{\widetilde{B}}B^\natural) + B\widehat{\widetilde{B}}B^\natural, \\ &F = A^3C - A^2C(\widehat{\widetilde{B}} + \widehat{B}^\triangle + \widetilde{B}^\triangle) + AC(\widehat{\widetilde{B}}\widehat{B}^\triangle + \widehat{\widetilde{B}}\widetilde{B}^\triangle + B^\natural) - C\widehat{\widetilde{B}}B^\natural. \end{aligned}$$

Proof. Multiplying both sides of (3.2) on the right by $-\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^\triangle$, we get

$$-AX\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^\triangle + XB\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^\triangle = -C\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^\triangle. \tag{3.8}$$

Multiplying both sides of (3.2) on the right by $\widehat{\widetilde{B}}\widehat{B}^\triangle + \widehat{\widetilde{B}}\widetilde{B}^\triangle + (\widehat{B}\widetilde{B})^\triangle$ and on the left by A, we get

$$\begin{aligned} &A^2X(\widehat{\widetilde{B}}\widehat{B}^\triangle + \widehat{\widetilde{B}}\widetilde{B}^\triangle + (\widehat{B}\widetilde{B})^\triangle) - AXB(\widehat{\widetilde{B}}\widehat{B}^\triangle + \widehat{\widetilde{B}}\widetilde{B}^\triangle + (\widehat{B}\widetilde{B})^\triangle) \\ &\quad = AC(\widehat{\widetilde{B}}\widehat{B}^\triangle + \widehat{\widetilde{B}}\widetilde{B}^\triangle + (\widehat{B}\widetilde{B})^\triangle). \end{aligned} \tag{3.9}$$

Multiplying both sides of (3.2) on the right by $-(\widehat{\widetilde{B}} + \widehat{B}^{\triangle} + \widetilde{B}^{\triangle})$ and on the left by A^2, we get

$$\begin{aligned}
&-A^3X(\widehat{\widetilde{B}} + \widehat{B}^{\triangle} + \widetilde{B}^{\triangle}) + A^2XB(\widehat{\widetilde{B}} + \widehat{B}^{\triangle} + \widetilde{B}^{\triangle})\\
&\quad = -A^2C(\widehat{\widetilde{B}} + \widehat{B}^{\triangle} + \widetilde{B}^{\triangle}).
\end{aligned} \tag{3.10}$$

Multiplying both sides of (3.2) on the left by A^3, we get

$$A^4X - A^3XB = A^3C. \tag{3.11}$$

Summing (3.8), (3.9), (3.10), and (3.11), and using the following explicit formulas for the characteristic polynomial coefficients from [25]

$$\begin{aligned}
b_{(1)} &:= B + \widehat{\widetilde{B}} + \widehat{B}^{\triangle} + \widetilde{B}^{\triangle} \in C\ell^0_{p,q},\\
b_{(2)} &:= -(B\widehat{\widetilde{B}} + B\widehat{B}^{\triangle} + B\widetilde{B}^{\triangle} + \widehat{\widetilde{B}}\widehat{B}^{\triangle} + \widehat{\widetilde{B}}\widetilde{B}^{\triangle} + (\widehat{B}\widetilde{B})^{\triangle}) \in C\ell^0_{p,q},\\
b_{(3)} &:= B\widehat{\widetilde{B}}\widehat{B}^{\triangle} + B\widehat{\widetilde{B}}\widetilde{B}^{\triangle} + B(\widehat{B}\widetilde{B})^{\triangle} + \widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^{\triangle} \in C\ell^0_{p,q},\\
b_{(4)} &:= -\mathrm{Det}(B) = -B\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^{\triangle} \in C\ell^0_{p,q},
\end{aligned}$$

we get

$$\begin{aligned}
&(A^4 - A^3(B + \widehat{\widetilde{B}} + \widehat{B}^{\triangle} + \widetilde{B}^{\triangle})\\
&\quad + A^2(B\widehat{\widetilde{B}} + B\widehat{B}^{\triangle} + B\widetilde{B}^{\triangle} + \widehat{\widetilde{B}}\widehat{B}^{\triangle} + \widehat{\widetilde{B}}\widetilde{B}^{\triangle} + (\widehat{B}\widetilde{B})^{\triangle})\\
&\quad - A(B\widehat{\widetilde{B}}\widehat{B}^{\triangle} + B\widehat{\widetilde{B}}\widetilde{B}^{\triangle} + B(\widehat{B}\widetilde{B})^{\triangle} + \widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^{\triangle}) + B\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^{\triangle})X\\
&\quad = A^3C - A^2C(\widehat{\widetilde{B}} + \widehat{B}^{\triangle} + \widetilde{B}^{\triangle})\\
&\qquad + AC(\widehat{\widetilde{B}}\widehat{B}^{\triangle} + \widehat{\widetilde{B}}\widetilde{B}^{\triangle} + (\widehat{B}\widetilde{B})^{\triangle}) - C\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^{\triangle}.
\end{aligned}$$

Denoting (3.5) and (3.6), and using the formula for the inverse in $C\ell_{p,q}$ with $n = p + q = 4$

$$\mathrm{Adj}(D) = \widehat{\widetilde{D}}(\widehat{D}\widetilde{D})^{\triangle}, \qquad \mathrm{Det}(D) = D\widehat{\widetilde{D}}(\widehat{D}\widetilde{D})^{\triangle}, \qquad D^{-1} = \frac{\mathrm{Adj}(D)}{\mathrm{Det}(D)},$$

we obtain (3.4). □

Let us present other formulas for the characteristic polynomial coefficients $b_{(1)}$, $b_{(2)}$, $b_{(3)}$, $b_{(4)}$ in the case $n = 4$. We use the same expressions in the case $n = 5$ (see Theorem 4.1). We have

