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On the quantitative quasi-isometry problem: Transport of Poincaré inequalities and different types of quasi-isometric distortion growth

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We consider a quantitative form of the quasi-isometry problem. We discuss several arguments which lead us to a number of results and bounds of quasi-isometric distortion: comparison of volumes, connectivity, etc. Then we study the transport of Poincaré constants by quasi-isometries and we give sharp lower and upper bounds for the homotopy distortion growth for a certain class of hyperbolic metric spaces, a quotient of a Heintze group $\mathbb{R} \ltimes \mathbb{R}^n$ by $\mathbb{Z}^n$. We also prove the linear distortion growth between hyperbolic space $\mathbb{H}^n, n \geq 3$ and a tree.

1. Introduction

In this article we study a quantitative form of the quasi-isometry problem: we give lower and upper bounds of quasi-isometry constants λ and c for different classes of spaces. Along the way, we give a method to transport Poincaré inequalities by quasi-isometries. This result allows to get sharp bounds of quasi-isometry constants for certain spaces.

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The quantitative quasi-isometry problem consists in evaluating how close two metric spaces can be at various scales, a problem stated in the Shchur's PhD thesis [29]. Specifically, let E, F be two metric spaces. Consider a ball of radius R in the first space E and take a (λ, c)-quasi-isometric embedding of this ball in F. We are interested in the behaviour of the infimum of the sum $\lambda + c$ of quasi-isometry constants as a function of R. We call this quantity *a quasi-isometric distortion growth*.

Our approach is related to the quasi-isometry rigidity and classification problems (see [10], p. 101, and [29], p. 24, for the detailed surveys), but we consider the quasi-isometry classification from the quantitative point of view. Indeed, as soon as two spaces are known to be not quasi-isometric, it would be desirable to give a quantitative measurement of this fact. For this purpose, we suggest studying quasi-isometric embeddings of balls of varying radius from one space into the other.

One of the first appearances of quasi-isometries is the proof of the famous Mostow's theorem [20] which claims that every isomorphism of two lattices $\Gamma, \Gamma' \subset Isom(\mathbb{H}^n)$ is induced by an isometry of $\mathbb{H}^n$. In the course of the proof Mostow established that two quasi-isometric rank 1 symmetric Riemannian spaces of negative curvature are homothetic. This theorem was followed by generalisations of P. Pansu [21] (case of rank one) and B. Kleiner and B. Leeb [19] (higher ranks) which allow proceeding towards a quasi-isometry classification of some important classes of metric space (for example, irreducible thick Euclidean Tits buildings).

Interesting and profound results were achieved for the quasi-isometric classification of 3-manifolds. M. Kapovich and B. Leeb [18] pose the question of whether the fundamental groups of all (closed) graph manifolds are quasi-isometric. Their result reduces this problem to the case of non-positively curved manifolds. Further, J.A. Behrstock and W.D. Neumann [2] proved that the fundamental groups of any two closed irreducible non-geometric graph-manifolds are quasi-isometric. They develop their study by classifying the quasi-isometry types of fundamental groups of graph-manifolds with boundary in terms of certain finite two-coloured graphs. They extend their study in [3] by describing the quasi-isometric classification of fundamental groups of irreducible non-geometric 3-manifolds which do not have "too many" arithmetic hyperbolic geometric components.

There are many results in the case of solvable groups. In particular, P. Pansu [22] proved that if two finitely-generated nilpotent groups are quasi-isometric, then the associated graded Lie groups are isomorphic. Also Y. Shalom [27] proved that quasi-isometric finitely generated nilpotent groups have the same Betti numbers.

Furthermore, A. Eskin, D. Fisher and K. Whyte launched a huge study of quasi-isometries of Lie groups of the form $\mathbb{R}^m \ltimes_M \mathbb{R}^n$; the work is still in progress. The current result states that for $m = 1$ and for two diagonalisable matrices with no eigenvalues on the unit circle, the corresponding Lie groups are quasi-isometric if and only if there exists $\alpha \in \mathbb{R}$ such that M^α and M' have the same absolute Jordan form. Parts and special cases of this theorem can be found in a series of papers by A. Eskin, D. Fisher, K. Whyte [11–13], T. Dymarz [9] and I. Peng [25,26].

Our problem is reminiscent of the Ramsey-type theory,[2] which generally states that a large proportion of a given metric space can be embedded with small distortion into some well-structured family of metric spaces. The metric Ramsey-type embeddings of finite metric spaces into finite dimensional normed spaces was studied in [1]. The main result of the mentioned paper is that for every $\varepsilon > 0$, any n-point metric space has a subset of size $n^{1-\varepsilon}$ which embeds into Hilbert space with distortion $\left(\frac{\log(1/\varepsilon)}{\varepsilon}\right)$. Notice that every space can be discretised with a $(1, 1)$-quasi-isometry, hence it is possible to pass to a study of finite metric spaces. It suggests another potential direction of the quantitative study of quasi-isometries. Proposition 2.4 is an example of such type of results: we build a quasi-isometry from a ball in a tree into a $\sqrt{R}$-dense subset of a ball in $\mathbb{H}^2$ with quasi-isometry distortion growth $\sqrt{R}$.

1.1. Basic definitions

In this part we will give the definitions of the main objects in our study: quasi-isometry, quasi-isometric embedding and quasi-isometric distortion growth.

Definition 1.1. Two metric spaces X and Y are said to be roughly quasi-isometric if there exists a pair of maps $f : X \to Y$, $g : Y \to X$ and two constants $\lambda > 0$ and $c \geq 0$ such that

- $|f(x) - f(y)| \leq \lambda|x - y| + c$ for every $x, y \in X$,
- $|g(x') - g(y')| \leq \lambda|x' - y'| + c$ for every $x', y' \in Y$,
- $|g(f(x)) - x| \leq c$ for every $x \in X$,
- $|f(g(x')) - x'| \leq c$ for every $x' \in Y$.

The word *rough* is often omitted.

The first two conditions mean that f and g are nearly Lipschitz if one is looking from afar, or at large scales. The additive constant c allows ignoring the local geometry of spaces. The two latter conditions provide that f and g are nearly inverse of each other. It is easy to check that the composition of two quasi-isometries is also a quasi-isometry. So, quasi-isometries provide an equivalence relation on the class of metric spaces.

Remark 1.1. Definition 1.1 is invariant under taking inverse maps.

Definition 1.2. A map $f : E \to F$ between metric spaces is a *rough* $(\lambda_1, c_1, \lambda_2, c_2)$-*quasi-isometric embedding* if for any two points x, y of E

$$\frac{1}{\lambda_2}(|x - y|_E - c_2) \leq |f(x) - f(y)|_F \leq \lambda_1|x - y|_E + c_1.$$

[2] We are thankful to the reviewer for pointing out paper [1] to us.

This definition includes quasi-isometries (with $\lambda_1 = \lambda_2$ and $c_1 = c_2$), but it does not require the existence of a nearly inverse map. The introduction of four constants instead of two allows tracing the role of each inequality in this definition, which leads to a better understanding of quantitative issues. It is easy to see that if the space E is continuous, then its image in F is c_1-connected. It means that, for every two points $z_1, z_2 \in f(E)$, there exists a chain of points $\tilde{z}_1 = z_1, \tilde{z}_2, \dots, \tilde{z}_n = z_2$, such that $\tilde{z}_i \in f(E)$ for $1 \le i \le n$ and $d(z_i, z_{i+1}) \le c_1$ for any $i < n$.

The following definition formalises the quantitative problem (see [29], pp. 17–24, for the explanation of such a choice).

Definition 1.3. Let X, Y be metric spaces, x_0, y_0 their base points respectively. The quasi-isometric distortion growth is the function

$$D_G(X, x_0, Y, y_0)(R) = \inf\{d | \exists f : B_X(x_0, R) \to Y \text{ a } (\lambda_f, c_f)\text{-quasi-isometric embedding such that } f(x_0) = y_0 \text{ and } d = \lambda_f + c_f\}.$$

We will study the growth of D_G as a function of R.

In this paper, we usually speak about quasi-isometric embeddings of balls in one space into another space. But some constructions of quasi-isometries (for example, Propositions 2.3 and 2.4) will provide quasi-isometries between balls in two spaces. Of course such constructions are limited to a narrower class of maps, but still lead to interesting results and answer some quantitative questions.

1.2. First examples

Let us start the discussion with some relatively basic, but important properties of metric spaces and quasi-isometries. They should be borne in mind when studying the quantitative quasi-isometry problem. For example, volume considerations show that any space with polynomial volume growth deviates linearly from any space of exponential volume growth (see Proposition 2.1).

Note, that since one may always take $\lambda = 1$ and $c = R$, the deviation between any two spaces is at most linear.

Connectedness considerations provide a lower bound of $\sqrt{R}$ for embeddings of Euclidean or hyperbolic balls into trees. In the hyperbolic case, this bound is sharp (see Proposition 2.3). Probably, in the family of Gromov hyperbolic metric spaces, deviations should be of the order $\sqrt{R}$. Indeed, we show (see Proposition 2.4) that given two hyperbolic metric spaces that are thick enough, one can map a $\sqrt{R}$-dense subset of an R-ball of the first space into the second one with $\sqrt{R}$ distortion. However, we have been unable to extend such embeddings to the full R-ball while preserving the same distortion.

There seems to be a rather subtle obstruction for constructing such an extension. For instance, we show in Theorem 7.1 that mapping a tree into hyperbolic space requires linear distortion. The proof is based on the notion of separation, cf. [4,5,16].

1.3. Main result

Our main result is another step towards capturing such obstructions. We consider a class of negatively curved locally homogeneous Riemannian manifolds which are not simply connected, but nevertheless hyperbolic. We prove a sharp linear lower bound on the distortion of embeddings which are homotopy equivalences.

Let $\mathbb{T}^n$ denote the n-dimensional torus. Given positive numbers $\mu_1 \leq \cdots \leq \mu_n$, denote $Z_\mu = \mathbb{T}^n \times \mathbb{R}$, where the product space is equipped with the Riemannian metric $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$. The universal cover of Z_μ is a Riemannian homogeneous space. Z_μ is a hyperbolic metric space. Its ideal boundary is a product of circles, and each of them has a metric which is a power of the usual metric. Thus Z_μ can be viewed as a hyperbolic cone over this fractal torus. Essentially, our theorem states that the quasi-isometric distortion growth function between such spaces is linear if one requires maps to be isomorphic on fundamental groups.

Theorem 1.1. *(For a detailed quantitative statement, see Theorem 6.2.) For R large enough, every (λ, c)-quasi-isometric embedding Θ of an R-ball of Z_μ into $Z_{\mu'}$ (up to replacing the spaces with connected 2-sheeted coverings) which is a homotopy equivalence satisfying*

$$\lambda + c \geq \mathit{const} \left(\frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_n}{\mu'_n} \right) R.$$

Conversely, there exist homotopy equivalences with linearly growing distortion,

$$\lambda + c \leq \mathit{const} \max |\mu_i - \mu'_i| R,$$

from an R-ball of Z_μ into $Z_{\mu'}$. This is a special case of a more general result which we describe next.

Further, a distance in a hyperbolic metric space can be approximated by the visual distance on the ideal boundary (Lemma 8.3). Using this approximation we find quasi-isometry constants for the restriction on balls of a map Θ between X and Y, where the map Θ is a kind of radial extension of a homeomorphism θ between ideal boundaries. For the detailed and more technical statement of the following theorem, see Theorem 8.1 in Section 8.

Theorem 1.2. *Let X, Y be hyperbolic metric spaces. Let $\theta : \partial X \to \partial Y$ be a homeomorphism on their boundaries. We define the following function for $R > 0$,*

$$K(R) = \sup \left\{ \left| \log \frac{d_{y_0}(\theta(\xi_1), \theta(\xi_2))}{d_{x_0}(\xi_1, \xi_2)} \right| \, | d_{x_0}(\xi_1, \xi_2) \geq e^{-R} \vee d_{y_0}(\theta(\xi_1), \theta(\xi_2)) \geq e^{-R} \right\}.$$

Here d_{x_0}, d_{y_0} denote visual metrics on ideal boundaries. Then there exists a $(K(R), K(R))$-quasi-isometry between $B_X(x_0, R)$ and $B_Y(y_0, R)$.

For spaces Z_μ, we show that $K(R) = R \cdot \max_i |\mu_i/\mu_i' - 1|$. We also give an example of a pair of non-quasi-isometric negatively curved locally homogeneous manifolds and a homeomorphism θ between their ideal boundaries with $K(R) \lesssim \log R$. This result shows that sublinear (possibly logarithmic) distortion growths also occur in the world of hyperbolic metric spaces.

1.4. Stability of Poincaré inequalities and proof of Theorem 1.1

The proof of Theorem 1.1 involves several results which could have an independent interest and more applications. In particular, we study the transport of Poincaré inequalities by quasi-isometries. For this purpose, kernels are introduced to regularise transported functions. Kernels allow transporting functions from Y to X, while controlling quantitatively their Poincaré constants.

One of the first studies of such type of problem was made by M. Kanai in [17]. In this work the stability of isoperimetric inequalities under quasi-isometries is discussed in the case of Riemannian manifolds. Consider a bounded domain $\Omega \subset X$ with a sufficiently smooth boundary, so that $\partial\Omega$ is measured. The m-dimensional isoperimetric constant $I_m(X)$ of a space X is the infimum over all such domains of the ratio of an area of a boundary $\partial\Omega$ to the volume of Ω to the power $(m-1)/m$

$$I_m(X) = \inf_{\Omega \subset X} \frac{Area(\partial\Omega)}{(Vol(\Omega))^{(m-1)/m}}.$$

Let X and X' be two quasi-isometric spaces with the Ricci curvature bounded from below by $-(\dim X - 1)\cdot const$ and $-(\dim X' - 1)\cdot const$ respectively with some positive constant. Then for $m > \max\{\dim X, \dim X'\}$ the isoperimetric constants are either strictly positive or vanish together.

In [7] the stability of Poincaré and Sobolev type inequalities for weighted graphs are studied. Further, R. Tessera used kernels to define a coarse Laplacian at a scale h in [30]. This allows defining a coarse Sobolev inequality which ignores local geometry of Riemannian manifolds. The important result of this work is that if X and X' are two quasi-isometric spaces satisfying the locally doubling property (the volume of a ball is controlled from above by the volume of a ball with a halted radius), and if X satisfies a Sobolev inequality at scale h, then X' also satisfies a Sobolev inequality at scale h'. Moreover, h' depends only on h and quasi-isometry constants.

In our approach we are also interested in the stability of Poincaré inequalities under quasi-isometries. But, we would like to transport them by quasi-isometries while controlling the dependencies of Poincaré constants on the quasi-isometry constants and initial kernel's constants. This approach entirely agrees with our quantitative study.

Now let us give more details on the proof of the theorem itself. It has several steps. Firstly, we introduce non-trivial double-covering spaces $\tilde{Z}$ and $\tilde{Z}'$ of $Z = Z_\mu$ and $Z' = Z_{\mu'}$. The quasi-isometric embedding Θ lifts to a $(\lambda_1, 2c_1)$-coarse Lipschitz map. Now consider the test-function $e^{\pi i x_n}$ on $\tilde{Z}'$ which depends only on one coordinate x_n. It varies very slowly outside a ball, so the absolute value of the transported and regularised function v on $\tilde{Z}$ stays close to 1. Lemmas 4.3 and 4.4 control how the lower bound of Poincaré constant changes under transport. It helps to get a lower bound for the Poincaré constant of $\tilde{Z}$ in terms of $\{\mu_i\}, \{\mu_i'\}$ and the constants of quasi-isometric embedding. At the same time Theorem 5.1 provides an upper bound for the Poincaré constant of $\tilde{Z}$. The combination of these results leads to a lower bound for the homotopy distortion growth for Z and Z'.

1.5. Paper structure

The results are organised in the following way.

