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MATHEMATICAL PHYSICS

Hyperbolic Singular Value Decomposition in the Study of Yang–Mills and Yang–Mills–Proca Equations

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Abstract—The hyperbolic singular value decomposition is used for studying the Yang–Mills equations with SU(2) gauge symmetry and the Yang–Mills–Proca equations in a pseudo-Euclidean (or Euclidean) space of an arbitrary finite dimension and signature. An explicit form of all constant solutions to the system of Yang–Mills–Proca equations in the case of the Lie group SU(2) is obtained. Nonconstant solutions to the Yang–Mills–Proca equations are considered as perturbation theory series.

Keywords: Yang–Mills equations, Yang–Mills–Proca equations, hyperbolic singular value decomposition, singular value decomposition, SU(2), constant solutions, pseudo-Euclidean space

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INTRODUCTION

In [1], a new formulation of the hyperbolic singular value decomposition for an arbitrary real (or complex) matrix using matrices from the pseudo-orthogonal group $\mathrm{O}(p,q)$ (or the pseudo-unitary group $\mathrm{U}(p,q)$, respectively) was proposed. The previous versions of the hyperbolic singular value decomposition worked in the general case only with the use of hyperexchange matrices, which do not form a group (see [2–4] and the discussion of them in [1]). In this paper, we apply the hyperbolic singular value decomposition in the new formulation for studying the Yang–Mills equations with the SU(2) gauge symmetry and the Yang–Mills–Proca equations in the pseudo-Euclidean space $\mathbb{R}^{p,q}$ of arbitrary dimension and signature.

The Yang–Mills equations were proposed by Yang and Mills in 1954 [5] as a mathematical generalization of the Maxwell equations to the non-Abelian case. Later (in 1960–1970) a theory was proposed in which these equations describe electroweak interactions in the case of the Lie group $\mathrm{SU}(2)\times\mathrm{U}(1)$ and strong interactions in the case of the Lie group $\mathrm{SU}(3)$. The Maxwell equations describing electromagnetic interactions are a special case of the Yang–Mills equations for the Abelian Lie group U(1). In this paper, we restrict ourselves to considering the case of the Lie group $\mathrm{SU}(2)$. Pay attention to the classical works on some known classes of particular solutions to the Yang–Mills equations [6–11] and review [12].

In one of my previous works [13] an explicit form of all constant solutions to the Yang–Mills equations with the SU(2) gauge symmetry and an arbitrary current in the Eucledean space $\mathbb{R}^n$ of an arbitrary finite dimension was obtained using the ordinary singular value decomposition.

In this paper, we obtain an explicit form of all constant solutions to the Yang–Mills–Proca equations in the case of the Lie group SU(2) in an arbitrary pseudo-Euclidean space $\mathbb{R}^{p,q}$ (or in the Euclidean space $\mathbb{R}^n$) using the hyperbolic singular value decomposition, which is a generalization of the ordinary singular value decomposition. Nonconstant solutions to the Yang–Mills–Proca equations are considered as perturbation theory series with constant solutions taken as the zero approximation.

1. YANG–MILLS–PROCA EQUATIONS

Let p and q be nonnegative integers and $n = p + q$ be a natural number. Consider the pseudo-Euclidean space $\mathbb{R}^{p,q}$ (or, as a special case, the Euclidean space $\mathbb{R}^n$ for $n = p$ and $q = 0$) with the Cartesian coordinates x^μ, $\mu = 1, \ldots, n$. The metric in $\mathbb{R}^{p,q}$ is specified by the diagonal matrix

$$\eta = (\eta_{\mu\nu}) = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}), \quad p + q = n. \tag{1.1}$$

The partial derivatives are denoted by $\partial_\mu = \partial/\partial x^\mu$. We assume that all functions of $x \in \mathbb{R}^{p,q}$ considered below are sufficiently smooth.

Let G be a semi-simple Lie group and $\mathfrak{g}$ be a real Lie algebra of the Lie group G. The Lie algebra $\mathfrak{g}$ is a real vector space of dimension N with the basis $\tau^1, \ldots, \tau^N$. The multiplication of elements of $\mathfrak{g}$ is defined by the Lie bracket $[A, B] = -[B, A]$, which satisfies the Jacoby identity. The multiplication of the basis elements is defined using the real structure constants $c_l^{rs} = -c_l^{sr}$ $(r, s, l = 1, \ldots, N)$ of the Lie algebra $\mathfrak{g}$:

$$[\tau^r, \tau^s] = c_l^{rs}\tau^l. \tag{1.2}$$

We assume that the elements of the Lie algebra $\mathfrak{g}$ and the Lie group G are represented by square matrices of the corresponding size. The Lie bracket is defined by the commutator $[A, B] = AB - BA$, where the multiplication on the right-hand side is matrix multiplication.

Denote by $\mathfrak{g}\mathrm{T}_b^a$ the set of tensor fields in the (pseudo)Euclidean space $\mathbb{R}^{p,q}$ of the type (a, b) and of rank $a + b$ with values in the Lie algebra $\mathfrak{g}$.

Consider the following equations in the pseudo-Euclidean space $\mathbb{R}^{p,q}$:

$$\partial_\mu A_\nu - \partial_\nu A_\mu - \rho[A_\mu, A_\nu] = F_{\mu\nu}, \tag{1.3}$$

$$\partial_\mu F^{\mu\nu} - \rho[A_\mu, F^{\mu\nu}] = J^\nu, \tag{1.4}$$

where $A_\mu \in \mathfrak{g}\mathrm{T}_1$, $J^\nu \in \mathfrak{g}\mathrm{T}^1$, $F_{\mu\nu} = -F_{\nu\mu} \in \mathfrak{g}\mathrm{T}_2$, and ρ is a real (coupling) constant. These equations are called the *Yang–Mills* equations (system of Yang–Mills equations). Usually, it is assumed that A_μ and $F_{\mu\nu}$ are unknowns, and J^ν is a given vector with values in the Lie algebra $\mathfrak{g}$. Equations (1.3), (1.4) determine the *Yang–Mills field* $(A_\mu, F_{\mu\nu})$, where A_μ is the *potential* and $F_{\mu\nu}$ is the *strength* of the Yang–Mills field. The vector J^ν is called the *non-Abelian current* (in the case of the Abelian group G, the vector J^ν is called *current*).

