DOI 10.1007/s10958-022-05821-z
Journal of Mathematical Sciences, Vol. 262, No. 3, April, 2022

EXACT CONTROL OF A DISTRIBUTED SYSTEM DESCRIBED BY THE WAVE EQUATION WITH INTEGRAL MEMORY

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UDC 517.95

We consider the distributed control problem for the wave equation with memory, where the kernel is the sum of decreasing exponential functions and the control is bounded in modulus. We prove that the oscillations of the system can be brought to the state of rest in a finite time. Bibliography: 14 *titles.*

1 Introduction

We study the control problem for the system of integrodifferential equations

$$\theta_{tt}(t,x) - K(0)\Delta\theta(t,x) - \int\limits_0^t K'(t-s)\Delta\theta(s,x)ds = u(t,x), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$\theta|_{t=0} = \varphi_0(x), \quad \theta_t|_{t=0} = \varphi_1(x), \tag{1.2}$$

$$\theta|_{\partial\Omega} = 0; \tag{1.3}$$

here,

$$K(t) = \sum_{j=1}^{N} \frac{c_j}{\gamma_j} e^{-\gamma_j t}, \quad N \geqslant 2,$$

where c_j and γ_j are given positive constants such that $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_N$, $u(t,x)$ is a

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Translated from *Problemy Matematicheskogo Analiza* **115**, 2022, pp. 111-125.

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control defined in a bounded (with respect to x) domain Ω, and $|u(t,x)| \leqslant M$, where $M > 0$ is a given constant. It is required to bring the system to the state of rest in a finite time.

The reasoning below can be modified to the rather simple case $N = 1$.

We say that a system can be *brought to rest* if for all initial values φ_0, φ_1 it is possible to find a control $u(t,x)$ and a time $T > 0$ such that $u(t,x)$ and the corresponding solution $\theta(t,x,u)$ to the problem (1.1)–(1.3) identically vanish for any $t > T$.

Similar problems for membranes and plates were studied in [1], where it was shown that vibrations of such mechanical systems can be brought to the state of rest by means of a bounded (in modulus) and volume-distributed control. For a survey on boundary control of distributed systems we refer to [3]. A condition under which the solution to the heat equation with memory cannot be brought to rest in a finite time can be found in [4], where some control problems for mechanical systems close to (1.1) were considered. This condition requires the existence of zeros of some analytic function of complex variable in its domain of holomorphy. Problems similar to (1.1)–(1.3) for integrodifferential equations were studied in many works. For example, Equation (1.1) was derived in [5]. The solvability of the problem and asymptotic behavior of a solution to an abstract equation of such a type was investigated, for example, in [6] and [7]. As proved in [8], the energy of some dissipative system decreases polynomially as the kernel of the integral term of the equation decreases exponentially. The solvability of a problem of the form (1.1)–(1.3) was studied in [9], where it was proved that the solution belongs to some Sobolev space on the half-axis (with respect to t) provided that the kernel $K(t)$ is a series of exponential functions converging to zero as $t \to +\infty$. Formulas for the solution to the problem (1.1)–(1.3) were obtained in [11]. In this case, the kernel $K(t)$ is also represented as a series of decreasing exponential functions. These formulas mean that the solution converges to zero as $t \to +\infty$. The problem of bringing one-dimensional string vibrations to the state of rest was studied in [12]. In this case, the kernel is identically equal to 1 and the control is concentrated on a compact set (part of the string) moving at a constant velocity.

The controllability to rest is closely connected with the problem of bringing a system to zero, i.e., for all initial data φ_0 and φ_1 there exists a control $u(t,x)$ and a time $T > 0$ such that the solution $\theta(t,x,u)$ and its first order t-derivative vanish at $t = T$. For systems with memory the problem of bringing to zero differs from the problem of bringing to rest. In many situations, it is impossible to bring a system to the state of rest. We discuss this question in detail. Let $K(t)$ be a linear combination of two or more decreasing exponential functions. Using the methods of [13], we can show that the problem (1.1)–(1.3) cannot be brought to rest if the control is applied to a part of the domain Ω. This means that there exists an initial condition such that for any control $u(t,x)$ (in the corresponding space) the solution to the problem (1.1)–(1.3) cannot be brought to rest. However, we show below that the problem (1.1)–(1.3) can be brought to rest if the control is distributed over the whole domain Ω and the boundary of Ω is rigidly fixed.

2 Statement of the Problem

Let $A := -\Delta$ be an operator acting on the space $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$, where $\Omega \subset R^s$ ($s \in \mathbf{N}$) is a bounded connected domain with infinitely smooth boundary. Let $\{\psi_n(x)\}_{n=1}^{+\infty}$ and $\{\alpha_n^2\}_{n=1}^{+\infty}$ be the corresponding orthonormal system of eigenfunctions and the corresponding eigenvalues:

$$-\Delta\psi_n(x) = \alpha_n^2\psi_n(x).$$

We denote by $W^2_{2,\gamma}(R_+, A)$ the linear space of functions $f : R_+ = (0, +\infty) \to D(A)$ equipped with the norm

$$\|\theta\|_{W^2_{2,\gamma}(R_+,A)} = \left(\int\limits_0^{+\infty} e^{-2\gamma t}(\|\theta^{(2)}(t)\|^2_{L_2(\Omega)} + \|A\theta(t)\|^2_{L_2(\Omega)})dt \right)^{\frac{1}{2}}, \quad \gamma \geqslant 0.$$

Definition 2.1. A function $\theta(t, x)$ is called a *strong solution* to the problem (1.1)–(1.3) if for some $\gamma \geqslant 0$ it belongs to the space $W^2_{2,\gamma}(R_+, A)$ and satisfies Equation (1.1) almost everywhere (with respect to t) on the positive half-axis R_+ and the initial conditions (1.2).

We introduce the functions of complex variable λ by

$$l_n(\lambda) := \lambda^2 + \alpha_n^2 \lambda \widehat{K}(\lambda),$$

where

$$\widehat{K}(\lambda) = \sum_{k=1}^{N} \frac{c_k}{\gamma_k(\lambda + \gamma_k)}.$$

We formulate two assertions in [11] on representation of the solution to the problem (1.1)–(1.3) in the form of a series.