$$\begin{aligned}
B_{(1)} &:= B,\\
b_{(1)} &= 4\langle B_{(1)}\rangle_0 = B + \widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle},\\
B_{(2)} &:= B(B - b_{(1)}) = -B(\widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle}),\\
b_{(2)} &= 2\langle B_{(2)}\rangle_0 = -2\langle B(\widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle})\rangle_0\\
&= -\frac{1}{2}(B\widetilde{B} + B\widehat{B}^{\triangle} + B\widetilde{\widehat{B}}^{\triangle} + B\widetilde{B} + \widetilde{\widehat{B}}^{\triangle}\widetilde{B} + \widehat{B}^{\triangle}\widetilde{B} + (\widehat{B}\widetilde{\widehat{B}})^{\triangle}\\
&\qquad + (\widehat{B}B^{\triangle})^{\triangle} + (\widehat{B}\widetilde{B}^{\triangle})^{\triangle} + (\widehat{B}\widetilde{\widehat{B}})^{\triangle} + (\widetilde{B}^{\triangle}\widetilde{\widehat{B}})^{\triangle} + (B^{\triangle}\widetilde{\widehat{B}})^{\triangle})
\end{aligned}$$

$$
\begin{aligned}
&= -(B\widetilde{B} + B\widehat{B}^{\triangle} + B\widetilde{\widehat{B}}^{\triangle} + \widetilde{B}\widehat{B}^{\triangle} + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + (\widehat{B}\widetilde{\widehat{B}})^{\triangle}),\\
B_{(3)} &:= B(B_{(2)} - b_{(2)}) = B(\widetilde{B}\widehat{B}^{\triangle} + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + (\widehat{B}\widetilde{\widehat{B}})^{\triangle}),\\
b_{(3)} &= \frac{4}{3}\langle B_{(3)}\rangle_0 = \frac{1}{3}(B\widetilde{B}\widehat{B}^{\triangle} + B\widetilde{B}\widetilde{\widehat{B}}^{\triangle} + B(\widehat{B}\widetilde{\widehat{B}})^{\triangle} + \widetilde{\widehat{B}}^{\triangle}B\widetilde{B}\\
&\quad + \widehat{B}^{\triangle}B\widetilde{B} + (\widehat{B}\widetilde{\widehat{B}})^{\triangle}\widetilde{B} + (\widehat{B}\widetilde{\widehat{B}}B^{\triangle})^{\triangle} + (\widehat{B}\widetilde{\widehat{B}}\widetilde{B}^{\triangle})^{\triangle}\\
&\quad + (\widehat{B}(B\widetilde{B})^{\triangle})^{\triangle} + (\widetilde{B}^{\triangle}\widehat{B}\widetilde{\widehat{B}})^{\triangle} + (B^{\triangle}\widehat{B}\widetilde{\widehat{B}})^{\triangle} + ((B\widetilde{B})^{\triangle}\widetilde{\widehat{B}})^{\triangle})\\
&= B\widetilde{B}\widehat{B}^{\triangle} + B\widetilde{B}\widetilde{\widehat{B}}^{\triangle} + B(\widehat{B}\widetilde{\widehat{B}})^{\triangle} + \widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle},\\
\mathrm{Det}(B) &= -B_{(4)} := B(b_{(3)} - B_{(3)}) = B\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle} = -b_{(4)},
\end{aligned}
$$

where we used two times computer calculations[2] to simplify the expressions for $b_{(2)}$ and $b_{(3)}$, because of nontrivial properties of the operation $\triangle$.

Instead of (3.4), we obtain another equivalent form of basis-free solution to the Sylvester equation in the case $n = 4$. If

$$Q := D\widetilde{D}(\widehat{D}\widetilde{\widehat{D}})^{\triangle} \neq 0, \tag{3.12}$$

then

$$X = \frac{\widetilde{D}(\widehat{D}\widetilde{\widehat{D}})^{\triangle}F}{Q}, \tag{3.13}$$

where

$$
\begin{aligned}
D := {}& A^4 - A^3(B + \widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle})\\
&+ A^2(B\widetilde{B} + B\widehat{B}^{\triangle} + B\widetilde{\widehat{B}}^{\triangle} + \widetilde{B}\widehat{B}^{\triangle} + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + (\widehat{B}\widetilde{\widehat{B}})^{\triangle})\\
&- A(B\widetilde{B}\widehat{B}^{\triangle} + B\widetilde{B}\widetilde{\widehat{B}}^{\triangle} + B(\widehat{B}\widetilde{\widehat{B}})^{\triangle} + \widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}) + B\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}, \qquad (3.14)\\
F := {}& A^3C - A^2C(\widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle})\\
&+ AC(\widetilde{B}\widehat{B}^{\triangle} + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + (\widehat{B}\widetilde{\widehat{B}})^{\triangle}) - C\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}. \qquad (3.15)
\end{aligned}
$$

We use the same expressions in the case $n = 5$ (see the next section).

Note that the formulas (3.12), (3.13), (3.14), and (3.15) can be rewritten using the new operation

$$B^{\sharp} := (\widehat{B}\widetilde{\widehat{B}})^{\triangle} \tag{3.16}$$

in the form

$$
\begin{aligned}
&Q = D\widetilde{D}D^{\sharp}, \qquad X = \frac{\widetilde{D}D^{\sharp}F}{Q},\\
&D = A^4 - A^3(B + \widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle}) + A^2(B\widetilde{B} + B\widehat{B}^{\triangle} + B\widetilde{\widehat{B}}^{\triangle} + \widetilde{B}\widehat{B}^{\triangle}\\
&\quad + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + B^{\sharp}) - A(B\widetilde{B}\widehat{B}^{\triangle} + B\widetilde{B}\widetilde{\widehat{B}}^{\triangle} + BB^{\sharp} + \widetilde{B}B^{\sharp}) + B\widetilde{B}B^{\sharp},
\end{aligned}
$$

[2] Analytic proof is also possible using the methods from [1].

$$F = A^3C - A^2C(\widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle}) + AC(\widetilde{B}\widehat{B}^{\triangle} + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + B^{\sharp}) - C\widetilde{B}B^{\sharp}.$$

