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A quantitative version of the Morse lemma and quasi-isometries fixing the ideal boundary

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Abstract

The Morse lemma is fundamental in hyperbolic group theory. Using exponential contraction, we establish an upper bound for the Morse lemma that is optimal up to multiplicative constants, which we demonstrate by presenting a concrete example. We also prove an "anti" version of the Morse lemma. We introduce the notion of a geodesically rich space and consider applications of these results to the displacement of points under quasi-isometries that fix the ideal boundary.

Keywords: Morse lemma; Quasi-isometry; Quasi-geodesic; Hyperbolic space; Hyperbolic group

1. Introduction

Roughly speaking, the Morse lemma states that in a hyperbolic metric space, a λ-quasi-geodesic γ belongs to a λ^2-neighborhood of every geodesic σ with the same endpoints. Our aim is to prove the optimal upper bound for the Morse lemma.

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Theorem 1 *(Morse lemma). Let γ be a (λ, c)-quasi-geodesic in a δ-hyperbolic space E and σ be a geodesic segment connecting its endpoints. Then γ belongs to an H-neighborhood of σ, where*

$$H = \lambda^2(A_1 c + A_2\delta),$$

where A_1 and A_2 are universal constants.

We prove this theorem with $A_1 = 4 \cdot 78 = 312$ and

$$A_2 = 4\left(78 + \frac{133}{\ln 2} e^{157 \ln 2/28}\right)$$

in Section 5.2. This result is optimal up to the value of these constants, i.e., there exists an example of a quasi-geodesic such that H is the distance of the farthest point of γ from σ (see Section 6).

The Morse lemma plays an important role in the geometry of hyperbolic spaces. For example, it is used to prove that hyperbolicity is invariant under quasi-isometries between geodesic spaces [4] (see Chapter 5.2, Theorem 12): let E and F be δ_1- and δ_2-hyperbolic geodesic spaces. If there exists a (λ, c)-quasi-isometry between these two spaces, then

$$\delta_1 \leqslant 8\lambda(2H + 4\delta_2 + c).$$

Hyperbolic metric spaces have recently appeared in discrete mathematics and computer science (see, e.g., [3]). The δ-hyperbolicity turns out to be more appropriate than other previously used notions of approximation by trees (e.g., tree width). This motivates our search for optimal bounds for a cornerstone of hyperbolic group theory like the Morse lemma.

Gromov's quasi-isometry classification problem for groups [5] provides another motivation. When two groups are shown to be non-quasi-isometric, it would be desirable to give a quantitative measure of this, such as a lower bound on the distortion of maps between balls in these groups (we thank Itai Benjamini for bringing this issue to our attention). We expect our optimal bound in the Morse lemma to be instrumental in proving such lower bounds. As an indication of this, we show that the center of a ball in a tree cannot be moved very far by a self-quasi-isometry.

Proposition 1. *Let O be a center of a ball of radius R in a d-regular metric tree T ($d \geqslant 3$). Let f be (λ, c)-self-quasi-isometry of this ball. Then for any image $f(O)$ of the center O,*

$$d\big(f(O), O\big) \leqslant \min\big\{R, H + c + \lambda(c+1)\big\}.$$

Because $\delta = 0$ for a tree, we have $d(f(O), O) \leqslant 2A_1\lambda^2 c$ for sufficiently large λ. We prove this proposition in Section 6.

We present an example of a (λ, c)-quasi-isometry of a ball in a d-regular tree that moves the center a distance λc. We are currently unable to fill the gap between λc and $\lambda^2 c$.

We give a second illustration. In certain hyperbolic metric spaces, self-quasi-isometries fixing the ideal boundary move points a bounded distance. Directly applying the Morse lemma yields a bound of $H \sim \lambda^2 c$, while the examples that we know achieve merely λc. For this problem, we can fill the gap partially. Our argument relies on the following theorem, which we call the anti-Morse lemma.

Theorem 2 *(anti-Morse lemma). Let γ be a (λ, c)-quasi-geodesic in a δ-hyperbolic metric space and σ be a geodesic connecting the endpoints of γ. Let $4\delta \ll \ln\lambda$. Then σ belongs to an $A_3(c+\delta)\ln\lambda$-neighborhood of γ, where A_3 is some constant.*

We prove Theorem 2 in Section 7. In Section 9, we define the class of geodesically rich hyperbolic spaces (it contains all Gromov hyperbolic groups), for which we can prove the following statement.

Theorem 3. *Let X be a geodesically rich δ-hyperbolic metric space and f be a (λ, c)-self-quasi-isometry fixing the boundary ∂X. Then for any point $O \in X$, the displacement $d(O, f(O)) \leqslant \max\{r_0, (A_4 + c)\lambda\ln\lambda\}$, where r_0, A_4 are constants depending on the space X.*

We first discuss the geometry of hyperbolic spaces and prove a lemma on the exponential contraction of lengths of curves with projections on geodesics. We then discuss the invariance of the Δ-length of geodesics under quasi-isometries. Using these results, we prove the quantitative version of the Morse and anti-Morse lemmas. We define the class of geodesically rich spaces; for this class, we estimate the displacement of points by self-quasi-isometries that fix the ideal boundary. Finally, we show that this class includes all Gromov hyperbolic groups.

2. The geometry of δ-hyperbolic spaces

Let E be a metric space with the metric d. We also write $|x - y|$ for the distance $d(x, y)$ between two points x and y of the space E. For a subset A of E and a point x, $d(x, A)$ denotes the distance from x to A.

There are several equivalent definitions of hyperbolic metric spaces. We first present the most general definition, given by Gromov [6,4], although another definition is more convenient for us.

Definition 1. Gromov's product of two points x and y at a point z is

$$(x, y)_p = \frac{1}{2}\big(|x - p| + |y - p| - |x - y|\big).$$

Definition 2. A metric space E with a metric d is said to be δ-hyperbolic if for every four points p, x, y, and z,

$$(x, z)_p \geqslant \min\big\{(x, y)_p, (y, z)_p\big\} - \delta.$$

Definition 3. A geodesic (geodesic segment, geodesic ray) σ in a metric space E is a isometric embedding of a real line (real interval I, real half-line $\mathbb{R}_+$) in E.

We write xy for a geodesic segment between two points x and y (in general, there could exist several geodesic paths between two points; we assume any one of them by this notation). A geodesic triangle xyz is a union of three geodesic segments xy, yz, and xz.

Definition 4. A geodesic triangle xyz is said to be δ-thin if for any point $p \in xy$,

$$d(p, xz \cup yz) \leqslant \delta.$$

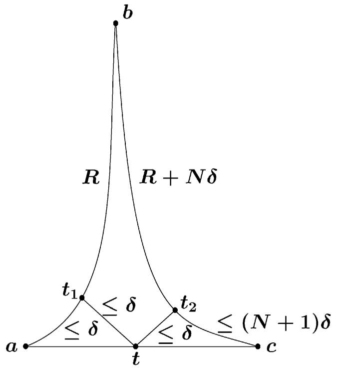

Fig. 1. Illustration for Lemma 1.

A geodesic metric space is a space such that there exists a geodesic segment xy between any two points x and y. It can be easily shown that for a geodesic space, Definition 2 is equivalent to the following definition.

Definition 5. A geodesic metric space E is δ-hyperbolic if and only if every geodesic triangle is $\delta/2$-thin (hereafter, we omit the factor $1/2$).

According to Bonk and Schramm [2], every δ-hyperbolic metric space embeds isometrically into a geodesic δ-hyperbolic metric space. Without loss of generality, we therefore consider only geodesic δ-hyperbolic spaces in what follows.

Definition 6. In a metric space, a *perpendicular* from a point to a curve (in particular, a geodesic) is a shortest path from this point to the curve.

Of course, a perpendicular is not necessarily unique.

Lemma 1. *In a geodesic δ-hyperbolic space, let b be a point and σ be a geodesic such that $d(b,\sigma) = R$. Let ba be a perpendicular from b to σ, where $a \in \sigma$. Let c be a point of σ such that $|b-c| = R + 2\Delta$. Then $|a-c| \leqslant 2\Delta + 4\delta$.*

Proof. The triangle abc (see Fig. 1) is δ-thin by the definition of a δ-hyperbolic space. Hence, there exists a point $t \in \sigma$ such that $d(t, ba) \leqslant \delta$ and $d(a, bc) \leqslant \delta$. Let t_1 and t_2 be the respective projections of t on ba and bc. By hypothesis, R is the minimum distance from b to the points of σ. Therefore, $R = |b-a| \leqslant |b-t_1| + |t_1-t| \leqslant |b-t_1| + \delta$ and $R \leqslant |b-t_2| + |t_2-t| \leqslant |b-t_2| + \delta$. Hence, $|a-t_1| \leqslant \delta$ and $|c-t_2| \leqslant 2\Delta + \delta$. By the triangle inequality, we obtain $|a-c| \leqslant |a-t_1| + |t_1-t| + |t-t_2| + |t_2-c| \leqslant 2\Delta + 4\delta$. □

Remark 1. In particular, all the orthogonal projections of a point to a geodesic lie in a segment of length 4δ.

Lemma 2. *In a δ-hyperbolic space, let two points b and d be such that $|b-d| = \Delta$. Let σ be a geodesic and a and c be the respective orthogonal projections of b and d on σ. Let*

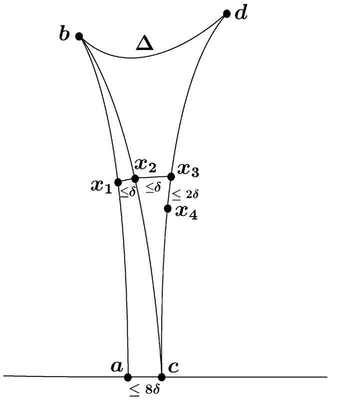

Fig. 2. Illustration for Lemma 2.

$|a-b| > 3\Delta + 6\delta$, and let $d(d,\sigma) > d(b,\sigma)$. Let two points $x_1 \in ab$ and $x_4 \in cd$ be such that $2\Delta + 5\delta < d(x_1,\sigma) = d(x_4,\sigma) < |a-b| - (\Delta + 2\delta)$. Then $|x_1 - x_4| \leqslant 4\delta$ and $|a-c| \leqslant 8\delta$.

Proof. (See Fig. 2.) By the triangle inequality and because cd is a perpendicular to σ, $|c-d| \leqslant |a-b| + |b-d|$, whence $|b-c| \leqslant |c-d| + |b-d| \leqslant |a-b| + 2|b-d|$. By Lemma 1, $|a-c| \leqslant 2\Delta + 4\delta$. The triangle abc is δ-thin, $|a-x_1| > |a-c| + \delta$. Therefore, by the triangle inequality, $d(x_1, ac) > \delta$, and hence $d(x_1, bc) \leqslant \delta$. Let x_2 denote the point of bc nearest x_1. Because the triangle bcd is also δ-thin and $|b-x_2| \geqslant |b-x_1| - |x_1-x_2| \geqslant \Delta + \delta$, there exists a point $x_3 \in cd$ such that $|x_3 - x_3| \leqslant \delta$. It follows from the triangle cx_1x_3 that $|x_3 - c| \geqslant |x_1 - c| - 2\delta \geqslant |x_1 - a| - 2\delta$. On the other hand, because x_5c is a perpendicular to σ, $|x_3 - c| \leqslant |x_3 - x_1| + |x_1 - a|$. Now, $|a - x_1| = |c - x_4|$, and hence $|x_4 - x_3| \leqslant 2\delta$. Finally, we obtain the statement in the lemma: $|x_1 - x_4| \leqslant 4\delta$.