Note that the Yang–Mills equations (1.4) can be obtained in the conventional way on the basis of the variational principle. Consider the action $\mathcal{S} = \int \mathcal{L} dx$ for the Lagrangian

$$\mathcal{L} = -\frac{1}{4}\mathrm{tr}(F^2), \quad F^2 := F^{\mu\nu}F_{\mu\nu}, \tag{1.5}$$

where $F_{\mu\nu}$ are the components of the curvature 2-form with respect to the connection A_μ, i.e., they are coupled by definition by Eqs. (1.3). By varying the action, we obtain Eqs. (1.4) with a zero current ($J^\nu = 0$). The current J^ν in Eqs (1.4) appears when terms related to other (e.g., scalar or spinor) fields are added to Lagrangian (1.5).

The components of the skew-symmetric tensor field $F_{\mu\nu}$ determined by Eq. (1.3) can be substituted into the second equation in (1.4) to obtain a single second-order equation for the covector potential of the Yang–Mills field

$$\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu - \rho[A^\mu, A^\nu]\right) - \rho\left[A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu - \rho[A^\mu, A^\nu]\right] = J^\nu. \tag{1.6}$$

Consider Eqs. (1.3) and (1.4) from another viewpoint. Let $A_\mu \in \mathfrak{g}\mathrm{T}_1$ be an arbitrary covector with values in $\mathfrak{g}$ that smoothly depends on $x \in \mathbb{R}^{p,q}$. Denote by $F_{\mu\nu}$ the expression

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - \rho[A_\mu, A_\nu], \tag{1.7}$$

and denote by J^ν the expression

$$J^\nu := \partial_\mu F^{\mu\nu} - \rho[A_\mu, F^{\mu\nu}].$$

It is easy to verify that

$$\partial_\nu J^\nu - \rho[A_\nu, J^\nu] = 0. \tag{1.8}$$

This identity is called the *non-Abelian conservation law* (in the case of the Abelian Lie group G, we have the ordinary conservation law $\partial_\nu J^\nu = 0$; i.e., the divergence of the vector J^ν is zero). Therefore, the non-Abelian conservation law (1.8) is a consequence of the Yang–Mills equations (1.3), (1.4).

Consider the tensor fields A_μ, $F_{\mu\nu}$, and J^ν that satisfy the Yang–Mills equations (1.3), (1.4). Take a scalar field with values in the Lie group $S = S(x) \in \mathrm{G}$, and consider the transformed tensor fields

$$\begin{aligned} A'_\mu &= S^{-1}A_\mu S - S^{-1}\partial_\mu S, \\ F'_{\mu\nu} &= S^{-1}F_{\mu\nu}S, \\ J'^\nu &= S^{-1}J^\nu S. \end{aligned} \tag{1.9}$$

These tensor fields satisfy the same Yang–Mills equations

$$\partial_\mu A'_\nu - \partial_\nu A'_\mu - \rho[A'_\mu, A'_\nu] = F'_{\mu\nu},$$

$$\partial_\mu F'^{\mu\nu} - \rho[A'_\mu, F'^{\mu\nu}] = J'^\nu;$$

i.e., Eqs. (1.3), (1.4) are invariant under transformations (1.9). Transformation (1.9) is called the *gauge transformation* (or gauge symmetry), and the Lie group G is called the *gauge group* of the Yang–Mills equations (1.3), (1.4).

2. THE CASE OF THE LIE GROUP SU(2)

Below in this paper, we will consider the particular case of the Lie group SU(2) that is important for describing weak interactions. Theorem 1 about the symmetry of SU(2) Yang–Mills equations substantially uses the two-sheet cover of the orthogonal group SO(3) by the spin group Spin(3) ≅ SU(2). Thus, the methods proposed in this paper do not directly apply to another (important from the physical viewpoint) case of the Lie group SU(3).

We consider the special unitary group

$$\mathrm{G} = \mathrm{SU}(2) = \{A \in \mathrm{Mat}(2, \mathbb{C}) \mid A^\dagger A = I, \det A = 1\}, \quad \dim \mathrm{G} = 3, \tag{2.1}$$

and the corresponding Lie algebra of anti-Hermitian matrices with zero trace

$$\mathfrak{g} = \mathfrak{su}(2) = \{A \in \mathrm{Mat}(2, \mathbb{C}) \mid A^\dagger = -A, \operatorname{tr} A = 0\}. \tag{2.2}$$

Here and below, the identity matrix of appropriate size is denoted by I. It is known that the Pauli matrices σ^a ($a = 1, 2, 3$)

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.3}$$

satisfy the relations

$$(\sigma^a)^\dagger = \sigma^a, \quad \operatorname{tr}\sigma^a = 0, \quad \{\sigma^a, \sigma^b\} = 2\delta^{ab}I, \quad [\sigma^a, \sigma^b] = 2i\epsilon^{ab}_c\sigma^c,$$

where $\epsilon^{ab}_c = \epsilon^{abc}$ is the completely antisymmetric identity tensor (Levi–Civita symbol) and $\epsilon^{123} = 1$. As a basis of the Lie algebra $\mathfrak{su}(2)$, we may take

$$\tau^1 = \frac{1}{2i}\sigma^1, \quad \tau^2 = \frac{1}{2i}\sigma^2, \quad \tau^3 = \frac{1}{2i}\sigma^3. \tag{2.4}$$

For the basis elements, we have

$$(\tau^a)^\dagger = -\tau^a, \quad \operatorname{tr}\tau^a = 0, \quad [\tau^a, \tau^b] = \epsilon_c^{ab}\tau^c; \tag{2.5}$$

i.e., the structure constants of the Lie algebra $\mathfrak{su}(2)$ in this case are the Levi–Civita symbols.

Let us write the decomposition of the Yang–Mills potential and current in the basis of the Lie algebra $\mathfrak{su}(2)$:

$$A^\mu = A_a^\mu \tau^a, \quad J^\mu = J_a^\mu \tau^a, \quad A_a^\mu, J_a^\mu \in \mathbb{R}. \tag{2.6}$$

Here and below, the Latin subscripts run through the values $a = 1, 2, 3$ (since the dimension of the Lie group SU(2) is three) and the Greek superscripts run through the values $\mu = 1, 2, \ldots, n$ (since the dimension of the pseudo-Euclidean space $\mathbb{R}^{p,q}$ is $p + q = n$).