- In Section 2, we discuss some basic arguments: volume consideration and rough connectedness lead to first bounds on quasi-isometric distortion growth.
- In Section 3, we give definitions of kernels and generalised Poincaré inequality associated with kernels. Further we discuss the idea of the proof of Theorem 1.1 which relies on the transport of Poincaré inequalities by quasi-isometries.
- In Section 4, we study the transport of Poincaré inequalities by quasi-isometries in a quantitative way.
- In Section 5, we give an upper bound for the Poincaré constant of balls in spaces of the form $\tilde{X} = \mathbb{R}_+ \times \mathbb{R}^n$ with metric $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$ quotiented by $\mathbb{Z}^n$.
- In Section 6, we prove Theorem 1.1.
- In Section 7, we prove a linear lower bound of a quasi-isometry distortion growth for a hyperbolic space $\mathbb{H}^n, n \geq 3$ and a regular tree based on the notion of coarse separation.
- In Section 8, we give an approximation of distance in Gromov hyperbolic spaces using visual distance on its ideal boundary. Using this result we construct a map induced by a boundary homeomorphism which provides a non-trivial upper bound on quasi-isometry distortion growth.
- In Section 9, we apply the results of the previous section to several examples. In the case of unipotent locally homogeneous spaces, we obtain logarithmic quasi-isometry distortion growth.
- In Appendix A, we discuss the hypothesis of Theorem 6.2 that the quasi-isometric embeddings under consideration are a homotopy equivalence. We will show that if $\dim(Z) \geq 3$, one may believe that the assumption that Θ is isomorphic on fundamental groups is not that restrictive. Yet this result is not effective, so it is not valuable for our study.

2. General discussion

Here we provide basic arguments which provide lower bounds on quasi-isometry constants.

2.1. Comparison of volumes

To begin with, let us show that the comparison of volumes in the domain and in the range plays an important role.

An image of a ball by quasi-isometry is not necessarily a measurable set. Though, for the sake of simplicity we will speak about volumes of such sets in the proof of the following theorem. Here a volume can be considered as the infimum of volumes of sets of 1-balls covering the corresponding image.

Consider a space X with an exponential volume growth (for example, hyperbolic plane $\mathbb{H}^2$) and a space Y with a polynomial volume growth (for example, Euclidean space $\mathbb{R}^n$), then constants of quasi-isometric embedding of X into Y grow linearly in R: $\lambda_R + c_R = \Omega(R)$ due to the limitations provided by the additive constant c_R.

Proposition 2.1. *Let X be a space with exponential volume growth and Y be a space with polynomial volume growth. Then for any (λ, c)-quasi-isometric embedding of a ball $B_X(R)$ into Y, the additive constant c grows linearly in R, or*

$$c \geq const \cdot R.$$

For the sake of simplicity, in the proof, assume that the volume of a ball $B_X(R)$ in X is e^R and the volume of a ball $B_Y(R)$ in Y is R^α.

Proof. Let $B_X(R)$ be a ball in X, $f : B_X(R) \to Y$ be a (λ, c)-quasi-isometric embedding. Then the diameter of the image $f(B_X(R))$ is $\leq 2\lambda R + c$. Consider a maximal set S of points in $B_X(R)$ such that pairwise distances between these points are at least $2c$. The cardinality of S can be estimated as follows: $\#(S) \sim Vol(B_X(R)/ Vol(B_X(2c)))$. For any two points $s_1, s_2 \in S$ the distance between their images is at least c/λ. Hence, the volume of $f(B_X(R))$ is at least $\#(S) \times Vol(B_Y(c/\lambda))$.

So, on the one hand $Vol(f(B_X(R))) \leq Vol(B_Y(2\lambda R + c))$ and on the other hand $Vol(f(B_X(R))) \geq Vol(B_Y(c/\lambda))\, Vol(B_X(R)/ Vol(B_X(2c))$. We get

$$(c/\lambda)^\alpha e^{R-2c} \leq (2\lambda R + c)^\alpha.$$

For R big enough, this inequality can be satisfied only if exponential term disappears, that is $c = R/2$. □

Remark 2.1. The same argument yields lower bounds on quasi-isometry constants between balls of the same radius in spaces of different exponential growths. This does not prevent such spaces from being quasi-isometric. For instance, P. Papasoglu [24] shows that two regular trees of degrees at least 4 are always quasi-isometric. The quasi-isometry provided in [24] does not preserve the distance to a fixed point.

2.2. Connectedness

Another property which can detect a difference in the coarse geometry of two spaces is connectedness. For example, if one cuts a ball from a tree then it will fall into several components, but cutting a ball from a hyperbolic plane still give only one connected component. First, we define coarse connectivity.

Definition 2.1. A map $f : X \to Y$ between two metric spaces is called c-connected if for any point $x \in X$ and any real number $\delta > 0$ there exists $\varepsilon > 0$ such that if a point $x' \in X$ satisfies $d(x, x') < \varepsilon$ then $d(f(x), f(x')) < c + \delta$.

Definition 2.2. A metric space X is called c-connected if for any two open sets $U, V \subset X$ such that $X = U \cup V$, the intersection of a c-neighbourhood of U and V is not empty: $(U + c) \cap V \neq \emptyset$.

This definition is evidently equivalent to the following.

Definition 2.3. A metric space X is c-connected if for any two points $x, x' \in X$ there exists a c-connected map $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = x'$.

We are ready to illustrate our idea how connectedness can be used for obtaining quantitative results on quasi-isometries. In the following proposition one can take, for example, hyperbolic plane as the space X.

Proposition 2.2. *Let X be a geodesic metric space. Suppose that for any points x, y and any positive real numbers R and $R' \leq R/2$ the set $B_x(R) \setminus B_y(R')$ is connected and non-empty. Let Y be a tree, let $f : B_x(R) \to Y$ be a $(\lambda_1, \lambda_2, c_1, c_2)$-quasi-isometric embedding. Then $R \leq 12\lambda_2 c_1 + 4c_2$.*

Proof. We are going to prove that there exist three points x_1, x_2 and x such that $x_1, x_2 \in B_x(R)$ and the distance $d(x_1, x_2)$ is at least R. Consider a ball of radius $2R$ centred in x_1. By hypothesis, the set $B_{x_1}(2R) \setminus B_{x_1}(R)$ is non-empty, hence there exists a point x_2 such that $2R > d(x_1, x_2) \geq R$. The space X is geodesic, hence now we can take the midpoint of x_1x_2 as x.

Denote $y_i = f(x_i)$ for $i = 1, 2$.

For any point y of a geodesic $(y_1, y_2) \subset Y$ there exists a point $z \in B_x(R)$ such that $d(f(z), y) \leq c_1$. This follows from the fact that the image of (x_1, x_2) is c_1-connected by the definition of a quasi-isometric embedding and every c_1-connected path between y_1 and y_2 includes the geodesic (y_1, y_2) in its c_1-neighbourhood.

Now consider a chain of points $\{\tilde{x}_i\}$ connecting x_1, x_2 and such that $d(\tilde{x}_i, \tilde{x}_{i+1}) < c_1/\lambda_1$. Hence, in the image $d(f(\tilde{x}_i), f(\tilde{x}_{i+1})) < 2c_1$ and so there exists i such that $d(f(\tilde{x}_i), y) \leq 2c_1$. Notice that $Y \setminus B_y(2c_1)$ has several $(4c_1 - 2)$-connected components and the distance between these components is at least $4c_1$.

Suppose that a point z is rather far from both x_1 and x_2: $d(z, x_i) > 4\lambda_2 c_1 + c_2$, $i = 1, 2$. Suppose also that $R > 2(4\lambda_2 c_1 + c_2)$ (if not there is nothing to prove). In the set $B_x(R) \setminus B_z(4\lambda_2 c_1 + c_2)$, also find a c_1/λ_1-chain. Hence, there exists a point $z' \notin B_z(4\lambda_2 c_1 + c_2)$ of this path such that $d(f(z'), y) \leq 2c_1$. Hence, $d(f(z), f(z')) \leq 4c_1$ and by property of quasi-isometry $d(z, z') \leq 4\lambda_2 c_1 + c_2$, so $z' \in B_z(4\lambda_2 c_1 + c_2)$. This leads to a contradiction with the hypothesis of the proposition. Hence, for any $y \in (y_1, y_2)$ there exists $z' \in B_{x_1}(4\lambda_2 c_1 + c_2) \cup B_{x_2}(4\lambda_2 c_1 + c_2)$ such that $d(f(z'), y) \leq 2c_1$.

Consider two points y', y'' on the geodesic (y_1, y_2) which are close enough to each other (more precisely $d(y', y'') \leq c_2/\lambda_2$) and such that respective points z' and z'' (which minimise distances to y' and y'', that is $d(y', f(z')) \leq 2c_1$ and $d(y'', f(z'')) \leq 2c_1$) lie in different balls $z' \in B_{x_1}(4\lambda_2 c_1 + c_2)$ and $z'' \in B_{x_2}(4\lambda_2 c_1 + c_2)$. So, on the one hand $d(z', z'') \geq R - 8\lambda_2 c_1 - 2c_2$ and on the other hand, by triangle inequality $d(f(z'), f(z'')) \leq c_2/\lambda_2 + 4c_1$. Hence $R - 8\lambda_2 c_1 - 2c_2 \leq \lambda_2(c_2/\lambda_2 + 4c_1) + c_2 = 4\lambda_2 c_1 + 2c_2$. So we get $R \leq 12\lambda_2 c_1 + 4c_2$. □

Proposition 2.2 implies that any quasi-isometric embedding of an R-ball in hyperbolic plane into a tree has a distortion of at least $\sqrt{R}$. We wonder whether this conclusion is sharp. A partial answer is given by Proposition 2.3. In this proposition we construct an example of a $(\sqrt{R}, \sqrt{R}, \sqrt{R}, \sqrt{R})$-quasi-isometry of an R-ball in a geodesic metric space X to a $\sqrt{R}$-ball in a tree, up to taking a $\sqrt{R}$-dense subset. The essential point here is that we consider trees of variable degree which depends on R.

Proposition 2.3. *Let X be a geodesic metric space. For any $R > 0$ there exists a $\sqrt{R}$-dense subset $S(R) \subset B_X(R)$, a tree $T(R)$ and a $(\sqrt{R}, \sqrt{R}, \sqrt{R}, \sqrt{R})$-quasi-isometric embedding $f_R : S(R) \to T(R)$.*

Proof. Consider a ball $B_X(R, z_0)$ centred at z_0. Let us construct a discrete set of points $S(R)$ generation by generation in the following way. The 0-generation is the origin z_0. For each k we pick a maximal $\sqrt{R}$-separated subset in the sphere of radius $k\sqrt{R}$. The resulting set $S(R)$ is $\sqrt{R}$-separated. It is also $3\sqrt{R}$-dense. Indeed, any point in $B((k+1)\sqrt{R})$ is $\sqrt{R}$-close to some point of the sphere of radius $k\sqrt{R}$, in which the k-th generation is $2\sqrt{R}$-dense, by maximality. In particular, every point of the $(k+1)$th-generation is at distance $\leq 3\sqrt{R}$ from at least one point of the kth-generation. This provides us with a tree $T(R)$ with vertex set $S(R)$: we connect each point of the $(k+1)$th-generation to the closest point of kth-generation (if the choice is not unique, choose the ancestor arbitrarily). Finally, set the lengths of all edges of the constructed tree $T(R)$ equal to 1. The diameter of $T(R)$ is $\sim \sqrt{R}$.

Now we sketch the proof that the induced map f is a $(\sqrt{R}, \sqrt{R}, \sqrt{R}, \sqrt{R})$-quasi-isometry. The right-hand quasi-isometric inequality $d(f(x), f(y)) \leq O(\sqrt{R})d(x, y)$ is automatically verified because the diameter of $T(R)$ is $O(\sqrt{R})$. Conversely, given points $x, y \in S(R)$, z_0, $f(x)$ and $f(y)$ form a tripod with median point u. The distance $d(f(x), f(y))$ is achieved by an arc from $f(x)$ to u followed by an arc from u to $f(y)$

in the tree. The descending arcs from $f(x)$ to u (resp. from $f(y)$ to u) consist of jumps in $S(R)$ from generation to generation, each of distance at most $3\sqrt{R}$. Therefore $d(x,y) \le 3\sqrt{R} d(f(x), f(y))$. $\square$

Similarly to the previous proposition, we can construct a $(\sqrt{R}, \sqrt{R}, \sqrt{R}, \sqrt{R})$-quasi-isometry between a ball $B_T(R)$ of radius R in a regular tree T of degree $d \ge 2$ and a $\sqrt{R}$-dense subset in a ball $B_{\mathbb{H}^2}(k \ln d, z_0)$ in $\mathbb{H}^2$.

Proposition 2.4. *For any $R > 0$, there exist a $\sqrt{R}$-dense subset S_R of a ball $B_{\mathbb{H}^2}(R)$ in the hyperbolic plane $\mathbb{H}^2$ and a $(\sqrt{R}, \sqrt{R}, \sqrt{R}, \sqrt{R})$-quasi-isometry $f_R : B_T(R) \to B_{\mathbb{H}^2}(R)$.*

Proof. Firstly, let us construct the set S_R and the quasi-isometry f_R. Next, we will prove that it is indeed a $(\sqrt{R}, \sqrt{R}, \sqrt{R}, \sqrt{R})$-quasi-isometry. Consider kth generation G_k of vertices in B_T (that is, points at distance k from the base point), there are $(d+1)d^{k-1}$ points in it. Consider a circle centred in z_0 of radius R_k (its exact value will be calculated soon) and take a subset S_k of this circle consisting of $(d+1)d^k$ points, such that the distance between them is at least $\sqrt{R_k}$. So we have the following relation (up to some multiplicative constants) which appears from the consideration of volumes

$$Vol(\text{ball of radius } \sqrt{R_k})(d+1)d^k = Vol(\text{circle of radius } R_k).$$

For big R_k the following relation holds approximately

$$e^{\sqrt{R_k}}(d+1)d^k = e^{R_k}.$$

Set $R_0 = 0$. Then it follows that $R_k \approx k \ln d$. The points from G_k are mapped to S_k naturally. Now we need to add edges between points of successive sets S_k. Connect points of S_k to the nearest points from S_{k-1}. If there are two possibilities, we choose one arbitrary.

Let us show that this is a $(\sqrt{R}, \sqrt{R}, \sqrt{R}, \sqrt{R})$-quasi-isometry. First of all, for any two points $t_1, t_2 \in S$, the distance between their images is at least $\sqrt{R}$. We always have $d(t_1, t_2) \le R \le \sqrt{R} d(f_R(t_1), f_R(t_2)) + \sqrt{R}$ and this inequality is checked automatically. Now, let $u_0 = t_1, u_1, \ldots, u_{n-1}, u_n = t_2$ be a geodesic path between t_1 and t_2. Notice that $d(u_i, u_{i+1}) = 1 \ge d(f(u_i), f(u_{i+1}))/\sqrt{R}$ for $i = 0, 1, \ldots, n-1$. Then

$$d(t_1,t_2) = \sum_{i=0}^{n-1} d(t_i, t_{i+1}) \ge \sum_{i=0}^{n-1} d(f(u_i), f(u_{i+1}))/\sqrt{R} \ge$$
$$d(f(t_1), f(t_2))/\sqrt{R} \ge \left(d(f(t_1), f(t_2)) - \sqrt{R}\right)/\sqrt{R},$$

which finishes the proof. $\square$

We have just presented a quasi-isometry from a ball in a tree to a ball in a discrete subset of a ball in a hyperbolic plane $\mathbb{H}^2$. But we do not know if we can extend this quasi-isometry to the whole ball $B_{\mathbb{H}^2}(R)$. The first idea is to do a projection of $B_{\mathbb{H}^2}(R)$ on a discrete subset, but this projection is a $(1,1,\sqrt{R},\sqrt{R})$-quasi-isometry itself, hence the resulting map is an (R,R,R,R)-quasi-isometry. So, the sharp bound on the quasi-isometric distortion growth between a tree and $\mathbb{H}^2$ is still an open problem.

3. Poincaré inequalities and quasi-isometries

We start this section with a definition of a quasi-isometric numerical invariant $p_{\neq 0}$, which will be involved in the sharp lower bound for quasi-isometric distortion growth for a certain family of metric spaces (Theorem 1.1). Then we proceed with a definition of Poincaré inequality, as well as of its generalisation based on the use of kernels. Finally, we discuss how Poincaré inequalities and the invariant $p_{\neq 0}$ appear in the proof of Theorem 1.1.

3.1. The critical exponent $p_{\neq 0}$ for L^p-cohomology

L^p-cohomology groups provide invariants for quasi-isometries. The continuous first L^p-cohomology group of a hyperbolic metric space X is

$$L^pH^1_{cont}(X) := \left\{[f] \in L^pH^1(X) | f \text{ extends continuously to } X \cup \partial X\right\},$$

where $X \cup \partial X$ is Gromov's compactification of X. Following results of Pierre Pansu [23], and Marc Bourdon and Bruce Kleiner [6], define the following quasi-isometrical numerical invariant of X

$$p_{\neq 0}(X) = \inf\left\{p \geq 1 | L^pH^1_{cont}(X) \neq 0\right\}.$$

If $p_{\neq 0}$ achieves different values for two spaces X and Y, then X and Y are not quasi-isometric. We expect that the difference $|p_{\neq 0}(X) - p_{\neq 0}(Y)|$ also bounds from below the quasi-isometrical distortion growth. We are able to prove this only for a family of examples, and under certain restrictions on maps.