By the triangle inequality and because $d(x_1,\sigma) = d(x_4,\sigma)$, we have $|x_1 - c| \leqslant |c - x_4| + |x_4 - x_1| \leqslant |a - x_1| + 4\delta$. Hence, using Lemma 1, we conclude that $|a-c| \leqslant 8\delta$. □

Lemma 3. *Let σ be a geodesic segment, a be a point not on σ, and c be a projection of a on σ. Let $b \in \sigma$ be arbitrary, and let d denote the projection of b on ac. Then $|c-d| \leqslant 2\delta$.*

Proof. By hypothesis, bd minimizes the distance from any its points to ac, and because the triangle bcd is δ-thin, there exists a point $e \in bd$ such that $d(e,ac) = |e-d| \leqslant \delta$ and $d(e,bc) \leqslant \delta$. Because ac is a perpendicular to σ, $|a-c| \leqslant |a-d| + |d-e| + d(e,bc) \leqslant |a-d| + 2\delta$. Hence $|c-d| \leqslant 2\delta$. □

Lemma 4. *As in the preceding lemma, let σ be a geodesic segment, a be a point not on σ, c be a projection of a on σ, and b be some point on σ. Let d denote a point on ac such that $|d-c| = \delta$ and e denote a point on bc such that $|e-c| = 3\delta$. Then*

- *$d(d,ab) \leqslant \delta$, $d(e,ab) \leqslant \delta$, $d(c,ab) \leqslant 2\delta$, and*
- *the length of ab differs from the sum of the lengths of the two other sides by at most 8δ,*

$$|a-c| + |b-c| - 2\delta \leqslant |a-b| \leqslant |a-c| + |b-c| + 8\delta.$$

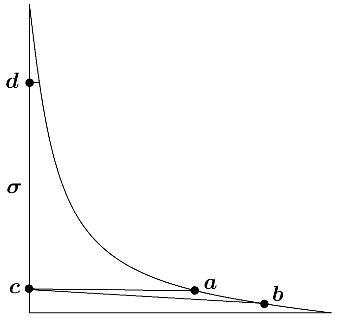

Fig. 3. Illustration for Remark 2.

Proof. The triangle abc is δ-thin. Therefore, obviously, $d(d, ab) \leqslant \delta$ (the distance from a point of ac to ab is a continuous function). We take a point $x \in bc$ such that $d(x, ca) \leqslant \delta$. Using Lemma 3, we obtain $|b - x| + d(x, ca) \geqslant |b - c| - 2\delta$, and hence $|c - x| \leqslant d(x, ca) + 2\delta \leqslant 3\delta$.

We now let d_1 and e_1 denote the respective projections of d and e on ab. Then by the triangle inequality, we have

- $|a - d| - \delta \leqslant |a - d_1| \leqslant |a - d| + \delta$,
- $|b - e| - \delta \leqslant |b - e_1| \leqslant |b - e| + \delta$, and
- $0 \leqslant |d_1 - e_1| \leqslant |d_1 - d| + |d - c| + |c - e| + |e - e_1| \leqslant 6\delta$.

Combining all these inequalities, we obtain the second point in the lemma. □

Lemma 5. *Let σ be a geodesic and a and b be two points not on σ. Further, let a and b have a common projection c on σ. Let d be a point of σ and c_1 be the projection of d on ab. Then*

$$|d - c| \leqslant |d - c_1| + 6\delta.$$

Remark 2. Lemma 5 deals with a geodesic segment. The statement is not true for a complete geodesic passing through a and b, as can be seen from Fig. 3.

Proof of Lemma 5. We take a point $e \in bc$ such that $|c - e| = \delta$ and consider the triangle bcd (see Fig. 4). Because bc is a perpendicular to dc, $d(e, bd) \leqslant \delta$. Let e_1 denote a projection of e on bd. Let e_2 and e_3 be the respective projections of e_1 on the geodesic segments dc_1 and bc_1. Because the triangle dbc_1 is δ-thin, either $|e_1 - e_2| \leqslant \delta$ or $|e_1 - e_3| \leqslant \delta$.

I. If $|e_1 - e_2| \leqslant \delta$, then $|d - c| \leqslant |c - e| + |e - e_1| + |e_1 - e_2| + |e_2 - d| \leqslant |d - c_1| + 3\delta$.

II. If $|e_1 - e_2| > \delta$, then the length of the path cee_3 is at most 3δ. We apply the same arguments to ad (we assume that this is possible; otherwise, we could apply the first case to it). We obtain the points g, g_1, and g_3 and the length of the path cgg_3 is also at most 3δ. If neither of these paths intersects cc_1, then its length does not exceed 6δ (which follows from consideration of the triangle ce_3g_3). □

Lemma 6. *Let E be a δ-hyperbolic metric space and abc be a triangle in E. Then the diameter of the set S of points of the side ab such that distance to bc and ac does not exceed $2d$ is not greater than $C(d + \delta)$, where C is a constant.*

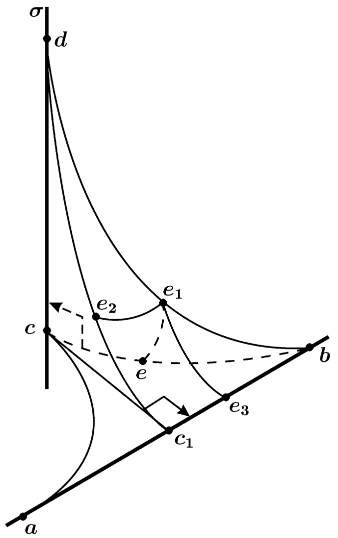

Fig. 4. Illustration for Lemma 5.

Proof. Let x be a point of ab such that $d(x, bc) \leqslant \delta$ and $d(x, ac) \leqslant \delta$ and y be a point of ab such that $d(y, bc) \leqslant d$ and $d(y, ac) < d$. Without loss of generality, we assume that $y \in (a, x)$. Because the triangle abc is δ-thin, one of these two distances does not exceed δ.

We first assume that $d(y, ac) \leqslant \delta$. Let x' and y' be points of ac such that $d(x, x') \leqslant \delta$ and $d(y, y') \leqslant \delta$. We let t, t', s, and s' denote the respective projections of x, x', y, and y' on bc. Because $x't'$ is a perpendicular to bc, $|x' - t'| \leqslant |x' - x| + |x - t| \leqslant 2\delta$, and hence $|t - t'| \leqslant 4\delta$. If y and y' are sufficiently far from bc, i.e., if $d \geqslant 9\delta$, then $|s - s'| \leqslant 6\delta$ by Lemma 2. Otherwise, we can give a rough estimate by the triangle inequality: $|s - s'| \leqslant |s - y| + |y - y'| + |y' - s'| \leqslant 19\delta$. Hence, in any case, $|s - s'| \leqslant 19\delta$. We consider two cases.

If s is in the segment $[b, t']$, then by applying the triangle inequality several times, we obtain

$$
\begin{aligned}
|b - y| &\leqslant |b - s| + |s - y| \leqslant \left|b - t'\right| + |s - y| \leqslant |b - x| + |x - t| + \left|t - t'\right| + |s - y| \\
&\leqslant |b - x| + 5\delta + d.
\end{aligned}
$$

And because $|b - y| = |b - x| + |x - y|$, we have $|x - y| \leqslant 5\delta + d$.

The same arguments we apply if $s \in [t', c]$. We merely note that we can replace y with y' and t with t' with respective errors less than δ and 19δ:

$$
\begin{aligned}
\left|c - y'\right| &\leqslant \left|c - s'\right| + \left|s' - y'\right| \leqslant \left|c - s'\right| + \left|s' - y'\right| \leqslant |c - s| + 19\delta + |s - y| + \delta \\
&\leqslant \left|c - t'\right| + 20\delta + d.
\end{aligned}
$$

Now, because $|c - t'| \leqslant |c - x'| + |x' - t'| \leqslant |c - x'| + 2\delta$, we have

$$
\left|c - x'\right| + \left|x' - y'\right| = \left|c - y'\right| \leqslant \left|c - x'\right| + 22\delta + d.
$$

Finally, $|x - y| \leqslant |y - y'| + |y' - x'| + |x - x'| \leqslant 24\delta + d$.

The case $d(y, bc) \leqslant \delta$ is treated identically with d and δ interchanged. □

3. Quasi-geodesics and Δ-length

Definition 7. A map $f : E \to F$ between metric spaces is a (λ, c)-*quasi-isometry* if

$$\frac{1}{\lambda}|x-y|_E - c \leqslant \bigl|f(x) - f(y)\bigr|_F \leqslant \lambda|x-y|_E + c$$

for any two points x and y of E.

Definition 8. A (λ, c)-quasi-geodesic in F is a (λ, c)-quasi-isometry from a real interval $I = [0, l]$ to F.

Let $\gamma : I \to F$ be a curve. We assume that the interval $I = [x_0, x_n]$ of length $|I| = l$ gives the parameterization of the quasi-geodesic γ. We take a subdivision $T_n = (x_0, x_1, \dots, x_n)$ and let y_i, $i = 0, 1, \dots, n$, denote $\gamma(x_i)$. The *mesh* of T_n is $d(T_n) = \min_{0<i\leqslant n} |y_i - y_{i-1}|$.

Definition 9 *(Δ-length).* Let $\gamma : I \to F$ be a curve. The value

$$L_\Delta(\gamma) = \sup_{T_n:\ d(T_n)\geqslant\Delta} \sum_{i=1}^{n} |y_i - y_{i-1}|$$

is called the Δ-*length* of the quasi-geodesic γ.

We note that the values of the Δ-length and the classical length are the same for a geodesic.

Lemma 7. *Let $\gamma : I \to F$ be a (λ, c)-quasi-geodesic. For $\Delta \geqslant 2c$,*

$$L_\Delta(\gamma) \leqslant 2\lambda l.$$

Proof. By the definition of the Δ-length, $\Delta \leqslant |y_i - y_{i-1}| \leqslant \lambda|x_i - x_{i-1}| + c$. Hence, because $\Delta \geqslant 2c$, we obtain $|x_i - x_{i-1}| \geqslant (\Delta - c)/\lambda \geqslant c/\lambda$.

Now, by the definition of a quasi-geodesic (and a quasi-isometry in particular), we have

$$\sup_{T_n} \sum_i |y_i - y_{i-1}| \leqslant \sup_{T_n} \sum_i \bigl(\lambda|x_i - x_{i-1}| + c\bigr) \leqslant \sup_{T_n} \sum_i 2\lambda|x_i - x_{i-1}| = 2\lambda l,$$

where the last equality follows because the sum of $|x_i - x_{i-1}|$ for every subdivision of the interval I is exactly equal to the length of I. □

Lemma 8. *Let $\gamma : I \to F$ be a (λ, c)-quasi-geodesic. Let $R \geqslant c$ be the distance between the endpoints of γ, and let $\Delta \geqslant 2c$. Then $L_\Delta(\gamma) \leqslant 4\lambda^2 R$.*

Proof. By the definition of a quasi-isometry, $l/\lambda - c \leqslant R \leqslant \lambda l + c$. Hence, $l \leqslant \lambda(R + c)$. And by Lemma 7, $L_\Delta(\gamma) \leqslant 2\lambda^2(R + c)$. In particular, $L_\Delta(\gamma) \leqslant 4\lambda^2 R$ for $R \geqslant c$. □

The next lemma allows replacing arbitrary quasi-geodesics with continuous ones.

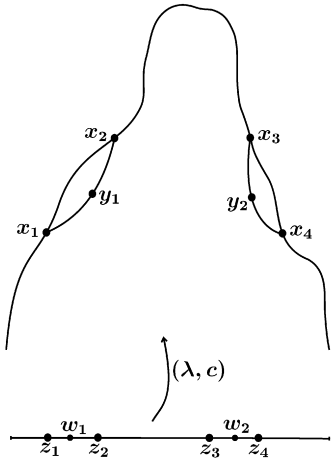

Fig. 5. Construction of the continuous arc $\tilde{\gamma}$ from the quasi-geodesic γ.

Lemma 9. *Let γ be a (λ, c)-quasi-geodesic, and let $\Delta \geqslant c$. Let $T = t_0, t_1, \ldots, t_n \subset \gamma$ be the set of points on γ such that T gives the Δ-length value L_Δ.*

1. *Then the curve $\tilde{\gamma}$ consisting of the geodesic segments $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$, is a $(\lambda, 12\Delta + 3c)$-geodesic with the (classical) length L_Δ.*
2. *Let y and y' be points of $\tilde{\gamma}$ such that $d(y, y') \geqslant 6\Delta + c$. Let $\tilde{\gamma}_0$ be the part of $\tilde{\gamma}$ between y and y'. Then the (classical) length of $\tilde{\gamma}_0$ is not greater than $L_\Delta(\tilde{\gamma}_0) \leqslant 4\lambda^2(R + 6\Delta)$.*

Proof. We first note that for every $i = 0, 1, \ldots, n-1$, the length of the interval $|[t_i, t_{i+1}]| \leqslant 3\Delta$. Indeed, if $|[t_i, t_{i+1}]| > 3\Delta$, then we can add a point t_i' to the partition T. Such a point exists because the gaps on a quasi-geodesic cannot be greater than c.