Theorem 2.1. *Let $u(t, x) \equiv 0$, $t \in R_+$. We assume that $\theta(t, x) \in W^2_{2,\gamma}(R_+, A)$, $\gamma > 0$, is a strong solution to the problem (1.1)–(1.3). Then for any $t \in R_+$*

$$\theta(t, x) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{(\varphi_{1n} + \lambda_n^+ \varphi_{0n}) e^{\lambda_n^+ t} \psi_n(x)}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{(\varphi_{1n} + \lambda_n^- \varphi_{0n}) e^{\lambda_n^- t} \psi_n(x)}{l_n^{(1)}(\lambda_n^-)}$$

$$+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left(\sum_{k=0}^{N-1} \frac{(\varphi_{1n} - q_{k,n}\varphi_{0n}) e^{-q_{k,n} t}}{l_n^{(1)}(-q_{k,n})} \right) \psi_n(x), \tag{2.1}$$

where $-q_{k,n}$ are real zeros of $l_n(\lambda)$ ($q_{0,n} = 0$, $q_{k,n} > 0$, $k = 1, \dots, N-1$), $\lambda_n^\pm$ is a pair of complex conjugate zeros, $l_n^{(1)}$ is the first order derivative of l_n, and the series (2.1) converges in the $L_2(\Omega)$-norm.

Theorem 2.2. *We assume that $u(t, x) \in C([0, T], L_2(\Omega))$ for any $T > 0$ and $\theta(t, x) \in W^2_{2,\gamma}(R_+, A)$ is a strong solution to the problem (1.1)–(1.3) for some $\gamma > 0$, $\varphi_0 = \varphi_1 = 0$. Then for any $t \in R_+$*

$$\theta(t, x) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \omega_n(t, \lambda_n^+) \psi_n(x) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \omega_n(t, \lambda_n^-) \psi_n(x)$$

$$+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left(\sum_{k=0}^{N-1} \omega_n(t, -q_{k,n}) \right) \psi_n(x), \tag{2.2}$$

where

$$\omega_n(t, \lambda) = \frac{\int\limits_0^t u_n(s) e^{\lambda(t-s)} ds}{l_n^{(1)}(\lambda)},$$

$u_n(t)$ are the Fourier coefficients of $u(t, x)$, and the series (2.2) converges in the $L_2(\Omega)$-norm.

Lemma 2.1. *For any natural number* n

$$\frac{1}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{l_n^{(1)}(\lambda_n^-)} + \sum_{k=0}^{N-1} \frac{1}{l_n^{(1)}(-q_{k,n})} = 0.$$

Proof. We consider the solution to the problem (1.1)–(1.3) in the case $\varphi_0 = \varphi_1 = 0$. By Theorem 2.2, this solution has the form (2.2) and $u(t,x)$ is a function satisfying the assumptions of the theorem. Taking the partial t-derivative of $\theta(t,x)$, we get

$$\frac{\partial \theta(t,x)}{\partial t} = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left(\frac{1}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{l_n^{(1)}(\lambda_n^-)} + \sum_{k=0}^{N-1} \frac{1}{l_n^{(1)}(-q_{k,n})} \right) u_n(t)\psi_n(x)$$
$$+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \lambda_n^+ \omega_n(t,\lambda_n^+)\psi_n(x) + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \lambda_n^- \omega_n(t,\lambda_n^-)\psi_n(x)$$
$$+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{N-1} (-q_{k,n})\omega_n(t,-q_{k,n}) \right) \psi_n(x). \tag{2.3}$$

Since $\theta_t(t,x)|_{t=0} = 0$, for any natural number n from (2.3) we find

$$\left(\frac{1}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{l_n^{(1)}(\lambda_n^-)} + \sum_{k=0}^{N-1} \frac{1}{l_n^{(1)}(-q_{k,n})} \right) u_n(0) = 0. \tag{2.4}$$

We can choose $u(t,x)$ such that all its Fourier coefficients $u_n(t)$ are nonzero at $t = 0$. Dividing (2.4) by $u_n(0)$, we arrive at the required assertion. □

We consider the space l_β of number sequences $\{c_n\}_{n=1}^{+\infty}$ such that the series $\sum\limits_{n=1}^{+\infty} |c_n|^2 \alpha_n^{2\beta}$ converges. Then we introduce the space

$$D(A^{\frac{\beta}{2}}) = \left\{ f(x) = \sum_{n=1}^{+\infty} f_n \psi_n(x) : \{f_n\}_{n=1}^{+\infty} \in l_\beta \right\}.$$

3 The Main Results

In this section, we prove the main theorem asserting that the mechanical system under consideration can be brought to rest in a finite rime. This means that the solution and its first order time-derivative can be brought to zero and is left at this state after the control stops.

Theorem 3.1. *We assume that* $\varphi_0 \in D(A^{\beta+1/2})$ *and* $\varphi_1 \in D(A^\beta)$*, where* $\beta > s/2$ *and* $M > 0$ *is a constant. Then, depending on the value of* M*, there exists a control* $u(t,x) \in C([0,T] \times \Omega)$ *and a time* $T > 0$ *such that the solution to the problem* (1.1)–(1.3) *satisfies the equality*

$$\theta(T,x) = \theta_t'(T,x) = 0, \tag{3.1}$$

and the condition

$$|u(t,x)| \leqslant M$$

for any $t \in (0,T]$, $x \in \Omega$*. If we extend* $u(t,x)$ *by zero for* $t > T$*, then the problem* (1.1)–(1.3) *is left at the zero state for* $t > T$*.*

Proof. We assume that $u(t,x)$ is a function satisfying the assumptions of the theorem and T is a given time. Following Theorems 2.1 and 2.2, we represent the solution to the problem (1.1)–(1.3) by formulas (2.1) and (2.2). Hence

$$\theta(t,x) = \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(\varphi_{1n}+\lambda_n^+\varphi_{0n})e^{\lambda_n^+ t}\psi_n(x)}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(\varphi_{1n}+\lambda_n^-\varphi_{0n})e^{\lambda_n^- t}\psi_n(x)}{l_n^{(1)}(\lambda_n^-)}$$
$$+\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\sum_{k=0}^{N-1}\Big(\frac{(\varphi_{1n}-q_{k,n}\varphi_{0n})e^{-q_{k,n}t}}{l_n^{(1)}(-q_{k,n})}\Big)\psi_n(x) + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\int_0^t u_n(s)e^{\lambda_n^+(t-s)}ds}{l_n^{(1)}(\lambda_n^+)}\psi_n(x)$$
$$+\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\int_0^t u_n(s)e^{\lambda_n^-(t-s)}ds}{l_n^{(1)}(\lambda_n^-)}\psi_n(x) + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\sum_{k=0}^{N-1}\left(\frac{\int_0^t u_n(s)e^{-q_{k,n}(t-s)}ds}{l_n^{(1)}(-q_{k,n})}\right)\psi_n(x). \quad (3.2)$$