Example. Let us consider the Sylvester equation (3.2) in $C\ell_{1,3}$ with

$$A = 3e - 5e_1 + 2e_2 - 2e_3 - 4e_4 + e_{12} + 3e_{13} + 5e_{14} + 2e_{23} + 2e_{24} - 5e_{34} \\ +2e_{123} - 4e_{124} + e_{134} + 4e_{234} + 2e_{1234},$$
$$B = 2e + 5e_1 - e_2 - 2e_3 - e_4 + e_{12} + 2e_{13} + 5e_{14} - 5e_{23} + 2e_{24} - 3e_{34} \\ +4e_{123} - 3e_{124} + 4e_{134} + 3e_{234} + e_{1234},$$
$$C = 4e + e_1 - 3e_2 - 2e_3 + 4e_4 - e_{12} + 5e_{13} + e_{14} + 3e_{23} + e_{24} - 4e_{34} \\ +2e_{123} - 3e_{124} - 2e_{134} - 5e_{234} + 5e_{1234}.$$

Using the formulas (3.12), (3.13), (3.14), (3.15), and computer calculations in Wolfram Mathematica, we get

$$D = -3331e + 16960e_1 - 2736e_2 + 5228e_3 + 11276e_4 - 4372e_{12} - 6740e_{13} \\ -17764e_{14} - 4208e_{23} - 4520e_{24} + 12072e_{34} - 8664e_{123} + 8664e_{124} \\ -2128e_{134} - 14968e_{234} - 5868e_{1234},$$
$$Q = 818014056354052817e \neq 0,$$
$$F = -3654e - 3114e_1 - 4238e_2 - 12909e_3 - 629e_4 - 7164e_{12} - 5583e_{13} \\ -9442e_{14} - 14155e_{23} - 1197e_{24} + 3316e_{34} - 9352e_{123} - 2768e_{124}, \\ -2570e_{134} + 6614e_{234} - 6485e_{1234},$$
$$X = \frac{1}{Q}(-119559672248263574e - 243271127103539030e_1 \\ -45110505690078854e_2 + 102025493907271711e_3 \\ -237419769499231033e_4 - 230234896037415164e_{12} \\ -631822395022405163e_{13} + 354830063944470830e_{14} \\ -248262081322178503e_{23} + 381628355781437695e_{24} \\ +242277961566965860e_{34} + 175205777213912492e_{123} \\ +85615763017907532e_{124} - 78264759152759606e_{134} \\ +121735560355563862e_{234} + 268142275333252559e_{1234}). \tag{3.17}$$

Substituting (3.17) into (3.2), we get equality.

4. The Case $n = 5$

Theorem 4.1. *Let us consider the Sylvester equation in $C\ell_{p,q}$, $p + q = 5$,*

$$AX - XB = C \tag{4.1}$$

for known $A, B, C \in C\ell_{p,q}$ and unknown $X \in C\ell_{p,q}$.

If

$$Q := D\widetilde{D}(\widehat{D}\widetilde{\widehat{D}})^{\triangle}(D\widetilde{D}(\widehat{D}\widetilde{\widehat{D}})^{\triangle})^{\triangle} \neq 0, \tag{4.2}$$

then

$$X = \frac{\widetilde{D}(\widehat{D}\widetilde{\widehat{D}})^{\triangle}(D\widetilde{D}(\widehat{D}\widetilde{\widehat{D}})^{\triangle})^{\triangle}F}{Q}, \tag{4.3}$$

where

$$
\begin{aligned}
D &:= A^4 - A^3(B + \widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle}) \\
&\quad + A^2(B\widetilde{B} + B\widehat{B}^{\triangle} + B\widetilde{\widehat{B}}^{\triangle} + \widetilde{B}\widehat{B}^{\triangle} + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + (\widehat{B}\widetilde{\widehat{B}})^{\triangle}) \\
&\quad - A(B\widetilde{B}\widehat{B}^{\triangle} + B\widetilde{B}\widetilde{\widehat{B}}^{\triangle} + B(\widehat{B}\widetilde{\widehat{B}})^{\triangle} + \widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}) + B\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}, \qquad (4.4)
\end{aligned}
$$

$$
\begin{aligned}
F &:= A^3C - A^2C(\widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle}) \\
&\quad + AC(\widetilde{B}\widehat{B}^{\triangle} + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + (\widehat{B}\widetilde{\widehat{B}})^{\triangle}) - C\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}. \qquad (4.5)
\end{aligned}
$$

Note that the formulas (4.2), (4.3), (4.4), and (4.5) can be rewritten using the operation (3.16) in the form

$$
Q = D\widetilde{D}D^{\sharp}(D\widetilde{D}D^{\sharp})^{\triangle}, \qquad X = \frac{\widetilde{D}D^{\sharp}(D\widetilde{D}D^{\sharp})^{\triangle}F}{Q},
$$

$$
\begin{aligned}
D &= A^4 - A^3(B + \widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle}) + A^2(B\widetilde{B} + B\widehat{B}^{\triangle} + B\widetilde{\widehat{B}}^{\triangle} + \widetilde{B}\widehat{B}^{\triangle} \\
&\quad + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + B^{\sharp}) - A(B\widetilde{B}\widehat{B}^{\triangle} + B\widetilde{B}\widetilde{\widehat{B}}^{\triangle} + BB^{\sharp} + \widetilde{B}B^{\sharp}) + B\widetilde{B}B^{\sharp}, \\
F &= A^3C - A^2C(\widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle}) + AC(\widetilde{B}\widehat{B}^{\triangle} + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + B^{\sharp}) - C\widetilde{B}B^{\sharp}.
\end{aligned}
$$

Proof. In the case $C\ell_{p,q}$, $p+q=5$, we have 8 characteristic polynomial coefficients[3] $b_{(1)}, \ldots, b_{(8)}$ for an arbitrary element $B \in C\ell_{p,q}$. Instead of them, let us consider the following 4 expressions (which are scalars in the case $n=4$)

$$
\begin{aligned}
b'_{(1)} &= B + \widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle}, \\
b'_{(2)} &= -(B\widetilde{B} + B\widehat{B}^{\triangle} + B\widetilde{\widehat{B}}^{\triangle} + \widetilde{B}\widehat{B}^{\triangle} + \widetilde{B}\widetilde{\widehat{B}}^{\triangle} + (\widehat{B}\widetilde{\widehat{B}})^{\triangle}), \\
b'_{(3)} &= B\widetilde{B}\widehat{B}^{\triangle} + B\widetilde{B}\widetilde{\widehat{B}}^{\triangle} + B(\widehat{B}\widetilde{\widehat{B}})^{\triangle} + \widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}, \\
b'_{(4)} &= -B\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}. \qquad (4.6)
\end{aligned}
$$