We assume that γ is parameterized by an interval I; $t_i^{-1} \in I$ are the parameters of t_i, $i = 0, 1, \ldots, n$ (see Fig. 5). Let $[t_i^{-1}, t_{i+1}^{-1}]$ be the affine parameterization of the geodesic segments $[t_i, t_{i+1}]$. Then the conditions for being a $(\lambda, 4c)$-geodesic are satisfied automatically for the points of the same segment.

To simplify the notation, we let $[x_1, x_2]$ and $[x_3, x_4]$ denote two different intervals of $\tilde{\gamma}$ and $[z_1, z_2]$ and $[z_3, z_4]$ denote their parameters. We take two points $y_1 \in [x_1, x_2]$ and $y_2 \in [x_3, x_4]$, where w_1 and w_2 are their parameters. By the triangle inequality and by the definition of a quasi-isometry,

$$|y_1 - y_2| \leqslant |x_2 - x_3| + |y_1 - x_2| + |y_2 - x_3| \leqslant |x_2 - x_3| + 6\Delta \leqslant \lambda|z_2 - z_3| + c + 6\Delta.$$

Similarly, we obtain the lower bound

$$|y_1 - y_2| \geqslant |x_2 - x_3| - |y_1 - x_2| - |y_2 - x_3| \geqslant |x_2 - x_3| - 6\Delta \geqslant \frac{1}{\lambda}|z_2 - z_3| - c - 6\Delta.$$

By the definition of a quasi-isometry, $|z_k - z_{k+1}| \leqslant \lambda(|x_k - x_{k+1}| + c) \leqslant \lambda(3\Delta + c)$ with $k = 1, 3$. Hence,

$$|w_1 - w_2| - 2\lambda(3\Delta + c) \leqslant |z_2 - z_3| \leqslant |w_1 - w_2|.$$

Therefore,

$$\frac{1}{\lambda}|w_1 - w_2| - \frac{2\lambda(3\Delta + c)}{\lambda} - 6\Delta - c \leqslant |y_1 - y_2| \leqslant \lambda|w_1 - w_2| + 6\Delta + c.$$

Consequently, $\tilde{\gamma}$ is a quasi-geodesic with the constants λ and $12\Delta + 3c$ and statement 1 in the lemma is proved.

To prove statement 2, we need merely note that if $|y_1 - y_2| \geqslant 6\Delta + c$, then $c \leqslant |x_1 - x_4| \leqslant |y_1 - y_2| + 6\Delta$ by the triangle inequality. The left-hand inequality allows applying Lemma 8 to the part γ_0 between x_1 and x_4 of the initial quasi-geodesic γ, and we use the right-hand part to obtain the upper bound,

$$L(\tilde{\gamma}_0) \leqslant L_\Delta(\gamma_0) \leqslant 4\lambda^2(R + 6\Delta). \qquad \square$$

4. Exponential contraction

Lemma 10 *(Exponential contraction). Let* $\Delta > 0$. *In a geodesic* δ*-hyperbolic space* E*, let* γ *be a connected curve at a distance not less than* $R \geqslant \Delta + 58\delta$ *from a geodesic* σ*. Let* L_Δ *be the* Δ*-length of* γ*. Let* $r = \lfloor (R - \Delta - 58\delta)/19\delta \rfloor 19\delta$*. Then the length of the projection of* γ *on* σ *is not greater than*

$$\max\left(\frac{4\delta}{\Delta} e^{-Kr/\delta}(L_\Delta + \Delta), 8\delta\right).$$

In other words,

- *if* $R \leqslant \Delta + 58\delta + (\delta/K)\ln((L_\Delta + \Delta)/2\Delta)$*, then the length of the projection of* γ *on* σ *is not greater than* $(4\delta/\Delta)e^{-Kr/\delta}(L_\Delta + \Delta)$;
- *otherwise, it is not greater than* 8δ.

Proof. Let $y_0, y_1, \ldots, y_n$ be points on γ such that $|y_i - y_{i-1}| = \Delta$ for $i = 1, 2, \ldots, n-1$, $|y_n - y_{n-1}| \leqslant \Delta$, and y_0 and y_n are the endpoints of γ. Let y_k be the point of this set that is nearest σ. We take a perpendicular from y_k to σ and a point x_k on it with $|y_k - x_k| = \Delta + 3\delta$. Now, on the perpendiculars from all other points y_i, we take points x_i such that $d(x_i, \sigma) = d(x_k, \sigma)$ (see Fig. 6). By Lemma 2, $|x_i - x_{i-1}| \leqslant 4\delta$ for $i = 1, 2, \ldots, n$. Therefore,

$$\sum_{i=1}^{n} |x_i - x_{i-1}| \leqslant n4\delta \leqslant n\Delta\frac{4\delta}{\Delta} \leqslant \frac{4\delta}{\Delta}(L_\Delta + \Delta).$$

We set $\bar{x}_0 = x_0$ and $\bar{x}_{n^1} = x_n$ and select points $\bar{x}_i \in \{x_1, x_2, \ldots, x_{n-1}\}$ such that $8\delta \leqslant |x_i - x_{i-1}| \leqslant 16\delta$. For each $i = 0, 1, \ldots, n^1$, we choose a perpendicular from $\bar{x}_i$ to σ, move $\bar{x}_i$ along it a distance $16\delta + 3\delta = 19\delta$ toward σ, and obtain x_i^1. By Lemma 2, $|x_i^1 - x_{i-1}^1| \leqslant 4\delta$ and

$$\sum_{i=1}^{n^1} \left|x_i^1 - x_{i-1}^1\right| \leqslant n^1 4\delta \leqslant \frac{1}{2}\sum_{i=1}^{n^1} |\bar{x}_i - \bar{x}_{i-1}| \leqslant \frac{1}{2}\sum_{i=1}^{n} |x_i - x_{i-1}| \leqslant \frac{1}{2}\frac{4\delta}{\Delta}(L_\Delta + \Delta).$$

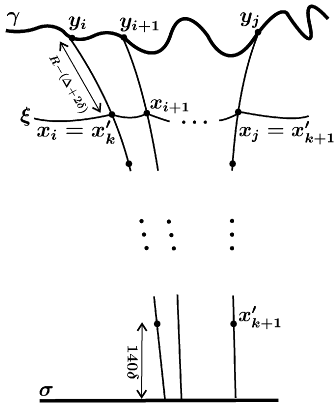

Fig. 6. Exponential contraction of the length of a curve γ under projection on a geodesic σ.

We can continue such a process while the distance from the set of points $\{x_i^m,\ i = 0, 1, \ldots, n^m\}$ to σ is not less than 19δ and $|x_0^m - x_{n^m}^m| \geqslant 8\delta$. After k steps, we have

$$\sum_{i=1}^{n^k} \left|x_i^k - x_{i-1}^k\right| \leqslant \frac{1}{2^k}\frac{4\delta}{\Delta}(L_\Delta + \Delta) = \frac{4\delta}{\Delta} e^{-((\ln 2)/19\delta)(19\delta k)}(L_\Delta + \Delta).$$

We set $r = 19\delta k$ and $K = (\ln 2)/19$. We need $8\delta \leqslant (4\delta/\Delta)e^{-Kr/\delta}(L_\Delta + \Delta)$ and hence $r \leqslant (\delta/K)\ln((L_\Delta + \Delta)/2\Delta)$. Now, if the distance between the projections of the endpoints $|x_0^m - x_{n^m}^m|$ is not less than 8δ at some step m, then we use Lemma 2 to do the last projection on σ, and its length does not exceed 8δ. Otherwise, we must do the last descent to the distance 55δ using Lemma 2 (the estimate for the projection on a geodesic with $\Delta = 16\delta$ gives the necessary distance from the set of points to the geodesic to be greater than $3 * 16\delta + 6\delta = 54\delta$) and intervals of a length not less than 8δ contract to intervals of a length not more than δ, and we hence have a contraction factor of unity at the last step. □

5. Quantitative version of the Morse lemma

We are now ready to prove our main result. In a δ-hyperbolic space E, any (λ, c)-quasi-geodesic γ belongs to an H-neighborhood of a geodesic σ connecting its endpoints, where the constant H depends only on the space E (in particular, on the constant δ) and the quasi-isometry constants λ and c.

5.1. Attempts

To motivate our method, we describe a sequence of arguments yielding sharper and sharper estimates. We start with the proof in [4, Chapter 5.1, Theorem 6 and Lemma 8], where the upper bound $H \leqslant \lambda^8 c^2 \delta$ was obtained (up to universal constants, factors of the order $\log_2(\lambda c \delta)$). The first weak step in this proof is replacing a (λ, c)-quasi-geodesic with a discrete (λ', c)-quasi-geodesic γ' parameterized by an interval $[1, 2, \ldots, l]$ of integers, where $\lambda' \sim \lambda^2 c$. For a suitable $R \sim \lambda'^2$, we take an arc $x_u x_v$ of γ' and introduce a partition of that arc $x_u, x_{u+N}, x_{u+2N}, \ldots, x_v$ for some well-chosen $N \sim \lambda'$. The approximation of a δ-hyperbolic space by a tree (see

[4, Chapter 2.2, Theorem 12.ii]) is used to obtain an estimate of the form $|y_{u+iN} - y'_{u+(i+1)N}| \leqslant c' \sim \ln \lambda'$. By the triangle inequality, $|x_u - x_v| \leqslant |x_u - y_u| + |y_u - y_{u+N}| + \cdots + |y_v - x_u| \leqslant 2(R + \lambda') + (N^{-1}|u - v| + 1)c'$. On the other hand, $\lambda'^{-1}|u - v| \leqslant |x_u - x_v|$. Combining these two inequalities, we obtain an estimate for $|u - v|$ and hence for a distance from any point of the arc $x_u x_v$ to the point x_u. The second weak step in this argument is in the estimate of the length of projections, which can be improved significantly.

Another proof was given in [1]. It allows obtaining the estimate $\lambda^2 H_{\mathrm{am}}$, where H_{am} is the constant of the anti-Morse lemma (see Section 7) and is given by the equation $H_{\mathrm{am}} \simeq \ln \lambda + \ln H_{\mathrm{am}}$.[1] It is very close to an optimal upper bound but still not sharp. Also we need to notice that the sharp estimate for $H_{\mathrm{am}} \simeq \ln \lambda$. The proof uses the notion of "exponential geodesic divergence."

Definition 10. Let F be a metric space. We call $e : \mathbb{N} \to \mathbb{R}$ a *divergence function* for the space F if for any point $x \in F$ and any two geodesic segments $\gamma = (x, y)$ and $\gamma' = (x, z)$, the length of a path σ from $\gamma(R + r)$ to $\gamma'(R + r)$ in the closure of the complement of a ball $B_{R+r}(x)$ (i.e., in $\overline{X \setminus B_{R+r}(x)}$) is not greater than $e(r)$ for any $R, r \in \mathbb{N}$ such that $R + r$ does not exceed the lengths of γ and γ' if $d(\gamma(R), \gamma'(R)) > e(0)$.

The divergence function is exponential in a hyperbolic space. The next step is to prove the anti-Morse lemma. The authors of [1] take a point p of the geodesic σ that is the distant from the quasi-geodesic γ and construct a path α between two points of γ such that α is in the complement of the ball of radius $d(p, \gamma)$ with the center p. Finally, they compare two estimates of the length: one estimate follows from the hypothesis that α is a quasi-geodesic, and the other is given by the exponential geodesic divergence. To prove the Morse lemma, they take a (connected) part γ_1 of γ that belongs to the complement of the H_{am}-neighborhood of the geodesic σ, and they show that the length of γ_1 does not exceed $2\lambda^2 H_{\mathrm{am}}$ by the definition of a quasi-geodesic. In [1], they also use another definition of a quasi-geodesic, which is less general than our definition because, in particular, it assumes that a quasi-geodesic is a continuous curve. Consequently, some technical work is needed to generalize their results.

To improve these bounds, we use Lemma 10 (exponential contraction) instead of exponential geodesic convergence and Lemma 8, which do not require discretization as in [4] and provide a much more precise estimate for a length of a projection. We can then take $R = \ln \lambda$ and obtain $H \leqslant O(\lambda^2 \ln \lambda)$ by a similar triangle inequality.

Below, we prove the Morse and anti-Morse lemmas independently. We only mention that arguments in [1] can be used to deduce the optimal bound for the Morse lemma from the anti-Morse lemma. We can also obtain an optimal upper bound for H from Lemma 11.

We now sketch the proof of a stronger result (but still not optimal): $H \leqslant O(\lambda^2 \ln^* \lambda)$, where $\ln^* \lambda$ is the minimal number n of logarithms such that $\underbrace{\ln \cdots \ln}_{n} \lambda \leqslant 1$.