After substitution (2.6), the left-hand side of the Yang–Mills equations (1.6) takes the form

$$\begin{aligned}
&\tau^a\partial_\mu(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) - \rho\partial_\mu(A_a^\mu A_b^\nu)[\tau^a, \tau^b] - \rho\eta_{\mu\alpha}A_a^\alpha(\partial^\mu A_b^\nu - \partial^\nu A_b^\mu)[\tau^a, \tau^b] \\
&\quad + \rho^2\eta_{\mu\alpha}A_c^\alpha A_a^\mu A_b^\nu[\tau^c, [\tau^a, \tau^b]] \\
&= \tau^a\partial_\mu(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) - \rho\epsilon_c^{ab}\tau^c(\partial_\mu(A_a^\mu A_b^\nu) + \eta_{\mu\alpha}A_a^\alpha(\partial^\mu A_b^\nu - \partial^\nu A_b^\mu)) \\
&\quad + \rho^2\eta_{\mu\alpha}A_c^\alpha A_a^\mu A_b^\nu\epsilon_d^{ab}[\tau^c, \tau^d] \\
&= \tau^a\partial_\mu(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) - \rho\epsilon_c^{ab}\tau^c(\partial_\mu(A_a^\mu A_b^\nu) + \eta_{\mu\alpha}A_a^\alpha(\partial^\mu A_b^\nu - \partial^\nu A_b^\mu)) \\
&\quad + \rho^2\eta_{\mu\alpha}A_c^\alpha A_a^\mu A_b^\nu\epsilon_d^{ab}\epsilon_k^{cd}\tau^k.
\end{aligned}$$

Finally, Eqs. (1.6) take the form

$$\begin{aligned}
&\partial_\mu(\partial^\mu A_k^\nu - \partial^\nu A_k^\mu) - \rho\epsilon_k^{ab}(\partial_\mu(A_a^\mu A_b^\nu) + \eta_{\mu\alpha}A_a^\alpha(\partial^\mu A_b^\nu - \partial^\nu A_b^\mu)) \\
&\quad + \rho^2\eta_{\mu\alpha}A_c^\alpha A_a^\mu A_b^\nu\epsilon_d^{ab}\epsilon_k^{cd} = J_k^\nu.
\end{aligned} \tag{2.7}$$

System (2.7) is a system of $3n$ equations ($k = 1, 2, 3, \nu = 1, 2, \ldots, n$) for $3n$ functions A_k^ν and $3n$ functions J_k^ν. It is convenient of interpret (2.7) as a system of equations for the elements of two matrices $A = (A_k^\mu)$ and $J = (J_k^\nu)$ of size $n \times 3$. Below, we will often assume that the matrix of current J is given or depends on the unknown potential matrix A in a certain given way (e.g., in the case of the Yang–Mills–Proca equations, we have $J = -m^2 A$).

Theorem 1. *The system of equations* (2.7) *is invariant under the transformations*

$$A \to \hat{A} = QA, \quad J \to \hat{J} = QJ, \quad Q \in \mathrm{O}(p, q), \tag{2.8}$$

and the transformation

$$A \to A' = AP + \Omega, \quad J \to J' = JP, \quad P = (p_b^a) \in \mathrm{SO}(3), \tag{2.9}$$

where

$$\Omega = \Omega(P) = (\omega_d^\mu), \quad \omega_d^\mu = \frac{1}{8}\delta_{ac}\epsilon_d^{bk}(p_k^c\partial^\mu p_b^a - p_k^a\partial^\mu p_b^c).$$

Proof. The invariance of the first type holds due to the invariance of the Yang–Mills equations under pseudo-orthogonal changes of coordinates of the space $\mathbb{R}^{p,q}$. More precisely, consider a change of coordinates $x^\mu \to \hat{x}^\mu = q_\nu^\mu x^\nu$, where $Q = (q_\nu^\mu) \in \mathrm{O}(p, q)$. The quantities A^μ and J^μ in Eqs. (1.6) are tensor quantities; i.e., they are transformed by the rule

$$A^\mu \to \hat{A}^\nu = q_\mu^\nu A^\mu = q_\mu^\nu A_a^\mu \tau^a = \hat{A}_a^\nu \tau^a, \quad A_a^\nu \to \hat{A}_a^\nu = q_\mu^\nu A_a^\mu,$$

$$J^\mu \to \hat{J}^\nu = q_\mu^\nu J^\mu = q_\mu^\nu J_a^\mu \tau^a = \hat{J}_a^\nu \tau^a, \quad J_a^\nu \to \hat{J}_a^\nu = q_\mu^\nu J_a^\mu.$$

Therefore, Eqs. (2.7) are invariant under transformation (2.8).

The system of Yang–Mills equations is invariant under the gauge transformations (1.9). The quantities of interest are transformed by the rules

$$A'_\mu = S^{-1}A_\mu S - S^{-1}\partial_\mu S, \quad J'^\nu = S^{-1}J^\nu S.$$

By the theorem about the two-sheet cover of the group SO(3) by the spin group Spin(3) ≅ SU(2), we have

$$S^{-1}\tau^a S = p_b^a\tau^b, \quad S \in \mathrm{SU}(2), \quad P = (p_b^a) \in \mathrm{SO}(3). \tag{2.10}$$

For each matrix $P \in \mathrm{SO}(3)$, there exist exactly two matrices $\pm S \in \mathrm{SU}(2)$ related by formula (2.10) (e.g., see [14–16]).

For the transformed current, we obtain

$$J'^\mu = S^{-1}J_a^\mu\tau^a S = J_a^\mu S^{-1}\tau^a S = J_a^\mu p_b^a\tau^b = J_b'^\mu\tau^b, \quad J_b'^\mu = J_a^\mu p_b^a.$$

For the potential, the transformation also contains the term with the spin connection $C_\mu = -S^{-1}\partial_\mu S$. We may use the expression for this quantity found in [17, 18]

$$C_\mu = -S^{-1}\partial_\mu S = \frac{1}{4}(\partial_\mu h^a)h_a, \quad h^a := S^{-1}\tau^a S = p_b^a\tau^b$$

to obtain

$$C_\mu = -S^{-1}\partial_\mu S = \frac{1}{4}\partial_\mu(p_b^a\tau^b)\delta_{ac}p_k^c\tau^k = \frac{1}{8}\delta_{ac}\epsilon_d^{bk}(p_k^c\partial_\mu p_b^a - p_k^a\partial_\mu p_b^c)\tau^d = \omega_{\mu d}\tau^d,$$

where we used the fact that $P^\mathrm{T}P = I$, i.e.,

$$p_b^a p_k^c\delta_{ac} = \delta_{bk}, \quad (\partial_\mu p_b^a)p_k^c\delta_{ac} + p_b^a(\partial_\mu p_k^c)\delta_{ac} = 0.$$

Therefore, Eqs. (2.7) are invariant under transformation (2.9). Theorem 1 is thus proved.