Let Z_μ and $Z_{\mu'}$ be two variants of the space $\mathbb{T}^n \times (-\infty, \infty)$ with metrics $dt^2 + \sum e^{2\mu_i t}dx_i^2$ and $dt^2 + \sum e^{2\mu'_i t}dx_i^2$ respectively. One of the main results of this paper is a sharp lower bound for the quasi-isometrical distortion growth between Z_μ and $Z_{\mu'}$, of the form

$$const\,(p_{\neq 0}(Z_{\mu'}) - p_{\neq 0}(Z_\mu))\,R.$$

3.2. Definition of Poincaré constants

Constants in Poincaré inequalities are the quantitative incarnation of L^p-cohomology. On Riemannian manifolds, Poincaré inequality is defined as follows.

Definition 3.1. Let X be a Riemannian manifold. We say that X satisfies *Poincaré inequality* if there exists a real number C, such that for any real valued function f on X, there exists a real number m_f, such that

$$\|f - m_f\|_p \leq C \, \|\nabla f\|_p.$$

The best constant C, denoted by $C_p(X)$, is called *Poincaré constant* of X.

We are not satisfied by this definition as we want to work with a wider class of metric spaces. The generalisation of Poincaré inequality involves semi-norms induced by kernels (see Definitions 3.2, 3.4). For a kernel ψ on X, define a semi-norm $N_{p,\psi}(f)$, which is an analogue of the $\mathbb{L}^p$-norm of the gradient on a Riemannian manifold.

Let us recall what are kernels on geodesic metric spaces.

Definition 3.2. Let X be a geodesic space, dx a Radon measure on X. A kernel ψ is a measurable non-negative function on $X \times X$ such that

- ψ is bounded, $\psi \leq S^\psi$;
- for every $x \in X$ $\int_X \psi(x, x')dx' = 1$;
- the support of ψ is concentrated near the diagonal: there exist constants $\varepsilon^\psi > 0$, $\tau^\psi > 0$ and $R^\psi < \infty$ such that $\psi(x, y) > \tau^\psi$ if $d(x, y) \leq \varepsilon^\psi$; $\psi(x, y) = 0$ if $d(x, y) > R^\psi$.

R^ψ is called the *width*, ε^ψ – the *radius of positivity*, S^ψ – the *supremum* and τ^ψ – the *margin* of ψ.

Definition 3.3. A *cocycle* on Y is a measurable map $a : Y \times Y \to \mathbb{R}$, such that for every y_1, y_2, y_3 in Y,

$$a(y_1, y_2) = a(y_1, y_3) + a(y_2, y_3).$$

A convolution of a cocycle with a kernel is defined by

$$a * \phi(x, x') = \int\limits_{Y \times Y} a(y, y')\phi(x, y)\phi(x', y') \, dy \, dy'.$$

Definition 3.4. Let ψ be a kernel and a a cocycle on X. The semi-norm $N_{p,\psi}$ is defined by

$$N_{p,\psi}(a) = \left(\int\limits_{X \times X} |a(x_1, x_2)|^p \psi(x_1, x_2) \, dx_1 \, dx_2 \right)^{1/p}.$$

For a measurable function f on X,

$$N_{p,\psi}(f) = \left(\int\limits_{X \times X} |f(x_1) - f(x_2)|^p \psi(x_1, x_2) \, dx_1 \, dx_2 \right)^{1/p}.$$

Definition 3.5. Poincaré inequality associated with a kernel ψ is

$$\|f - m_f\|_p \leq C_p(X, \psi) N_{p,\psi}(f).$$

3.3. Scheme of proof of Theorem 1.1

Let us explain how Poincaré inequalities will be used in the proof of a lower bound on distortion given by Theorem 1.1.

For the family of spaces Z_μ, it is known that $p_{\neq 0}(Z_\mu) = \frac{\sum \mu_i}{\max \mu_i}$ (unpublished result of P. Pansu, [23]). In Theorem 6.2 it is proved that

- if $p > p_{\neq 0}(Z_\mu)$, then the Poincaré constant for a ball of radius R satisfies

$$C_p(B^{Z_\mu}(R)) \geq const \cdot (\,Vol\, B(R))^{1/p};$$

- if $p \leq p_{\neq 0}(Z_\mu)$, then

$$C_p(B^{Z_\mu}(R)) = o\left((\,Vol\, B(R))^{1/p}\right).$$

Next, we show that under transport by a (λ, c)-quasi-isometry, C_p is multiplied by at most $e^{(\lambda+c)/a}$ for some positive constant a. Transport under quasi-isometric embeddings is more delicate, this is why our arguments work only for a family of examples. For these examples, we are able to get a lower bound. Roughly speaking, it states the following.

Assume that $p_{\neq 0}(Z_{\mu'}) < p < p_{\neq 0}(Z_\mu)$. If there exists a (λ, c)-quasi-isometric embedding $B^{Z_\mu}(R) \to Z_{\mu'}$, which induces an isomorphism on fundamental groups, then

$$C_p(B^{Z_\mu}(R)) \geq const \cdot e^{-(\lambda+c)/a} C_p(B^{Z_{\mu'}}(R)).$$

This yields

$$\begin{aligned}\lambda + c &\geq a(\log(C_p(B^{Z_{\mu'}}(R))) - \log(C_p(B^{Z_\mu}(R))) \\ &\sim (p_{\neq 0}(Z_{\mu'}) - p_{\neq 0}(Z_\mu))R,\end{aligned}$$

which is the announced lower bound on quasi-isometric distortion growth.

4. Regularisation and quasi-isometries

In this section we study how Poincaré inequalities are transformed under quasi-isometries. For this purpose kernels are used, which helps to regularise transported functions.

4.1. Kernels

Firstly, define a convolution of kernels and functions, which is the principal operation in our study.

Definition 4.1. A convolution of two kernels is

$$\psi_1 * \psi_2 = \int\limits_X \psi_1(x,z)\psi_2(z,y)\,dz.$$

A convolution of a kernel and a function is

$$g * \psi(x) = \int\limits_X g(z)\psi(x,z)\,dz.$$

It is easy to check that a convolution of two kernels is also a kernel.

Now we are ready to present several results on properties of kernels. The next lemma proves that there exists a kernel with a radius of positivity as large as wanted.

Lemma 4.1. *There exists a constant c_τ (which depends on the local geometry of the space X), such that for any $\varepsilon > 0$ there exists $\tau = c_\tau e^{-\varepsilon}$ and a kernel ψ on $X \times X$, such that for any two points x_1, x_2 with $d(x_1,x_2) < \varepsilon$, we have $\psi(x_1,x_2) > \tau$. In other words, for any given radius of positivity ε there exists a kernel with a margin controlled from below by $c_\tau e^{-\varepsilon}$.*

Proof. Let us start from kernel

$$\psi'(x,x') = Vol(B(x,1))^{-1}1_{\{d(x,x')\le 1\}}$$

with radius of positivity $\varepsilon' = 1$ and margin $\tau' = v(1)^{-1}$, where, for $r > 0$, $v(r)$ denotes the infimum of volumes of balls of radius r in X. It follows from the proof of Lemma 1.2 in [22] that the m-th convolution ψ'^{*m} has radius of positivity $\varepsilon'_m \ge m(\varepsilon'/2) = m/2$ and margin $\tau'_m \ge \tau'^m v(\frac{1}{2})^{m-1}$. Denote $v(\frac{1}{2})^{m-1}$ by c_τ which finishes the proof. □

The following two facts are known, see [22].

Lemma 4.2. *Let X be a geodesic metric space such that the infimum* $\inf\{Vol\,B(x,r)|x \in X\}$ *of volume of balls of radius r is positive. Semi-norms $N_{p,\psi}$ are pairwise equivalent. More precisely, let ψ_1 and ψ_2 be two kernels on X. Then*

$$N_{\psi_2} \le \hat{C} N_{\psi_1},$$

where

$$\hat{C} = \frac{\sup\psi_1 \sup\psi_2}{c_\tau}\frac{R^{\psi_2}}{\varepsilon^{\psi_1}}(2e)^{R^{\psi_2}/\varepsilon^{\psi_1}}.$$

Lemma 4.3. *Let the space X be a Riemannian manifold and have the following properties*

- *its injectivity radius is bounded below,*
- *its Ricci curvature is bounded from below.*

Then the volumes of balls are bounded from below (Croke inequality [8]) and from above (Bishop inequality).

1) For any function g define a cocycle $u(x, y) = g(x) - g(y)$. Then for any p and any kernel ψ' with bounded derivatives there exists a kernel ψ_1, such that the $\mathbb{L}^p$-norm of $\nabla(g*\psi')$ (we regularise g) is bounded from above by a ψ_1-seminorm of the corresponding cocycle u

$$\|\nabla(g * \psi')\|_p \leq N_{p,\psi_1}(u)$$

with the kernel ψ_1 defined as follows

$$\psi_1 = \frac{\sup \nabla\psi' \sup \psi'}{Vol(B(z', R^{\psi'}))} 1_{\{d(z,z')\leq R^{\psi'}\}}.$$

2) Conversely, there also exists a kernel ψ_2 such that

$$N_{p,\psi_2}(u) \leq C\|\nabla g\|_p,$$

where C depends only on dimension. Here the kernel ψ_2 can be taken as

$$\psi_2(x, y) = \max\{1, \Theta(x, y)^{-1}\} 1_{\{d(x,y)\leq R\}},$$

where $\Theta(x, y)$ is the density of the volume element in polar coordinates with origin at x

$$\Theta(x, y)^{-1} dy = dr d\theta$$

and $R > 0$ can be chosen arbitrarily.

For the kernel ψ_2, in the third hypothesis in kernel definition we suggest to set $R^{\psi_2} = 1$. Then ψ_2 is bounded by 1 and the width of its support is also 1. For reader's convenience, we include the proof of the first statement of the last lemma, following [22].

Proof. Denote by α the cocycle $u * \psi'$. Then for any y,

$$\nabla(u * \psi')(x) = \frac{\partial\alpha(x, y)}{\partial x} = \int (g(z') - g(z))\, d_x\psi'(z, x)\psi'(z', y)\, dz\, dz'.$$

Choose $y = x$. Then we obtain

$$|\nabla(g * \psi'(x))| \leq \sup \nabla\psi' \sup \psi \int\limits_{B(x,R^{\psi})\times B(x,R^{\psi})} |g(z') - g(z)|\, dz\, dz'.$$

Now applying Hölder inequality, the needed statement is fulfilled with the kernel

$$\psi_1 = \frac{\sup \nabla \psi' \sup \psi'}{Vol(B(z', R^{\psi'}))} 1_{\{d(z,z') \leq R^{\psi'}\}}. \qquad \square$$

This lemma gives an idea how to generalise Poincaré inequalities for the case of arbitrary metric spaces. Of course, such a Poincaré inequality depends on the choice of a kernel ψ. Let f be an $\mathbb{L}_p$-function on X, ψ a kernel on X. The Poincaré inequalities for f associated to ψ with constants c_f and $C_p(f)$ are

$$\|f - c_f\|_p \leq C_p(f) \|N_{p,\psi}(u)\|.$$

The Poincaré constant $C_p(X, \psi)$ is a constant, such that for any $\mathbb{L}_p$-function f Poincaré inequality is checked with $C_p(f) = C_p(X, \psi)$. It follows from Lemma 4.2 that the existence of Poincaré constant does not depend on the choice of a kernel.

4.2. Transporting functions by quasi-isometries

Let X, Y be two metric spaces, let $f : X \to Y$ and $f' : Y \to X$ be (K, c)-quasi-isometries between them, such that for any $x \in X$, $d(x, f' \circ f(x)) \leq c$ and vice versa (that is, they are inverse in the quasi-isometrical sense). Let g be a measurable function on Y. We want to find a way to transport and to regularise g by our quasi-isometry to obtain a similar measurable function on X. A function on X corresponding to g is

$$h(x) = \int_Y g(z) \psi(f(x), z)\, dz.$$

This integral exists for all x because ψ is measurable by the second variable by definition. Still we want h to be also measurable. For that, it will be sufficient if f is measurable too.

Proposition 4.1. *Let f be a $(\lambda_1, \lambda_2, c_1, c_2)$-quasi-isometric embedding between metric spaces X and Y. Then there exists a measurable $(\lambda_1, \lambda_2, 3c_1, c_2 + 2c_1/\lambda_1)$-quasi-isometric embedding g at distance $2c_1$ from f.*

Proof. Take a measurable partition P of X with a mesh c_1/λ_1. For each set $A \in P$, choose a base point x_A. Set g be constant on A

$$g|_A = f(x_A).$$

Take any two points $x, x' \in X$. Assume $x \in A$ and $x' \in A'$ where $A, A' \in P$. Then

$$\begin{aligned} d(g(x), g(x')) &= d(f(x_A), f(x_{A'})) \leq \lambda_1 d(x_A, x'_A) + c_1 \\ &\leq \lambda_1 (d(x, x') + d(x, x_A) + d(x', x_{A'})) + c_1 \leq \lambda_1 d(x, x') + 3c_1. \end{aligned}$$

In the same way we prove the right-hand inequality. $\square$

Proposition 4.1 gives the idea that one can always pass to measurable quasi-isometries without significant loss in constants. From now, we will consider only measurable quasi-isometries.

4.3. Transporting cocycles

Definition 4.2. Let a be a cocycle on Y, $f : X \to Y$ be a quasi-isometric embedding and ϕ be a kernel on Y. The transporting convolution of a with ϕ by f is the cocycle defined on X by

$$a *_t \phi(f)(x, x') = \int_{Y \times Y} a(y, y')\phi(f(x), y)\phi(f(x'), y')\, dy\, dy'.$$

Lemma 4.4. *Let X, Y be two metric space. Suppose also that X has a bounded geometry (that is for any $R > 0$ the supremum of volume of balls of radius R in X is bounded). Let ϕ be a kernel on Y, let a be a cocycle on Y and let ψ be a kernel on X. Let also f be a $(\lambda_1, \lambda_2, c_1, c_2)$-quasi-isometric embedding. Then there exists a kernel $\tilde{\psi}$ on Y such that*

$$N_\psi(a *_t \phi(f)) \le C N_{\tilde{\psi}}(a),$$

where

$$C \le \left(c_\tau^Y\right)^{-1} e^{R^{\psi'}} \sup \psi \left(\sup \phi \sup \mathit{Vol}\, B_X(2\lambda_2 R^\phi + c_2)\right)^2.$$

Proof. By definition,

$$(N_\psi(a *_t \phi(f)))^p = \int_{X \times X} |a *_t \phi(x, x')|^p \psi(x, x') dx dx' =$$

$$= \int_{X \times X} \left| \int_{Y \times Y} a(y, y')\phi(f(x), y)\phi(f(x'), y') dy dy' \right|^p \psi(x, x') dx dx'$$

applying Hölder inequality

$$\le \int_{X \times X} \int_{Y \times Y} |a(y, y')^p| \phi(f(x), y)\phi(f(x'), y') dy dy' \psi(x, x') dx dx'$$

denoting $\psi'(y, y') = \int_{X \times X} \phi(f(x), y)\phi(f(x'), y')\psi(x, x') dx dx'$

$$= \int_{Y \times Y} |a(y, y')|^p \psi'(y, y') dy dy'.$$

We need to show that ψ' is dominated by some kernel $\tilde{\psi}$.

Firstly, let us prove that $\psi'(y,y') = 0$ if $d(y,y') > R^{\psi'}$ for some $R^{\psi'} = 2R^\phi + \lambda R^\psi + c$. If $d(x,x') > R^\psi$ then by the definition of kernels $\psi(x,x') = 0$. Otherwise, suppose that $d(x,x') < R^\psi$. If $d(y,y') > R^{\psi'}$, then by triangle inequality either $\phi(f(x),y)$ or $\phi(f(x'),y')$ vanishes:

$$d(f(x), f(x')) \le \lambda d(x,x') + c \le \lambda R^\psi + c.$$

Hence, if, for example, $d(f(x),y) \le R^\phi$, then $d(f(x'),y') \ge R^{\psi'} - d(f(x),f(x')) - d(f(x),y) > R^\phi$ which leads to $\phi(f(x'),y') = 0$.