The preceding argument is used as the initial step. It allows assuming that the endpoints x and x' of γ satisfy $|x - x'| \leqslant O(\ln \lambda)$. Then comes an iterative step. We prove that if xx' is an arc on γ and $|x - x'| = d_1$, then there exist two points y and y' at distance at most $C_2(c, \delta)\lambda^2$ from a geodesic σ_1 connecting x and x' such that $d_2 := |y - y'| \leqslant C_3(c, \delta) \ln d_1$. Indeed, we choose a point z of the arc xx' that is farthest from σ_1 and let σ' denote a perpendicular from z to σ_1. If all points of the arc xx' (on either side of z) whose projection on σ' is at a distance $\leqslant \lambda^2$

[1] Be careful while reading [1] because a slightly different definition of quasi-geodesics is used there with $\lambda_1 = \lambda^2$; cf. Lemma 8.

from σ_1 are at a distance not less than $\ln d_1$ from σ', then Lemma 10 implies that the length of the arc is much greater than $\lambda^2 \ln d_1$, contradicting the quasi-geodesic assumption. Hence, there are points y and y' that are near σ'. We can arrange that their projections on σ' are near each other, which yields $|y - y'| \leqslant \ln d_1$. We apply this relation several times starting with $d_1 = C_1(c, \delta) \ln \lambda$ until $d_i \leqslant 1$ for some $i = \ln^* \lambda$.

In summary, we use two key ideas to improve the upper bound of H: exponential contraction and a consideration of a projection of γ on a different geodesic σ'.

5.2. *Proof of the Morse lemma*

We use the same ideas to prove the quantitative version of the Morse lemma, but we should do it more accurately. Let γ be a (λ, c)-quasi-geodesic in a δ-hyperbolic space E, and let σ be a geodesic segment connecting its endpoints. We prove that γ belongs to an H-neighborhood of σ, where

$$H = 4\lambda^2\left(78c + \left(78 + \frac{133}{\ln 2}e^{157\ln 2/28}\right)\delta\right). \tag{1}$$

Remark 3. It is easy to give an example where $H = \frac{\lambda^2 c}{2}$ (see Section 6.2).

Indeed, a path that goes back and forth along a geodesic segment of length $\lambda^2 c$ in a tree is a (λ, c)-quasi-geodesic (see Section 6 for details).

Proof of Theorem 1. Applying Lemma 9 to the quasi-geodesic γ with $\Delta = 2c$, we obtain a continuous $(\lambda, 27c)$-quasi-geodesic $\tilde{\gamma}$. By Lemma 8, γ belongs to a $4\lambda^2 \cdot 6c = 24\lambda^2 c$-neighborhood of $\tilde{\gamma}$. **Hereafter, we consider only the $(\lambda, 27c)$-quasi-geodesic $\tilde{\gamma}$, which for brevity is denoted simply by γ, and we set $\tilde{c} = 27c$.** The classical length of the part of this quasi-geodesic between two points separated by a distance R does not exceed $4\lambda^2(R + \tilde{c})$.

We introduce the following construction for subdividing the quasi-geodesic γ. We let z denote the point of our quasi-geodesic that is farthest from σ. Let $\sigma_0 = \sigma$ be the geodesic connecting the endpoints of γ. Let σ_0' be the geodesic minimizing the distance between z and σ_0 (because σ_0 is a geodesic segment, σ_0' is not necessarily perpendicular to the complete geodesic carrying σ_0). Let s_0 denote the point of intersection of σ_0 and σ_0'. Let s_0' be the point of σ_0' such that the length of the segment $[s_0, s_0']$ is equal to δ. We consider the set of points of γ whose projections on σ_0' belong to the segment $[s_0, s_0']$. The point z separates this set into two subsets γ_0^+ and γ_0^- (see Fig. 7).

Let $d_0^\pm$ denote the minimal distance of points of $\gamma_0^\pm$ to σ_0'. We also introduce the following notation:

- $d_0 = d_0^+ + d_0^- + \delta$;
- γ_1 is a connected component of $\gamma \setminus (\gamma_0^+ \cup \gamma_0^-)$ containing z and is also a quasi-geodesic with the same constants and properties as γ;
- σ_1 is a geodesic connecting the endpoints of the sub-quasi-geodesic γ_1;
- L_1 is the length of γ_1.

Applying the same idea to the curve γ_1, the same point z, and the geodesic σ_1, we obtain the geodesic σ_1', the parts $\gamma_1^\pm$ of the quasi-geodesic, and the distances $d_1^\pm$. We have

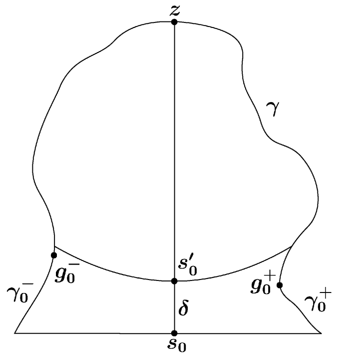

Fig. 7. Illustration of proof of Theorem 1.

$l(\sigma_0') \leqslant l(\sigma_1') + \delta + 6\delta$. To show this, we apply Lemma 5 assuming that $c = s_0'$, $d = z$, and a and b are the endpoints of γ_1. Continuing the process, we obtain a subdivision of γ by $\gamma_i^\pm$ and two families of geodesics σ_i and σ_i'. Finally, for some n, we obtain $d_n \leqslant \tilde{c} + \delta + 77\delta = 78\delta + \tilde{c}$.

The quantity L_i is the length of the subcurve γ_{i-1}, which is also a quasi-geodesic. Hence, $l(\sigma_n') \leqslant L_n \leqslant 4(d_n + \tilde{c})\lambda^2$ by construction. Therefore,

$$l(\sigma_0') \leqslant \sum_{i=1}^{n} 7\delta + 4(78\delta + 2\tilde{c})\lambda^2.$$

Our goal is to prove that for sufficiently large λ, $\sum d_i \leqslant C\lambda^2$, where C is a constant depending only on $\tilde{c}$ and δ.

Because the value of the classical length of a segment is not less than the value of its Δ'-length, by Lemma 10 (with $\Delta' = \delta$) and because $\lfloor (d_{i+1}^\pm - \delta - 58\delta)/19\delta \rfloor 19\delta \geqslant d_{i+1}^\pm - 78\delta$, we obtain

$$l(\gamma_i^+ \cup \gamma_i^-) \geqslant \delta \frac{\delta}{4\delta} \max\bigl(e^{K(d_{i+1}^+ - 78\delta)/\delta}, e^{K(d_{i+1}^- - 78\delta)/\delta}\bigr) \geqslant \frac{\delta}{4} e^{K(d_{i+1} - \delta - 156\delta)/2\delta}.$$

On the other hand, $l(\gamma_i^+ \cup \gamma_i^-) = L_i - L_{i+1}$. Hence, setting $C_0 = (\delta/4)e^{-157K/2}$, we have

$$C_0 e^{K d_{i+1}/2\delta} \leqslant L_i - L_{i+1}. \tag{2}$$

Let $g_i^\pm$ be a point of $\gamma_i^\pm$ that minimizes the distance to σ_i'. The part of the quasi-geodesic γ between g_i^+ and g_i^- is also a quasi-geodesic with the same constants and properties. By the triangle inequality, $|g_i^- - g_i^+| < d_i^+ + d_i^- + \delta$. Therefore, by construction (see the beginning of the proof) and because $d_i \geqslant 78\delta$,

$$L_i \leqslant 4\lambda^2(d_i + \tilde{c}) \leqslant 8\lambda^2 d_i. \tag{3}$$

The function e^{-d} is decreasing. Therefore, because $d_i \geqslant \frac{4}{\lambda^2} L_i$, we obtain

$$\frac{K}{2\delta} d_i e^{-K d_i/2\delta} \leqslant \frac{K}{2\delta}\frac{4}{\lambda^2} L_i e^{-(4K/2\delta\lambda^2)L_i}.$$

We are now ready to estimate n:

$$n=\sum_{i=1}^{n}1=\frac{1}{C_0}\sum_{i=1}^{n}e^{-Kd_i/2\delta}C_0e^{Kd_i/2\delta}\leqslant\frac{1}{C_0}\frac{\lambda^2\delta}{4K}\sum_{i=1}^{n}e^{-(8K/2\delta\lambda^2)L_i}\frac{4K}{\lambda^2\delta}(L_{i-1}-L_i).$$

Setting $X_i=(4K/\lambda^2\delta)L_i$, we have

$$\sum_{i=1}^{n}i\leqslant\frac{\lambda^2\delta}{4C_0K}\sum_{i=1}^{n}e^{-X_i}(X_{i-1}-X_i),$$

and because the function e^{-X} is decreasing for $X\geqslant 0$, we can use the estimate

$$\sum_{i=1}^{n}e^{-X_i}(X_{i-1}-X_i)\leqslant\int_0^\infty e^{-X}\,dX=-e^{-x}\Big|_0^\infty=1.$$

Summarizing all the facts, returning to the initial notation, and recalling that $K=\ln 2/19$, we finally obtain the claimed result

$$H=4\lambda^2\left(78c+\left(78+\frac{133}{\ln 2}e^{157\ln 2/38}\right)\delta\right).\qquad\square$$

6. Examples

6.1. *Proof of Proposition 1*

Here, we prove Proposition 1 (see the Introduction). We call any connected component of a ball with a deleted center O a *branch*. We call points that are sent to the branch containing the image of the center $f(O)$ green points and all other points of T red points.

Proof of Proposition 1. We show that there exist two red points r_1 and r_2 such that $d(O,r_1r_2)\leqslant r=c+1$.

By Definition 7, a c-neighborhood of every point of the border should contain a point of the image. We must have at least $(d-1)d^{R-c-1}$ red points near the border (we exclude the green part). The number of points in each connected component of the complement of the ball of radius r is less than d^{R-r}. Therefore, if $r\gg c$, then one component contains an insufficient number of points to cover the border of B. Hence, there exist two points r_1 and r_2 in different components of T, which means that the geodesic r_1r_2 passes at a distance less than r from the center O and the quasi-geodesic $f(r_1r_2)$ passes at a distance $\lambda r+c$ from $f(O)$ and belongs to an H-neighborhood of the geodesic $f(r_1)f(r_2)$. Because every path from $f(O)$ to $f(r_1)f(r_2)$ passes through O, we conclude that $d(O,f(0))<H+c+\lambda r$. We need only choose a good value for r. Simply calculating the number of points in a mentioned component gives the estimate $1+d+d^2+\cdots+d^{R-r}\leqslant(1/\ln d)d^{R-r+1}$. For $r=c+1$, we have $(1/\ln d)d^{R-r+1}\leqslant(d-1)d^{R-c-1}$, which completes the proof. $\square$

6.2. *Optimality of Theorem 1*

We present an example of a (λ, c)-quasi-geodesic γ in a tree with $H = \lambda^2 c/2$. We take a real interval $[a, b]$ of length $\lambda^2 c/2$ that is a subtree. We use an interval $I = [u, v]$ of length λc to parameterize γ. We define γ as follows:

- $\gamma(u) = \gamma(v) = a$,
- we set $\gamma(w) = b$ for the midpoint w of I, and
- we set $D = \min\{|u - x|, |v - x|\}$ and $|a - \gamma(x)| = \lambda D$ for any $x \in [a, b]$.

It is easy to verify that γ is a well-defined quasi-geodesic. On the half-intervals $[u, w]$ and $[w, v]$, γ just stretches the distances by λ. We now take any two points $x \in [u, w]$ and $y \in [w, v]$. Assuming that $|u - x| \leqslant |v - y|$, we obviously have $|x - y| = |u - v| - |u - x| - |v - y|$.

I. The lower bound of $|\gamma(x) - \gamma(y)|$ is given by

$$\frac{1}{\lambda}\big(|u - v| - |u - x| - |v - y|\big) - c \leqslant 0 \leqslant \big|\gamma(x) - \gamma(y)\big|.$$

II. The upper bound of $|\gamma(x) - \gamma(y)|$ is given by

$$\begin{aligned}
&\lambda\big(|u - v| - |u - x| - |v - y|\big) + c - \big(\big|a - \gamma(y)\big| - \big|a - \gamma(x)\big|\big) \\
&\quad = \lambda\big(|u - v| - |u - x| - |v - y|\big) + c - \lambda\big(|v - y| - |u - x|\big) \\
&\quad = \lambda^2 c - 2\lambda|v - y| + c \geqslant c \geqslant 0.
\end{aligned}$$

6.3. *Achieving the displacement λc*

We now describe a self-quasi-isometry f of a ball B in a tree that moves the center O a distance $\lambda c/2$. We assume that the radius of B is greater than λc. We note that the images of two points inside the ball B_1 of radius λc with a center O can be just the same point. Let the quasi-isometry f fix the boundary of B_1, and let $|O - f(O)| = \lambda c/2$. The segment $[O, f(O)]$ is sent to the only point $f(O)$. For any point a of ∂B_1, we let a' denote a projection of a on $[O, f(O)]$ and assume that the interval $[a, a']$ is linearly stretched and sent to the interval $[a, f(O)]$. Such a map f assigns only one image to any point. It is easy to verify that f is a quasi-isometry because the distances between points can be diminished up to 0 and are not increased more than λ times.