We have[4] $b'_{(1)}, b'_{(2)}, b'_{(3)}, b'_{(4)} \in \mathrm{cen}(C\ell_{p,q}) = C\ell^0_{p,q} \oplus C\ell^5_{p,q}$. We can easily verify that

$$
B + \widetilde{B} + \widehat{B}^{\triangle} + \widetilde{\widehat{B}}^{\triangle} = 4(\langle B\rangle_0 + \langle B\rangle_5) \in C\ell^0_{p,q} \oplus C\ell^5_{p,q}
$$

using definitions of the operations (2.1), (2.2), and (3.1). We have $B\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle} \in C\ell^0_{p,q} \oplus C\ell^5_{p,q}$ (see [25]). We verified $b'_{(2)}, b'_{(3)} \in C\ell^0_{p,q} \oplus C\ell^5_{p,q}$ using computer calculations[5] .

[3] Explicit formulas for the coefficients $b_{(1)}, \ldots, b_{(8)}$ are presented in [1].

[4] Note that the same is not true for the expressions for characteristic polynomial coefficients from Theorem 3.1. For example, $B\widehat{\widetilde{B}}(\widehat{B}\widetilde{B})^{\triangle} \in C\ell^0_{p,q} \oplus C\ell^1_{p,q} \oplus C\ell^4_{p,q} \neq \mathrm{cen}(C\ell_{p,q})$ in the case $n=5$, see the details in [25].

[5] Analytic proof is also possible using the methods from [1].

Multiplying both sides of (4.1) on the right by $-\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}$, we get

$$-AX\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}+XB\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}=-C\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}. \tag{4.7}$$

Multiplying both sides of (4.1) on the right by $\widetilde{B}\widehat{B}^{\triangle}+\widetilde{B}\widetilde{\widehat{B}}^{\triangle}+(\widehat{B}\widetilde{\widehat{B}})^{\triangle}$ and on the left by A, we get

$$\begin{aligned}&A^2X(\widetilde{B}\widehat{B}^{\triangle}+\widetilde{B}\widetilde{\widehat{B}}^{\triangle}+(\widehat{B}\widetilde{\widehat{B}})^{\triangle})-AXB(\widetilde{B}\widehat{B}^{\triangle}+\widetilde{B}\widetilde{\widehat{B}}^{\triangle}+(\widehat{B}\widetilde{\widehat{B}})^{\triangle})\\&\quad=AC(\widetilde{B}\widehat{B}^{\triangle}+\widetilde{B}\widetilde{\widehat{B}}^{\triangle}+(\widehat{B}\widetilde{\widehat{B}})^{\triangle}).\end{aligned} \tag{4.8}$$

Multiplying both sides of (4.1) on the right by $-(\widetilde{B}+\widehat{B}^{\triangle}+\widetilde{\widehat{B}}^{\triangle})$ and on the left by A^2, we get

$$\begin{aligned}&-A^3X(\widetilde{B}+\widehat{B}^{\triangle}+\widetilde{\widehat{B}}^{\triangle})+A^2XB(\widetilde{B}+\widehat{B}^{\triangle}+\widetilde{\widehat{B}}^{\triangle})\\&\quad=-A^2C(\widetilde{B}+\widehat{B}^{\triangle}+\widetilde{\widehat{B}}^{\triangle}).\end{aligned} \tag{4.9}$$

Multiplying both sides of (4.1) on the left by A^3, we get

$$A^4X-A^3XB=A^3C. \tag{4.10}$$

Summing (4.7), (4.8), (4.9), and (4.10), we get

$$\begin{aligned}&(A^4-A^3(B+\widetilde{B}+\widehat{B}^{\triangle}+\widetilde{\widehat{B}}^{\triangle})\\&\quad+A^2(B\widetilde{B}+B\widehat{B}^{\triangle}+B\widetilde{\widehat{B}}^{\triangle}+\widetilde{B}\widehat{B}^{\triangle}+\widetilde{B}\widetilde{\widehat{B}}^{\triangle}+(\widehat{B}\widetilde{\widehat{B}})^{\triangle})\\&\quad-A(B\widetilde{B}\widehat{B}^{\triangle}+B\widetilde{B}\widetilde{\widehat{B}}^{\triangle}+B(\widehat{B}\widetilde{\widehat{B}})^{\triangle}+\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle})+B\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle})X\\&=A^3C-A^2C(\widetilde{B}+\widehat{B}^{\triangle}+\widetilde{\widehat{B}}^{\triangle})\\&\quad+AC(\widetilde{B}\widehat{B}^{\triangle}+\widetilde{B}\widetilde{\widehat{B}}^{\triangle}+(\widehat{B}\widetilde{\widehat{B}})^{\triangle})-C\widetilde{B}(\widehat{B}\widetilde{\widehat{B}})^{\triangle}.\end{aligned}$$

Denoting (4.4) and (4.5), and using the formula for the inverse in the case $n=5$ (see [25]):

$$D^{-1}=\frac{\mathrm{Adj}(D)}{\mathrm{Det}(D)},\qquad \mathrm{Det}(D)=D\widetilde{D}(\widehat{D}\widetilde{\widehat{D}})^{\triangle}(D\widetilde{D}(\widehat{D}\widetilde{\widehat{D}})^{\triangle})^{\triangle}\in C\ell^0_{p,q}\equiv\mathbb{R},$$

$$\mathrm{Adj}(D)=\widetilde{D}(\widehat{D}\widetilde{\widehat{D}})^{\triangle}(D\widetilde{D}(\widehat{D}\widetilde{\widehat{D}})^{\triangle})^{\triangle},$$