$\psi'(y,y')$ is estimated from above in the following way. Write

$$\psi'(y,y') \le \sup\psi \int\limits_{X\times X} \phi(f(x),y)\phi(f(x'),y')dxdx'$$

and then integrate $\int_X \phi(f(x),y)dx$ and $\int_X \phi(f(x'),y')dx'$.

For any $y \in Y$, if $d(f(x),y) > R^\phi$ then $\phi(f(x),y) = 0$. Hence, the diameter of the set of points $X_y \in X$, such that for any $x \in X_y$, $d(f(x),y) \le R^\phi$, is at most $\lambda_2 2R^\phi + c_2$. Hence, $\int_X \phi(f(x),y)dx \le \left(\sup_{x\in X} Vol\, B_X(x, 2\lambda_2 R^\phi + c_2)\right) \sup_{Y\times Y}\phi$, that is $\sup_{x\in X} Vol\, B_X(x, 2\lambda_2 R^\phi + c_2)$ stands for the supremum of volumes of all balls of radius $2\lambda_2 R^\psi + c_2$ in X. So we come to the following upper-bound for $\psi'(y,y')$

$$\psi'(y,y') \le \sup\psi \left(\sup\phi \sup Vol\, B_X(2\lambda_2 R^\phi + c_2)\right)^2.$$

Lemma 4.1 helps to construct a kernel $\tilde\psi$, such that its radius of positivity is at least $R^{\psi'}$ and at the same time we control its margin from below. $\tilde\psi(y,y') \ge \tau = c^Y_\tau e^{-R^{\psi'}}$ whenever the distance between y, y' does not exceed $R^{\psi'}$. Hence,

$$\psi'(y,y') \le \tau^{-1}\tilde\psi(y,y') \sup\psi \left(\sup\phi \sup Vol\, B_X(2\lambda_2 R^\phi + c_2)\right)^2.$$

So, we obtain

$$C \le \left(c^Y_\tau\right)^{-1} e^{R^{\psi'}} \sup\psi \left(\sup\phi \sup Vol\, B_X(2\lambda_2 R^\phi + c_2)\right)^2. \qquad \square$$

5. Poincaré inequality for exponential metric

We will give an upper bound for the Poincaré constant in a ball of radius R in the space with the metric $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$.

Theorem 5.1. *Let $\tilde X = \mathbb{R}_+ \times \mathbb{R}^n$ with the metric $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$. Let $X = \tilde X/\Gamma$ where Γ is a lattice of translations in the factor $\mathbb{R}^n$. Then the Poincaré constant for a ball $B(R)$ in X is*

$$C_p(\mu) \le \frac{p}{\mu} + (A(\mu))^{1/p} C_p(\mathbb{T}^n) e^{\mu_n R},$$

where $\mu = \sum \mu_i$, $A(\mu)$ is a constant depending only on μ, $C_p(\mathbb{T}^n)$ is a Poincaré constant for a torus $\mathbb{T}^n$.

Let us fix the direction $\theta = (x_1, \dots, x_n)$.

5.1. Poincaré inequality in a fixed direction

Lemma 5.1. *Let $\tilde{X} = \mathbb{R}_+ \times \mathbb{R}^n$ with the metric $dt^2 + \sum_i e^{2\mu_i t} dx_i^2$. Let $X = \tilde{X}/\Gamma$ where Γ is a lattice of translations in the factor $\mathbb{R}^n$. Let $R \in \mathbb{R}^+ \cup \{\infty\}$. Then for any fixed direction $\theta = (x_1, \dots, x_n)$*

$$\left(\int_a^R |f(t) - c_\theta|^p e^{\mu t} dt \right)^{1/p} \leq \frac{p}{\mu} \left(\int_a^R |f'(t)|^p e^{\mu t} dt \right)^{1/p},$$

where $c_\theta = f(R, \theta)$ or $c_\theta = \lim_{R\to\infty} f(R, \theta)$.

Proof. Let f be a function, such that its partial derivative $\partial f/\partial t$ is in $\mathbb{L}^p(e^{\mu t} dt, [0, +\infty))$ where $p > 1$. By Hölder inequality we get

$$\int_0^{+\infty} \left| \frac{\partial f}{\partial t} \right| dt \leq \left(\int_0^{+\infty} \left| \frac{\partial f}{\partial t} \right|^p e^{\mu t} dt \right)^{1/p} \left(\int_0^{+\infty} e^{-(\mu t/p)(p/(p-1))} \right)^{1-1/p} < +\infty.$$

Hence, for every fixed direction θ there exists a limit $\lim_{t\to\infty} f(t, \theta)$.

If $R = \infty$, we prove that $|f(t) - c_\theta|^p e^{\mu t} \to 0$ as $t \to \infty$. Apply the Newton–Leibniz theorem and then Hölder inequality to $|f(t) - c_\theta|$. We have

$$\begin{aligned} |f(t) - c_\theta| &= \left| \int_t^\infty \frac{\partial f}{\partial s} ds \right| \leq \int_t^\infty \left| \frac{\partial f}{\partial s} \right| ds \leq \qquad (5.1) \\ &\leq \left(\int_t^\infty \left| \frac{\partial f}{\partial s} \right|^p e^{\mu u} du \right)^{1/p} \left(\int_t^\infty e^{-\mu s/(p-1)} ds \right)^{1-1/p}. \end{aligned}$$

Calculate the last integral

$$\int_t^\infty e^{-\mu s/(p-1)} ds = -\frac{p-1}{\mu} e^{-\frac{\mu s}{p-1}} |_t^\infty = \frac{p-1}{\mu} e^{-\frac{\mu t}{p-1}}.$$

With the notation $D_0 = \left(\frac{p-1}{\mu}\right)^{p-1}$,

$$|f(t) - c_\theta|^p \le D_0 e^{-\mu t} \int_t^{+\infty} \left|\frac{\partial f}{\partial s}\right|^p e^{\mu s} ds.$$

Hence

$$|f(t) - c_\theta|^p e^{\mu t} \le D_0 \int_t^{+\infty} \left|\frac{\partial f}{\partial s}\right|^p e^{\mu s} ds \to 0$$

as $t \to +\infty$.

Now integrate by parts

$$\int_a^R |f(t) - c_\theta|^p e^{\mu t} dt = \left[|f(t) - c_\theta|^p \frac{e^{\mu t}}{\mu}\right]_a^R - \int_a^R f'(t) p |f(t) - c_\theta|^{p-1} \frac{e^{\mu t}}{\mu} dt. \tag{5.2}$$

As $c_\theta = f(R)$

$$\int_a^R |f(t) - c_\theta|^p e^{\mu t} dt = -|f(a) - c_\theta|^p \frac{e^{\mu a}}{\mu} - p \int_a^R f'(t) |f(t) - c_\theta|^{p-1} \frac{e^{\mu t}}{\mu} dt.$$

Notice that the integral at the left is positive. On the right hand side, the first term is negative (for this reason we will drop it soon). Hence, the second term should be positive. By Hölder inequality,

$$\int_a^R (-f'(t)) |f(t) - c_\theta|^{p-1} \frac{e^{\mu t}}{\mu} dt \le \left(\int_a^R |f'(t)|^p \frac{e^{\mu t}}{\mu} dt\right)^{1/p} \left(\int_a^R |f(t) - c_\theta|^p \frac{e^{\mu t}}{\mu} dt\right)^{(p-1)/p}. \tag{5.3}$$

Introduce the following notations

$$X = \int_a^R |f(t) - c_\theta|^p e^{\mu t} dt, \quad Y = \int_a^R |f'(t)|^p e^{\mu t} dt.$$

Using these notations we return to Eq. (5.2). Drop the term $-|f(a) - c_\theta|^p e^{\mu a}/\mu$ and then apply Eq. (5.3)

$$X \le \frac{p}{\mu} Y^{1/p} X^{(p-1)/p}.$$

So, we get immediately that

$$X^{1/p} \le \frac{p}{\mu} Y^{1/p}$$

which proves Poincaré inequality in a fixed direction. □

5.2. Poincaré inequality for exponential metric

We are ready to finish the proof of Theorem 5.1. Introduce the following notations $\tilde{f}_r(t,\theta) = f(r,\theta)$ (the function is considered as a function of two variables), $f_r(\theta) = f(r,\theta)$ (the function is considered as a function of one variable).

We have already proved that for any $\theta \in \mathbb{T}^n$,

$$\int_0^R |f(t,\theta) - f(R,\theta)|^p e^{\mu t} dt \leq \left(\frac{p}{\mu}\right)^p \int_0^R \left|\frac{\partial f}{\partial t}\right|^p e^{\mu t} dt.$$

Integrate over θ and introduce the volume element for $\tilde{X}$, $dVol = drd\theta e^{\sum \mu_i r}$. We get

$$\int_{B(R)} |f - f_R|^p dVol \leq \left(\frac{p}{\mu}\right)^p \int_{B(R)} |\nabla f|^p dVol.$$

Denote the Euclidean gradient by ∇_e. By the form of the metric one can see that $e^{2\mu_i t}|dx_i^2| = 1$. Hence, $\|\nabla_e f_r\| \leq e^{\mu_n t}|\nabla f|$. Now notice that

$$\int_{R-1}^R \|\nabla_e f_r\|^p_{\mathbb{L}^p(\mathbb{T}^n)} e^{\mu t} dt \geq e^{\sum \mu_i (R-1)} \int_{R-1}^R \|\nabla_e f_r\|^p_{\mathbb{L}^p(\mathbb{T}^n)} dt.$$

So, we write

$$e^{\sum \mu_i (R-1)} \int_{R-1}^R \|\nabla_e f_r\|^p_{\mathbb{L}^p(\mathbb{T}^n)} dt \leq e^{p\mu_n R} \int_{B(R)\setminus B(R-1)} |\nabla f|^p dVol. \tag{5.4}$$

Fixing $r \in [R-1, R]$, write Poincaré inequality on the torus for the function $f_r(\theta)$. There exists a number c_r, such that

$$\int_{\mathbb{T}^n} |f_r(\theta) - c_r|^p d\theta \leq (C_p(\mathbb{T}^n))^p \int_{\mathbb{T}^n} |\nabla_e f_r(\theta)|^p d\theta,$$

where $C_p(\mathbb{T}^n)$ is a Poincaré constant for $\mathbb{T}^n$. Next consider the function $f_r(\theta)$ as a function on the ball $B(R)$ which does not depend on t. Integrate this inequality over t,

$$\begin{aligned}\int_{B(R)} |f_r(\theta) - c_r|^p dVol &\leq (C_p(\mathbb{T}^n))^p \int_0^R \int_{\mathbb{T}^n} |\nabla_e f_r(\theta)|^p d\theta e^{\sum \mu_i t} dt \\ &\leq \frac{e^{\sum \mu_i R}}{\sum \mu_i} (C_p(\mathbb{T}^n))^p \int_{\mathbb{T}^n} |\nabla_e f_r(\theta)|^p d\theta.\end{aligned}$$

Integrate over r from $R-1$ to R and exploit inequality (5.4). It gives

$$\int\limits_{R-1}^{R} \left(\int\limits_{B(R)} |f_r(\theta) - c_r|^p \, dVol \right) dr \leq A(\mu)(C_p(\mathbb{T}^n))^p e^{p\mu_n R} \int\limits_{B(R)\setminus B(R-1)} |\nabla f|^p \, dVol,$$

where $A(\mu)$ is a constant which depends only on $\mu_i, i = 1, \ldots, n$. Now apply Hölder inequality again,

$$\begin{aligned}
\int\limits_{R-1}^{R} \|f_r - c_r\|_{\mathbb{L}^p(B(R))} dr &\leq \left(\int\limits_{R-1}^{R} \int\limits_{B(R)} |f_r - c_r|^p \, dVol \, dr \right)^{1/p} \\
&\leq \left(A(\mu)(C_p(\mathbb{T}^n))^p e^{p\mu_n R} \int\limits_{B(R)\setminus B(R-1)} |\nabla f|^p \, dVol \right)^{1/p} \\
&\leq (A(\mu))^{1/p} \, C_p(\mathbb{T}^n) e^{\mu_n R} \|\nabla f\|_{\mathbb{L}^p(B(R))}
\end{aligned}$$

Set $c = \int_{R-1}^{R} c_r dr$. In the following chain of inequalities we will first apply triangle inequality and then we will use the fact that the norm of the integral is less than or equal to the integral of the norm (briefly $\|\int f dr\| = \int \|f\| dr$).

$$\begin{aligned}
\|f - c\|_{\mathbb{L}^p(B(R))} &= \left\| \int\limits_{R-1}^{R} (f - c_r) dr \right\|_{\mathbb{L}^p(B(R))} \\
&\leq \left\| \int\limits_{R-1}^{R} (f - f_r) dr \right\|_{\mathbb{L}^p(B(R))} + \left\| \int\limits_{R-1}^{R} (f_r - c_r) dr \right\|_{\mathbb{L}^p(B(R))} \\
&\leq \int\limits_{R-1}^{R} \left(\|f - f_r\|_{\mathbb{L}^p(B(R))} + \|f_r - c_r\|_{\mathbb{L}^p(B(R))} \right) dr \\
&\leq \frac{p}{\mu} \|\nabla f\|_{\mathbb{L}^p(B(R))} + (A(\mu))^{1/p} \, C_p(\mathbb{T}^n) e^{\mu_n R} \|\nabla f\|_{\mathbb{L}^p(B(R))}.
\end{aligned}$$

6. Lower bound on Poincaré constant

In this section we state in details and prove Theorem 1.1, this detailed version is given as Theorem 6.2. Theorem 6.1 plays an important role in that proof.

Let us start with the brief discussion of the class of considered spaces.

Let Z_μ denote $\mathbb{T}^n \times \mathbb{R}$ equipped with metrics $dt^2 + \sum e^{2\mu_i t} dx_i^2$, where we suppose $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$. Let O, $O' = (0, \ldots, 0)$ be base points of Z and Z' respectively.

Notice that the "width" of $\mathbb{T}^n \times (-\infty, 0]$ is finite, so it is at a finite distance from a ray $(-\infty, 0]$. So from now on, we focus our attention on the part of $B_Z(O, R)$ where $t \geq 0$. Indeed, we want to consider quasi-isometric embeddings of balls $\mathbb{T}^n \times [-R, R]$. The volume of $T^n \times (-\infty, 0]$ is finite, whereas the volume of $\mathbb{T}^n \times [0, R]$ is exponential in R. Hence, only a negligible part of $\mathbb{T}^n \times [-R, R]$ can be sent to the negative part $\mathbb{T}^n \times (-\infty, 0]$ (compare to Subsection 2.1).

Consider a ball $B_Z(O, R)$ in $Z = Z_\mu$ and its quasi-isometric embedding in $Z' = Z_{\mu'}$. We are going to give a lower bound for the sum of quasi-isometric constants $\lambda + c$ as a function of R, using our results on transported Poincaré inequalities. Notice that our method does not apply to a general quasi-isometric embedding. We will consider only quasi-isometric embeddings which are homotopy equivalences.

Here is some motivation for studying these spaces Z_μ. Following U. Hamenstädt [14] and X. Xie [31,28], there is a family of hyperbolic spaces whose quasi-isometric classification is known, these are spaces with transitive Lie groups of isometries. In this family (classified by E. Heintze [15]), the easiest spaces are X_μ. Their $\mathbb{L}^p$ cohomologies are also known (P. Pansu [22]). These spaces are still rather difficult because their $\mathbb{L}^p$ cohomology vanishes for a delicate global reason, which is hard to make quantitative, on balls. Fortunately, their quotients Z_μ by $\mathbb{Z}^n$ are easier to treat. One can also consider the spaces Z_μ as hyperbolic spaces with ideal boundaries being products of circles supplied with power of the standard metric.