We formally differentiate the last series with respect to t (the uniform convergence of the series for θ and θ_t will be shown in Section 5):

$$\frac{\partial\theta(t,x)}{\partial t} = \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\lambda_n^+(\varphi_{1n}+\lambda_n^+\varphi_{0n})e^{\lambda_n^+ t}\psi_n(x)}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\lambda_n^-(\varphi_{1n}+\lambda_n^-\varphi_{0n})e^{\lambda_n^- t}\psi_n(x)}{l_n^{(1)}(\lambda_n^-)}$$
$$+\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\sum_{k=1}^{N-1}\Big(\frac{(-q_{k,n})(\varphi_{1n}-q_{k,n}\varphi_{0n})e^{-q_{k,n}t}}{l_n^{(1)}(-q_{k,n})}\Big)\psi_n(x)$$
$$+\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\left(\frac{1}{l_n^{(1)}(\lambda_n^+)}+\frac{1}{l_n^{(1)}(\lambda_n^-)}+\sum_{k=0}^{N-1}\frac{1}{l_n^{(1)}(-q_{k,n})}\right)u_n(t)\psi_n(x)$$
$$+\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\lambda_n^+\int_0^t u_n(s)e^{\lambda_n^+(t-s)}ds}{l_n^{(1)}(\lambda_n^+)}\psi_n(x) + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\lambda_n^-\int_0^t u_n(s)e^{\lambda_n^-(t-s)}ds}{l_n^{(1)}(\lambda_n^-)}\psi_n(x)$$
$$+\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\sum_{k=1}^{N-1}\left(\frac{(-q_{k,n})\int_0^t u_n(s)e^{-q_{k,n}(t-s)}ds}{l_n^{(1)}(-q_{k,n})}\right)\psi_n(x). \quad (3.3)$$

We note that the fourth term in (3.3) vanishes in view of Lemma 2.1. Using (3.1) and (3.2), (3.3), we get

$$-\left(\frac{(\varphi_{1n}+\lambda_n^+\varphi_{0n})e^{\lambda_n^+ T}}{l_n^{(1)}(\lambda_n^+)}+\frac{(\varphi_{1n}+\lambda_n^-\varphi_{0n})e^{\lambda_n^- T}}{l_n^{(1)}(\lambda_n^-)}+\sum_{k=1}^{N-1}\frac{(\varphi_{1n}-q_{k,n}\varphi_{0n})e^{-q_{k,n}T}}{l_n^{(1)}(-q_{k,n})}\right)$$
$$=\frac{\int_0^T u_n(s)e^{\lambda_n^+(T-s)}ds}{l_n^{(1)}(\lambda_n^+)}+\frac{\int_0^T u_n(s)e^{\lambda_n^-(T-s)}ds}{l_n^{(1)}(\lambda_n^-)}+\sum_{k=0}^{N-1}\frac{\int_0^T u_n(s)e^{-q_{k,n}(T-s)}ds}{l_n^{(1)}(-q_{k,n})}, \quad (3.4)$$

$$
-\frac{\lambda_n^+(\varphi_{1n}+\lambda_n^+\varphi_{0n})e^{\lambda_n^+T}}{l_n^{(1)}(\lambda_n^+)}-\frac{\lambda_n^-(\varphi_{1n}+\lambda_n^-\varphi_{0n})e^{\lambda_n^-T}}{l_n^{(1)}(\lambda_n^-)}-\sum_{k=1}^{N-1}\frac{(-q_{k,n})(\varphi_{1n}-q_{k,n}\varphi_{0n})e^{-q_{k,n}T}}{l_n^{(1)}(-q_{k,n})}
$$

$$
=\frac{\lambda_n^+\int\limits_0^T u_n(s)e^{\lambda_n^+(T-s)}ds}{l_n^{(1)}(\lambda_n^+)}+\frac{\lambda_n^-\int\limits_0^T u_n(s)e^{\lambda_n^-(T-s)}ds}{l_n^{(1)}(\lambda_n^-)}+\sum_{k=1}^{N-1}\frac{(-q_{k,n})\int\limits_0^T u_n(s)e^{-q_{k,n}(T-s)}ds}{l_n^{(1)}(-q_{k,n})}, \quad (3.5)
$$

where $n=1,2,\dots$. We set

$$
a_n=-(\varphi_{1n}+\lambda_n^+\varphi_{0n}),\quad \overline{a}_n=-(\varphi_{1n}+\lambda_n^-\varphi_{0n}),\quad b_{k,n}=-(\varphi_{1n}+(-q_{k,n})\varphi_{0n}),
$$

where $k=0,1,2,\dots,N-1$. We equate the values for the same coefficients

$$
\frac{1}{l_n^{(1)}(\lambda_n^+)},\quad \frac{1}{l_n^{(1)}(\lambda_n^-)},\quad \frac{1}{l_n^{(1)}(-q_{k,n})},\quad k=0,1,2,\dots,N-1.
$$

on the left- and right-hand sides of (3.4) and (3.5). Then we obtain the new moment problem

$$
\begin{aligned}
&\int\limits_0^T u_n(s)e^{\lambda_n^+(T-s)}ds=a_ne^{\lambda_n^+T},\\
&\int\limits_0^T u_n(s)e^{\lambda_n^-(T-s)}ds=\overline{a}_ne^{\lambda_n^-T},\\
&\int\limits_0^T u_n(s)e^{-q_{k,n}(T-s)}ds=b_{k,n}e^{-q_{k,n}T},
\end{aligned}\qquad (3.6)
$$

where $k=0,1,2,\dots,N-1$, $n=1,2,\dots$. If the moment problem (3.6) is solvable, then the problem of moments (3.4), (3.5) is obviously solvable. Canceling on both sides of (3.6), we obtain the system

$$
\int\limits_0^T u_n(s)e^{-\lambda_n^+s}ds=a_n,\quad \int\limits_0^T u_n(s)e^{-\lambda_n^-s}ds=\overline{a}_n,\quad \int\limits_0^T u_n(s)e^{q_{k,n}s}ds=b_{k,n}, \qquad (3.7)
$$

where $k=0,1,2,\dots,N-1$, $n=1,2,\dots$. We substitute $-\lambda_n^+=\lambda_n$ and $-\lambda_n^-=\overline{\lambda}_n$ into (3.7) and note that $\operatorname{Re}\lambda_n>0$, and $q_{k,n}>0$, $k=1,2,\dots,N-1$ (cf. [10]). Finally, we obtain the system of $N+2$ moments for every natural number n:

$$
\int\limits_0^T u_n(s)e^{\lambda_ns}ds=a_n,\quad \int\limits_0^T u_n(s)e^{\overline{\lambda}_ns}ds=\overline{a}_n,\quad \int\limits_0^T u_n(s)e^{q_{k,n}s}ds=b_{k,n}, \qquad (3.8)
$$

where $k=0,1,2,\dots,N-1$, $n=1,2,\dots$. The solution of the system (3.8) is looked for in the form