we get (4.3). □

Example. Let us consider the Sylvester equation (4.1) in $C\ell_{4,1}$ with

$$\begin{aligned}A=&-e+e_1-3e_2+3e_3+2e_4-e_5+3e_{12}+2e_{13}-2e_{14}-3e_{15}-e_{23}\\&-3e_{24}-e_{25}-e_{34}-3e_{35}-3e_{45}-e_{123}-3e_{124}+e_{125}-e_{134}-3e_{135}\\&+e_{145}+2e_{234}+2e_{235}-2e_{245}+3e_{345}+3e_{1234}-2e_{1235}+2e_{1245}\\&-e_{1345}-2e_{2345}-2e_{12345},\\B=&-2e-e_1-3e_2-2e_3-e_4+e_5-2e_{12}+2e_{13}-e_{14}-2e_{15}+3e_{23}\\&+e_{24}-2e_{25}-3e_{34}+2e_{35}-3e_{45}-e_{123}+e_{124}+2e_{125}-2e_{134}+3e_{135}\end{aligned}$$

$$
\begin{aligned}
&+3e_{145} - 3e_{234} - e_{235} - 3e_{245} - e_{345} + e_{1234} + e_{1235} + e_{1245}\\
&+3e_{1345} + 2e_{2345} - 3e_{12345},\\
C = {}& 3e - 3e_1 + 2e_2 + e_3 + 3e_4 - 2e_5 - 3e_{12} - 2e_{13} + e_{14} - e_{15} + 2e_{23}\\
&+2e_{24} + 2e_{25} - 2e_{34} - 3e_{35} + e_{45} + e_{123} - 3e_{124} + e_{125} - e_{134} - 2e_{135}\\
&-2e_{145} - e_{234} - 2e_{235} - 3e_{245} - 2e_{345} - e_{1234} - 3e_{1235} + e_{1245}\\
&-2e_{1345} - e_{2345} - e_{12345}.
\end{aligned}
$$

Using the formulas (4.2), (4.3), (4.4), (4.5), and computer calculations in Wolfram Mathematica, we get

$$
\begin{aligned}
D = {}& -28e - 2784e_1 + 4088e_2 - 1584e_3 + 1432e_4 - 2528e_5 - 1688e_{12}\\
&-2496e_{13} - 3624e_{14} - 1392e_{15} + 2904e_{23} - 3392e_{24} - 648e_{25}\\
&+1664e_{34} + 3104e_{35} + 3088e_{45} + 1568e_{123} - 3056e_{124} - 4272e_{125}\\
&+2888e_{134} + 4280e_{135} + 296e_{145} + 416e_{234} - 936e_{235} + 560e_{245}\\
&+3064e_{345} - 208e_{1234} + 4648e_{1235} - 1632e_{1245} - 1528e_{1345}\\
&-1712e_{2345} - 1112e_{12345},\\
Q = {}& 26951763359342217682351456256 0e \neq 0,\\
F = {}& -4792e - 4250e_1 + 2398e_2 + 1168e_3 - 8208e_4 + 3268e_5 + 784e_{12}\\
&+4594e_{13} + 2108e_{14} - 4948e_{15} - 1454e_{23} + 606e_{24} + 2350e_{25}\\
&+7786e_{34} - 3102e_{35} + 8970e_{45} - 10044e_{123} - 4682e_{124} + 5594e_{125}\\
&+3822e_{134} - 1034e_{135} + 9688e_{145} + 6272e_{234} + 4448e_{235} + 5580e_{245}\\
&-222e_{345} + 3816e_{1234} + 2648e_{1235} + 7042e_{1245} + 6824e_{1345}\\
&+6496e_{2345} - 5290e_{12345},
\end{aligned}
$$

and

$$
\begin{aligned}
X = {}& \frac{1}{Q}(-2542637343026554833978318 52032e + 1243331611929224341222827 95008e_1\\
&+4254232860869616089214910464e_2 + 7759061400011677799555517 6448e_3\\
&-2747976893638733658908729 67168e_4 - 2516616565241405234695394 42688e_5\\
&+1721710778554950010584268 80000e_{12} + 4509746108317488985539010 56000e_{13}\\
&+1253628776859934886108037 77536e_{14} - 1050845623356278471976821 71904e_{15}\\
&-3693967572880512452400660 80768e_{23} + 2415280095496026983738903 7568e_{24}\\
&+4040377055345199776623298 80576e_{25} + 1540183456802404459941354 86464e_{34}\\
&-7574950071694641301908963 3280e_{35} - 6590783498544577118484060 5696e_{45}\\
&-3500451655246148422019126 39488e_{123} - 1030454686559123954141909 81120e_{124}\\
&+5909535999768187503399721 69728e_{125} + 7370777512106515038277405 9008e_{134}\\
&-2311264752154980042825825 32096e_{135} + 2718452906461239693593738 60864e_{145}\\
&-1066245446164377099700750 25408e_{234} + 1483926117248902600943643 52512e_{235}\\
&+8266584280294636422351585 2800e_{245} + 3780710763977003530182967 2960e_{345}\\
&+2206075265938980401509242 63424e_{1234} - 1226107969662957210096845 45536e_{1235}\\
&+2004665189631565384499659 73504e_{1245} + 2737316751747038486121701 70368e_{1345}\\
&+2496753434281086658382750 67904e_{2345} - 3455848110307459437964314 86976e_{12345}).
\end{aligned}
\tag{4.11}
$$

Substituting (4.11) into (4.1), we get equality.

5. The Case of Arbitrary n

Let us consider the general case of the real Clifford algebra $C\ell_{p,q}$ with arbitrary $n=p+q$. We use the following concepts of characteristic polynomial $\varphi_B(\lambda)$, determinant $\operatorname{Det}(B)$, adjugate $\operatorname{Adj}(B)$, and inverse B^{-1} in $C\ell_{p,q}$ (see the details in [25])[6]:

$$\varphi_B(\lambda) := \operatorname{Det}(\lambda e - B) = \lambda^N - b_{(1)}\lambda^{N-1} - \cdots - b_{(N)} \in C\ell^0_{p,q},$$
$$B_{(1)} := B, \quad B_{(k+1)} := B(B_{(k)} - b_{(k)}), \qquad N := 2^{[\frac{n+1}{2}]},$$
$$b_{(k)} = \frac{N}{k}\langle B_{(k)}\rangle_0 \in C\ell^0_{p,q} \equiv \mathbb{R}, \qquad k=1,\ldots,N,$$
$$\operatorname{Det}(B) = -B_{(N)} = -b_{(N)} = B(b_{(N-1)} - B_{(N-1)}) \in C\ell^0_{p,q} \equiv \mathbb{R},$$
$$\operatorname{Adj}(B) = b_{(N-1)} - B_{(N-1)}, \qquad B^{-1} = \frac{\operatorname{Adj}(B)}{\operatorname{Det}(B)}.$$

In the following theorem, we present recursive formulas for the basis-free solution to the Sylvester equation in the case of arbitrary $n=p+q$.