6.1. Statement of theorems

Theorem 6.1. *Let Z, Z' be two locally homogeneous hyperbolic metric spaces with metrics $dt^2 + \sum e^{2\mu_i t} dx_i^2$ and $dt^2 + \sum e^{2\mu'_i t} dx_i^2$ respectively, $0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ and $0 < \mu'_1 \leq \mu'_2 \leq \ldots \leq \mu'_n$. Assume also that $\sum \mu_i/\mu_n > \sum \mu'_i/\mu'_n$. Denote $a = \max\{\mu_i, \mu'_i, i = 1, 2, \ldots, n\}$ and $b = \min\{\mu_i, \mu'_i, i = 1, 2, \ldots, n\}$. Then there exists a constant $G_0(a, b)$, such that the following holds. Let $\Theta : B_Z(R) \to Z'$ be a continuous $(\lambda_1, \lambda_2, c_1, c_2)$-quasi-isometric embedding, inducing an isomorphism on fundamental groups. Suppose that Θ sends base point to base point, $\Theta(O) = O'$ and that $R \geq 8(\lambda_1 + c_1) + (\lambda_2 + c_2) + 1$. If $p > \sum \mu'_i/\mu'_n$, up to replacing Z with a connected 2-sheeted covering, the Poincaré constant $C_p(\mu)$ for a ball of radius R in the space Z is bounded from below by*

$$C_p(\mu) \geq (G_0(a, b))^{1/p} (\lambda_1 + c_1)^{-3/p - 2/p^2} e^{-(9/p + 3/p^2)(\lambda_1 + c_1)} e^{(\sum \mu_i/p)R} \left(p - \sum \mu'_i/\mu'_n\right)^{1/p}.$$

This theorem is not symmetric, it can be applied only in one direction: it does not give any lower bound to the quasi-isometric embeddings of Z_μ into $Z_{\mu'}$ and of $Z_{\mu'}$ into Z_μ at the same time.

As it has already been mentioned, we are able to treat the quantitative problem only for quasi-isometric embeddings which are homotopy equivalences. So, let us modify Definition 1.3 in the following way.

Definition 6.1. Let X, Y be metric spaces, x_0, y_0 their base points respectively. The homotopy quasi-isometric distortion growth is the function

$$\begin{aligned} &D_{hG}(X, x_0, Y, y_0)(R) \\ &\quad = \inf\{d | \exists f : B_X(x_0, R) \to Y \text{ a } (\lambda_f, c_f)\text{-quasi-isometric embedding,} \\ &\qquad\qquad \text{such that } f(x_0) = y_0 \text{ and } f \text{ is a homotopy equivalence}, d = \lambda_f + c_f\}. \end{aligned}$$

Theorem 6.2. *Let Z, Z' be two locally homogeneous hyperbolic metric spaces with metrics $dt^2 + \sum e^{2\mu_i t} dx_i^2$ and $dt^2 + \sum e^{2\mu'_i t} dx_i^2$ respectively, $0 < \mu_1 \le \mu_2 \le \ldots \le \mu_n$ and $0 < \mu'_1 \le \mu'_2 \le \ldots \le \mu'_n$. Assume also that $\sum \mu_i/\mu_n > \sum \mu'_i/\mu'_n$. Denote $a = \max\{\mu_i, \mu'_i, i = 1, 2, \ldots, n\}$ and $b = \min\{\mu_i, \mu'_i, i = 1, 2, \ldots, n\}$. Then there exist constants $G_1(a, b)$ and $G_2(a, b)$ such that the following holds. The homotopy distortion growth (see Definition 6.1) for quasi-isometrical embedding of $B_Z(R)$ into Z' is bounded from below by*

$$D_{hG}(R) \ge \min\left\{G_1\left(\frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_i}{\mu'_n}\right)R - G_2, \frac{1}{8}R\right\}.$$

Theorem 6.1 plays an important role in the proof of Theorem 6.2. Before proving these two theorems, let us discuss the double cover of the family of spaces under consideration and provide some preliminary lemmas.

6.2. Lifting to a double covering space

Let us introduce a double covering of Z'. Let $\tilde{Z}' = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1} \times \mathbb{R}/2\mathbb{Z} \times [0, +\infty)$ with the metric defined by the same formula as for Z': $dt^2 + \sum e^{2\mu_i t} dx_i^2$. Consider the map $\tilde{Z}' \to Z'$ defined by

$$(x_1, x_2, \ldots, x_n, t) \mapsto (x_1, x_2, \ldots, x_n \mod 1, t).$$

So we identify $(x_1, x_2, \ldots, x_n, t)$ and $(x_1, x_2, \ldots, x_n + 1, t)$ in $\tilde{Z}'$. Consider a complex function $u(x_1, x_2, \ldots, x_n, t) = e^{\pi i x_n}$ on $\tilde{Z}'$.

Composition of u with the deck transformation $\iota' : \tilde{Z}' \to \tilde{Z}'$

$$\iota' : (x_1, x_2, \ldots, x_n, t) \mapsto (x_1, x_2, \ldots, x_n + 1, t)$$

gives $u \circ \iota' = -u$.

By assumption, $\Theta : Z \to Z'$ is a continuous map inducing an isomorphism on fundamental groups, and $\tilde{Z}'$ is a covering space of Z'. It is necessary to show that there exists a non-trivial covering space $\tilde{Z} \to Z$ such that the following diagram commutes.

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\tilde{\Theta}} & \tilde{Z}' \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle \pi_{Z'}} \\ Z & \xrightarrow{\Theta} & Z' \end{array}$$

Define

$$\tilde{Z} = \left\{ (z, \tilde{z}') | z \in Z, \tilde{z}' \in \pi_{Z'}^{-1}(\Theta(z)) \right\},$$

that is $\tilde{Z} \subset Z \times \tilde{Z}'$. Let $[\gamma']$ be a loop in Z' which does not lift to a loop in $\tilde{Z}'$. By hypothesis, there exists a loop γ in Z such that $\Theta(\gamma)$ is homotopic to γ'. Then γ does not lift to a loop in $\tilde{Z}$. There exists an isometry ι of order 2 on $\tilde{Z}$ such that $\tilde{\Theta} \circ \iota = \iota' \circ \tilde{\Theta}$.

6.3. Lifting of Θ

Here we will prove that in the constructed double coverings Θ lifts to a map satisfying the right-hand inequality in the definition of quasi-isometry with constants λ_1 and $2c_1$. We start with two preliminary lemmas concerning distances in two-fold coverings.

Lemma 6.1. *Let $Z = Z_\mu$ be a locally homogeneous space. There is an effective constant $c_0(\mu)$ with the following effect. Let z be a point in Z in the region where $t \geq c_0$. Let $c = t(z)$. Every loop of length less than c based at z is null-homotopic.*

Proof. Let $\pi_s : Z \to \mathbb{T}^n \times \{s\} \subset Z$ denote projection onto the first factor $\mathbb{T}^n$. This is a homotopy equivalence. Note that π_s is length decreasing on $\{(t, x) \in Z\,;\, t \geq s\}$. Moreover, on $T^n \times \{t\}$, π_s decreases length by $e^{\mu_1(s-t)}$ at least. Let γ be a non-null-homotopic geodesic loop at z. Assume that its length is $\leq 2c$. Then $\gamma \subset \{(t, x) \in Z\,;\, t \geq \frac{c}{2}\}$, therefore

$$\mathrm{length}(\pi_{\frac{c}{2}}(\gamma)) \leq c,$$

thus

$$\mathrm{length}(\pi_0(\gamma)) \leq c\, e^{-\mu_1 \frac{c}{2}}.$$

Since $\pi_0(\gamma)$ is not null-homotopic, its length is at least 1, and this shows that

$$c \geq e^{\mu_1 \frac{c}{2}}.$$

This can happen only for $c \leq c_0(\mu_1)$. $\square$

Lemma 6.2. *Let z_1, z_2 be two points in Z such that $d(O', \Theta(z_1)) > c_1$ or $d(O', \Theta(z_2)) > c_1$ and $d(z_1, z_2) \leq c_1/\lambda_1$. Then $d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) = d(\Theta(z_1), \Theta(z_2))$.*

Proof. Let $\tilde{z}_1 \in \tilde{Z}$ be such that $d(\tilde{O}, \tilde{z}_1) > c_1$. Set

$$\begin{aligned} W &= \{\tilde{z}_2 \in \tilde{Z} |, d(\tilde{z}_1, \tilde{z}_2) \leq c_1\}, \\ U &= \{\tilde{z}_2 \in W | d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) = d(\Theta(z_1), \Theta(z_2))\} \subset W, \\ V &= \{\tilde{z}_2 \in W | d(\tilde{\Theta}(\tilde{z}_1), \iota' \circ \tilde{\Theta}(\tilde{z}_2)) = d(\Theta(z_1), \Theta(z_2))\} \subset W. \end{aligned}$$

By construction, $W = U \cup V$. Let us show that the intersection of U and V is empty

$$U \cap V = \{\tilde{z}_2 \in W | d(\tilde{\Theta}(\tilde{z}_1), \iota' \circ \tilde{\Theta}(\tilde{z}_2)) = d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2))\}.$$

If $\tilde{z}_2 \in U \cap V$, then the geodesic segments connecting $\tilde{\Theta}(\tilde{z}_1)$ with $\tilde{\Theta}(\tilde{z}_2)$ and $\tilde{\Theta}(\tilde{z}_1)$ with $\iota' \circ \tilde{\Theta}(\tilde{z}_2)$ induce a loop γ in Z' of length $2d(\Theta(z_1), \Theta(z_2)) \leq 2\left(\lambda_1(c_1/\lambda_1) + c_1\right) = 4c_1$ which is not homotopic to 0. According to Lemma 6.1, this is incompatible with the assumption that $d(O', \Theta(z_1)) > c_1$. Hence, $U \cap V$ is empty. Since U is non-empty (it contains at least $\tilde{z}_1$) and closed in W, V is closed in W and W is connected, we conclude that $U = W$, which finishes the proof. □

Lemma 6.3. *A $(\lambda_1, \lambda_2, c_1, c_2)$-quasi-isometric embedding $\Theta : Z \to Z'$ lifts to a "quasi-Lipschitz" map $\tilde{\Theta} : \tilde{Z} \to \tilde{Z}'$, that is, for any two points $\tilde{z}_1, \tilde{z}_2 \in \tilde{Z}$,*

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) \leq \lambda_1 d(\tilde{z}_1, \tilde{z}_2) + 2c_1.$$

Proof. Let $\tilde{\gamma} \subset \tilde{Z}$ be a geodesic between $\tilde{z}_1$ and $\tilde{z}_2$. Let t_1 be the first point such that $d(\tilde{\Theta}\gamma(t), \tilde{O}') \leq c_1$ and t_2 be the last point with such a property (if such points t_1, t_2 do not exist, then we can apply the following arguments directly to $d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2))$ instead of cutting the curve in three parts and considering $d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_1)) + d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_2)))$. Then

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) \leq d(\tilde{\Theta}\tilde{\gamma}(t_1), \tilde{\Theta}\tilde{\gamma}(t_2)) + d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_1)) + d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_2)).$$

By definition of t_1 and t_2, $d(\tilde{\Theta}\tilde{\gamma}(t_1), \tilde{\Theta}\tilde{\gamma}(t_2)) \leq 2c_1$. Now divide parts of γ between $\tilde{\Theta}(\tilde{z}_1)$ and $\tilde{\Theta}\tilde{\gamma}(t_1)$ and between $\tilde{\Theta}(\tilde{z}_1)$ and $\tilde{\Theta}\tilde{\gamma}(t_2)$ by segments of length c_1/λ_1. Apply the previous lemma to them, so

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_1)) + d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}\tilde{\gamma}(t_2)) \leq N\left(\lambda_1 \frac{c_1}{\lambda_1} + c_1\right),$$

where $N \leq d(\tilde{z}_1, \tilde{z}_2)/(c_1/\lambda_1)$ is the number of segments in the subdivision. So,

$$d(\tilde{\Theta}(\tilde{z}_1), \tilde{\Theta}(\tilde{z}_2)) \leq 2c_1 + 2\lambda_1 d(\tilde{z}_1, \tilde{z}_2). \qquad \square$$

6.4. Proof of Theorem 6.1 – Part 1

Let ψ' be a kernel on $\tilde{Z}$ which is invariant by isometry, that is, for any isometry ι,

$$\psi'(\iota(\tilde{z}_1), \iota(\tilde{z}_2)) = \psi'(\tilde{z}_1, \tilde{z}_2).$$

As an example of such a kernel, consider a kernel depending only on the distance between points. Let ζ be a kernel on $\tilde{Z}'$ which is also invariant by isometries. Define a complex function v on $\tilde{Z}$ as follows

$$v(\cdot) = \left(\int\limits_Y u(\tilde{z}')\zeta(\tilde{\Theta}(\tilde{z}), \tilde{z}')d\tilde{z}' \right) * \psi'(\cdot, \tilde{z}).$$

We will write shortly for the integral

$$u *_t \zeta(\tilde{\Theta})(\tilde{z}) = \int\limits_Y u(\tilde{z}')\zeta(\tilde{\Theta}(\tilde{z}), \tilde{z}')d\tilde{z}'.$$

Then $v \circ \iota = -v$. Indeed,

$$v \circ \iota = \big(u *_t \zeta(\tilde{\Theta})\big) * \psi' \circ \iota = \big(u *_t \zeta(\tilde{\Theta}) \circ \iota\big) * \psi'.$$

On the other hand, using both relations $\tilde{\Theta} \circ \iota = \iota' \circ \tilde{\Theta}$ and $(\iota')^2 = id$, we have

$$\begin{aligned} u *_t \zeta(\tilde{\Theta}) \circ \iota(\tilde{z}) &= \int u(\tilde{z}')\zeta(\tilde{\Theta}(\iota\tilde{z}), \tilde{z}')d\tilde{z}' = \int u(\tilde{z}')\zeta(\iota'\tilde{\Theta}(\tilde{z}), (\iota')^2\tilde{z}')d\tilde{z}' = \\ &= \int u(\tilde{z}')\zeta(\tilde{\Theta}(\tilde{z}), \iota'\tilde{z}')d\tilde{z}' = \int u(\iota'\tilde{z}')\zeta(\tilde{\Theta}(\tilde{z}), \tilde{z}')d\tilde{z}' = -u *_t \zeta(\tilde{\Theta}), \end{aligned}$$

hence v is skew-symmetric with respect to ι. We get immediately that $\int v = 0$. Now apply successively Lemma 4.3 and Lemma 4.4.

Step 1. By Lemma 4.3 there exists a kernel ψ_1 on $\tilde{Z}$ which is controlled by a and b and such that

$$\left(\int |\nabla(u *_t \zeta(\tilde{\Theta}) * \psi')|^p \right)^{1/p} \leq N_{\psi_1}\left(u *_t \zeta(\tilde{\Theta})\right),$$

where for ψ_1, the width of support is $R^{\psi_1} = R^{\psi'}$ and

$$\sup \psi_1 \leq \frac{\sup \nabla\psi' \sup \psi'}{\inf_z Vol\, B(\tilde{z}, R^{\psi})}.$$

Step 2. By Lemma 4.4 there exists a kernel ζ_1 on $\tilde{Z}'$ such that

$$N_{\psi_1}\left(u *_t \zeta(\tilde{\Theta})\right) \leq \tilde{C} N_{\zeta_1}(u),$$

where the width of support of ζ_1 is $2R^\zeta + \lambda_1 R^{\psi'} + c_1$, the supremum of ζ_1 is

$$\sup \zeta_1 = \frac{\sup \psi_1}{c_\tau^Y} e^{2R^\zeta + \lambda_1 R^{\psi'} + c_1} (2\lambda_1 R^\zeta + c_1)^2$$

and

$$\tilde{C} = \frac{1}{c_\tau^Y} (\sup \psi_1)^{3/p} e^{\left((2+\lambda_1)R^{\psi'} + c_1\right)/p} \left((2+\lambda_1)R^{\psi'} + c_1\right)^{2/p}.$$

Step 3. Applying Lemma 4.3 we get that there exists a kernel ζ_2 on $\tilde{Z}'$ such that

$$N_{\zeta_2}(u) \leq C(n)\|\nabla u\|_p,$$

we remind that the constant $C(n)$ depends only on the dimension of $\tilde{Z}'$ if the Ricci curvature is bounded from below, that is $\sup \mu_i$ is bounded.