$$
u_n(s)=C_{-2,n}e^{\lambda_ns}+C_{-1,n}e^{\overline{\lambda}_ns}+\sum_{j=0}^{N-1}C_{j,n}e^{q_{j,n}s},\quad n=1,2,\dots, \qquad (3.9)
$$

where $C_{-2,n}$, $C_{-1,n}$, and $C_{k,n}$ are unknown constants. Substituting (3.9) into (3.8), we obtain the system of $N+2$ algebraic equations for every natural number n:

$$\begin{aligned}
&C_{-2,n}\int_0^T e^{2\lambda_n s}ds + C_{-1,n}\int_0^T e^{(\lambda_n+\overline{\lambda}_n)s}ds + \sum_{k=0}^{N-1} C_{k,n}\int_0^T e^{(\lambda_n+q_{k,n})s}ds = a_n,\\
&C_{-2,n}\int_0^T e^{(\overline{\lambda}_n+\lambda_n)s}ds + C_{-1,n}\int_0^T e^{2\overline{\lambda}_n s}ds + \sum_{k=0}^{N-1} C_{k,n}\int_0^T e^{(\overline{\lambda}_n+q_{k,n})s}ds = a_n,\\
&C_{-2,n}\int_0^T e^{(\lambda_n+q_{k,n})s}ds + C_{-1,n}\int_0^T e^{(\overline{\lambda}_n+q_{k,n})s}ds + \sum_{j=0}^{N-1} C_{j,n}\int_0^T e^{(q_{j,n}+q_{k,n})s}ds = b_{k,n},
\end{aligned} \tag{3.10}$$

where $k=0,1,2,\dots,N-1$. We find the determinant Δ_n of the problem (3.10):

$$\begin{vmatrix}
\int_0^T e^{2\lambda_n s}ds & \int_0^T e^{(\lambda_n+\overline{\lambda}_n)s}ds & \int_0^T e^{\lambda_n s}ds & \int_0^T e^{(\lambda_n+q_{1,n})s}ds & \dots & \int_0^T e^{(\lambda_n+q_{N-1,n})s}ds\\
\int_0^T e^{(\overline{\lambda}_n+\lambda_n)s}ds & \int_0^T e^{2\overline{\lambda}_n s}ds & \int_0^T e^{\overline{\lambda}_n s}ds & \int_0^T e^{(\overline{\lambda}_n+q_{1,n})s}ds & \dots & \int_0^T e^{(\overline{\lambda}_n+q_{N-1,n})s}ds\\
\int_0^T e^{\lambda_n s}ds & \int_0^T e^{\overline{\lambda}_n s}ds & T & \int_0^T e^{q_{1,n} s}ds & \dots & \int_0^T e^{q_{N-1,n} s}ds\\
\int_0^T e^{(q_{1,n}+\lambda_n)s}ds & \int_0^T e^{(q_{1,n}+\overline{\lambda}_n)s}ds & \int_0^T e^{q_{1,n} s}ds & \int_0^T e^{2q_{1,n} s}ds & \dots & \int_0^T e^{(q_{1,n}+q_{N-1,n})s}ds\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\int_0^T e^{(q_{N-1,n}+\lambda_n)s}ds & \int_0^T e^{(q_{N-1,n}+\overline{\lambda}_n)s}ds & \int_0^T e^{q_{N-1,n} s}ds & \int_0^T e^{(q_{N-1,n}+q_{1,n})s}ds & \dots & \int_0^T e^{2q_{N-1,n} s}ds
\end{vmatrix}$$

We note that all determinants Δ_n differ from zero for any natural number n since Δ_n is the Gram determinant. Since

$$\int_0^T e^{(q_{i,n}+q_{j,n})s}ds = \frac{1}{q_{i,n}+q_{j,n}}e^{(q_{i,n}+q_{j,n})T} - \frac{1}{q_{i,n}+q_{j,n}}, \tag{3.11}$$

from (3.11) and the well-known property of determinants

$$\begin{vmatrix}
a_{11} & a_{12} & \dots & a_{1n}\\
a_{21} & a_{22} & \dots & a_{2n}\\
\vdots & \vdots & & \vdots\\
b_{i1}+c_{i1} & b_{i2}+c_{i2} & \dots & b_{in}+c_{in}\\
\vdots & \vdots & \ddots & \vdots\\
a_{n1} & a_{n2} & \dots & a_{nn}
\end{vmatrix} = \begin{vmatrix}
a_{11} & a_{12} & \dots & a_{1n}\\
a_{21} & a_{22} & \dots & a_{2n}\\
\vdots & \vdots & & \vdots\\
b_{i1} & b_{i2} & \dots & b_{in}\\
\vdots & \vdots & \ddots & \vdots\\
a_{n1} & a_{n2} & \dots & a_{nn}
\end{vmatrix} + \begin{vmatrix}
a_{11} & a_{12} & \dots & a_{1n}\\
a_{21} & a_{22} & \dots & a_{2n}\\
\vdots & \vdots & & \vdots\\
c_{i1} & c_{i2} & \dots & c_{in}\\
\vdots & \vdots & \ddots & \vdots\\
a_{n1} & a_{n2} & \dots & a_{nn}
\end{vmatrix}, \tag{3.12}$$

we find that Δ_n is equal to the following:

$$\begin{vmatrix} \frac{e^{2\lambda_n T}}{2\lambda_n} & \frac{e^{(\lambda_n+\overline{\lambda}_n)T}}{\lambda_n+\overline{\lambda}_n} & \frac{e^{\lambda_n T}}{\lambda_n} & \frac{e^{(\lambda_n+q_{1,n})T}}{\lambda_n+q_{1,n}} & \cdots & \frac{e^{(\lambda_n+q_{N-1,n})T}}{\lambda_n+q_{N-1,n}} \\ \frac{e^{(\overline{\lambda}_n+\lambda_n)T}}{\overline{\lambda}_n+\lambda_n} & \frac{e^{2\overline{\lambda}_n T}}{2\overline{\lambda}_n} & \frac{e^{\overline{\lambda}_n T}}{\overline{\lambda}_n} & \frac{e^{(\overline{\lambda}_n+q_{1,n})T}}{\overline{\lambda}_n+q_{1,n}} & \cdots & \frac{e^{(\overline{\lambda}_n+q_{N-1,n})T}}{\overline{\lambda}_n+q_{N-1,n}} \\ \frac{e^{\lambda_n T}}{\lambda_n} & \frac{e^{\overline{\lambda}_n T}}{\overline{\lambda}_n} & T & \frac{e^{q_{1,n}T}}{q_{1,n}} & \cdots & \frac{e^{q_{N-1,n}T}}{q_{N-1,n}} \\ \frac{e^{(q_{1,n}+\lambda_n)T}}{q_{1,n}+\lambda_n} & \frac{e^{(q_{1,n}+\overline{\lambda}_n)T}}{q_{1,n}+\overline{\lambda}_n} & \frac{e^{q_{1,n}T}}{q_{1,n}} & \frac{e^{2q_{1,n}T}}{2q_{1,n}} & \cdots & \frac{e^{(q_{1,n}+q_{N-1,n})T}}{q_{1,n}+q_{N-1,n}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{e^{(q_{N-1,n}+\lambda_n)T}}{q_{N-1,n}+\lambda_n} & \frac{e^{(q_{N-1,n}+\overline{\lambda}_n)T}}{q_{N-1,n}+\overline{\lambda}_n} & \frac{e^{q_{N-1,n}T}}{q_{N-1,n}} & \frac{e^{(q_{N-1,n}+q_{1,n})T}}{q_{N-1,n}+q_{1,n}} & \cdots & \frac{e^{2q_{N-1,n}T}}{2q_{N-1,n}} \end{vmatrix} + \beta_n(T), \tag{3.13}$$