Theorem 5.1. *Let us consider the Sylvester equation in $C\ell_{p,q}$, $p+q=n$,*

$$AX - XB = C \tag{5.1}$$

for known $A, B, C \in C\ell_{p,q}$ and unknown $X \in C\ell_{p,q}$.

Let us denote $N := 2^{[\frac{n+1}{2}]}$. If

$$Q := d_{(N)} \neq 0, \tag{5.2}$$

then

$$X = \frac{(D_{(N-1)} - d_{(N-1)})F}{Q}, \tag{5.3}$$

where

$$D := -\sum_{j=0}^{N} A^{N-j} b_{(j)}, \tag{5.4}$$

$$F := \sum_{j=1}^{N} A^{N-j} C(B_{(j-1)} - b_{(j-1)}), \tag{5.5}$$

and the following expressions are defined recursively [7] *:*

$$b_{(k)} = \frac{N}{k}\langle B_{(k)}\rangle_0, \quad B_{(k+1)} = B(B_{(k)} - b_{(k)}), \quad B_{(1)} = B,$$
$$d_{(k)} = \frac{N}{k}\langle D_{(k)}\rangle_0, \quad D_{(k+1)} = D(D_{(k)} - d_{(k)}), \quad D_{(1)} = D,$$
$$B_{(0)} = D_{(0)} := 0, \qquad b_{(0)} = d_{(0)} := -1, \qquad k=1,\ldots,N.$$

Note that D (5.4) is the characteristic polynomial of the element B with the substitution of A.

---

[6] Here and below we denote the integer part of the number $\frac{n+1}{2}$ by $[\frac{n+1}{2}]$.

[7] Note that using the recursive formulas $B_{(k+1)} = B(B_{(k)} - b_{(k)})$, the expression (5.5) can be reduced to the form $\sum_{i,j} b_{ij} A^i C B^j$ with some scalars $b_{ij} \in \mathbb{R}$.

Proof. Multiplying both sides of (5.1) on the right by $B_{(N-1)}-b_{(N-1)}$, on the right by $B_{(N-2)}-b_{(N-2)}$ and on the left by A, on the right by $B_{(N-3)}-b_{(N-3)}$ and on the left by $A^2, \ldots$, on the right by $B_{(2)}-b_{(2)}$ and on the left by A^{N-3}, on the right by $B-b_{(1)}$ and on the left by A^{N-2}, on the left by A^{N-1}, we get

$$\begin{aligned}
&AX(B_{(N-1)}-b_{(N-1)})-XB(B_{(N-1)}-b_{(N-1)})\\
&\quad =C(B_{(N-1)}-b_{(N-1)}),\\
&A^2X(B_{(N-2)}-b_{(N-2)})-AXB(B_{(N-2)}-b_{(N-2)})\\
&\quad =AC(B_{(N-2)}-b_{(N-2)}),\\
&A^3X(B_{(N-3)}-b_{(N-3)})-A^2XB(B_{(N-3)}-b_{(N-3)})\\
&\quad =A^2C(B_{(N-3)}-b_{(N-3)}),\\
&\ldots\\
&A^{N-2}X(B_{(2)}-b_{(2)})-A^{N-3}XB(B_{(2)}-b_{(2)})=A^{N-3}C(B_{(2)}-b_{(2)}),\\
&A^{N-1}X(B-b_{(1)})-A^{N-2}XB(B-b_{(1)})=A^{N-2}C(B-b_{(1)}),\\
&A^NX-A^{N-1}XB=A^{N-1}C.
\end{aligned}$$

Summing these equations and using

$$\begin{aligned}
&B_{(k+1)}=B(B_{(k)}-b_{(k)}), \qquad k=1,\ldots,N,\\
&B_{(N)}=B(B_{(N-1)}-b_{(N-1)})=b_{(N)}=-\mathrm{Det}(B)\in C\ell^0_{p,q},
\end{aligned}$$

we get

$$\begin{aligned}
&(A^N-b_{(1)}A^{N-1}X-b_{(2)}A^{N-2}X-\cdots-b_{(N-1)}AX-b_{(N)})X\\
&\quad =A^{N-1}C+A^{N-2}C(B-b_{(1)})+\cdots+C(B_{(N-1)}-b_{(N-1)}).
\end{aligned}$$

Denoting (5.4), (5.5), and using

$$D^{-1}=\frac{\mathrm{Adj}(D)}{\mathrm{Det}(D)}, \quad \mathrm{Adj}(D)=d_{(N-1)}-D_{(N-1)}, \quad \mathrm{Det}(D)=-d_{(N)},$$

we get (5.3). □

Note that in the case of odd n, the formulas (5.4) and (5.5) can be simplified. We can use instead of N characteristic polynomial coefficients some other $\frac{N}{2}$ expressions. We call them generalized characteristic polynomial coefficients. For example, in the case $n=5$ (see Theorem 4.1), we use the 4 expressions $b'_{(k)}$, $k=1,2,3,4$, (4.6), which are in the center of $C\ell_{p,q}$, instead of the 8 ordinary characteristic polynomial coefficients $b_{(k)}$, $k=1,\ldots,8$, which are in $C\ell^0_{p,q}$.