Step 4. Here we merely need to pass from N_{ζ_1} to N_{ζ_2}. Apply Lemma 4.3 once more

$$N_{\zeta_1} \leq \hat{C} N_{\zeta_2},$$

where

$$\hat{C} = \frac{\sup \zeta_1 \sup \zeta_2}{c_\tau^Y} \frac{R^{\zeta_2}}{\varepsilon^{\zeta_2}} (2e)^{(2R^\zeta + \lambda_1 R^{\psi'} + c_1)/\varepsilon^{\zeta_2}}.$$

Choose ψ' and ζ such that $R^{\psi'} = 1$ and $R^\zeta = 1$. Then $\sup \psi'$ and $\sup \zeta$ are controlled by a and b. Note also that $\varepsilon^{\zeta_2} = 1$. So combining all inequalities we get

$$\int_{B(R)} |\nabla v|^p \leq C_1(a,b)\,(\lambda_1 + c_1)^{3+2/p}\, e^{(9+3/p)(\lambda_1+c_1)} \int_{\mathbb{T}^n \times [0,+\infty]} |\nabla u|^p,$$

where $C_1(a,b)$ is a constant depending only on a, b and dimension n. Let $Q = \lambda_1 + c_1$ and

$$C(Q) = (\lambda_1 + c_1)^{3+2/p}\, e^{(9+3/p)(\lambda_1+c_1)}.$$

6.5. Proof of Theorem 6.1 – Part 2

We will give a lower bound for the $\mathbb{L}^p$-norm of the function $v = (u * \phi) * \psi'$. Our aim is to prove that the absolute value of v is nearly constant. For simplicity of notations, suppose first that the volume growth of Z_μ and $Z_{\mu'}$ is the same, that is $\sum \mu_i = \sum \mu'_i$. Denote these sums by $|\mu|$ and $|\mu'|$ respectively. We are going to show that there exists a subset A of the ball $B(z_0, R)$ such that on the one hand the volume of A is rather big, that is $Vol(A) \geq Vol(B(z_0, R))/2$ and on the other hand its image lies rather far from the base point $\Theta(A) \cap B(z'_0, R - (\lambda_1 + c_1 + \lambda_2 + c_2)) = \emptyset$.

Denote $r = \lambda_2 + c_2$. We will construct a finite subset J in $B(z_0, R) \subset Z_\mu$ and a partition of J into $e^{|\mu|r}$ subsets $\{J_k\}_{k=1,\dots,n}$, each of cardinality $|J_k| = e^{|\mu|(R-r)}$ with the following property

- **(P)** For any $k \in \{1, \dots, n\}$ if z_1 and z_2 are points of J_k then the open balls of radius r centred at these points are disjoint.

So, let $z_1, z_2 \in J_k$ be two different points. It follows from **(P)** that

$$2r \leq d(z_1, z_2) \leq \lambda_2 d(\Theta(z_1), \Theta(z_2)) + c_2,$$

hence $d(\Theta(z_1), \Theta(z_2)) \geq 2$, so the balls $B(\Theta(z_1), 1)$ and $B(\Theta(z_2), 1)$ are disjoint. Fix some $d > 0$ and denote by $J'_k \subset J_k$ the set of points whose images are not farther than $R - d$ from z'_0 that is if $z \in J'_k$ then $d(z'_0, \Theta(z)) \leq R - d$. We obtain

$$|J'_k| \, Vol(B(\Theta(z), 1)) \leq Vol(B(z'_0, R - d + 1)),$$

and we conclude that $|J'_k| \leq e^{|\mu|(R-d)}$. Denote the union of J'_k by J' then $|J'| \leq e^{|\mu|(R-d+r)}$. Hence, whenever $d \geq r + 1$,

$$\frac{|J'|}{|J|} \leq e^{|\mu|(r-d)} \leq \frac{1}{2}.$$

So, we choose $d = r + 1$. Now let A be the union of all 1-balls centred at points of $J \setminus J'$, $A = \cup_{z \in J \setminus J'} B(z, 1)$. The volume $Vol\, A \geq 1/2\, Vol(B(z_0, R))$. By definition of A, for any point $z \in A$ there exists a point $z' \in J \setminus J'$ at most 1-far away from z, $d(z, z') \leq 1$. Applying triangle inequality we get $d(z'_0, \Theta(z)) \geq d(z'_0, \Theta(z')) - (\lambda_1 + c_1) \geq R - (\lambda_1 + c_1 + \lambda_2 + c_2)$.

Here we describe the set $J \subset \{R\} \times \mathbb{R}^n/\mathbb{Z}^n$ (we fix the first coordinate $t = R$). This is the set of points $z = (R, x_1, \dots, x_n)$ such that for any $i = 1, \dots, n$, x_i is an integer multiple of $e^{-\mu_i R}$ modulo 1. J_0 is the subset of points such that for any i, x_i is a whole multiple of $e^{\mu_i(r-R)}$. Let K be the set of vectors $k = (0, k_1, \dots, k_n)$ such that for any i the number $e^{\mu_i R} k_i$ is an integer between 0 and $e^{\mu_i(r-R)} - 1$. For $k \in K$, we define

$J_k = J_0 + k$. Then for any two different points z_1, z_2 of J_k,

$$d(z_1, z_2) = \max \log \left(|x_i^1 - x_i^2|^{1/\mu_i} \right) \geq r.$$

We constructed the needed set. Now we notice that the lifting $\tilde{A} \subset \tilde{Z}$ of A has the same properties relatively to $\tilde{\Theta}$: the image $\tilde{\Theta}(\tilde{A})$ lies at distance at least $R - (\lambda_1 + c_1 + \lambda_2 + c_2)$ from the base point and the volume of $\tilde{A}$ is at least a half of the volume of the ball $B(\tilde{z}_0, R)$.

Now let us compute $|v(\tilde{z})|$ for $\tilde{z} \in \tilde{A}$ (in fact here we will give an upper bound for $|v|$ which is true for all $\tilde{z} \in B(z_0, R)$ and a lower bound for $\tilde{z} \in \tilde{A}$). We remind that by construction, $\tilde{z}$ is sent far from the base point, $d(\tilde{z}_0', \tilde{\Theta}(\tilde{z})) \geq R - (\lambda_1 + c_1 + \lambda_2 + c_2)$.

$$\begin{aligned}
|(u * \phi) * \psi'(\tilde{z})| &= \left| \int_X \int_Y u(\tilde{z}')\zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}')\psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\
&\geq \left| \int_X \int_Y (u(\tilde{z}') - u(\tilde{\Theta}(\tilde{z})) + u(\tilde{\Theta}(\tilde{z})))\zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}')\psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\
&\geq \left| \int_X \int_Y (u(\tilde{\Theta}(\tilde{z})))\zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}')\psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\
&\quad - \left| \int_X \int_Y (u(\tilde{z}') - u(\tilde{\Theta}(\tilde{z})))\zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}')\psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\
&\geq 1 - \int_X \int_Y |u(\tilde{z}') - u(\tilde{\Theta}(\tilde{z}))|\zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}')\psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1. \qquad (6.1)
\end{aligned}$$

For the last inequality we shall use the following facts: $|u| = 1$ and the integral of a kernel over the second argument is equal to 1.

$$\begin{aligned}
&\left| \int_X \int_Y u(\tilde{\Theta}(\tilde{z}))\zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}')\psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}' d\tilde{z}_1 \right| \\
&= \left| \int_X u(\tilde{\Theta}(\tilde{z}))\psi'(\tilde{z}, \tilde{z}_1) \left(\int_Y \zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}') d\tilde{z}' \right) d\tilde{z}_1 \right| \\
&= \left| \int_X u(\tilde{\Theta}(\tilde{z}))\psi'(\tilde{z}, \tilde{z}_1) d\tilde{z}_1 \right| = \left| u(\tilde{\Theta}(\tilde{z})) \right| = 1.
\end{aligned}$$

We need to estimate the double integral in Eq. (6.1). $\psi'(\tilde{z}, \tilde{z}_1)$ is non-zero if $d(\tilde{z}, \tilde{z}_1) \leq R^{\psi'} = 1$ and $\zeta(\tilde{\Theta}(\tilde{z}_1), \tilde{z}')$ is non-zero if $d(\tilde{z}', \tilde{\Theta}(z_1)) \leq R^{\zeta} = 1$. So the diameter of the set $\hat{S}$

of points $\tilde{z}'$ such that the integrand is non-zero, is at most $2\lambda_1+c_1+2 \leq 4(\lambda_1+c_1)$ because $\lambda_1 \geq 1$. Hence $\hat{S}$ is contained in a ball $B_{\hat{S}}$ of radius $4(\lambda_1+c_1)$. Assume $\hat{z}' = \tilde{\Theta}(\tilde{z}) \in \hat{S}$. Then by the mean value theorem, for any point $\tilde{z}' \in \hat{S}$,

$$\begin{aligned}|u(\tilde{z}') - u(\hat{z}')| &\leq |\tilde{z}' - \hat{z}'| \sup_{\tilde{z}' \in B_{\hat{S}}} |\nabla u(\tilde{z}')| \leq 8(\lambda_1 + c_1) \sup_{\tilde{z}' \in B_{\hat{S}}} \left|\frac{\partial u}{\partial \tilde{x}_n}\right| e^{-\mu_n' t} \\ &\leq 8\pi(\lambda_1 + c_1) e^{-\mu_n' t} \leq 8\pi(\lambda_1 + c_1) \sup_{\hat{z}' \in B_{\hat{S}}} e^{-\mu_n' d(O', \hat{z}')} \\ &\leq 8\pi(\lambda_1 + c_1) e^{-\mu_n'(R-(\lambda_1+c_1+\lambda_2+c_2)-2(\lambda_1+c_1))} \leq \frac{1}{2}\end{aligned}$$

for $R \geq 8(\lambda_1 + c_1) + (\lambda_2 + c_2) = R_0$. Hence we have proved that

$$\begin{aligned}&\frac{1}{2} \leq |(u * \phi) * \psi'(\tilde{z})| \ \text{ if } \ \tilde{z} \in \tilde{A} \\ &|(u * \phi) * \psi'(\tilde{z})| \leq 1 \ \text{ if } \ \tilde{z} \in B(\tilde{z}_0, R).\end{aligned}$$

And we conclude from this relation that for $R \geq R_0 + 1$,

$$\int_{B(R)} |v|^p \geq \frac{1}{2^p} Vol(B(R)) - Vol(B(R_0)) \geq e^{(\sum \mu_i)R}/2^{p+1}.$$

Let us compute the integral $\int |\nabla u|^p$.

$$\int |\nabla u|^p = \int \left|\frac{\partial u}{\partial x_n}\right|^p e^{-\mu_n' p t} e^{(\sum \mu_i')t} dt dx_n = \pi \int_0^{+\infty} e^{(\sum \mu_i' - p\mu_n')t} dt = \frac{\mu_n' \pi}{-\sum \mu_i'/\mu_n' + p}.$$

Hence the Poincaré constant $C_p(\mu)$ for Z satisfies

$$\begin{aligned}(C_p(\mu))^p &\geq \frac{\|v\|^p}{\|\nabla v\|^p} \geq \frac{\|v\|^p}{C_1(a,b)C(Q)\|\nabla u\|^p} \\ &\geq \left(\mu_n' \pi 2^{p+1} C_1(a,b) C(Q)\right)^{-1} e^{(\sum \mu_i)R} (p - \sum \mu_i'/\mu_n').\end{aligned}$$

This proves the claim in Theorem 6.1.

6.6. *Proof of Theorem 6.2*

Let $\Theta : B_Z(R) \to Z'$ be a $(\lambda_1, \lambda_2, c_1, c_2)$-quasi-isometric embedding. By hypothesis, Θ is isomorphic on fundamental groups. Lemma 6.1 implies that Θ moves the origin a bounded distance away. Indeed, a non-null-homotopic loop of length 1 based at O is mapped to a non-null-homotopic loop of length $\leq Q = \lambda_1 + c_1$ based at $\Theta(O)$. This implies that $t(\Theta(O)) \leq 4Q$ and $d(O', \Theta(O)) \leq 4Q + 1$.

The space $\tilde{Z}$ is of the form $\tilde{T} \times \mathbb{R}$ where $\tilde{T} \to T$ is a connected 2-sheeted covering space of torus, hence $\tilde{T}$ is also a torus. Hence we can apply Theorem 5.1. We have $C_p(\mu) \le C_2(a,b)e^{\mu_n R}$. If $R \le 8(\lambda_1 + c_1) + (\lambda_2 + c_2)$ there is nothing to prove. Otherwise we develop the latter inequality and arrive at

$$\left(\mu'_n \pi 2^{p+1} C_1(a,b) C(Q)\right)^{-1/p} e^{(\sum \mu_i/p)R} \left(p - \sum \mu'_i/\mu'_n\right)^{1/p} \le C_2(a,b)e^{\mu_n R}.$$

Hence with $C_3(a,b) = (\mu'_n \pi 2^{p+1} C_1(a,b))^{1/p} C_2(a,b)$,

$$C_3(a,b)C(Q) \ge e^{(\sum \mu_i/p - \mu_n)R} \left(p - \frac{\sum \mu'_i}{\mu'_n}\right)^{1/p}.$$

We have calculated that $C(Q) = Q^{3+2/p} e^{(9+3/p)Q}$. Combining these results and taking the logarithm (note that in the following calculations every constant depending on μ and μ' can be estimated using a and b), we get

$$\left(3 + \frac{2}{p}\right)\log Q + \left(9 + \frac{3}{p}\right) Q \ge G'(a,b) + \left(\frac{\sum \mu_i}{p} - \mu_n\right) R + \frac{1}{p}\log\left(p - \frac{\sum \mu'_i}{\mu'_n}\right)$$

with some constant G' depending only on a and b. Because $p \ge 1$, the left-hand size can be estimated as $5\log Q + 12Q < 24Q$. Setting $p = \sum \mu'_i/\mu'_n + 1/R$, we get

$$24Q \ge G'(a,b) + \frac{\mu_n\left(\frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_i}{\mu'_n} - \frac{1}{R}\right) R}{\frac{\sum \mu'_i}{\mu'_n} + \frac{1}{R}} + \frac{1}{p}\log\frac{1}{R}.$$

For $R \ge G''(a,b)$ with some well-chosen constant G'',

$$24Q \ge G'(a,b) + \frac{\mu_n \mu'_n}{4\sum \mu'_i}\left(\frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_i}{\mu'_n}\right) R - \frac{\mu'_n}{2\sum \mu'_i}\log R,$$

and finally we rewrite our inequality under the desired form

$$Q \ge G_1(a,b)\left(\frac{\sum \mu_i}{\mu_n} - \frac{\sum \mu'_n}{\mu'_n}\right) R - G_2(a,b)$$

with $G_1(a,b)$ and $G_2(a,b)$ being constants depending only on a and b.

This finishes the proof of Theorem 6.2.

7. Quasi-isometric distortion for regular trees

In this section, we prove that embedding hyperbolic balls into trees requires a linear distortion growth.

First we need to introduce coarse notions of volume and of separation (minimal volume of subsets dividing a metric space X into two pieces).

Definition 7.1. Let $a > 0$. The a-volume of a metric space X is the following quantity

$$Vol_a(X) = \sup\left\{v \middle| \text{for any family } B_j \text{ of balls of radius } a \text{ covering } X : \#\{B_j\} \geq v\right\}.$$

Definition 7.2. Let $a > 0$. The a-separation of X is the number

$$\begin{aligned} sep_a(X) = \sup \Big\{ N \Big| & \text{for any partition } X = U_1 \sqcup U_2 \text{ such that } Vol_a(U_i) \geq Vol_a(X)/3, \\ & i = 1, 2, \text{ for any family } B_j \text{ of pairwise disjoint balls of radius } a, \\ & \#\text{balls intersecting both } U_1 \text{ and } U_2 \geq N \Big\} \end{aligned}$$

Theorem 7.1. *Let X be a bounded metric space, and T be a tree of degree at most d. $S = sep_a(X)$ and $V = Vol_a(X)$. Suppose that for any subset Y of X of a-volume at least one third of V, the diameter of Y is at least $diam(X)/D$ for some constant D depending only on X. If $f : X \to T$ is a (λ, c)-quasi-isometric embedding then*

- *either $diam(X) \leq cD$,*
- *or*

$$\lambda 2a + c \geq \log_d \frac{S}{Vol_a(B(c))}.$$

Proof. Let $\{B_j\}$ be a maximal set of pairwise disjoint balls of radius a in X. We consider T as a finite discrete metric space. If there exists a vertex t of T such that at least one third of centres of B_j are sent to t then $diam(X) \leq cD$ because of the hypothesis on the space X. Otherwise, for any vertex t,

$$Vol_a(f^{-1}(t)) < Vol_a(X)/3.$$

We are going to find a vertex t which divides the tree into two components $T = T_1 \cup T_2, T_1 \cap T_2 = \{t\}$ such that $Vol_a f^{-1}(T_i)$, $i = 1, 2$, is at least one third of $Vol_a(X)$. To obtain such a vertex consider a subtree consisting of all vertices $t \in T$ with non-empty inverse image $f^{-1}(t)$. Now, start from some boundary vertex (we will call T_1 the component which contains the initial vertex) of the tree and to pass from one vertex to another. At every step we choose a vertex which increases $Vol_a f^{-1}(T_1)$. We finish when the accumulated volume is sufficient, that is $Vol_a f^{-1}(T_1) \geq Vol_a(X)/3$.