where $\beta_n(T)$ is the sum of all remaining determinants, which is a result of $N+2$ times application of the property (3.12) to each row of the determinant Δ_n. We extract $e^{\lambda_n T}$ from the first row of the determinant on the right-hand side of (3.13) and from the first column. Then we repeat this action for the second row and second column with $e^{\overline{\lambda}_n T}$, and so on. As a result, we get

$$\Delta_n = e^{2\lambda_n T} e^{2\overline{\lambda}_n T} \prod_{j=1}^{N-1} e^{2q_{j,n}T} \overline{\Delta}_n + \beta_n(T), \tag{3.14}$$

where

$$\overline{\Delta}_n = \begin{vmatrix} \frac{1}{2\lambda_n} & \frac{1}{\lambda_n+\overline{\lambda}_n} & \frac{1}{\lambda_n} & \frac{1}{\lambda_n+q_{1,n}} & \cdots & \frac{1}{\lambda_n+q_{N-1,n}} \\ \frac{1}{\overline{\lambda}_n+\lambda_n} & \frac{1}{2\overline{\lambda}_n} & \frac{1}{\overline{\lambda}_n} & \frac{1}{\overline{\lambda}_n+q_{1,n}} & \cdots & \frac{1}{\overline{\lambda}_n+q_{N-1,n}} \\ \frac{1}{\lambda_n} & \frac{1}{\overline{\lambda}_n} & T & \frac{1}{q_{1,n}} & \cdots & \frac{1}{q_{N-1,n}} \\ \frac{1}{q_{1,n}+\lambda_n} & \frac{1}{q_{1,n}+\overline{\lambda}_n} & \frac{1}{q_{1,n}} & \frac{1}{2q_{1,n}} & \cdots & \frac{1}{q_{1,n}+q_{N-1,n}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{q_{N-1,n}+\lambda_n} & \frac{1}{q_{N-1,n}+\overline{\lambda}_n} & \frac{1}{q_{N-1,n}} & \frac{1}{q_{N-1,n}+q_{1,n}} & \cdots & \frac{1}{2q_{N-1,n}} \end{vmatrix}.$$

Then

$$\Delta_n = e^{2\lambda_n T} e^{2\overline{\lambda}_n T} \prod_{j=1}^{N-1} e^{2q_{j,n}T} \left(\overline{\Delta}_n + e^{-2\lambda_n T} e^{-2\overline{\lambda}_n T} \prod_{j=1}^{N-1} e^{-2q_{j,n}T} \beta_n(T) \right).$$

We note that the sequence $\{|\lambda_n|\}$ converges to $+\infty$ as $n \to +\infty$, but $\operatorname{Re}\lambda_n = \mu + O(n^{-2})$ $(\mu > 0)$ and the sequence of real numbers $\{q_{k,n}\}_{n=1}^{\infty}$ converges to some positive number q_k; more exactly, $q_{k,n} = q_k + O(n^{-2})$ (cf. [10]). By the definition of $\beta_n(T)$,

$$\left| e^{-2\lambda_n T} e^{-2\overline{\lambda}_n T} \prod_{j=1}^{N-1} e^{-2q_{j,n}T} \beta_n(T) \right| \to 0, \quad T \to +\infty.$$

We represent $\overline{\Delta}_n$ in the form

$$\overline{\Delta}_n = \frac{1}{(2\operatorname{Re}\lambda_n)^2} \begin{vmatrix} T & \dfrac{1}{q_{1,n}} & \dots & \dfrac{1}{q_{N-1,n}} \\ \dfrac{1}{q_{1,n}} & \dfrac{1}{2q_{1,n}} & \dots & \dfrac{1}{q_{1,n}+q_{N-1,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{q_{N-1,n}} & \dfrac{1}{q_{N-1,n}+q_{1,n}} & \dots & \dfrac{1}{2q_{N-1,n}} \end{vmatrix} + \Lambda_n(T), \tag{3.15}$$

where $\Lambda_n(T) \to 0$, $n \to +\infty$ of T is fixed. We set

$$P_n = \begin{vmatrix} \frac{1}{2q_{1,n}} & \dfrac{1}{q_{1,n}+q_{2,n}} & \dots & \dfrac{1}{q_{1,n}+q_{N-1,n}} \\ \dfrac{1}{q_{2,n}+q_{1,n}} & \dfrac{1}{2q_{2,n}} & \dots & \dfrac{1}{q_{2,n}+q_{N-1,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{q_{N-1,n}+q_{1,n}} & \dfrac{1}{q_{N-1,n}+q_{2,n}} & \dots & \dfrac{1}{2q_{N-1,n}} \end{vmatrix},$$

where P_n is the Cauchy determinant. As known,

$$P_n = \frac{\prod\limits_{N-1 \geqslant i > j \geqslant 1} (q_{i,n} - q_{j,n})^2}{\prod\limits_{i,j=1}^{N-1} (q_{i,n} + q_{j,n})}.$$