The ordinary characteristic polynomial coefficients of Clifford algebra element corresponds to the characteristic polynomial coefficients of the corresponding matrix representation of dimension N (see the details in [25]). The generalized characteristic polynomial coefficients of Clifford algebra element corresponds to the characteristic polynomial coefficients of the corresponding matrix of dimension $\frac{N}{2}$ with entries in $\mathbb{C}$ or $\mathbb{R}\oplus\mathbb{R}$. In more details, the center of $C\ell_{p,q}$ with odd $n=p+q$ is $\mathrm{cen}(C\ell_{p,q})=C\ell^0_{p,q}\oplus C\ell^n_{p,q}$, which is isomorphic to $\mathbb{C}$ in the case $e_{1\dots n}^2 = -e$ (i.e. $p-q = 2, 3 \bmod 4$) and to $\mathbb{R}\oplus\mathbb{R}$ in the case $e_{1\dots n}^2 = e$ (i.e. $p-q = 0, 1 \bmod 4$). The Clifford algebra $C\ell_{p,q}$ with odd $n = p+q$ can be represented in the form (the same idea is used in [14])

$$C\ell_{p,q} = C\ell^{(0)}_{p,q} \oplus C\ell^{(1)}_{p,q} = C\ell^{(0)}_{p,q} \oplus e_{1\dots n} C\ell^{(0)}_{p,q},$$

where

$$C\ell^{(0)}_{p,q} = \bigoplus_{k=0 \bmod 2} C\ell^k_{p,q}, \qquad C\ell^{(1)}_{p,q} = \bigoplus_{k=1 \bmod 2} C\ell^k_{p,q}$$

are the even subalgebra and the odd subspace of $C\ell_{p,q}$. Thus any element $B \in C\ell_{p,q}$ can be written as an element of the even subalgebra $C\ell^{(0)}_{p,q}$ with complex (in the cases $p-q = 2, 3 \bmod 4$) or hyperbolic (in the cases $p-q = 0, 1 \bmod 4$) coefficients. Also we use the well-known isomorphisms (see, for example, [18,22])

$$C\ell^{(0)}_{p,q} \cong C\ell_{p,q-1}, \qquad q \geq 1; \qquad C\ell^{(0)}_{p,q} \cong C\ell_{q,p-1}, \qquad p \geq 1.$$

We obtain the following simplification of the statement of Theorem 5.1 in the case of odd $n = p+q$ (with $\frac{N}{2}$ steps in the corresponding recursive formulas for D and F instead of N steps for these expressions as in the previous theorem).

Let us consider the Sylvester equation in $C\ell_{p,q}$ with odd $p+q = n$,

$$AX - XB = C \tag{5.6}$$

for known $A, B, C \in C\ell_{p,q}$ and unknown $X \in C\ell_{p,q}$. Let us denote[8] $N := 2^{\frac{n+1}{2}}$. If

$$Q := d_{(N)} \neq 0, \tag{5.7}$$

then

$$X = \frac{(D_{(N-1)} - d_{(N-1)})F}{d_{(N)}},$$

where

$$D := \varphi'_B(A) = -\sum_{j=0}^{\frac{N}{2}} A^{\frac{N}{2}-j} b'_{(j)}, \tag{5.8}$$

$$F := \sum_{j=1}^{\frac{N}{2}} A^{\frac{N}{2}-j} C (B'_{(j-1)} - b'_{(j-1)}), \tag{5.9}$$

and the following expressions are defined recursively:

$$b'_{(k)} = \frac{N}{k}\langle B'_{(k)}\rangle_{\text{cen}}, \quad B'_{(k+1)} = B(B'_{(k)} - b'_{(k)}), \quad k = 1, \dots, \frac{N}{2},$$
$$B'_{(1)} = B, \qquad B'_{(0)} := 0, \qquad b'_{(0)} := -1,$$
$$d_{(m)} = \frac{N}{m}\langle D_{(m)}\rangle_0, \quad D_{(m+1)} = D(D_{(m)} - d_{(m)}), \quad m = 1, \dots, N,$$
$$D_{(1)} = D, \quad D_{(0)} := 0, \quad d_{(0)} := -1.$$

[8] In the case of odd n, the integer part of the number $\frac{n+1}{2}$ is equal to $[\frac{n+1}{2}] = \frac{n+1}{2} \in \mathbb{Z}$.

Note that expressions for generalized characteristic polynomial coefficients $b'_{(k)}$, $k=1,\ldots,\frac{N}{2}$ coincide with the expressions for the characteristic polynomial coefficients for the previous even $n-1$ (see the example for the case $n=5$ in Theorem 4.1).

Example. Let us present the explicit expressions for $\langle B\rangle_{\rm cen}$ in the case of small dimensions $n\leq 15$ using the operations of conjugation $\widetilde{\ }$, $\widehat{\ }$, ${}^{\triangle}$, ${}^{\square}$:

$$\frac{1}{2}(B+\widetilde{\widehat{B}})=\langle B\rangle_{\rm cen}=\begin{cases}\langle B\rangle_0, & \text{if } n=2,\\ \langle B\rangle_0+\langle B\rangle_3, & \text{if } n=3;\end{cases}$$

$$\frac{1}{4}(B+\widetilde{B}+\widehat{B}^{\triangle}+\widetilde{\widehat{B}}^{\triangle})=\langle B\rangle_{\rm cen}=\begin{cases}\langle B\rangle_0, & \text{if } n=4,\\ \langle B\rangle_0+\langle B\rangle_5, & \text{if } n=5;\end{cases}$$

$$\frac{1}{4}(B+\widehat{\widetilde{B}}+\widehat{B}^{\triangle}+\widetilde{B}^{\triangle})=\langle B\rangle_{\rm cen}=\begin{cases}\langle B\rangle_0, & \text{if } n=6,\\ \langle B\rangle_0+\langle B\rangle_7, & \text{if } n=7;\end{cases}$$

$$\frac{1}{8}(B+\widehat{B}^{\square}+\widetilde{B}+\widehat{\widetilde{B}}^{\square}+B^{\triangle}+\widehat{B}^{\triangle\square}+\widetilde{B}^{\triangle}+\widehat{\widetilde{B}}^{\triangle\square})$$
$$=\langle B\rangle_{\rm cen}=\begin{cases}\langle B\rangle_0, & \text{if } n=8,\\ \langle B\rangle_0+\langle B\rangle_9, & \text{if } n=9;\end{cases}$$

$$\frac{1}{8}(B+\widehat{B}^{\square}+\widetilde{B}^{\square}+\widehat{\widetilde{B}}+B^{\triangle}+\widehat{B}^{\triangle\square}+\widetilde{B}^{\triangle\square}+\widehat{\widetilde{B}}^{\triangle\square})$$
$$=\langle B\rangle_{\rm cen}=\begin{cases}\langle B\rangle_0, & \text{if } n=10,\\ \langle B\rangle_0+\langle B\rangle_{11}, & \text{if } n=11;\end{cases}$$