Denote $U_i = f^{-1}(T_i), i = 1, 2$. The number N_s of balls B_j which intersect both U_1 and U_2 is at least $N_s \geq sep_a(X) = S$. Let I be a set which contains a point of the intersection $U_1 \cap B_j$ for all such balls, denote the image of I by $I' = f(I)$.

$$|I'| \geq \frac{S}{Vol_a(B(c))}.$$

Because $I' \subset T_1$, volume consideration immediately implies that there exists $v_1 \in I'$ such that

$$d(v_1, T_2) \geq \log_d \frac{S}{Vol_a(B(c))}.$$

Thus $v_1 = f(u_1)$ and $u_1 \in B_j$ which intersects U_2, there exists $u_2 \in U_2 \cap B_j$ and $d(u_1, u_2) \leq 2a$, hence $d(v_1, f(u_2)) \leq \lambda 2a + c$. Hence,

$$\lambda 2a + c \geq \log_d \frac{S}{Vol_a(B(c))}. \qquad \square$$

Consider $\mathbb{H}^n$ with $n \geq 3$. For a ball of radius R in $\mathbb{H}^n$ we have $S \sim e^{(n-2)R}$ (see Lemma 7.1, which follows this proof), $V \sim e^{(n-1)R}$ and $D = 1$. Then the application of Theorem 7.1 to $B(R) \subset \mathbb{H}^n$ with $n \geq 3$ proves the linear quasi-isometric distortion between $\mathbb{H}^n$ and a regular tree.

Corollary 7.1. *The quasi-isometric distortion growth for hyperbolic space $\mathbb{H}^n$, $n \geq 3$, and a regular tree is linear in R.*

Lemma 7.1. *Let $B(R) := B_{\mathbb{H}^n}(R) = A_1 \sqcup A_2$ be a partition of an R-ball of hyperbolic n-space. Suppose that both pieces have large volume: $Vol\, A_i \geq 1/3\, Vol\, B(R)$, $i = 1, 2$. Then for R large enough the volume of the common boundary of A_1 and A_2, $S_{12} = \partial A_1 \cap \partial A_2$ is at least*

$$Vol\, S_{12} \geq const(n) e^{(n-2)R},$$

where the multiplicative constant depends only on dimension n.

This lemma nearly follows from Proposition 4.1 in [5]. Indeed, in the course of the proof of that proposition, it is demonstrated that balls maximise the cut size[3] of subsets of $\mathbb{H}_n$ with a given volume. Hence, the cut size of a ball is of the order of $e^{(n-2)R}$.

Question 7.1. What is the quasi-isometric distortion between a d-regular tree and hyperbolic plane $\mathbb{H}^2$.

8. Approximation of distances and radial quasi-isometries

The goal of this section is to construct a map between geodesic Gromov hyperbolic spaces with homeomorphic ideal boundary, such that its restrictions on balls of variable radius prove a non-trivial upper bound on quasi-isometry distortion growth. The first step in this construction is an approximation of distance in Gromov hyperbolic spaces using visual distance on its ideal boundary. Then we prove that the map of interest is induced by a boundary homeomorphism.

[3] Cut size of a set X is the infimum of volume of a subset $A \subset X$, where A divides X in several components with volumes at most a half of $Vol(X)$.

8.1. Orthogonal triangles in hyperbolic spaces

At the beginning of this subsection we give two lemmas on the geometry of orthogonal triangles in hyperbolic spaces. The second lemma will be used to establish an approximation of distances in hyperbolic spaces which allow to control a quasi-isometric action.

Lemma 8.1. *Let σ be a geodesic segment, a be a point not on σ, and c be a projection of a on σ. Let $b \in \sigma$ be arbitrary, and let d denote a projection of b on ac. Then $|c - d| \leq 2\delta$.*

Proof. By hypothesis, bd minimises the distance of b to any point of ac, and because the triangle bcd is δ-thin, there exists a point $e \in bd$ such that $d(e, ac) = |e - d| \leq \delta$ and $d(e, bc) \leq \delta$. Because ac is a perpendicular to σ, $|a - c| \leq |a - d| + |d - e| + d(e, bc) \leq |a - d| + 2\delta$. Hence $|c - d| \leq 2\delta$. □

Lemma 8.2. *As in the preceding lemma, let σ be a geodesic segment, a be a point not on σ, c be a projection of a on σ, and b be some point on σ. Let d denote a point on ac such that $|d - c| = \delta$ and e denote a point on bc such that $|e - c| = 3\delta$. Then*

- *$d(d, ab) \leq \delta$, $d(e, ab) \leq \delta$, $d(c, ab) \leq 2\delta$, and*
- *the length of ab differs from the sum of the lengths of the two other sides by at most 8δ,*

$$|a - c| + |b - c| - 2\delta \leq |a - b| \leq |a - c| + |b - c| + 8\delta.$$

Proof. The triangle abc is δ-thin. Therefore, obviously, $d(d, ab) \leq \delta$ (the distance from a point of ac to ab is a continuous function). We take a point $x \in bc$ such that $d(x, ca) \leq \delta$. Using Lemma 8.1, we obtain $|b - x| + d(x, ca) \geq |b - c| - 2\delta$, and hence $|c - x| \leq d(x, ca) + 2\delta \leq 3\delta$.

We now let d_1 and e_1 denote respective projections of d and e on ab. Then by the triangle inequality, we have

- $|a - d| - \delta \leq |a - d_1| \leq |a - d| + \delta$,
- $|b - e| - \delta \leq |b - e_1| \leq |b - e| + \delta$, and
- $0 \leq |d_1 - e_1| \leq |d_1 - d| + |d - c| + |c - e| + |e - e_1| \leq 6\delta$.

Combining all these inequalities, we obtain the second statement in the lemma. □

8.2. Approximation of distances in hyperbolic metric spaces

Let X, Y be two geodesic hyperbolic metric spaces with base points $x_0 \in X$, $y_0 \in Y$. Let $\theta : \partial X \to \partial Y$ be a homeomorphism between ideal boundaries.

Hypothesis 8.1. Assume that there exists a constant D such that for any $x \in X$ there exists a geodesic ray γ from the base point $\gamma(0) = x_0$ and passing near x: $d(x, \gamma) < D$.

We are going to construct approximately (up to D) a map $\Theta : X \to Y$ extending the boundary homeomorphism θ. Take some point x and a geodesic ray γ from x_0 passing near x: $d(\gamma, x) < D$. Then $\gamma(\infty)$ is a point on ideal boundary ∂X. The corresponding point $\theta(\gamma(\infty)) \in \partial Y$ defines a geodesic ray γ' such that $\gamma'(0) = y_0$ and $\gamma'(\infty) = \theta(\gamma(\infty))$. Set $\Theta(x) = \gamma'(d(x_0, x))$. So, by construction, Θ preserves the distance to the base point. Still, it depends on the choices of γ and γ'.

Definition 8.1. Define the following quantity

$$K(R) = \sup\left\{\left|\log \frac{d_{y_0}(\theta(\xi_1), \theta(\xi_2))}{d_{x_0}(\xi_1, \xi_2)}\right| |d_{x_0}(\xi_1, \xi_2) \geq e^{-R} \vee d_{y_0}(\theta(\xi_1), \theta(\xi_2)) \geq e^{-R}\right\}.$$

We are going to prove that the restriction of Θ on the ball $B(R) \subset X$ of radius R is a $\left(1 + 2\frac{K(R)}{D+\delta}, D + \delta + 2K(R)\right)$-quasi-isometry. The important step in the proof of this result is Lemma 8.3 which gives an approximation (up to an additive constant) of the distance between two points in a hyperbolic metric space.

Lemma 8.3. *Let P_1, P_2 be two points in a hyperbolic metric space Z. Let P_0 be a base point (possibly at infinity). Let distances (horo-distances if P_0 is at infinity) from P_1 and P_2 to P_0 be $d(P_1, P_0) = t_1$ and $d(P_2, P_0) = t_2$. Assume that there exist points P_1^∞ and P_2^∞ such that P_1 (resp. P_2) belongs to the geodesic ray defined by P_0 and P_1^∞ (resp. P_2^∞). Denote by*[4]

$$t_\infty = -\log visdist_{P_0}(P_1^\infty, P_2^\infty)$$

the logarithm of visual distance seen from P_0. Then up to adding a multiple of δ,

$$d(P_1, P_2) = t_1 + t_2 - 2\min\{t_1, t_2, t_\infty\}.$$

In the proof of this lemma, all equalities hold with a bounded additive error depending linearly on δ which we drop for the sake of simplicity of notations.

Proof. Let P_0' be a projection of P_0 on the geodesic $P_1^\infty P_2^\infty$. By Lemma 8.2, P_0' lies at distance at most 2δ from both $P_0P_1^\infty$ and $P_0P_2^\infty$. Hence, up to an additive constant bounded by 4δ the distance between P_0 and P_0' is equal to Gromov's product of P_1^∞ and P_2^∞. It follows that $t_\infty = d(P_0, P_0') = -\log visdist(P_1, P_2)$.

[4] We define $visdist(P_1^\infty, P_2^\infty)$ of two points P_1^∞, P_2^∞ at the ideal boundary as the exponential of minus Gromov's product of these points $e^{-(P_1^\infty|P_2^\infty)}$. Indeed, it is not a distance as it does not satisfy triangle inequality. But we will never have more than two points at infinity at the same time in our setting, so we will not use the triangle inequality property.

The triangle $P_0P_1^\infty P_2^\infty$ is δ-thin. Notice that if P_1 (or P_2) lies near the side $P_1^\infty P_2^\infty$ then $t_1 \geq t_\infty$. Otherwise, $t_1 \leq t_\infty$ (both inequalities are understood up to an additive error δ). This follows from the definition of the point P_0' as a projection and Lemma 8.2.

Hence, if $t_1, t_2 \geq t_\infty$, $d(P_1, P_2) = d(P_1, P_0) + d(P_2, P_0) - 2d(P_0, P_0') = t_1 + t_2 - 2t_\infty$.

If $t_1 \leq t_\infty \leq t_2$, $d(P_1, P_2) = d(P_1, P_0') + d(P_0', P_2) = t_2 - t_1$.

Finally, if $t_1, t_2 \leq t_\infty$, we get $d(P_1, P_2) = |t_1 - t_2| = t_1 + t_2 - 2\min\{t_1, t_2\}$ as P_1 lies near $P_0P_2^\infty$. □

8.3. Construction of quasi-isometry

The quasi-isometry which are going to construct in this subsection, allows us to establish an example of logarithmic quasi-isometric distortion in Subsection 9.2.

Lemma 8.4. *Let Z and Z' be two hyperbolic metric spaces. Let Θ be the radial extension of a boundary homeomorphism θ, as described at the beginning of this section. Then for any two points P_1, $P_2 \in B(P_0, R) \subset Z$ such that $d(P_1, P_2) > c$, the upper bound holds*

$$\frac{d_{Z'}(\Theta(P_1), \Theta(P_2))}{d_Z(P_1, P_2)} \leq 1 + 2\frac{K(R)}{c}.$$

If $d(P_1, P_2) < c$,

$$d_{Z'}(\Theta(P_1), \Theta(P_2)) < 2K(R) + c.$$

Proof. We will use the same notations as in Lemma 8.3. Visual distance d_Z^∞ between P_1^∞ and P_2^∞ and the (horo-)distance t_∞ from P_0 to $P_1^\infty P_2^\infty$ are connected by the relation $e^{-t_\infty} = d_\infty(P_1^\infty, P_2^\infty)$. In the same way we define t'_∞ as the (horo-)distance for corresponding images.

By Lemma 8.3 we know that $d(P_1, P_2) = t_1 + t_2 - 2\min\{t_1, t_2, t_\infty\}$.

Assume first $d(P_1, P_2) > c$. We will write $d_Z = d(P_1, P_2)$ for the distance between P_1 and P_2 and $d_{Z'} = d(\Theta(P_1), \Theta(P_2))$ for the distance between their images.

There are four cases to consider. These cases depend on the relative sizes of t_1, t_2, t_0 and t'_∞ as these variables determine values of minima defining d_Z and $d_{Z'}$. Without loss of generality, one may assume that $t_1 \leq t_2$.

1st case. If both $t_1 < t_\infty$ and $t_1 < t'_\infty$, then

$$\frac{d_{Z'}}{d_Z} = \frac{t_2 - t_1}{t_2 - t_1} = 1,$$

and this case is trivial.

2nd case. If $t_\infty < t_1$ and $t'_\infty < t_1$. Let us give an upper bound for

$$\frac{d_{Z'}}{d_Z} = \frac{t_1 + t_2 - 2t'_\infty}{t_1 + t_2 - 2t_\infty}.$$

Consider

$$t'_\infty - t_\infty = \log \frac{d_\infty(\theta(P_1^\infty), \theta(P_2^\infty))}{d_\infty(P_1^\infty, P_2^\infty)}.$$

Because $d_Z > c$, we have $t_1 + t_2 - 2t_\infty > c$ hence $e^{(t_1+t_2)/2} e^{-t_\infty} > e^{c/2}$. And as $t_1, t_2 \leq R$ we obtain for visual distance $d_Z^\infty \geq e^{c/2} e^{-R} \geq e^{-R}$. We conclude that

$$|t'_\infty - t_\infty| \leq K(R).$$

Finally,

$$\frac{d_{Z'}}{d_Z} = \frac{d_{Z'} - d_Z + d_Z}{d_Z} = 1 + \frac{t'_\infty - t_\infty}{t_1 + t_2 - t_\infty} \leq 1 + \frac{1}{c}|t'_\infty - t_\infty|.$$

3rd case. Now let $t_\infty < t_1 < t'_\infty$. Then

$$d_{Z'} - d_Z = t_2 - t_1 - (t_1 + t_2 - 2t_\infty) = 2(t_\infty - t_1) \leq 0,$$

which leads to

$$\frac{d_{Z'}}{d_Z} \leq 1.$$

4th case. Finally if $t'_\infty < t_1 < t_\infty$ then

$$d_{Z'} - d_Z = (t_1 + t_2 - 2t'_\infty) - (t_2 - t_1) = 2(t_1 - t'_\infty) \leq 2(t_\infty - t'_\infty).$$

We know that $t_1 \leq R$ and at the same time we have $t'_\infty < t_1$, hence $t'_\infty < R$ and visual distance between $P_1^{\infty\prime}$ and $P_2^{\infty\prime}$ is at least e^{-R}. Now in the same manner as in the 2nd case, we obtain that $t_\infty - t'_\infty \leq K(R)$ and hence

$$\frac{d_{Z'}}{d_Z} \leq 1 + 2\frac{K(R)}{c}.$$

Now assume that $d_Z(P_1, P_2) \leq c$ (we still suppose $t_1 \leq t_2$), hence the distance $t_\infty > t_2$ and we are either in first or fourth situation. In the first case, $t_1 < t_\infty$ and $t_1 < t'_\infty$ so $d_{Z'} = d_Z \leq c$. In the fourth case, we have still $d_{Z'} - d_Z \leq 2K(R)$ and hence $d'_Z \leq c + 2K(R)$. □

Applying the lemma both to Θ and Θ^{-1}, we get the following theorem.

Theorem 8.1. *Let X, Y be two geodesic hyperbolic metric spaces with base points $x_0 \in X$, $y_0 \in Y$. Assume that there exists a constant D such that for any $x \in X$ there exists a geodesic ray γ from the base point $\gamma(0) = x_0$ and passing near x: $d(x, \gamma) < D$*

(Hypothesis 8.1). Let the restriction of $\Theta : \partial X \to \partial Y$ be a homeomorphism between ideal boundaries. Then the restriction of Θ on a ball $B(x_0, R) \subset X$ of radius R is a (λ, C_q)-quasi-isometry to $B(y_0, R) \subset Y$, where $\lambda = 1 + 2^{\frac{K(R)}{c}}$ and $C_q = 2K(R) + c$. The constant c can be chosen as $c = D + \delta$ where δ is the hyperbolicity constant.

This theorem is based on the construction of the radial extension of a boundary homeomorphism. The restriction of this homeomorphism on a ball in X induces a quasi-isometry of balls in X and Y (which is a stronger limitation than a general quasi-isometric embedding). This construction might seem too naive, but it allows to prove the existence of sublinear quasi-isometric growth and hence allows to get non-trivial results. That is why we believe it is useful to include this result in this paper.