Since $q_{i,n}$, $i = 1, 2, \dots, N-1$, are pairwise disjoint for any n (cf. [10]), we conclude that P_n are nonzero. It is obvious that

$$\begin{vmatrix} T & \dfrac{1}{q_{1,n}} & \dots & \dfrac{1}{q_{N-1,n}} \\ \dfrac{1}{q_{1,n}} & \dfrac{1}{2q_{1,n}} & \dots & \dfrac{1}{q_{1,n}+q_{N-1,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{1}{q_{N-1,n}} & \dfrac{1}{q_{N-1,n}+q_{1,n}} & \dots & \dfrac{1}{2q_{N-1,n}} \end{vmatrix} = T(P_n + \xi_n(T)),$$

where $\xi_n(T) \to 0$, $T \to +\infty$. Consequently,

$$\Delta_n = e^{2\lambda_n T} e^{2\overline{\lambda}_n T} \prod_{j=1}^{N-1} e^{2q_{j,n}T}\Big(\frac{1}{(2\,\mathrm{Re}\ \lambda_n)^2} T(P_n + \xi_n(T))$$

$$+ \Lambda_n(T) + e^{-2\lambda_n T} e^{-2\overline{\lambda}_n T} \prod_{j=1}^{N-1} e^{-2q_{j,n}T}\beta_n(T)\Big)$$

$$= \frac{TP_n}{(2\,\mathrm{Re}\ \lambda_n)^2} e^{2(\lambda_n+\overline{\lambda}_n)T} \prod_{j=1}^{N-1} e^{2q_{j,n}T}\Big(1 + \frac{\xi_n(T)}{P_n} + \frac{(2\,\mathrm{Re}\ \lambda_n)^2}{TP_n}\Lambda_n(T)$$

$$+ \frac{(2\,\mathrm{Re}\ \lambda_n)^2}{TP_n} e^{-2(\lambda_n+\overline{\lambda}_n)T} \prod_{j=1}^{N-1} e^{-2q_{j,n}T}\beta_n(T)\Big).$$

We set

$$\overline{\xi}_n(T) = \frac{\xi_n(T)}{P_n},$$

$$\overline{\Lambda}_n(T) = \frac{(2\,\mathrm{Re}\ \lambda_n)^2}{TP_n}\Lambda_n(T),$$

$$\overline{\beta}_n(T) = \frac{(2\,\mathrm{Re}\ \lambda_n)^2}{TP_n} e^{-2(\lambda_n+\overline{\lambda}_n)T} \prod_{j=1}^{N-1} e^{-2q_{j,n}T}\beta_n(T).$$

Then

$$\Delta_n = \frac{TP_n}{(2\,\mathrm{Re}\ \lambda_n)^2} e^{2\lambda_n T} e^{2\overline{\lambda}_n T} \prod_{j=1}^{N-1} e^{2q_{j,n}T}(1 + \overline{\xi}_n(T) + \overline{\Lambda}_n(T) + \overline{\beta}_n(T)). \tag{3.16}$$

We note that $\overline{\Lambda}_n(T) \to 0$ as $n \to +\infty$ (uniformly with respect to $T \in [T_*, +\infty)$ for any $T_* > 0$) and $\overline{\xi}_n(T)$, $\overline{\beta}_n(T) \to 0$ as $T \to +\infty$; more exactly, for any $\varepsilon > 0$ there exist $T > 0$ and n_* such that $|\overline{\xi}_n(T)| < \varepsilon$ and $|\overline{\beta}_n(T)| < \varepsilon$ for any $n > n_*$. Let $\Delta_{-2,n}$ be equal to the following:

$$\begin{vmatrix} a_n & \int_0^T e^{(\lambda_n+\overline{\lambda}_n)s}ds & \int_0^T e^{\lambda_n s}ds & \int_0^T e^{(\lambda_n+q_{1,n})s}ds & \dots & \int_0^T e^{(\lambda_n+q_{N-1,n})s}ds \\ \overline{a}_n & \int_0^T e^{2\overline{\lambda}_n s}ds & \int_0^T e^{\overline{\lambda}_n s}ds & \int_0^T e^{(\overline{\lambda}_n+q_{1,n})s}ds & \dots & \int_0^T e^{(\overline{\lambda}_n+q_{N-1,n})s}ds \\ b_{0,n} & \int_0^T e^{\overline{\lambda}_n s}ds & T & \int_0^T e^{q_{1,n}s}ds & \dots & \int_0^T e^{q_{N-1,n}s}ds \\ b_{1,n} & \int_0^T e^{(q_{1,n}+\overline{\lambda}_n)s}ds & \int_0^T e^{q_{1,n}s}ds & \int_0^T e^{2q_{1,n}s}ds & \dots & \int_0^T e^{(q_{1,n}+q_{N-1,n})s}ds \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{N-1,n} & \int_0^T e^{(q_{N-1,n}+\overline{\lambda}_n)s}ds & \int_0^T e^{q_{N-1,n}s}ds & \int_0^T e^{(q_{N-1,n}+q_{1,n})s}ds & \dots & \int_0^T e^{2q_{N-1,n}s}ds \end{vmatrix}.$$

We assume that there exists $\Delta_{k,n}$, where $k = -1, 0, 1, 2, \ldots, N-1$, such that

$$\Delta_{k,n} = \begin{vmatrix} \int\limits_0^T e^{2\lambda_n s} ds & \ldots & a_n & \ldots & \int\limits_0^T e^{(\lambda_n + q_{N-1,n})s} ds \\ \int\limits_0^T e^{(\overline{\lambda}_n + \lambda_n)s} ds & \ldots & \overline{a}_n & \ldots & \int\limits_0^T e^{(\overline{\lambda}_n + q_{N-1,n})s} ds \\ \int\limits_0^T e^{(q_{0,n} + \lambda_n)s} ds & \ldots & b_{0,n} & \ldots & \int\limits_0^T e^{(q_{0,n} + q_{N-1,n})s} ds \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \int\limits_0^T e^{(q_{N-1,n} + \lambda_n)s} ds & \ldots & b_{N-1,n} & \ldots & \int\limits_0^T e^{2q_{N-1,n}s} ds \end{vmatrix},$$

where $\{a_n, \overline{a}_n, b_{0,n}, b_{1,n}, \ldots, b_{N-1,n}\}$ is the kth column. Applying the Cramer rule, we get

$$C_{-2,n} = \frac{\Delta_{-2,n}}{\Delta_n}, \quad C_{-1,n} = \frac{\Delta_{-1,n}}{\Delta_n}, \quad C_{k,n} = \frac{\Delta_{k,n}}{\Delta_n}, \ k = 0, 1, 2, \ldots, N-1.$$

Then the solution (3.8) at time t has the form

$$u_n(t) = \frac{\Delta_{-2,n}}{\Delta_n} e^{\lambda_n t} + \frac{\Delta_{-1,n}}{\Delta_n} e^{\overline{\lambda}_n t} + \sum_{k=0}^{N-1} \frac{\Delta_{k,n}}{\Delta_n} e^{q_{k,n} t}.$$

Let $\lambda_n = \mu_n - i\nu_n$. As proved in [10], $\mu_n, \nu_n > 0$ for any natural number n. We write the estimate for the absolute value of $u_n(t)$ with any natural number n

$$|u_n(t)| \leqslant \frac{|\Delta_{-2,n}|}{|\Delta_n|} e^{\mu_n T} + \frac{|\Delta_{-1,n}|}{|\Delta_n|} e^{\mu_n T} + \sum_{k=0}^{N-1} \frac{|\Delta_{k,n}|}{|\Delta_n|} e^{q_{k,n} T}. \tag{3.17}$$