7. Anti-Morse lemma

We have already proved that any quasi-geodesic γ in a hyperbolic space is at distance not more than $\lambda^2(A_1 c + A_2\delta)$ from a geodesic segment σ connecting its endpoints. This estimate cannot be improved. But the curious thing is that this geodesic belongs to a $\ln\lambda$-neighborhood of the quasi-geodesic! We can therefore say that any quasi-geodesic is $\ln\lambda$-quasiconvex. This upper bound can be improved in some particular spaces: for example, any quasi-geodesic is c-quasiconvex in a tree.

The proof of Theorem 2 (see the Introduction) that we give below is based on using

- Lemma 10 (exponential contraction) to prove that at the distance $\ln\lambda$ from the geodesic σ is at most $\lambda^2 \ln\lambda$ and

- an analogue of Lemma 10 to prove that the length of a circle of radius R is at least e^R (up to some constants).

Lemma 11. *Let X be a hyperbolic metric space, γ be a (λ, c)-quasi-geodesic, and σ be a geodesic connecting the endpoints of γ. Let (y_u, y_v) be an arc of γ such that no point of this arc is at distance less than $C_1 \ln \lambda + C_2$ from σ and y_u and y_v are the points of the arc nearest σ. Then the length of the projection of the arc (y_u, y_v) on σ does not exceed* $\max(8\delta, C_3 \ln \lambda)$ (*with some well-chosen constants C_1, C_2, and C_3 depending linearly on c).*

Proof. By the definition of a quasi-geodesic, we have

$$\frac{|u-v|}{\lambda} - c \leqslant |y_u - y_v| \leqslant \lambda |u-v| + c.$$

On the other hand,

$$|y_u - y_v| \leqslant \left|y_u - y_u'\right| + \left|y_u' - y_v'\right| + \left|y_v' - y_v\right|,$$

where y_u' and y_v' are the projections of y_u and y_v on σ. We adjust the constants C_1 and C_2 such that

$$C_1 \ln \lambda + C_2 = \frac{19\delta^2}{K} \ln \frac{8\delta\lambda^4}{\Delta} + \Delta + 58\delta,$$

where $\Delta = 2c$ (such a choice allows applying Lemma 8). We apply the lemma on exponential contraction (we assume that the length of the arc is rather large for using the estimate with an exponential factor and not to treat the obvious case where the length of the projection is 8δ). We let $l(y_u, y_v)$ denote the Δ-length of the arc (y_u, y_v):

$$\left|y_u' - y_v'\right| \leqslant l(y_u, y_v) e^{-K(r-\Delta-58\delta)/\delta} = \frac{1}{2\lambda^4} l(y_u, y_v).$$

Combining all these inequalities and using Lemma 8, we obtain

$$\begin{aligned}\frac{|u-v|}{\lambda} - c \leqslant |y_u - y_v| &\leqslant \frac{8}{K} \ln \sqrt[4]{2}\lambda + \frac{1}{8\lambda^4} l(y_u, y_v) \\ &\leqslant \frac{8}{K} \ln \sqrt[4]{2}\lambda + 4\lambda^2 \frac{1}{8\lambda^4} |y_u - y_v| \\ &\leqslant \frac{8}{K} \ln \sqrt[4]{2}\lambda + \frac{1}{2\lambda^2}\big(\lambda|u-v| + c\big).\end{aligned}$$

We therefore conclude that $|y_u - y_v| \leqslant C_3 \lambda^2 \ln \lambda$, hence $l(y_u, y_v) \leqslant C_3 \lambda^4 \ln \lambda$, and, finally, the length of the projection of the arc (y_u, y_v) of γ does not exceed $\max(8\delta, C_3 \ln \lambda)$. □

Proof of Theorem 2. The proof follows directly from Lemma 11. Because we have already proved that for every point $z' \in \sigma$, there exists a point $z \in \gamma$ such that the projection of z on σ is at distance not more than several times $c + \delta$ from z'. For simplicity, we therefore assume that for any point of σ, there exists a point of γ projecting on this point.

If the distance between z and z' is less than $C_1 \ln \lambda$ for some constant $C_1 = C_1(c, \delta)$ (the value of C_1 can be found from Lemma 11), then the statement is already proved. If not, then we take an arc (y_u, y_v) of γ containing the point z such that the endpoints y_u and y_v are at the distance $C_1 \ln \lambda$ from σ and these points are the points of this arc that are nearest σ. Hence, by Lemma 11, the length of the projection (which includes z) of the arc (y_u, y_v) does not exceed $C_4 \ln \lambda$. Therefore, the distance from z to y_u (and y_v) is not greater than $(C_1 + C_4) \ln \lambda$. □

8. Geodesically rich spaces

Definition 11. A metric space X is said to be geodesically rich if there exist constants r_0, r_1, r_2, r_3, and r_4 such that

- for every pair of points p and q with $|p - q| \geqslant r_0$, there exists a geodesic γ such that $d(p, \gamma) < r_1$ and $|d(q, \gamma) - |q - p|| < r_2$ and
- for any geodesic γ and any point $p \in X$, there exists a geodesic γ' passing in an r_3-neighborhood of the point p and such that $d(p, \gamma)$ differs from the distance between γ' and γ by not more than r_4.

Example 1. A line and a ray are not geodesically rich. Both of them satisfy the second condition in the definition, but not the first.

Example 2. Nonelementary hyperbolic groups are geodesically rich. We prove this later.

Any δ-hyperbolic metric space H can be embedded isometrically in a geodesically-rich δ-hyperbolic metric space G (with the same constant of hyperbolicity). We take a 3-regular tree with a root (T, O), assume that $G = H \times T$, and set the metric analogously to a real tree:

- the distance between points in the subspace (H, O) equals the distance between the corresponding points in H;
- the distance between other points equals the sum of the three distances from the points to their projections on (H, O) and between their projections on (H, O).

It is easy to show that the space G is δ-hyperbolic and geodesically rich. But such a procedure completely changes the ideal boundary of the space. We therefore ask another question:

Question 1. Is it possible to embed a δ-hyperbolic metric space H isometrically in a geodesically rich δ-hyperbolic metric space G with an isomorphic boundary?

Lemma 12. *Let G be a nonelementary hyperbolic group. Then there exist constants c_1 and c_2 such that for any point $p \in G$ and any geodesic $\gamma \in G$ such that $d(p, \gamma) \geqslant c_1$, there exists a geodesic γ' with a point q minimizing (up to a constant times δ) the distance to γ and $|p - q| \leqslant c_2$.*

Proof. By symmetry, we can assume that p is the unity of the group G. We supply the ideal boundary $G(\infty)$ with a visual distance. Because G is a nonelementary group, its ideal boundary $G(\infty)$ has at least three points (hence, infinitely many points).

We first prove by contradiction that there exists an ε such that for every pair of points ξ and η of $G(\infty)$, the union of the two balls of radius ε with the centers ξ and η does not cover the whole ideal boundary. On the contrary, we suppose that there exist two sequences of points ξ_n and η_n such that the union of $B(\xi_n, 1/n)$ and $B(\eta_n, 1/n)$ includes $G(\infty)$. By compactness, we can assume that $\xi_n \to \xi$ and $\eta_n \to \eta$, and we find that $G(\infty)$ belongs to the union of $B(\xi, 2/n)$ and $B(\eta, 2/n)$. Hence, the ideal boundary contains only the two points ξ and η, which contradicts the assumption that G is nonelementary.

Let c_1 be a constant such that if a geodesic γ is at a distance at least c_1 from the point p, then the visual distance between its endpoints (at infinity) is less than $\varepsilon/2$. We now take two points ξ and η of $G(\infty)$ outside an $\varepsilon/4$-neighborhood of $\gamma(\infty)$ such that $|\xi - \eta| > \varepsilon$ (the preceding argument established that such a choice is possible). Let γ' be a geodesic with the endpoints ξ and η. Hence, $d(p, \gamma') < c_1$. Applying Lemma 13 completes the proof. □

Lemma 13. *Let X be a δ-hyperbolic space. Then for every $\varepsilon > 0$, there exist constants c_1 and c_2 such that for every pair of geodesics γ and γ' and every point p such that $d(p, \gamma) < c_1$ and visual distance between the endpoints $\gamma(\infty)$ and $\gamma'(\infty) \geqslant \varepsilon$, there exists a point q on γ minimizing the distance to γ' up to some constant times δ and such that $|p - q| \leqslant c_2$.*

Proof. By Lemma 15, we can replace the point p with its projection p' on the geodesic γ. Let a' and b' be the projections on γ of the endpoint $a = \gamma'(-\infty)$ and the point b of γ' that minimizes the distance from γ' to γ.

We consider two sequences x_n and y_n of points respectively on aa' and $a'\gamma(+\infty)$ such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = \gamma(+\infty)$. We let a'_n denote the projections of x_n. Obviously, $a'_n \to a'$ as $n \to \infty$. By the definition of Gromov's product, $(x|y)_{p'} = \lim_{n\to\infty}(x_n|y_n)_{p'}$. Using Lemma 4, we now estimate $(x_n|y_n)_{p'}$:

$$
\begin{aligned}
(x_n|y_n)_{p'} &= \frac{1}{2}\big(\big|p' - x_n\big| + \big|p' - y_n\big| - |x_n - y_n|\big) \\
&\leqslant \frac{1}{2}\big(\big|p' - a'_n\big| + \big|a'_n - x_n\big| + 8\delta + \big|p' - y_n\big| - \big|a'_n - x_n\big| - \big|a'_n - y_n\big| + 2\delta\big).
\end{aligned}
$$

Now, if p' is between a' and b', then $(x_n|y_n)_{p'} \leqslant 5\delta$; otherwise (we assume that p' is closer to a', i.e., the order of points on γ is p', a', b'), $(x_n|y_n)_{p'} \leqslant |p' - a'| + 5\delta$.

Therefore, to finish the proof, we must now prove that the point a' is not far from ab. We apply Lemma 4 once more to the triangle $aa'b'$ and obtain $d(a', ab') \leqslant 2\delta$. Hence, because the triangle abb' is δ-thin, the distance from a' to ab or bb' is not greater than 3δ. In the first case, the statement is proved immediately. In the second case, we note that bb' is a perpendicular to ab' and hence $d(a'b') \leqslant 2d(a', bb') \leqslant 6\delta$. Therefore, a' in this case is near the projection of the point of ab that is nearest ab', which completes the proof. □

Lemma 14. *Let G be a nonelementary hyperbolic group. Then there exist constants c_0, c_1, and c_2 such that for every two points p and q in the group G with $|p - q| > r_0$, there exists a geodesic γ such that $d(p, \gamma) \leqslant r_1$ and $||p - q| - d(q, \gamma)| \leqslant r_2$.*

Proof. We first assume that p is the unity of the group. We argue by contradiction: we suppose that the statement is false, i.e., there exists a sequence of points q_n such that $|q_n - p| \to \infty$ as $n \to \infty$, and all pairs p and q_n do not satisfy the conditions in the lemma. We suppose that ξ

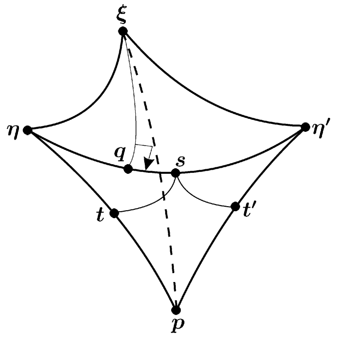

Fig. 8. Illustration for Lemma 14.

is a limit point of this sequence. As in the proof of Lemma 12, we supply the boundary of the group with a visual metric. And the same arguments provide that there exist $\varepsilon > 0$ and points η and η' on the ideal boundary $G(\infty)$ such that the pairwise visual distances between ξ, η, and η' are greater than ε (see Fig. 8). We show that the geodesic γ with the endpoints η and η' satisfies the conditions in the lemma, which leads to a contradiction.