$$\frac{1}{8}(B+\widehat{B}^{\square}+\widetilde{B}+\widehat{\widetilde{B}}^{\square}+B^{\triangle\square}+\widehat{B}^{\triangle}+\widetilde{B}^{\triangle\square}+\widehat{\widetilde{B}}^{\triangle})$$
$$=\langle B\rangle_{\rm cen}=\begin{cases}\langle B\rangle_0, & \text{if } n=12,\\ \langle B\rangle_0+\langle B\rangle_{13}, & \text{if } n=13;\end{cases}$$

$$\frac{1}{8}(B+\widehat{B}^{\square}+\widetilde{B}^{\square}+\widehat{\widetilde{B}}+B^{\triangle\square}+\widehat{B}^{\triangle}+\widetilde{B}^{\triangle}+\widehat{\widetilde{B}}^{\triangle\square})$$
$$=\langle B\rangle_{\rm cen}=\begin{cases}\langle B\rangle_0, & \text{if } n=14,\\ \langle B\rangle_0+\langle B\rangle_{15}, & \text{if } n=15,\end{cases}$$

where we use the additional operation of conjugation ${}^{\square}$ (compare with (2.1), (2.2), and (3.1), see also [25])

$$B^{\square}=\sum_{k=0}^{n}(-1)^{\frac{k(k-1)(k-2)(k-3)(k-4)(k-5)(k-6)(k-7)}{8!}}\langle U\rangle_k$$
$$=\sum_{k=0,1,\ldots,7\bmod 16}\langle B\rangle_k-\sum_{k=8,9,\ldots,15\bmod 16}\langle B\rangle_k,\qquad\forall B\in C\!\ell_{p,q}.$$

In the general case, we have $(UV)^{\square}\neq U^{\square}V^{\square}$ and $(UV)^{\square}\neq V^{\square}U^{\square}$.

Let us return to the case of arbitrary n. The scalar part operation $\langle B\rangle_0$ and the projection onto the center $\langle B\rangle_{\rm cen}$ can always be realized as linear combinations of operations of conjugation

$$B \mapsto \sum_{k=0}^{n} \lambda_k \langle B\rangle_k, \qquad \lambda_k = \pm 1.$$

This fact is discussed in [25] (see Theorem 1 and Footnote 7). We need[9] $[\log_2 n] + 1$ different operations of conjugation to do this[10].

Let us denote $m := [\log_2 n] + 1$. The formulas from Theorem 5.1 and the following formulas[11] from [25]

$$\langle B\rangle_0 = \frac{1}{2^m}(B + B^{\triangle_1} + B^{\triangle_2} + \cdots + B^{\triangle_m} + B^{\triangle_1\triangle_2} + \cdots + B^{\triangle_1\cdots\triangle_m}),$$

$$B^{\triangle_j} = \sum_{k=0}^{n}(-1)^{C_k^{2^{j-1}}}\langle B\rangle_k, \qquad C_k^i = \frac{k!}{i!(k-i)!}, \qquad j = 1, \ldots, m,$$

give us the basis-free solution (which involve only the operations of Clifford product, summation, and the operations of conjugation) to the Sylvester equation in $C\ell_{p,q}$ with arbitrary n. Thus, our approach works in the case of arbitrary n.

6. Conclusions

In this paper, we present the basis-free solution to the Sylvester equation in Clifford (geometric) algebra of arbitrary dimension. Note that we discuss the most important (nondegenerate) case when the element Q (2.4), (2.6), (2.7), (3.3), (4.2), (5.2), (5.7) is non-zero and the corresponding Sylvester equation (2.3) has a unique solution X. The degenerate case $Q = 0$ (with zero divisors) can also be studied.

An interesting task is to generalize results of this paper to the case of general linear equations in geometric algebras

$$\sum_{j=1}^{k} A_j X B_j = C, \qquad A_j, B_j, C, X \in C\ell_{p,q} \tag{6.1}$$

in the case of arbitrary $n = p + q$. The basis-free solution to the equation (6.1) in the case of quaternions $\mathbb{H} \cong C\ell_{0,2}$ is given in [21] (see also the papers [15, 20]).

Note that all results of this paper remain true for the complexified Clifford algebras $\mathbb{C} \otimes C\ell_{p,q}$ because characteristic polynomial coefficients are the same and are defined for $C\ell_{p,q}$ and $\mathbb{C} \otimes C\ell_{p,q}$ in the same manner using the matrix representation of $\mathbb{C} \otimes C\ell_{p,q}$ of dimension $N = 2^{[\frac{n+1}{2}]}$ (see [25]).

The real Clifford algebras are isomorphic to the matrix algebras over $\mathbb{R}$, $\mathbb{R} \oplus \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{H} \oplus \mathbb{H}$ depending on $p - q \bmod 8$ and the complex Clifford algebras (and complexified Clifford algebras) are isomorphic to the matrix algebras over $\mathbb{C}$ or $\mathbb{C} \oplus \mathbb{C}$ depending on $n \bmod 2$. In the opinion of the author, the

[9] Here and below we denote the integer part of $\log_2 n$ by $[\log_2 n]$.

[10] In the above example, we use the three operations $\widehat{B}$, $\widetilde{B}$, and $B^{\triangle}$ in the cases $4 \le n \le 7$, the four operations $\widehat{B}$, $\widetilde{B}$, $B^{\triangle}$, and $B^{\square}$ in the cases $8 \le n \le 15$.

[11] Note that $B^{\triangle_1} = \widehat{B}$, $B^{\triangle_2} = \widetilde{B}$, $B^{\triangle_3} = B^{\triangle}$, and $B^{\triangle_4} = B^{\square}$.

structure of naturally defined fundamental subspaces (the subspaces of fixed grades, the even subalgebra, and the odd subspace) and the corresponding operations of conjugation (the grade involution, the reversion, and the others) favourably distinguishes Clifford algebras from the corresponding matrix algebras, when we use them for different applications – in physics, engineering, robotics, computer vision, control theory, stability analysis, model reduction, image and signal processing.

The explicit formulas for solutions to the Sylvester and Lyapunov equations may be useful in applications, in particular, in solving algebraic noncommutative linear equations in quantum physics.

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