If we combine two transformations in Theorem 1, then we obtain invariance under the transformation

$$A \to QAP + \Omega, \quad J \to QJP, \quad Q \in \mathrm{O}(p,q), \quad P \in \mathrm{SO}(3), \quad \Omega = \Omega(P). \tag{2.11}$$

Multiplication of a matrix by a pseudo-orthogonal matrix on the left and by an orthogonal matrix on the right makes it possible to transform the first matrix to the canonical form with a large number of zeros. To this end, we use the new formulation of the hyperbolic singular value decomposition proposed in [1].

3. HYPERBOLIC SINGULAR VALUE DECOMPOSITION

Recall the formulation of the hyperbolic singular value decomposition of an arbitrary real matrix proposed in [1]. This theorem generalizes the results obtained in [2–4], where the pseudo-orthogonal matrices were replaced by hyperexchange matrices, which do not form a group. Here and below, we denote by O the zero blocks of matrices of appropriate size.

Theorem 2 (hyperbolic singular value decomposition, see [1]). *Fix a matrix* (1.1). *For an arbitrary real matrix* $A \in \mathrm{Mat}_{n\times N}(\mathbb{R})$, *there exist matrices* $R \in \mathrm{O}(N)$ *and* $L \in \mathrm{O}(p,q)$ *such that*

$$L^\mathrm{T}AR = \Sigma^A, \quad \Sigma^A = \left(\begin{array}{cccc} X_x & O & O & O \\ O & O & I_d & O \\ O & O & O & O \\ \hline O & Y_y & O & O \\ O & O & I_d & O \\ O & O & O & O \end{array}\right)\begin{array}{l} \left.\vphantom{\begin{array}{c}X\\O\\O\end{array}}\right\} p \\ \left.\vphantom{\begin{array}{c}O\\O\\O\end{array}}\right\} q \end{array} \in \mathrm{Mat}_{n\times N}(\mathbb{R}), \tag{3.1}$$

where the first block of Σ^A *has* p *rows, the second block has* q *rows,* X_x *and* Y_y *are diagonal matrices of appropriate sizes* x *and* y, *respectively, with positive unambiguously defined diagonal elements* (*up to a permutation*), *and* I_d *is the identity matrix of size* d.

Moreover, by choosing R, *it is possible to exchange the columns of* Σ^A. *By choosing* L, *one can exchange rows in each block but not between blocks. Therefore, we can always arrange the diagonal elements of the matrices* X_x *and* Y_y *in decreasing order.*

We have

$$d = \operatorname{rank}(A) - \operatorname{rank}(A^{\mathrm{T}}\eta A), \quad x + y = \operatorname{rank}(A^{\mathrm{T}}\eta A),$$

where x is the number of positive eigenvalues of the matrix $A^{\mathrm{T}}\eta A$ and y is the number of negative eigenvalues of $A^{\mathrm{T}}\eta A$.

The matrix Σ^A (3.1) is called the *canonical form* of the matrix A, and the elements of the diagonal blocks X and Y are called *hyperbolic singular values*. Below, we assume that the elements of each of these blocks are arranged in decreasing order.

In [1], an algorithm for calculating the matrices Σ^A, L, and R was described. The hyperbolic singular values are square roots of the absolute values of the eigenvalues of matrix $A^{\mathrm{T}}\eta A$. The columns of the matrix R are the eigenvectors of the matrix $A^{\mathrm{T}}\eta A$. The columns of the matrix L are the eigenvectors of the matrix ηAA^{T} (in the case $d = 0$) and the eigenvectors and generalized eigenvectors of the matrix ηAA^{T} (in the case $d \neq 0$). The matrices L and R are not unique.

Note that the standard singular value decomposition is a special case of the hyperbolic singular value decomposition. In the case $n = p$ and $q = 0$, the parameter d is always zero: $d = \operatorname{rank}(A) - \operatorname{rank}(A^{\mathrm{T}}A) = 0$. In this particular case, we obtain the classical singular value decomposition theorem. The singular value decomposition was first independently discovered by Beltrami [19] in 1873 and by Jordan [20, 21] in 1874. Below, we give the modern formulation of this theorem, which can be found, e.g., in [22, 23].

Theorem 3 (singular value decomposition). *For an arbitrary real matrix $A \in \operatorname{Mat}_{n\times N}(\mathbb{R})$, there exist orthogonal matrices $Q \in \mathrm{O}(n)$ and $P \in \mathrm{O}(N)$ such that*

$$L^{\mathrm{T}}AR = D,$$

where

$$D = \operatorname{diag}(\mu_1, \ldots, \mu_s) \in \operatorname{Mat}_{n\times N}(\mathbb{R}), \quad s = \min(n, N), \quad \mu_1 \geq \mu_2 \geq \ldots \geq \mu_s \geq 0.$$

The numbers $\mu_1, \ldots, \mu_s$ are called *singular values*; they are square roots of the eigenvalues of the matrix $A^{\mathrm{T}}A$. The columns of the matrix R are called *right singular vectors*, and they are eigenvectors of the matrix $A^{\mathrm{T}}A$; the columns of L are called *left singular vectors,* and they are eigenvectors of the matrix AA^{T}.