In what follows, we write ξ, η, and η' but assume that we consider three sequences of points converging to the corresponding points of the ideal boundary. The triangle $p\eta\eta'$ is δ-thin. We take a point s of $\eta\eta'$ such that $d(s, p\eta) \leqslant \delta$ and $d(s, p\eta') \leqslant \delta$. We let t and t' denote projections of s respectively on $p\eta$ and $p\eta'$. By the triangle inequality, we have

$$|\eta - t| + |\eta' - t'| - 2\delta \leqslant |\eta - \eta'| \leqslant |\eta - t| + |\eta' - t'| + 2\delta.$$

By hypothesis,

$$\operatorname{visdist}_p(\eta, \eta') = e^{-(\eta|\eta')_p} > \varepsilon.$$

Hence,

$$|p - \eta| + |p - \eta'| - |\eta - \eta'| < 2\varepsilon_0,$$

where $\varepsilon_0 = -\ln \varepsilon$

Combining the two inequalities, we obtain $|p - t| + |p - t'| \leqslant 2(\varepsilon_0 + \delta)$ and $d(p, \eta\eta') \leqslant 2\varepsilon_0 + 3\delta$. The same arguments applied to the triangles $p\eta\xi$ and $p\eta'\xi$ show that the distance from the point p to the geodesics $\eta\xi$ and $\eta'\xi$ also does not exceed $2\varepsilon_0 + 3\delta$. We let p_1, p_2, and p_3 denote the respective projections of p on $\eta\eta'$, $\eta\xi$, and $\eta'\xi$ and q denote the projection of ξ on $\eta\eta'$. By the triangle inequality, $|p_1 - p_2| \leqslant |p_1 - p| + |p - p_2| \leqslant 2(2\varepsilon_0 + 3\delta)$. Applying Lemma 4 to the triangles $q\xi\eta$ and $q\xi\eta'$, we find that the point q is not farther than 2δ from both $\eta\xi$ and $\eta'\xi$. Therefore, both p_1 and q are at bounded distances from $\eta\xi$ and $\eta'\xi$, and we can apply Lemma 6, whence it follows that p_1 and q are near each other at a distance of the order $\varepsilon_0 + \delta$. □

Lemma 15. *Let X be a δ-hyperbolic space, ξ and η be two points of the ideal boundary ∂X, and p and p' be two points such that $d(p, p') = D$. Then the visual distances between ξ and η from the points p and p' satisfy the inequality*

$$\operatorname{visdist}_{p'}(\xi, \eta) \leqslant e^D \operatorname{visdist}_p(\xi, \eta).$$

Proof. By definition, Gromov's product of x and y in p is

$$(x|y)_p = \frac{1}{2}\big(|p-x| + |p-y| - |x-y|\big).$$

We have the same equality for x, y, and p'. Hence,

$$\big|(x|y)_{p'} - (x|y)_p\big| = \left|\frac{1}{2}\big(\big|p'-x\big| + \big|p'-y\big| - |p-x| - |p-y|\big)\right| \leqslant \big|p-p'\big|.$$

The last inequality follows from the triangle inequality. Therefore, by the definition of a visual metric,

$$\text{visdist}_{p'}(\xi,\eta) = e^{(\xi|\eta)_{p'}} \leqslant e^{(\xi|\eta)_p + |p-p'|} = e^D \,\text{visdist}_p(\xi,\eta). \qquad \square$$

9. Quasi-isometries fixing the ideal boundary

We now give some estimates of the displacement of points in geodesically rich spaces under quasi-isometries that fix the ideal boundary. We do not yet know whether these results are optimal.

Remark 4. Let X be a metric space satisfying the first condition in the definition of geodesically rich. Let $f : X \to X$ be a (λ, c)-self-quasi-isometry fixing the boundary ∂X. Then for sufficiently large λ and any point $O \in X$, $d(f(O), O) \leqslant H(\lambda, c, \delta) + r_2$, where the constant C_1 depends only on the space X.

Proof. For any point O, $r_1 \leqslant H(\lambda, c, \delta)$ for sufficiently large λ if $d(O, f(O)) < r_0$. Otherwise, let γ be a geodesic such that $d(O,\gamma) \leqslant r_1$ and $d(f(O),\gamma) > d(O, f(O)) - r_2$. Because $f(\gamma)$ is a quasi-geodesic with the same endpoints as γ, the quasi-geodesic lies near γ: $f(\gamma) \subset U_H(\gamma)$. Combining all the arguments, we obtain

$$d\big(O, f(O)\big) \leqslant d\big(f(O),\gamma\big) + r_2 \leqslant H + r_2. \qquad \square$$

We do not know if it is possible to improve this upper bound in the general case. But in the case of a geodesically rich space, we can improve the bound from λ^2 to $\lambda \ln \lambda$.

Theorem 1. *(See Theorem 3 in the Introduction.) Let X be an (r_1, r_2)-geodesically rich δ-hyperbolic metric space and f be a (λ, c)-self-quasi-isometry fixing a boundary ∂X. Then for any point $O \in X$, $d(O, f(O)) \leqslant \max(r_0, \lambda(r_3 + c + c_1 \ln \lambda) + r_1 + r_2 + r_4)$.*

Proof. Because f fixes the boundary of X and by the anti-Morse lemma, a $(c_1 \ln \lambda)$-neighborhood (where $c_1 = c + \delta$) of an image $f(\sigma)$ of any geodesic σ includes σ: $\sigma \subset V_{c_1 \ln \lambda}(f(\sigma))$. All the constants r_0, r_1, r_2, r_3, and r_4 are the same constants as in the definition of a geodesically rich space. We take an arbitrary point $O \in X$. We assume that $d(O, f(O)) \geqslant r_0$ because otherwise there is nothing to prove. There exists a geodesic γ such that $d(\gamma, O) \leqslant r_1$ and $|d(O, f(O)) - d(f(O),\gamma)| \leqslant r_2$, and there also exists a geodesic γ' such that $f(O)$ lies in r_3-neighborhood of γ' and such that $f(O)$ is (up to r_4) the point of γ' that is nearest γ.

Because $\gamma' \subset V_{c_1 \ln\lambda}(f(\gamma'))$, there exists a point O' of γ' such that $|f(O') - f(O)| \leqslant r_3 + c_1 \ln\lambda$. Now, $d(f(O), \gamma) \leqslant d(O', \gamma) + r_4 \leqslant |O' - O| + r_1 + r_4$, and by the definition of a quasi-isometry, $|O' - O| \leqslant \lambda(|f(O') - f(O)| + c) \leqslant \lambda(r_3 + c + c_1 \ln\lambda)$. Hence, $d(f(O), \gamma) \leqslant \lambda(r_3 + c + c_1 \ln\lambda) + r_1 + r_4$. Finally, we conclude that $d(O, f(O)) \leqslant d(f(O), \gamma) + r_2 \leqslant \lambda(r_3 + c + c_1 \ln\lambda) + r_1 + r_2 + r_4$. $\square$

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Corrigendum

A corrected quantitative version of the Morse lemma

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A R T I C L E I N F O

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A B S T R A C T

There is a gap in the proof of the main theorem in the article [5] on optimal bounds for the Morse lemma in Gromov-hyperbolic spaces. We correct this gap, showing that the main theorem of [5] is true. We also describe a computer certification of this result.

1. Introduction

The Morse lemma is a fundamental result in the theory of Gromov-hyperbolic spaces. It asserts that, in a δ-hyperbolic space, the Hausdorff distance between a (λ, C)-quasi-geodesic and a geodesic segment sharing the same endpoints is bounded by a constant $A(\lambda, C, \delta)$ depending only on λ, C and δ, and not on the length of the geodesic. Many

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proofs of this result have been given, with different expressions for A. An optimal value for A (up to a multiplicative constant) has only been found recently in the article [5] by the second author, giving $A(\lambda, C, \delta) = K\lambda^2(C+\delta)$ for an explicit constant $K = 4(78 + 133/\log(2) \cdot \exp(157\log(2)/28)) \sim 37723$.

Unfortunately, there is a gap in the proof of this theorem in [5], which was noticed by the first author while he was developing a library [4] on Gromov-hyperbolic spaces in the computer assistant Isabelle/HOL. In such a process, all proofs are formalized on a computer, and checked starting from the most basic axioms. The degree of confidence reached after such a formal proof is orders of magnitude higher than what can be obtained by even the most diligent reader or referee, and indeed this process shed the light on the gap in [5]. The gap is on Page 829: the inequality $\sum_{i=1}^{n} e^{-X_i}(X_{i-1} - X_i) \leq \int_0^\infty e^{-t}\,\mathrm{d}t$ goes in the wrong direction as the sequence X_i is decreasing.

In this paper, we fix this gap. Here is the estimate we get.

Theorem 1.1. *Consider a (λ, C)-quasi-geodesic Q in a δ-hyperbolic space X, and G a geodesic segment between its endpoints. Then the Hausdorff distance $HD(Q,G)$ between Q and G satisfies*

$$HD(Q,G) \leq 92\lambda^2(C+\delta).$$

Let us specify precisely the terms used in this statement, as there are small variations in the definitions in the literature. For us, a (λ, C)-quasi-geodesic is the image of a map f from a compact interval to X satisfying for all x, y the inequalities

$$\lambda^{-1}|y-x| - C \leq d(f(x), f(y)) \leq \lambda|y-x| + C.$$

A map satisfying these inequalities is also called a (λ, C)-quasi-isometry. We also require $\lambda \geq 1$ and $C \geq 0$ in the definition. A geodesic segment is by definition a $(1,0)$-quasi-geodesic. We say that the space X is δ-hyperbolic if the Gromov product $(x,y)_w = (d(x,w) + d(y,w) - d(x,y))/2$ satisfies for all points x, y, z, w the inequality

$$(x,z)_w \geq \min((x,y)_w, (y,z)_w) - \delta.$$

Finally, the Hausdorff distance $HD(Q,G)$ is the smallest number r such that G is included in the r-neighborhood of Q, and conversely.

Remark 1.2. For any $\lambda \geq 3$, $C \geq 0$ and $\delta \geq 0$, one can construct an example of a (λ, C)-quasi-geodesic Q in a δ-hyperbolic space which satisfies $HD(Q,G) \geq \lambda^2(C+\delta)/9$ where G is a geodesic segment joining the endpoints of Q. This shows that Theorem 1.1 is optimal, up to the value of the multiplicative constant. Such examples for $\delta = 0$ are already given in [5], and the following is a variation around these examples.

Example 1.3. Let $\lambda \geq 3$, $C \geq 0$ and $\delta \geq 0$. Take $X = \mathbb{R} \times [0, \delta]$ with the L^1 distance. This is a δ-hyperbolic space. Let $\bar{\lambda} = \lambda/3 \geq 1$. Define a quasi-geodesic $f : [0, 2\bar{\lambda}(C+\delta) + \delta/\bar{\lambda}] \to$

X by going always at speed $\bar{\lambda}$ from $(0,0)$ to $(\bar{\lambda}^2(C+\delta),0)$, then to $(\bar{\lambda}^2(C+\delta),\delta)$, then to $(0,\delta)$. The Hausdorff distance between the quasi-geodesic Q defined by f and the geodesic G joining $(0,0)$ and $(0,\delta)$ is $\bar{\lambda}^2(C+\delta) = \lambda^2(C+\delta)/9$. We claim that f is a (λ,C)-quasi-geodesic. The upper bound $d(f(x),f(y)) \leq \lambda\,|y-x|+C$ is obvious as f is $\bar{\lambda}$-Lipschitz by construction. For the lower bound $d(f(x),f(y)) \geq \lambda^{-1}\,|y-x|-C$, the most demanding points are the endpoints of the interval $x=0$ and $y = 2\bar{\lambda}(C+\delta)+\delta/\bar{\lambda}$: we should check that

$$d(f(x),f(y)) = \delta \geq \lambda^{-1}\cdot\left(2(C+\delta)\bar{\lambda}+\delta/\bar{\lambda}\right) - C.$$

This follows from the choice $\bar{\lambda} = \lambda/3$.