Calculating the determinants $\Delta_{-2,n}$, $\Delta_{-1,n}$, $\Delta_{k,n}$, $k = 0, 1, 2, \ldots, N-1$, we see that some terms are the products of different exponential functions. We note that the product with the largest number of exponential factors in $\Delta_{-2,n}$ has the form

$$e^{\lambda_n T} e^{2\overline{\lambda}_n T} e^{2q_{1,n} T} e^{2q_{2,n} T} \ldots e^{2q_{N-1,n} T}.$$

For $\Delta_{-1,n}$ we have similar formulas. For $\Delta_{k,n}$ ($k \neq 0$) we find

$$e^{2\lambda_n T} e^{2\overline{\lambda}_n T} e^{2q_{1,n} T} e^{2q_{2,n} T} \ldots e^{q_{k,n} T} \ldots e^{2q_{N-1,n} T}.$$

In $\Delta_{0,n}$, we have

$$e^{2\lambda_n T} e^{2\overline{\lambda}_n T} e^{2q_{1,n} T} e^{2q_{2,n} T} \ldots e^{2q_{N-1,n} T},$$

which means that u_n decreases like T^{-1} as $T \to +\infty$. Consequently, it is possible to make the modulus of $u_n(t)$ (consequently, the control $u(t)$ itself) sufficiently small by increasing the control time. Using (3.16) and (3.17), we get

$$|u_n(t)| \leqslant \frac{4\mu_n^2}{T|P_n|e^{4\mu_n T}\prod\limits_{j=1}^{N-1} e^{2q_{j,n}T}(1-|\overline{\xi}_n(T)|-|\overline{\Lambda}_n(T)|-|\overline{\beta}_n(T)|)} e^{\mu_n T}(|\Delta_{-2,n}|+|\Delta_{-1,n}|)$$
$$+\sum_{k=0}^{N-1} \frac{4\mu_n^2|\Delta_{k,n}|}{T|P_n|e^{4\mu_n T}\prod\limits_{j=1}^{N-1} e^{2q_{j,n}T}\big(1-|\overline{\xi}_n(T)|-|\overline{\Lambda}_n(T)|-|\overline{\beta}_n(T)|\big)} e^{q_{k,n}T}, \quad t\in[0,T]. \tag{3.18}$$

Strictly speaking, the estimate (3.18) is obtained for n larger than some n_*. It is obvious that the first terms of the series are estimated from above in modulus by the constant c_*/T.

Using (3.17) and (3.18), we prove that there exists a time required for stabilization of the system such that the control $u(t,x)$ satisfies the conditions

$$|u(t,x)| \leqslant M, \tag{3.19}$$

where M is an arbitrary constant.

Since the sequences $\{\mu_n\}$, $\{\nu_n\}$, $\{q_{k,n}\}$ such that $\mu_n = \mu + O(n^{-2})$, $\nu_n = D\alpha_n$, $q_{k,n} = q_k + O(n^{-2})$, where μ, D, q_k are positive numbers (cf. [10]), and the sequences $\{|a_n|\}$, $\{|b_{k,n}|\}$, $\{|\Lambda_n|\}$ converge to zero, we obtain the estimate

$$|u(t,x)| \leqslant \frac{c}{T}\sqrt{\sum_{n=1}^{\infty}\alpha_n^{2\beta}\left(|a_n|^2+|\overline{a}_n|^2+\sum_{k=0}^{N-1}|b_{k,n}|^2\right)}\sqrt{\sum_{n=1}^{\infty}\alpha_n^{-2\beta}\psi_n^2(x)}, \tag{3.20}$$

where c is a constant and T is sufficiently large. It is known (cf. [14]) that

$$\sum_{n=1}^{\infty}\alpha_n^{-2\beta}\psi_n^2(x) \leqslant \text{const}\,, \quad 2\beta > s.$$

Furthermore, the series $\sum\limits_{n=1}^{\infty}\alpha_n^{-2\beta}\psi_n^2(x)$ is a continuous function (if $2\beta > s$) and

$$\sum_{n=1}^{\infty}\alpha_n^{2\beta+2}\varphi_{0n}^2 = \int_{\Omega}\left(A^{\frac{\beta+1}{2}}\varphi_0(x)\right)^2 dx,$$

$$\sum_{n=1}^{\infty}\alpha_n^{2\beta}\varphi_{1n}^2 = \int_{\Omega}\left(A^{\frac{\beta}{2}}\varphi_1(x)\right)^2 dx.$$

The last series converges if $A^{(\beta+1)/2}\varphi_0(x) \in L_2(\Omega)$ and $A^{\beta/2}\varphi_1(x) \in L_2(\Omega)$. Then $\varphi_0(x) \in D(A^{(\beta+1)/2})$ and $\varphi_1(x) \in D(A^{\beta/2})$. However, these conditions in the formulation of the theorem are imposed on the initial data since $D(A^{\beta+1/2}) \subset D(A^{(\beta+1)/2})$ and $D(A^{\beta}) \subset D(A^{\beta/2})$. Then

$$|u(t,x)| \leqslant \frac{C_1}{T} \leqslant M, \tag{3.21}$$

where C_1 is a constant and T is sufficiently large. □

4 Proof of Continuity of Control

We prove that $u(t,x) \in C([0,T]\times\Omega)$. We have

$$|u(t,x)| \leqslant \sum_{n=1}^{\infty} |u_n(t)||\psi_n(x)| \leqslant \frac{1}{2}\Bigg(\sum_{n=1}^{\infty} \alpha_n^{2\beta}|u_n(t)|^2 + \sum_{n=1}^{\infty} \alpha_n^{-2\beta}\psi_n^2(x)\Bigg). \tag{4.1}$$

We note that the series $\sum\limits_{n=1}^{\infty} \alpha_n^{-2\beta}\psi_n^2(x)$ converges uniformly by the Dini theorem. Using the Weierstrass criterion for uniform convergence, we obtain the required assertion.

5 Existence of Solutions and Convergence of Series

Now, we show that $\theta(t,x)$ can be understood as a solution to the problem (1.1)–(1.3) for $t>0$. For this purpose we use the results of [10]. Following [10], for the existence and uniqueness of a solution to the problem (1.1)–(1.3) for $t \in [0,+\infty)$ the following smoothness conditions should be satisfied: $\varphi_0 \in D(A)$, $\varphi_1 \in D(A^{\frac{1}{2}})$, and $A^{\frac{1}{2}}u(t,x) \in L_2(R_+,H)$. In this case, the solution $\theta(t,x)$ belongs to the space $W^2_{2,\gamma}(R_+,A)$ for any $\gamma>0$.