In the general case, by considering the system of Yang–Mills equations (2.7) and choosing appropriate matrices $Q \in \mathrm{O}(p,q)$ and $P \in \mathrm{SO}(3)$ in transformation (2.11), we can locally (in a neighborhood of the point $x \in \mathbb{R}^{p,q}$) transform the matrix of current J to the canonical form described in Theorem 2 (in the case of the Euclidean space, it is described in Theorem 3). In Theorems 2 and 3, we take as R and L the matrices $P \in \mathrm{SO}(3)$ and $Q^{\mathrm{T}} \in \mathrm{O}(p,q)$ appearing in transformation (2.11), respectively. Note that we can always choose the matrix R from the special orthogonal group $\mathrm{SO}(3)$. If the matrix R in Theorems 2 and 3 has the determinant equal to -1, then, we may simultaneously change the sign of all columns in the matrices L and R, and then their determinant becomes equal to $+1$. Then, the matrix A is not necessarily canonical because transformation (2.11) includes the matrix Ω, which depends on the matrix P.

Let us discuss the particular case of the system of Yang–Mills–Proca equations (1.6) for constant (independent of $x \in \mathbb{R}^{p,q}$) solutions. In this case, system (1.6) takes the form

$$[A_\mu, [A^\mu, A^\nu]] = \frac{1}{\rho^2} J^\nu. \tag{3.2}$$

Equations (2.7) take the form

$$\rho^2 \eta_{\mu\alpha} A_c^\alpha A_a^\mu A_b^\nu \epsilon_d^{ab} \varepsilon_k^{cd} = J_k^\nu. \tag{3.3}$$

Consider (global) transformations that are similar to the gauge transformations (1.9) but with a matrix $S \in \mathrm{SU}(2)$ that is independent of $x \in \mathbb{R}^{p,q}$:

$$A'_\mu = S^{-1}A_\mu S, \quad F'_{\mu\nu} = S^{-1}F_{\mu\nu}S, \quad J'^\nu = S^{-1}J^\nu S. \tag{3.4}$$

In transformations (2.9) and (2.11), we obtain $\Omega = 0$, and they become global symmetries. Using the global symmetry

$$A \to QAP, \quad J \to QJP, \quad Q \in \mathrm{O}(p,q), \quad P \in \mathrm{SO}(3), \tag{3.5}$$

we can reduce the matrices A and J to the canonical form simultaneously. This fact is proved in [13], which also gives the general solution to the system of Yang–Mills equations for the constant solutions (3.3) in the case of the Euclidean space $\mathbb{R}^n$ and arbitrary current J using the ordinary singular expansion. In the case of a pseudo-Euclidean space, a similar problem can be solved using the hyperbolic singular value decomposition; this will be done in a later paper.

Note that the case of zero current for constant solutions to the Yang–Mills equations was studied in [24, 25]. Plane wave solutions to the SU(2) Yang–Mills equations in an arbitrary pseudo-Euclidean space are discussed in [26]; they are reduced to the problem about the constant solutions described in the current paper. Some special cases of the Yang–Mills equations in the formalism of Clifford algebras and Atiyah–Kahler algebras related to constant solutions are discussed in [17, 27, 28].

In this paper, we find all constant solutions to the system of SU(2) Yang–Mills–Proca equations, which can be interpreted as the system of Yang–Mills equations with the current depending on potential.

4. SOLUTIONS TO SU(2) YANG–MILLS–PROCA EQUATIONS

The Proca equations were proposed in [29] in 1936 as a generalization of the Maxwell equations. They differ from the Maxwell equations by the inclusion of the term with mass squared. It is assumed that the Proca equations describe massive particles with spin 1. The Yang–Mills–Proca equations are a natural analog of the Proca equations in the non-Abelian case; i.e., they are simultaneously a generalization of the Yang–Mills and Proca equations, and they are considered, e.g, in [30, 31].

The *system of Yang–Mills–Proca equations* in the pseudo-Euclidean space $\mathbb{R}^{p,q}$ (or, in particular, in the Euclidean space $\mathbb{R}^n$) has the form

$$\partial_\mu A_\nu - \partial_\nu A_\mu - \rho[A_\mu, A_\nu] = F_{\mu\nu}, \tag{4.1}$$

$$\partial_\mu F^{\mu\nu} - \rho[A_\mu, F^{\mu\nu}] + m^2 A^\nu = 0. \tag{4.2}$$

These equations differ from the Yang–Mills equations (1.3), (1.4) by the term m^2A^ν with the mass $m \in \mathbb{R}$. We have $A_\mu \in \mathfrak{g}\mathrm{T}_1$, $J^\nu \in \mathfrak{g}\mathrm{T}^1$, $F_{\mu\nu} = -F_{\nu\mu} \in \mathfrak{g}\mathrm{T}_2$, and $\rho \in \mathbb{R}$. If the mass is zero $m = 0$, then Eqs. (4.1), (4.2) coincide with the Yang–Mills equations (1.3), (1.4) with the zero current $J^\nu = 0$. Below, we consider the case $m \neq 0$.

The Lagrangian of the Yang–Mills–Proca field is

$$\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu a} + \frac{1}{2}m^2 A_a^\nu A_{\nu a}, \tag{4.3}$$

where the components $F^{\mu\nu} = F_a^{\mu\nu}\tau^a$ have form (4.1). By varying the action $\mathcal{S} = \int \mathcal{L}dx$, we obtain Eqs. (4.2).

For the potential A^μ, we obtain from Eqs. (4.1), (4.2) the generalized gauge condition

$$\partial_\mu A^\mu = 0. \tag{4.4}$$

Note that this condition is an analog of the non-Abelian conservation law (1.8) for the Yang–Mills equations (1.3), (1.4). The Yang–Mills–Proca equations (4.1), (4.2) can be interpreted as the Yang–Mills equations with the current $J^\nu = -m^2A^\nu$ depending on the potential. By substituting $J^\nu = -m^2A^\nu$ into (1.8), we obtain (4.4).