The new proof of Theorem 1.1 has been completely formalized in Isabelle/HOL in [4]. Therefore, the above theorem is certified. Here is this statement as proved in Isabelle/HOL.

```
theorem (in Gromov_hyperbolic_space) Morse_Gromov_theorem':
  fixes f::"real ⇒ 'a"
  assumes "lambda C-quasi_isometry_on {a..b} f"
          "geodesic_segment_between G (f a) (f b)"
  shows "hausdorff_distance (f`{a..b}) G ≤ 92 * lambda^2 * (C + deltaG(TYPE('a)))"
```

In this formal statement, `'a` is a type of class `Gromov_hyperbolic_space`. It corresponds to the space X of Theorem 1.1, and the associated hyperbolicity constant is `deltaG(TYPE('a))`. Instead of talking of the quasi-geodesic Q, the formal statement is made in terms of its parametrization f, as the notion of endpoint of a quasi-geodesic is not really well defined. With this correspondence, the two statements directly correspond to each other.

Although the proof is more involved than the original argument in [5], the constant we get in the end is much better (92 instead of 37724). Indeed, we have tried to optimize the constant as much as we could, contrary to [5], keeping in mind the foundational nature of the library [4]. This optimization owes a lot to the formalization process. It makes it possible to optimize locally one part of the proof, and see if it breaks other parts of the proof by checking if the proof assistant complains that the proof is not correct any more, or if everything goes through. The certainty of the result also makes the optimization worth it, as we are sure not to have forgotten for example an edge case that would spoil the estimates.

Having a formalized certified proof raises interesting questions about the way to write mathematics. We do not need to convince a reader (or a referee!) that the result is correct, as we have already done the much more demanding task of convincing a computer, and the proof with all details can be read by the interested reader in [4]. Rather, we have to convey the interesting ideas. We have decided to give all the precise statements we use (in their traditional version, but the very same statements have been formalized in [4]), but skip their proofs if they are small variations around results that are already available in the literature. For the main proof, we will explain (with as many details as in a traditional mathematical paper) a simplified version of the proof that gives the same statement as Theorem 1.1 but not caring much about the universal constants (this simplified argument gives the constant 2460 instead of 92 in Theorem 1.1). Then we will comment without entering in too many details on the various optimizations that can be done, leading to the above statement.

Remark 1.4. The proof of Theorem 1.1 is delicate. However, we would like to emphasize that this is not due to our desire to formalize the proof on computer: the argument we give in this article is the simplest one we have been able to come up with, without any attempt to get an easy to formalize proof. And indeed this proof was not easy to formalize, but the mere fact that this was possible shows how powerful proof assistants already are today.

2. Proof of the main theorem

The proof uses the notion of quasiconvexity. We say that a subset $Y \subseteq X$ is K-quasiconvex if, for any $y_1, y_2 \in Y$, there exists a geodesic between y_1 and y_2 which is included in the K-neighborhood of Y. For instance, geodesics are 0-quasiconvex. The r-neighborhood of a 0-quasiconvex set is always 8δ-quasiconvex, see [3, Proposition 10.1.2].

We follow the global strategy of [5] to prove Theorem 1.1, with a new more involved argument at a key technical step. Thanks to [1], we can assume without loss of generality that the space X is geodesic. The quasi-geodesic Q is by definition the image of a (λ, C)-quasi-isometric map $f : [u^-, u^+] \to X$. The statement for a general quasi-isometric map f reduces to the one for a continuous quasi-isometric map f thanks to the following approximation lemma, which is a version of [5, Lemma 9] or [2, Lemma III.H.1.11].

Lemma 2.1. *Consider a (λ, C)-quasi-isometry from a compact interval to a geodesic metric space, whose endpoints are at distance at least $2C$. Then it is within Hausdorff distance $2C$ of a $(\lambda, 4C)$-quasi-geodesic with the same endpoints which is moreover 2λ-Lipschitz.*

The proof of this lemma is very classical: assume that the initial quasi-geodesic is defined on an interval $[u^-, u^+]$. Then the assumptions ensure that $u^+ - u^- \geq C/\lambda$. Split suitably the interval $[u^-, u^+]$ into subintervals with length in $[C/\lambda, 2C/\lambda]$. The new quasi-geodesic will coincide with the initial one on the endpoints of these subintervals, and be geodesic in between. The facts that this new function is a $(\lambda, 4C)$-quasi-geodesic, within Hausdorff distance $2C$ of the original one, and 2λ-Lipschitz, follow from direct computations.

Replacing the original quasi-geodesic by the new one given by Lemma 2.1 and C by $4C$, we will assume from this point on that the (λ, C)-quasi-geodesic f is also continuous. Replacing the original hyperbolicity constant δ_0 by a slightly larger constant δ (and letting δ tend to δ_0 at the end of the argument), we can assume that the space is hyperbolic for a constant strictly smaller than δ, and also that $\delta > 0$.

Consider $z \in [u^-, u^+]$. We want to estimate $d(f(z), G)$. We will prove an estimate of the form

$$d(f(z), G) \leq K_0 + \frac{K_1}{K_2} \int_0^{u^+ - u^-} e^{-K_2 t}\, \mathrm{d}t = K_0 + K_1 \cdot (1 - e^{-K_2(u^+ - u^-)}), \tag{2.1}$$

where K_0, K_1 and K_2 are suitable parameters that do not depend on u^- and u^+. Both K_0 and K_1 will be of the form $K_i = k_i \lambda^2 (C + \delta)$, while K_2 will be of the form $K_2 = k_2/(\delta\lambda)$ where k_0, k_1, k_2 are explicit positive real constants. They will be defined in (2.4), (2.7) and (2.6). This estimate is proved inductively over the size of $u^+ - u^-$, reducing the estimate over $[u^-, u^+]$ to the estimate over a shorter interval $[v^-, v^+]$. We will have to show that the loss in this reduction process is controlled in terms of $K_1 e^{-K_2(v^+ - v^-)} - K_1 e^{-K_2(u^+ - u^-)}$, to conclude the proof of (2.1) by induction.

Let us first explain why this estimate concludes the proof. It implies that $d(f(z), G) \leq K_0 + K_1$. This proves that the image Q of f is included in the $(k_0 + k_1)\lambda^2(C + \delta)$-neighborhood of G. To get the estimate on the Hausdorff distance, one needs to show that G is also included in a $k\lambda^2(C + \delta)$-neighborhood of Q for some k. This follows from the previous estimate and a standard argument (see [2]) that we recall now. Consider a point $g \in G$. Denote by Q^- the set of points on Q that are within distance $(k_0 + k_1)\lambda^2(C + \delta)$ of a point of G in $[f(u^-), g]$, and by Q^+ the set of points on Q that are within distance $(k_0 + k_1)\lambda^2(C + \delta)$ of a point of G in $[g, f(u^+)]$. The previous estimate implies that $Q = Q_1 \cup Q_2$. As Q is connected, it follows that $Q_1 \cap Q_2 \neq \emptyset$. Denote by $f(z)$ a point in this intersection, and by g^- and g^+ two points before and after g on G, at distance at most $(k_0 + k_1)\lambda^2(C + \delta)$ of $f(z)$. Using hyperbolicity in a triangle with vertices at $g^-, g^+, f(z)$ and the fact that g is on a geodesic between g^- and g^+, it follows that the distance between g and $f(z)$ is at most $(k_0 + k_1)\lambda^2(C + \delta) + \delta$. As $\lambda \geq 1$, this expression is bounded by $(k_0 + k_1 + 1)\lambda^2(C + \delta)$. This concludes the argument, for the constant $k = k_0 + k_1 + 1$. We remind that [6] contains a stronger result (Theorem 3) claiming that the geodesic G is included in an $A(\delta \log \lambda + C + \delta)$-neighborhood of the quasi-geodesic Q with some universal constant A.

It remains to prove the estimate (2.1). The proof will use two parameters L and D. For simplicity, let us take

$$L = D = 100\delta. \tag{2.2}$$

We keep separate notations for L and D because we will want to optimize the choice of their values later.

Case 1. The case where $d(f(z), G) \leq L$ is trivial, as the estimate (2.1) holds if one takes K_0 large enough.

Case 2. Let us therefore assume $d(f(z), G) > L$. We will construct several points along $[u^-, z]$. To ease the reading, their order will correspond to the alphabetical order when possible.

Consider a projection π_z of $f(z)$ on G, and a geodesic segment H from π_z to $f(z)$. Denote by $p : X \to H$ a closest-point projection on H. The idea is to project the quasi-geodesic Q on H and to consider the subpart Q' of Q that projects at distance at least L of π_z. If one could show that Q' is quantitatively shorter than Q and that the distance from $f(z)$ to π_z is controlled in terms of the distance from $f(z)$ to a geodesic joining the endpoints of Q', then we would be in good shape to prove (2.1) inductively, deducing the estimate for Q from the estimate for Q'. The real argument will be built around this naive idea, but in a more subtle way.

More precisely, consider two points $y^- \in [u^-, z]$ and $y^+ \in [z, u^+]$ such that the projections $p(f(y^-))$ and $p(f(y^+))$ are at distance roughly L of π_z. In general, p is not uniquely defined and not continuous, but this is almost the case up to $O(\delta)$ thanks to the hyperbolicity of the space. With the following standard lemma and recalling that H is 0-quasiconvex as it is a geodesic, one can find y^- and y^+ such that

$$d(p(f(y^\pm)), \pi_z) \in [L - 4\delta, L]. \tag{2.3}$$

Lemma 2.2. *A closest-point projection of a connected set on a K-quasiconvex subset Y of X has gaps of size at most $4\delta + 2K$. More precisely, if $f : [a, b] \to X$ is a continuous function and $p(f(t))$ denotes a closest point projection of $f(t)$ on Y, then for any $\tau \leq d(p(f(a)), p(f(b)))$, there exists $t \in [a, b]$ such that $d(p(f(a)), p(f(t))) \in [\tau - 4\delta - 2K, \tau]$. Moreover, one can ensure that $d(p(f(a)), p(f(s))) \leq d(p(f(a)), p(f(t)))$ for all $s \leq t$.*

Denote by d^- (respectively d^+) the minimal distance of a point in $f([u^-, y^-])$ (respectively $f([y^+, u^+])$) to H. These distances are realized by two points $f(m^-)$ and $f(m^+)$, by continuity of f.

Case 2.1. Assume that $\max(d^-, d^+)$ is not large, say $\leq D + C$ where $D = 100\delta$ is the constant we have chosen in (2.2) and C is the quasi-isometry parameter. This is again an easy case. Indeed, as the projections of $f(m^-)$ and $f(m^+)$ are within distance L of π_z, one gets $d(f(m^-), f(m^+)) \leq 2D + 2C + L$. By quasi-isometry,

$$d(m^-, m^+) \leq \lambda(d(f(m^-), f(m^+)) + C) \leq \lambda(2D + 3C + L).$$

As z is between m^- and m^+, one gets in particular $d(m^-, z) \leq \lambda(2D + 3C + L)$. Then

$$\begin{aligned} d(f(z), \pi_z) &\leq d(f(z), f(m^-)) + d(f(m^-), p(f(m^-))) + d(p(f(m^-)), \pi_z) \\ &\leq (\lambda d(z, m^-) + C) + (D + C) + L \leq \lambda^2(3D + 5C + 2L). \end{aligned}$$

This is compatible with the inequality (2.1) if one takes

$$K_0 = 500\lambda^2(\delta + C). \tag{2.4}$$

Case 2.2. Assume now that $\max(d^-, d^+) \geq D + C$, and $d^- \geq d^+$ for instance. This is the interesting case. The main step in the proof is the following lemma.

Lemma 2.3. *There exist two points $v \leq x$ in $[u^-, y^-]$ and a real number $d' \geq d^-$ such that*

$$L - 74\delta \leq 4\sqrt{2}\lambda(x - v)e^{-d' \log(2)/(10\delta)} \tag{2.5}$$

and $d(f(v), p(f(v))) \leq 4d'$.

The numerology in the lemma (74 and $4\sqrt{2}$ and $\log(2)/10$ and 4) is of no importance: what only matters is that $L - 74\delta$ is positive, thanks to the choice of L in (2.2), and that the other numbers are positive and fixed.