9. Examples

In this section we present examples of the application of the results from the previous section. In the first example, we consider a special case of a boundary homeomorphism which is a bi-Hölder map. The calculations show that for spaces Z and Z' in particular, which are two copies of $T^n \times [0, +\infty)$ with metrics $dt^2 + \sum e^{2\mu_i t} dx_i^2$ and $dt^2 + \sum e^{2\mu_i' t} dx_i^2$ respectively, $K(R) = |\max_i(\mu_i/\mu_i') - 1|\, R$. In the second example, for unipotent locally homogeneous spaces we obtain a logarithmic quasi-isometry distortion growth.

9.1. Bi-Hölder maps

Here we consider a special case of the boundary homeomorphism. Let X and Y be again two hyperbolic metric spaces and let the homeomorphism θ between their boundaries be a bi-Hölder map with parameters $c \geq 1, \alpha < 1$ and $\beta > 1$

$$d\big(\theta(\xi_1), \theta(\xi_2)\big) \leq c d(\xi_1, \xi_2)^\alpha,$$
$$d\big(\theta(\xi_1), \theta(\xi_2)\big) \geq \frac{1}{c} d(\xi_1, \xi_2)^\beta.$$

Assume first that for two points ξ_1, ξ_2 of the ideal boundary ∂X, the visual distance $d(\xi_1, \xi_2) > e^{-R}$. Then we have

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \leq \log c d(\xi_1, \xi_2)^{\alpha - 1} = -(1 - \alpha) \log d(\xi_1, \xi_2) \lesssim (1 - \alpha) R.$$

Now, if the visual distance between images of ξ_1 and ξ_2 satisfy $d(\theta(\xi_1), \theta(\xi_2)) > e^{-R}$, we get

$$d(\xi_1, \xi_2) \geq \frac{1}{c^{1/\alpha}} e^{-R/\alpha}$$

and hence

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \gtrsim \frac{1-\alpha}{\alpha} R.$$

We obtain the lower bound for the logarithm of the ratio $\log \frac{d(\theta(\xi_1),\theta(\xi_2))}{d(\xi_1,\xi_2)}$ just in the same way as the upper-bound. If $d(\xi_1, \xi_2) > e^{-R}$

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \geq \log \frac{1}{c} d(\xi_1, \xi_2)^{\beta-1} = -(1-\beta)\log d(\xi_1, \xi_2) \lesssim (1-\beta)R.$$

If $d(\theta(\xi_1), \theta(\xi_2)) > e^{-R}$

$$\log \frac{d(\theta(\xi_1), \theta(\xi_2))}{d(\xi_1, \xi_2)} \geq \log \frac{1}{c} d(\theta(\xi_1), \theta(\xi_2))^{(\beta-1)/\beta} = -\frac{1-\beta}{\beta} \log d(\theta(\xi_1), \theta(\xi_2)) \gtrsim \frac{1-\beta}{\beta} R.$$

This gives

$$K(R) \lesssim \max\{1-\alpha, 1-\beta\} R.$$

In particular, let us compute $K(R)$ for two spaces which are variants of $T^n \times [0, +\infty)$, denoted by Z and Z', with metrics $dt^2 + \sum e^{2\mu_i t} dx_i^2$ and $dt^2 + \sum e^{2\mu_i' t} dx_i^2$ respectively. The visual distance between points P_1 and P_2 is given by

$$d_\infty(P_1, P_2) \sim \max_{1 \leq i \leq n} |x_i^1 - x_i^2|^{1/\mu_i}.$$

Pick the identity map $\theta : \partial Z \to \partial Z'$. Then

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \sim \frac{\max_i |x_i^1 - x_i^2|^{1/\mu_i'}}{\max_i |x_i^1 - x_i^2|^{1/\mu_i}} \leq \max_i |x_i^1 - x_i^2|^{1/\mu_i' - 1/\mu_i}.$$

Suppose that $d(P_1, P_2) > e^{-R}$. Then

$$\begin{aligned} \left|\log \frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)}\right| &\leq \left|\log \max_i |x_i^1 - x_i^2|^{1/\mu_i' - 1/\mu_i}\right| \\ &= \max_i \left(\mu_i \left| \frac{1}{\mu_i'} - \frac{1}{\mu_i} \right| \left| \log |x_i^1 - x_i^2|^{1/\mu_i} \right| \right) \leq \max_i \left| \frac{\mu_i}{\mu_i'} - 1 \right| R. \end{aligned}$$

So, we conclude that $K(R) = |\max_i(\mu_i/\mu_i') - 1|\, R$.

Remark 9.1. More generally, such bi-Hölder maps exist between boundaries of arbitrary simply connected Riemannian manifolds with bounded negative sectional curvature. The Hölder exponent is controlled by sectional curvature bounds.

9.2. Unipotent locally homogeneous space

Now assume that the space Z is a quotient $\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R}$ of the space $\mathbb{R}^2 \times \mathbb{R}$ with the metric $dt^2 + e^{2t}(dx^2 + dy^2)$. Consider the space $Z' = \mathbb{R}^2/\mathbb{Z}^2 \ltimes_\alpha \mathbb{R}$, quotient of the space $\mathbb{R}^2 \rtimes_\alpha \mathbb{R}$, where α is the 2×2 matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The locally homogeneous metric is of the form $dt^2 + g_t$ where $g_t = (e^{t\alpha})^* g_0$

$$e^{t\alpha}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t x + te^t y \\ e^t y \end{pmatrix}$$

and so $g_t = d(e^t x + te^t y)^2 + d(e^t y)^2 = e^{2t}(dx^2 + 2tdxdy + (t^2+1)dy^2)$.

Let $\theta : \partial Z \to \partial Z'$ be the identity. Consider two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in Z. We will write $x = x_1 - x_2$ and $y = y_1 - y_2$. For the visual distance between P_1, P_2 we have

$$d_\infty(P_1, P_2) = \max\{|x|, |y|\}.$$

The distance between their images $\theta(P_1)$ and $\theta(P_2)$ is (see Section 5 of [28] and [31])

$$d_\infty(\theta(P_1), \theta(P_2)) = \max\{|y|, |x - y\log|y|\}.$$

We begin with giving an upper-bound for $\log(d_\infty(\theta(P_1), \theta(P_2))/d_\infty(P_1, P_2))$. We have to explore four different cases.

1st case. If $|x| < |y|$ and $|x - y\log|y|\,| < |y|$,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = 1.$$

2nd case. If $|x - y\log|y|\,| < |y| < |x|$,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} < 1.$$

3rd case. If $|x| < |y| < |x - y\log|y|\,|$,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y\log y|}{|y|} \leq \frac{|x|}{|y|} + |\log|y|\,|.$$

If $d_\infty(P_1, P_2) > e^{-R}$ we have $e^{-R} < |y| \le 1$ (the upper bound follows from the fact that y is a coordinate of a point of a torus) and hence $|\log|y|\,| \le R$ and we finish as follows:

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \le \frac{|x|}{|y|} + |\log|y|\,| \le 1 + R.$$

If $d_\infty(\theta(P_1), \theta(P_2)) > e^{-R}$ we will consider two situations.

- If $|x| > |y \log|y|\,|$ then $|x - y\log y| < 2|x|$ and as $|x| < |y|$,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \le 2.$$

- If $|x| < |y \log|y|\,|$ then $e^{-R} < |x - y\log|y|\,| < 2|y\log|y|\,|$ and hence $|\log|y|\,| < R$, so

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \le 1 + R.$$

4th case. Let now $|y| < |x|$ and $|y| < |x - y\log|y|\,|$

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y\log|y|\,|}{|x|} \le 1 + \frac{|y\log|y|\,|}{|x|}.$$

We will check two possibilities.

- If $|y| \le |x|^2$ then

$$\frac{|y\log|y|\,|}{|x|} = \frac{|y|^{1/2}}{|x|}\left|\,|y|^{1/2}\log|y|\right| \le 1.$$

- Now suppose that $|y| \ge |x|^2$. If $d_\infty(P_1, P_2) > e^{-R}$, we see easily that $|y| \ge e^{-2R}$ and hence

$$\frac{|y\log|y|\,|}{|x|} \le \frac{|x\log|y|\,|}{|x|} \le |\log|y|\,| \le 2R.$$

If $d_\infty(\theta(P_1), \theta(P_2)) > e^{-R}$ we use the fact that $|a + b| \ge 2\max\{|a|, |b|\}$. Hence, either $|x| > e^{-R}/2$ or $|y\log|y|\,| > e^{-R}/2$ and so $|y| \gtrsim e^{-R}$ and we finish the estimation as earlier.

So in the fourth case we have also

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \le 2R.$$

We have proved that $\log \frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} \le \log R$ in all four cases. Now we proceed to give a lower bound for the same expression.

1st case. If $|x| < |y|$ and $|x - y \log |y|\,| < |y|$,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = 1.$$

2nd case. If $|x - y \log |y|\,| < |y| < |x|$,

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|y|}{|x|}.$$

Without loss of generality, assume $x > 0$. By the construction of Z, $|y| < 1$ hence $\log |y| < 0$. If $0 < x \le y \log |y|$, we have $y < 0$. Now transform $x \le y \log |y|$ as $1 \le -\log |y|(-y)/x$, hence

$$-\frac{y}{x} \ge -\frac{1}{\log |y|}.$$

Now either $d_\infty(\theta(P_1), \theta(P_2)) = |y| > e^{-R}$ or $e^{-R} \le d_\infty(P_1, P_2) = |x| \le y \log |y|$ which also means that $|y| \gtrsim e^{-R}$. So,

$$\frac{|y|}{|x|} \ge \frac{1}{R}.$$

If on the contrary $y \log |y| \le x$ we have

$$x - y \log |y| < |y| < x. \tag{9.1}$$

First we notice that $y \log |y| > x - |y| > 0$. As $|y| < 1$ for any point of our space, $\log |y| < 0$ and we conclude that $y < 0$. Now from (9.1) we obtain that $x < -y(1 - \log |y|)$. As $1 - \log |y| > 0$ we obtain

$$-\frac{y}{x} > \frac{1}{1 - \log |y|}.$$

If $d_\infty(\theta(P_1), \theta(P_2)) = |y| > e^{-R}$, we trivially get that

$$\frac{|y|}{|x|} > \frac{1}{R}.$$

If $e^{-R} \le d_\infty(P_1, P_2) = |x|$ we write $e^{-R} < x < -y(1 - \log |y|)$ and hence $y \gtrsim e^{-R}$, so we obtain the same result. So, in both cases we come to the same result

$$\left| \log \frac{|y|}{|x|} \right| < R.$$

3rd case. Assume $|x| < |y| < |x - y\log|y||$, this case is trivial as

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y\log y|}{|y|} \geq 1.$$

4th case. Let now $|y| < |x|$ and $|y| < |x - y\log|y||$. We also suppose that $x > 0$ to save notation.

$$\frac{d_\infty(\theta(P_1), \theta(P_2))}{d_\infty(P_1, P_2)} = \frac{|x - y\log|y||}{|x|} = \left|1 - \frac{y\log|y|}{x}\right|. \tag{9.2}$$

If (9.2) is greater than 1/2 then we have nothing to prove. So suppose that (9.2) is less than 1/2

$$-\frac{x}{2} \leq x - y\log|y| \leq \frac{x}{2},$$

and so

$$\frac{x}{2} \leq y\log|y| \leq \frac{3x}{2}.$$

The last inequality shows that if either $d_\infty(\theta(P_1), \theta(P_2)) \geq e^{-R}$ or $d_\infty(P_1, P_2) \geq e^{-R}$, $|y| \gtrsim e^{-R}$ and so we have

$$\frac{|y\log|y||}{x} \geq \frac{|y\log|y||}{y} = |\log|y|| \geq \frac{1}{R},$$

which completes our discussion of this example. We have proved that

$$K(R) \lesssim \log R.$$

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Appendix A. Quasi-isometric embeddings and fundamental groups

Here we would like to discuss the hypothesis of Theorem 6.2 that quasi-isometric embeddings under consideration are a homotopy equivalence. We will show that if $\dim(Z) \geq 3$, one may believe that the assumption that Θ is isomorphic on fundamental groups, is not that restrictive. Indeed, in Proposition A.1, we shall show that this is automatic, but unfortunately, the argument introduces an ineffective constant R_0, which makes it useless. For instance, if it turns out that $R_0 = \lambda_1^2$, Proposition A.1 does not help to remove the homotopy assumption in Theorem 6.2. Nevertheless, it is included for the sake of completeness.

Proposition A.1. *Let Z, Z' be two locally homogeneous hyperbolic metric spaces with metrics $dt^2 + \sum e^{2\mu_i t} dx_i^2$ and $dt^2 + \sum e^{2\mu'_i t} dx_i^2$ respectively, $0 < \mu_i$ and $0 < \mu'_i$ for any $1 \leq i \leq n$. Let their dimension to be at least 3: $n + 1 \geq 3$. Then for any $\lambda_1 \geq 1, \lambda_2 \geq 1, c_1 \geq 0, c_2 \geq 0$ there exists $R_0 = R_0(\lambda_1, \lambda_2, c_1, c_2)$ such that if $R > R_0$ and a continuous map $f : B_{Z_\mu}(O, R_0) \to Z_{\mu'}$ is a $(\lambda_1, \lambda_2, c_1, c_2)$-quasi-isometric embedding, then f induces an isomorphism on fundamental groups $\pi_1(Z_\mu) \to \pi_1(Z_{\mu'})$.*

Proof. We provide a proof by contradiction. Assume that for arbitrarily large values of R, there exists a map $f_R : B_Z(R) \to Z'$ which is a $(\lambda_1, \lambda_2, c_1, c_2)$-quasi-isometric embedding which is not isomorphic on fundamental groups. Pick a $2c_1/\lambda_1$-dense and c_1/λ_1-discrete subset Λ of Z. Notice that if f_R is a $(\lambda_1, \lambda_2, c_1, c_2)$-quasi-isometry, then f_R is bi-Lipschitz on $B_Z(R) \cap \Lambda$. Conversely, if a map defined on $B(R) \cap \Lambda$ is bi-Lipschitz, then it can be continuously extended on $B(R)$ as a quasi-isometric embedding. Indeed, away from a ball, Z' is contractible up to scale c_1.

Set $\rho = d(O', f_R(O))$. First, consider the case when $\rho \to \infty$. Set $\sigma = (\rho/4 - c_1)/\lambda_1$. Then $f_R(B(O, \sigma))$ is contained in a ball $B(f_R(O), \rho/4)$ which lies in the complement of $B(O', \rho/2)$

$$f_R(B(O, \sigma)) \subset B(f_R(O), \rho/4) \subset B(O', \rho/2)^c.$$

The diameter of the image of any loop in $B(O, \sigma)$ is at most $\lambda_1\sigma + c_1$. Because $\lambda_1\sigma + c_1 < \rho/4$, these loops are homotopic to 0 (diameters of loops are too short relatively to $B(O', \rho/2)^c$). Hence, the restriction of f_R on $B(0, \sigma)$ is homotopic to 0. Hence f_R lifts to $\tilde{f}_R : B_Z(\sigma) \to \tilde{Z}' = X_{\mu'}$ which is homogeneous. Now up to composing $\tilde{f}_R$ with an isometry we can suppose that it preserves the centre $\tilde{f}_R(O) = O'$. By Ascoli's theorem, we can find a sequence $\tilde{f}_{R_j}|_\Lambda$ which uniformly converges to $\tilde{f}|\Lambda : Z \cap \Lambda \to \tilde{Z}'$ which is also bi-Lipschitz. We continuously extend $\tilde{f}_{|\Lambda}$ to $\tilde{f} : Z \to \tilde{Z}'$, $\tilde{f}$ is a quasi-isometric embedding. Its extension to ideal boundaries is continuous and injective. By the theorem of invariance of domain, $\partial\tilde{f} : T^n \simeq \partial X_\mu = S^n$ is open, and thus a homeomorphism. This provides a contradiction if $n \geq 2$.

If $\rho = d(O', f_R(O))$ stays bounded, we can directly use Ascoli's theorem, and get a limiting continuous quasi-isometric embedding f. Again, f extends to the ideal boundary, $\partial f : \partial Z \to \partial Z'$, the map ∂f is continuous and injective. Because ∂Z and $\partial Z'$ have the same dimension, ∂f is an open map by the theorem of invariance of domain and ∂f is a homeomorphism. Hence, ∂f induces an isomorphism on fundamental groups. If R_j is sufficiently large, then f_{R_j} is at bounded distance from f and hence f_{R_j} also induces an isomorphism $\pi_1(B_Z(R)) \to \pi_1(Z')$. This contradiction completes the proof. $\square$

Remark A.1. The proof does not provide an effective value of R_0.

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