Substitute (4.1) into (4.2) to obtain

$$\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu - \rho[A^\mu, A^\nu]\right) - \rho\left[A_\mu, \partial^\mu A^\nu - \partial^\nu A^\mu - \rho[A^\mu, A^\nu]\right] + m^2 A^\nu = 0, \tag{4.5}$$

which, taking into account (4.4), can be written as

$$\partial_\mu\partial^\mu A^\nu - 2\rho[A^\mu, \partial_\mu A^\nu] + \rho[A_\mu, \partial^\nu A^\mu] + \rho^2[A_\mu, [A^\mu, A^\nu]] + m^2 A^\nu = 0. \tag{4.6}$$

Note that the system of equations (4.1), (4.2) is not gauge invariant under transformations (1.9) (as well as the Proca equations [29], which are a generalization of the Maxwell equations, are not gauge invariant). However, the system of Yang–Mills–Proca equations (4.1), (4.2) is invariant under the global (independent of $x \in \mathbb{R}^{p,q}$) transformation

$$A_\mu \to A'_\mu = S^{-1}A_\mu S, \quad F_{\mu\nu} \to F'_{\mu\nu} = S^{-1}F_{\mu\nu}S, \quad S \in \mathrm{G}. \tag{4.7}$$

While finding constant solutions to the Yang–Mills–Proca equations, we obtain the system of equations

$$[A_\mu, [A^\mu, A^\nu]] = -\lambda A^\nu, \quad \lambda = \frac{m^2}{\rho^2} > 0, \tag{4.8}$$

which can be interpreted as the system of Yang–Mills equations for constant solutions with the current $J^\nu = -\lambda A^\nu$ depending on the potential A^ν.

Next, we consider the case of the Lie group $\mathrm{G} = \mathrm{SU}(2)$ and the corresponding (real) Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$. Fix basis (2.4) of the Lie algebra $\mathfrak{su}(2)$, and write the system of equations (4.6) as

$$\partial_\mu\partial^\mu A_k^\nu - 2\rho\epsilon_k^{ab}A_a^\mu\partial_\mu A_b^\nu + \rho\epsilon_k^{ab}\eta_{\mu\alpha}A_a^\alpha\partial^\nu A_b^\mu + \rho^2\eta_{\mu\alpha}A_c^\alpha A_a^\mu A_b^\nu\epsilon_d^{ab}\epsilon_k^{cd} + m^2 A_k^\nu = 0. \tag{4.9}$$

The system of equations (4.8) for the constant solutions takes the form

$$\eta_{\mu\alpha}A_c^\alpha A_a^\mu A_b^\nu\epsilon_d^{ab}\epsilon_k^{cd} = -\lambda A_k^\nu, \quad \lambda = \frac{m^2}{\rho^2} > 0. \tag{4.10}$$

The system of equations (4.10) is invariant under the transformation

$$A \to QAP, \quad Q \in \mathrm{O}(p,q), \quad P \in \mathrm{SO}(3), \tag{4.11}$$

where the orthogonal matrix $P = \left(p_b^a\right) \in \mathrm{SO}(3)$ is related to the matrix $S \in \mathrm{SU}(2)$ in the global transformation (4.7) as the two-sheet cover

$$S^{-1}\tau^a S = p_b^a\tau^b.$$

The matrix $Q = (q_\nu^\mu) \in \mathrm{O}(p,q)$ corresponds to the change of coordinates $x^\mu \to q_\nu^\mu x^\nu$ of the space $\mathbb{R}^{p,q}$ (similarly to how it was for the Yang–Mills equations in Theorem 1).

After finding all solutions to system (4.10), we can calculate the components of strength

$$F^{\mu\nu} = -\rho[A^\mu, A^\nu] = -\rho[A_a^\mu\tau^a, A_b^\nu\tau^b] = -\rho A_a^\mu A_b^\nu\epsilon_c^{ab}\tau^c = F_c^{\mu\nu}\tau^c \tag{4.12}$$

and the invariant F^2:

$$F^2 = F_{\mu\nu}F^{\mu\nu} = -\frac{\rho^2}{2}\sum_{\mu<\nu}\eta^{\mu\mu}\eta^{\nu\nu}(F_c^{\mu\nu})^2 I_2. \tag{4.13}$$

Let us formulate and prove a theorem about all solutions to system (4.10), i.e., about all constant solutions of the system of Yang–Mills–Proca equations in the case of the Lie group SU(2).

Theorem 4. *Any solution* $A = (A_a^\mu)$ *to the system of Yang–Mills–Proca equations* (4.10) *in the pseudo-Euclidean space* $\mathbb{R}^{p,q}$ (*or in the Euclidean space* $\mathbb{R}^n$) *can be reduced, by choosing matrices* $Q \in \mathrm{O}(p,q)$ *and* $P \in \mathrm{SO}(3)$ *in symmetry* (4.11), *to the solution of one of the following forms*:

(1) *in the cases* $\mathbb{R}^{p,q}$, $p \geq 3$, $q \geq 0$:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \end{pmatrix}, \quad a := \sqrt{\frac{\lambda}{2}}, \tag{4.14}$$

i.e.,

$$A_a^\mu = \begin{cases} \sqrt{\dfrac{\lambda}{2}} & \text{for} \quad \mu = a = 1, 2, 3; \\ 0, & \text{otherwise} \end{cases}$$

with the following nonzero strength components:

$$F^{12} = -F^{21} = -\frac{\rho\lambda}{2}\tau^3, \quad F^{23} = -F^{32} = -\frac{\rho\lambda}{2}\tau^1, \quad F^{31} = -F^{13} = -\frac{\rho\lambda}{2}\tau^2, \tag{4.15}$$

and invariant

$$F^2 = -\frac{3\rho^2\lambda^2}{8} I_2 \neq 0; \tag{4.16}$$

(2) *in the cases* $\mathbb{R}^{p,q}$, $p \geq 2$, $q \geq 0$:

$$A = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \\ \dots & \dots & \dots \\ 0 & 0 & 0 \end{pmatrix}, \quad b := \sqrt{\lambda}, \tag{4.17}$$

i.e.,

$$A_a^\mu = \begin{cases} \sqrt{\lambda} & \text{for} \quad \mu = a = 1, 2; \\ 0, & \text{otherwise} \end{cases}$$

with the following nonzero strength components:

$$F^{12} = -F^{21} = -\rho\lambda\tau^3, \tag{4.18}$$

and invariant

$$F^2 = -\frac{\rho^2\lambda^2}{2} I_2 \neq 0; \tag{4.19}$$

(3) *in the cases* $\mathbb{R}^{p,q}$, $p \geq 0$, $q \geq 0$:

$$A = 0, \quad F = 0, \quad F^2 = 0 \quad \forall \lambda > 0. \tag{4.20}$$

Proof. We use the invariance of Eqs. (4.10) under transformations (4.11) and the hyperbolic singular value decomposition (Theorem 2). Note that we can always choose the matrix P from the special orthogonal group SO(3). If the determinant of this matrix is -1, then we can simultaneously change the sign of all columns in P and Q to make the determinant equal to $+1$.