Let us show how to conclude the proof using the lemma. We have

$$\begin{aligned} m^+ - v = d(v, m^+) &\leq \lambda(d(f(v), f(m^+)) + C) \\ &\leq \lambda\Big(d(f(v), p(f(v))) + d(p(f(v)), p(f(m^+))) + d(p(f(m^+)), f(m^+)) + C\Big) \\ &\leq \lambda(4d' + L + d^+ + C) \leq 6\lambda d', \end{aligned}$$

as $L + C = D + C \leq d^- \leq d'$ and $d^+ \leq d^- \leq d'$. Therefore, taking

$$K_2 = \log(2)/(60\delta\lambda), \tag{2.6}$$

the inequality (2.5) gives

$$\begin{aligned} L - 74\delta &\leq 4\sqrt{2}\lambda(x - v)e^{-(m^+ - v)\cdot\log(2)/(60\delta\lambda)} = \frac{4\sqrt{2}\lambda}{K_2} \cdot K_2(x - v)e^{-K_2(m^+ - v)} \\ &\leq \frac{4\sqrt{2}\lambda}{K_2}(e^{K_2(x-v)} - 1)e^{-K_2(m^+ - v)} = \frac{4\sqrt{2}\lambda}{K_2}(e^{-K_2(m^+ - x)} - e^{-K_2(m^+ - v)}) \\ &\leq \frac{4\sqrt{2}\lambda}{K_2}(e^{-K_2(m^+ - x)} - e^{-K_2(u^+ - u^-)}). \end{aligned}$$

Consider a new geodesic G' between $f(x)$ and $f(m^+)$. Arguing by induction, we can assume that the estimate (2.1) has already been proved for G', and we want to deduce it for G. Since both endpoints of G' project within distance L of π_z, one checks that the distance from $f(z)$ to G is controlled by the distance from $f(z)$ to G' (this is a version of [5, Lemma 5]). More specifically,

$$d(f(z), G) \leq d(f(z), G') + L + 4\delta.$$

Bounding $d(f(z), G')$ thanks to the induction assumption, and plugging in the estimate from the previous equation, we get

$$d(f(z), G) \leq K_0 + K_1(1 - e^{-K_2(m^+ - x)}) + \frac{L + 4\delta}{L - 74\delta} \cdot \frac{4\sqrt{2}\lambda}{K_2}(e^{-K_2(m^+ - x)} - e^{-K_2(u^+ - u^-)}).$$

Let us take

$$K_1 = \frac{L + 4\delta}{L - 74\delta} \cdot \frac{4\sqrt{2}\lambda}{K_2}. \tag{2.7}$$

Then the terms $K_1 e^{-K_2(m^+ - x)}$ simplify in this equation, and we are left with

$$d(f(z), G) \leq K_0 + K_1(1 - e^{-K_2(u^+ - u^-)}).$$

This is (2.1), as desired. This concludes the proof of Theorem 1.1. □

It remains to prove Lemma 2.3. The argument relies on the contracting properties of closest-point projections on quasiconvex sets. The first such basic statement is the following variation around [3, Proposition 10.2.1].

Lemma 2.4. *Consider a K-quasiconvex subset Y of X. Then projections p_x and p_y on Y of two points x and y satisfy*

$$d(p_x, p_y) \leq \max(5\delta + 2K, d(x, y) - d(x, p_x) - d(y, p_y) + 10\delta + 4K).$$

This result expresses the classical fact that a geodesic from x to y essentially follows a geodesic from x to p_x, then from p_x to p_y, then from p_y to y.

The second result we need is more sophisticated. Instead of a linear gain in terms of the distance to the set one projects on, as in the previous lemma, it gives an exponential gain in the upper bound, by a successive reduction process. It is proved by putting points along the path with gaps of size 10δ. Then, move by 5δ towards Y: this reduces the distance between the points by 5δ essentially thanks to the previous lemma. Then, discard half the points: this shows that by moving towards Y by 5δ the length of the path has been divided by 2. One can iterate this argument to get the exponential gain. We give a statement for the projection on quasiconvex sets as this is what we will need later on. This statement is proved in [5, Lemma 10] for the projection on a geodesic segment, but the case of a general quasiconvex set is analogous.

Lemma 2.5. *Consider a (λ, C)-quasi-geodesic path $f : [a, b] \to X$, everywhere at distance at least D of a K-quasiconvex subset Y. Then, if $D \geq 15/2 \cdot \delta + K + C/2$, projections p_a of $f(a)$ and p_b of $f(b)$ on Y satisfy the inequality*

$$d(p_a, p_b) \leq 2K + 8\delta + \max\left(5\delta, 4\sqrt{2}\lambda(b-a)\exp\Big(-(D - K - C/2)\log(2)/(5\delta)\Big)\right).$$

Using these results, we can prove Lemma 2.3.

Proof of Lemma 2.3. For $k \geq 0$, let V_k denote the $(2^k - 1)d^-$-neighborhood of H. These sets are all 8δ-quasiconvex. We recall that $p(f(x))$ is a projection of $f(x)$ on H. Let $p_k(x)$ denote the point on a fixed geodesic between $p(f(x))$ and $f(x)$ at distance $\min((2^k - 1)d^-, d(p(f(x)), f(x)))$ of $p(f(x))$. Then $p_k(x)$ is a projection of $f(x)$ on V_k, and moreover these projections are compatible in the following sense: for $k \leq \ell$, then $p_k(x)$ is a projection of $p_\ell(x)$ on V_k. Moreover, $p_0(x) = p(f(x))$.

We will do an inductive construction over k. This construction will have to stop at some step, where it will give the desired points. Until the argument stops, we will construct a point $x_k \in [u^-, y^-]$ such that

$$d(p_k(u^-), p_k(x_k)) \geq L - 8\delta \tag{2.8}$$

and

$$\text{for all } w \in [u^-, x_k],\ d(f(w), p_0(w)) \geq (2^{k+1} - 1)d^-. \tag{2.9}$$

Let us first check that this property holds for $k = 0$. Take $x_0 = y^-$. The point π_z is a projection of $f(z)$ on the geodesic G between $f(u^-)$ and $f(u^+)$. This does not imply that the projection $p_0(u^-)$ of $f(u^-)$ on the geodesic H between π_z and $f(z)$ is exactly at π_z (contrary to the situation in the Euclidean plane), but by hyperbolicity one checks that $d(\pi_z, p_0(u^-)) \leq 4\delta$ (this is a version of [5, Lemma 3]). Since $d(\pi_z, p_0(y^-)) \in [L - 4\delta, L]$ by (2.3) and $x_0 = y^-$, we deduce that $d(p_0(u^-), p_0(x_0)) \geq L - 8\delta$. This is (2.8). Moreover, by definition of d^-, the inequality (2.9) holds for $k = 0$.

Assume now that (2.8) and (2.9) hold at k. We will show that either we can find a pair of points that satisfy the conclusion of the lemma, or we can construct a point x_{k+1} such that (2.8) and (2.9) hold at $k + 1$.

As V_k is 8δ-quasiconvex, we deduce from Lemma 2.2 that the gaps of the closest-point projection p_k are bounded by 20δ. Therefore, we can find a point $x_{k+1} \in [u^-, x_k]$ whose projection on V_k satisfies

$$d(p_k(u^-), p_k(x_{k+1})) \in [22\delta, 42\delta], \tag{2.10}$$

and moreover all points $w \in [u^-, x_{k+1}]$ satisfy

$$d(p_k(u^-), p_k(w)) \leq 42\delta. \tag{2.11}$$

There are two cases to consider:

If there exists $v \in [u^-, x_{k+1}]$ *with* $d(f(v), p_0(v)) \leq (2^{k+2} - 1)d^-$. Then we claim that the pair (v, x_k) satisfies the conclusion of Lemma 2.3, for $d' = 2^k d^-$. First, the inequalities $d' \geq d^-$ and $d(f(v), p_0(v)) \leq 4d'$ hold by construction. Moreover, $d(p_k(v), p_k(x_k)) \geq L - 50\delta$ as $p_k(x_k)$ is far from $p_k(u^-)$ by (2.8), and $p_k(v)$ is close to $p_k(u^-)$ by (2.11). As all intermediate points are at distance at least $(2^{k+1} - 1)d^-$ of V_0 by (2.9), they are at distance at least $2^k d^-$ of V_k and we can apply the exponential contraction Lemma 2.5 with $D = 2^k d^-$. As V_k is 8δ-quasiconvex, we get

$$\begin{aligned} L - 50\delta &\leq d(p_k(v), p_k(x_k)) \\ &\leq 24\delta + \max\left(5\delta, 4\sqrt{2}\lambda(x_k - v)\exp\left(-(2^k d^- - 8\delta - C/2)\log(2)/(5\delta)\right)\right). \end{aligned}$$

As $L - 50\delta > 29\delta$, the maximum has to be realized by the second term. Moreover, $2^k d^- - 8\delta - C/2 \geq (2^k d^-)/2 = d'/2$, as $d^- \geq D + C = 100\delta + C$. We obtain

$$L - 74\delta \leq 4\sqrt{2}\lambda(x_k - v)\exp\left(-d'\log(2)/(10\delta)\right). \tag{2.12}$$

This concludes the proof in this case.

Otherwise, $d(f(w), p_0(w)) \geq (2^{k+2} - 1)d^-$ *for all* $w \in [u^-, x_{k+1}]$. In this case, (2.9) holds for $k+1$. Let us check that (2.8) also holds for $k+1$, by applying the projection Lemma 2.4 to the points $p_{k+1}(u^-)$ and $p_{k+1}(x_{k+1})$, which project respectively to $p_k(u^-)$ and $p_k(x_{k+1})$ on V_k. As V_k is 8δ-quasiconvex, this lemma gives

$$\begin{aligned} &d(p_k(u^-), p_k(x_{k+1})) \leq \max(21\delta, \\ &\qquad d(p_{k+1}(u^-), p_{k+1}(x_{k+1})) - d(p_{k+1}(u^-), p_k(u^-)) - d(p_{k+1}(x_{k+1}), p_k(x_{k+1})) + 42\delta). \end{aligned}$$

As $d(p_k(u^-), p_k(x_{k+1})) \geq 22\delta$ by (2.10), the maximum has to be realized by the second term. Both distances $d(p_{k+1}(u^-), p_k(u^-))$ and $d(p_{k+1}(x_{k+1}), p_k(x_{k+1}))$ are equal to $2^k d^-$. We obtain

$$2 \cdot 2^k d^- - 20\delta \leq d(p_{k+1}(u^-), p_{k+1}(x_{k+1})).$$

As $d^- \geq D = 100\delta$, the left hand side is $\geq L - 8\delta = 92\delta$. This concludes the proof of (2.8), and of the induction.

Finally, if the conclusion of the lemma does not hold, then the induction will go on forever. Taking in particular $w = u^-$ in (2.9), we get $d(f(u^-), p_0(u^-)) \geq (2^{k+1} - 1)d^-$ for all k, a contradiction. □

Here are some ways to optimize the proof to get better constants. In addition to multiple minor optimizations, let us mention the main ones:

- The set V_0 is 0-quasiconvex, not only 8δ-quasiconvex. This means that estimates in the proof of Lemma 2.3 are better for $k = 0$. There is a different source of gain for $k > 0$, thanks to the factor 2^k. Separating the two cases improves the final constant.
- There is an exponential gain in (2.12). One can spend some part of this gain, say $\exp(-(1-\alpha)d' \log(2)/(10\delta)) \leq \exp(-(1-\alpha)D \log(2)/(10\delta))$ to improve the multiplicative constant, and use the remaining part $\exp(-\alpha d' \log(2)/(10\delta))$ for the induction (for a suitable value of α).
- Instead of formulating the induction in terms of the distance from $f(z)$ to a geodesic G between $f(u^-)$ and $f(u^+)$, it is more efficient to induce over the Gromov product $(f(u^-), f(u^+))_{f(z)}$ (which coincides with the distance $d(f(z), G)$ up to 2δ) as most inequalities are done in terms of Gromov products. The main interest of this change is that, with the current argument, the point $f(u^-)$ projects on H between π_z and $f(z)$ within distance 4δ of π_z, which means there is a small loss. With the Gromov product approach, let m denote the point on G which is opposite to $f(z)$ in the triangle $[f(z), f(u^-), f(u^+)]$, i.e., it is on G at distance $(f(z), f(u^+))_{f(u^-)}$ of $f(u^-)$ and at distance $(f(z), f(u^-))_{f(u^+)}$ of $f(u^+)$. Let π_z denote the point on a geodesic H from $f(z)$ to m at distance $(f(u^-), f(u^+))_{f(z)}$ of $f(z)$. This point is within distance 2δ of m. It turns out that the projection of $f(u^-)$ on H is between m and π_z, i.e., opposite from $f(z)$. The above loss is suppressed in this approach.
- Finally, one can choose freely L, D and α within some range. In particular, L and D do not have to coincide. One can optimize numerically over these parameters to get the best possible bound. In the end, we take $L = 18\delta$ and $D = 55\delta$ and $\alpha = 12/100$ to get the value 92 in Theorem 1.1.

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