Let the elements of the matrix A satisfy the system of equations (4.10) in the pseudo-Euclidean space $\mathbb{R}^{p,q}$ (or in the Euclidean space $\mathbb{R}^n$). Then, there exist matrices $P \in SO(3)$ and $Q \in O(p,q)$ (or $Q \in O(n)$, respectively) such that the matrix QAP has the canonical form

$$\left.\begin{pmatrix} X_x & O & O & O \\ O & O & I_d & O \\ O & O & O & O \\ \hline O & Y_y & O & O \\ O & O & I_d & O \\ O & O & O & O \end{pmatrix}\begin{matrix} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} p \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} q \end{matrix}\right. .$$

System (4.10) takes a new form with the unknowns—hyperbolic singular values of the matrix A (elements of the diagonal blocks X and Y). Next, we should consider various cases of the canonical form of the matrix A depending on the values of the parameters x, y, and d and solve the corresponding systems of equations. The elements of each diagonal block X and Y are assumed to be positive and arranged in decreasing order. In total, there are 20 different cases of values of the parameters (d, x, y) of the matrix A:

$$(0, 3, 0), \quad (0, 0, 3), \quad (0, 2, 1), \quad (0, 1, 2), \quad (0, 2, 0), \quad (0, 0, 2), \quad (0, 1, 1),$$
$$(0, 1, 0), \quad (0, 0, 1), \quad (0, 0, 0), \quad (1, 2, 0), \quad (1, 0, 2), \quad (1, 1, 1), \quad (1, 1, 0),$$
$$(1, 0, 1), \quad (1, 0, 0), \quad (2, 1, 0), \quad (2, 0, 1), \quad (2, 0, 0), \quad (3, 0, 0).$$

In the first case ($d = 0$, $x = 3$, $y = 0$), only nonzero diagonal terms remain in system (4.10); therefore, $\mu = \alpha = a = c$, $\nu = b = k$, and the product of two Levi–Chivita symbols gives -1. Then, we obtain the following system of equations, in which the positive elements of the diagonal block X arranged in decreasing order are denoted by a_1, a_2, and a_3, respectively:

$$a_1(a_2^2 + a_3^2) = \lambda a_1, \quad a_2(a_1^2 + a_3^2) = \lambda a_2, \quad a_3(a_1^2 + a_2^2) = \lambda a_3, \quad a_1, a_2, a_3, \lambda > 0. \tag{4.21}$$

This system has the only solution

$$a_1 = a_2 = a_3 = \sqrt{\frac{\lambda}{2}}. \tag{4.22}$$

In the second, third, and fourth cases, we obtain, respectively, the following systems each of which has no solutions:

$$-a_1(a_2^2 + a_3^2) = \lambda a_1, \quad -a_2(a_1^2 + a_3^2) = \lambda a_2, \quad -a_3(a_1^2 + a_2^2) = \lambda a_3, \quad a_1, a_2, a_3, \lambda > 0;$$
$$a_1(a_2^2 - a_3^2) = \lambda a_1, \quad a_2(a_1^2 - a_3^2) = \lambda a_2, \quad a_3(a_1^2 + a_2^2) = \lambda a_3, \quad a_1, a_2, a_3, \lambda > 0;$$
$$-a_1(a_2^2 + a_3^2) = \lambda a_1, \quad a_2(a_1^2 - a_3^2) = \lambda a_2, \quad a_3(a_1^2 - a_2^2) = \lambda a_3, \quad a_1, a_2, a_3, \lambda > 0.$$

In the fifth case ($d = 0$, $x = 2$, $y = 0$), we obtain the system for two diagonal elements a_1 and a_2 of the block X

$$a_1 a_2^2 = \lambda a_1, \quad a_2 a_1^2 = \lambda a_2, \quad a_1, a_2, \lambda > 0, \tag{4.23}$$

with the general solution

$$a_1 = a_2 = \sqrt{\lambda}. \tag{4.24}$$

In the sixth and the seventh cases, we have, respectively, the following systems each of which has no solutions:

$$-a_1 a_2^2 = \lambda a_1, \quad -a_2 a_1^2 = \lambda a_2, \quad a_1, a_2, \lambda > 0;$$
$$a_1 a_2^2 = \lambda a_1, \quad -a_2 a_1^2 = \lambda a_2, \quad a_1, a_2, \lambda > 0.$$

In the eighth and ninth cases, we have the following system for the only nonzero element a_1 of the matrix A, which has no solutions:

$$0 = \lambda a_1, \quad a_1, \lambda > 0. \tag{4.25}$$

In the tenth case $d = x = y = 0$, we have the trivial solution $A = 0$ for any $\lambda > 0$.

In cases 11–13, we obtain, respectively, the systems

$$a_1 a_2^2 = \lambda a_1, \quad a_2 a_1^2 = \lambda a_2, \quad a_1^2 + a_2^2 = \lambda, \quad a_1, a_2, \lambda > 0;$$

$$-a_1 a_2^2 = \lambda a_1, \quad -a_2 a_1^2 = \lambda a_2, \quad -(a_1^2 + a_2^2) = \lambda, \quad a_1, a_2, \lambda > 0;$$

$$-a_1 a_2^2 = \lambda a_1, \quad a_2 a_1^2 = \lambda a_2, \quad a_1^2 - a_2^2 = \lambda, \quad a_1, a_2, \lambda > 0,$$

which have no solutions. In the remaining cases 14–20 the resulting systems are also inconsistent.

For the three types of solutions, we calculate the strength components using (4.12) and the invariant F^2 using (4.13). This completes the proof of the theorem.

5. NONCONSTANT SOLUTIONS TO THE SU(2) YANG–MILLS–PROCA EQUATIONS IN THE FORM OF A PERTURBATION THEORY SERIES

In Theorem 4, we obtained an explicit form of all constant solutions to the SU(2) Yang–Mills–Proca equations (4.1), (4.2) in an arbitrary pseudo-Euclidean (or Euclidean) space $\mathbb{R}^{p,q}$. The constant solutions to the equations thus obtained allow us to construct nonconstant solutions to the Yang–Mills–Proca equations in the form of a perturbation theory series. More precisely, expand the solution to Eqs. (4.1), (4.2) in a small parameter $\varepsilon \ll 1$

$$A^\mu = \sum_{k=0}^{\infty} \varepsilon^k \overset{k}{A}{}^\mu = \overset{0}{A}{}^\mu + \varepsilon \overset{1}{A}{}^\mu + \varepsilon^2 \overset{2}{A}{}^\mu + \dots = (\overset{0}{A}{}^\mu_a + \varepsilon \overset{1}{A}{}^\mu_a + \varepsilon^2 \overset{2}{A}{}^\mu_a + \dots)\tau^a, \tag{5.1}$$

where the zero-order approximations $\overset{0}{A}{}^\mu$ are the constant (independent of $x \in \mathbb{R}^{p,q}$) solutions to the Yang–Mills–Proca equations (4.1), (4.2). Substitute (5.1) into Eqs. (4.6) to obtain the equation

$$\sum_{k=0}^{\infty} \varepsilon^k \overset{k}{Q}{}^\nu = 0,$$

where $\overset{k}{Q}{}^\nu$ are differential expressions depending on $\overset{0}{A}{}^\mu, \dots, \overset{k}{A}{}^\mu$ for each $k = 0, 1, \dots$. Since $\overset{0}{A}{}^\mu$ are constant solutions to Eqs. (4.6), it is easy to verify that

$$\overset{0}{Q}{}^\nu = [\overset{0}{A}_\mu, [\overset{0}{A}{}^\mu, \overset{0}{A}{}^\nu]] + \lambda \overset{0}{A}{}^\nu = 0.$$

Next, we obtain a system of linear partial differential equations $\overset{1}{Q}{}^\nu = 0$ with constant coefficients (depending on $\overset{0}{A}{}^\mu$) for finding $\overset{1}{A}{}^\mu$. After the solutions $\overset{1}{A}{}^\mu$ to this system have been found, we substitute them and the solutions $\overset{0}{A}{}^\mu$ into the system $\overset{2}{Q}{}^\nu = 0$. This gives a system of linear partial differential equations with variable coefficients (which depend on $x \in \mathbb{R}^{p,q}$) for finding $\overset{2}{A}{}^\mu$. Next, we substitute these solutions into $\overset{3}{Q}{}^\nu = 0$, and so on. By continuing this process, we find $\overset{k}{A}{}^\mu$ for all $k = 0, 1, \dots$, and thus find nonconstant solutions to the Yang–Mills–Proca equations in the form of series (5.1).

This algorithm for finding nonconstant solutions reduces the solutions of the nonlinear (cubic) Yang–Miils–Proca equations to solving systems of linear partial differential equations.

Now, we discuss the system for the first-order approximation $\overset{1}{Q}{}^\mu = 0$ in more detail. To derive an explicit form of this system, we set $A^\mu = \overset{0}{A}{}^\mu + \varepsilon \overset{1}{A}{}^\mu(x) = (\overset{0}{A}{}^\mu_a + \varepsilon \overset{1}{A}{}^\mu_a(x))\tau^a$ with $\varepsilon^2 = 0$. We can choose matrices $P \in \mathrm{SO}(3)$ and $Q \in \mathrm{O}(p,q)$ in symmetry (4.11) such that the matrix composed of $\overset{0}{A}{}_a^{\mu}$ is diagonal (see Theorem 4). This gives the system of $3n$ linear partial differential equations

$$\begin{aligned} &\partial_\mu \partial^\mu u_k^\nu - 2\rho \epsilon_k^{ab} h_a^\mu \partial_\mu u_b^\nu + \rho \epsilon_k^{ab} \eta_{\mu\alpha} h_a^\alpha \partial^\nu u_b^\mu \\ &+ \rho^2 \eta_{\mu\alpha} \epsilon_d^{ab} \epsilon_k^{cd} (h_c^\alpha h_a^\mu u_b^\nu + h_c^\alpha u_a^\mu h_b^\nu + u_c^\alpha h_a^\mu h_b^\nu) + m^2 u_k^\nu = 0, \end{aligned} \tag{5.2}$$

for the unknown functions $u_a^\mu := \overset{1}{A}{}_a^\mu(x)$ with known constant coefficients $h_a^\mu := \overset{0}{A}{}_a^\mu$ depending on the parameter $\lambda = \frac{m^2}{\rho^2}$. These coefficient h_a^μ are the elements of one of the diagonal matrices (4.14), (4.17) or the completely zero matrix (4.20), depending on the type of constant solutions.

In the case of the zero matrix, we have $h_a^\mu = 0$, and obtain the following system for the first-order solution:

$$\partial_\mu \partial^\mu u_k^\nu + m^2 u_k^\nu = 0. \tag{5.3}$$

Note that Eqs. (5.3) are the Klein–Gordon–Fock equations for each component u_k^ν.

In the case of solution (4.14), we can substitute into system (5.2) the expressions

$$h_a^\mu = \begin{cases} \sqrt{\frac{\lambda}{2}} & \text{for} \quad \mu = a = 1, 2, 3; \\ 0, \text{ otherwise.} \end{cases}$$

In the case of solution (4.17) we can substitute into system (5.2) the expressions

$$h_a^\mu = \begin{cases} \sqrt{\lambda} & \text{for} \quad \mu = a = 1, 2; \\ 0, \text{ otherwise.} \end{cases}$$

The resulting systems of equations for the unknown functions u_a^μ can be investigated using known numerical methods and methods of the theory of linear partial differential equations.

CONCLUSIONS

In this paper, we showed how methods of computational mathematics (singular value decomposition and hyperbolic singular value decomposition) can be used for studying the Yang–Mills and Yang–Mills–Proca equations in the case of the Lie group SU(2), which is important for describing electroweak interactions. An explicit form of all constant solutions to the system of Yang–Mills–Proca equations in the case of the Lie group SU(2) is obtained and these solutions are classified. Nonconstant solutions to the Yang–Mills–Proca equations are considered as series of the perturbation theory. It would be interesting to further study the first-order approximation linear systems of equations. The results can be useful for describing physical vacuum [32–34].

Note that the methods considered in this paper cannot be directly applied to the case of the Lie group SU(3), which is important for describing strong interactions, since we substantially use the two-sheet cover of the orthogonal group SO(3) by the spin group SU(2). The extension of the proposed methods to the case of the Lie group SU(3) is an interesting topic of further research.

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