The initial data satisfy the smoothness conditions by the assumptions of Theorem 3.1. We need to show that the right-hand side is sufficiently smooth. We have

$$\int\limits_{\Omega} |A^{\frac{1}{2}}u(t,x)|^2 dx = \sum_{n=1}^{\infty} \alpha_n^2 |u_n(t)|^2.$$

Using estimates similar to (3.20) for $u_n^2(t)$, we get

$$\int\limits_{\Omega} |A^{\frac{1}{2}}u(t,x)|^2 dx \leqslant C_2, \quad t\in[0,T],$$

because

$$\sum_{n=1}^{\infty} \alpha_n^{4\beta+2}|\varphi_{0n}|^2 = \int\limits_{\Omega} |A^{\beta+\frac{1}{2}}\varphi_0(x)|^2 dx,$$

$$\sum_{n=1}^{\infty} \alpha_n^{4\beta}|\varphi_{1n}|^2 = \int\limits_{\Omega} |A^{\beta}\varphi_1(x)|^2 dx,$$

and $4\beta > 2s \geqslant 2$. Since $u(t,x)$ is equal to zero for $t>T$, we have $A^{1/2}u(t,x) \in L_2(R_+,H)$.

Now, we establish the uniform convergence of the series for θ_t with respect to $t \in [0,T]$ in the $L_2(\Omega)$-norm. For θ the proof is similar and even easier. We have

$$l_n(\lambda) = \lambda^2 + \alpha_n^2 \sum_{k=1}^{N} \frac{c_k}{\gamma_k} - \alpha_n^2 \sum_{k=1}^{N} \frac{c_k}{\gamma_k(\lambda+\gamma_k)}.$$

Then the first order derivative has the form

$$l_n^{(1)}(\lambda) = 2\lambda + \alpha_n^2 \sum_{k=1}^{N} \frac{c_k}{\gamma_k(\lambda+\gamma_k)^2}.$$

We consider the series (3.3), $t \in [0,T]$. Using the Parseval identity, we find

$$\sum_{n=1}^{\infty}\Big|\frac{\lambda_n^+(\varphi_{1n}+\lambda_n^+\varphi_{0n})e^{\lambda_n^+ t}}{l_n^{(1)}(\lambda_n^+)}\Big|^2 \leqslant C_3\sum_{n=1}^{\infty}\frac{\alpha_n^2(|\varphi_{1n}|^2+\alpha_n^2|\varphi_{0n}|^2)}{\alpha_n^2} = C_3\sum_{n=1}^{\infty}(|\varphi_{1n}|^2+\alpha_n^2|\varphi_{0n}|^2).$$

The last number sequence converges due to the choice of the space of initial data. The same argument is valid for $\lambda = \lambda_n^-$. Further,

$$\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{N-1}\frac{(-q_{k,n})(\varphi_{1n}-q_{k,n}\varphi_{0n})e^{-q_{k,n}t}}{l_n^{(1)}(-q_{k,n})}\Big|^2$$

$$\leqslant C_4\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N-1}\Big|\frac{(-q_{k,n})(\varphi_{1n}-q_{k,n}\varphi_{0n})e^{-q_{k,n}t}}{l_n^{(1)}(-q_{k,n})}\Big|\right)^2 \leqslant C_5\sum_{n=1}^{\infty}(|\varphi_{1n}|^2+|\varphi_{0n}|^2).$$

It is obvious that the last series converges. We have

$$\sum_{n=1}^{\infty}\Big|\frac{\lambda_n^+\int\limits_0^t u_n(s)e^{\lambda_n^+(t-s)}ds}{l_n^{(1)}(\lambda_n^+)}\Big|^2 \leqslant C_6\sum_{n=1}^{\infty}\frac{\alpha_n^2\int\limits_0^t|u_n(s)|^2ds\int\limits_0^t|e^{\lambda_n^+(t-s)}|^2ds}{\alpha_n^2} \leqslant C_7\sum_{n=1}^{\infty}\int\limits_0^T|u_n(s)|^2ds.$$

The last number series converges since $u(t,x) \in C([0,T]\times\Omega)$ by the above. For $\lambda = \lambda_n^-$ we can argue in a similar way. We have

$$\sum_{n=1}^{\infty}\Big|\sum_{k=1}^{N-1}\frac{(-q_{k,n})\int\limits_0^t u_n(s)e^{-q_{k,n}(t-s)}ds}{l_n^{(1)}(-q_{k,n})}\Big|^2 \leqslant \sum_{n=1}^{\infty}\left(\sum_{k=1}^{N-1}\Big|\frac{(-q_{k,n})\int\limits_0^t u_n(s)e^{-q_{k,n}(t-s)}ds}{l_n^{(1)}(-q_{k,n})}\Big|\right)^2$$

$$\leqslant C_8\sum_{n=1}^{\infty}\frac{1}{\alpha_n^4}\int\limits_0^t|u_n(s)|^2ds \leqslant C_8\sum_{n=1}^{\infty}\int\limits_0^T|u_n(s)|^2ds.$$

Thus, the uniform convergence of the series for θ_t with respect to $t \in [0,T]$ in the $L_2(\Omega)$-norm is proved.

6 Bringing to Rest

We show that the control constructed in the proof of the theorem brings the system to rest. For this purpose we use formula (3.2) and the integral equation (3.7). The function $u(t,x)$ can be extended by zero for $t > T$. Then the system (3.7) can be written (for $t > T$) in the form

$$\int\limits_0^t u_n(s)e^{-\lambda_n^+ s}ds = a_n, \quad \int\limits_0^t u_n(s)e^{-\lambda_n^- s}ds = \overline{a}_n, \quad n = 1,2,\dots,$$

$$\int\limits_0^t u_n(s)e^{q_{k,n}s}ds = b_{k,n}, \quad k = 0,1,2,\dots,N-1, \quad n = 1,2,\dots. \tag{6.1}$$

This means that the values of the Laplace transform of $u_n(t)$ at zeros of the functions $l_n(\lambda)$ are equal to given numbers. Further, taking $t > T$ in (3.2) and using (6.1), we see that $\theta(t, x) \equiv 0$ for any $t > T$.

7 Estimate for Time Required to Bring to Rest

Let the control $u(t, x)$ identically vanish. If the initial data φ_1 in the problem (1.1)–(1.3) is equal to zero, then from (2.1) it follows that the solution θ of Equation (1.1) converges exponentially to zero as $t \to +\infty$. In a sense, this fact helps the control process, and one can show that the upper estimate for the time T_* spent on control (if $\varphi_1 = 0$) has order $\ln \varepsilon^{-1}$ provided that the absolute value of $u(t, x)$ is bounded by a parameter ε close to zero, i.e., $|u(t, x)| \leqslant \varepsilon$. In the case $\varphi_1 \neq 0$, the order of T_* is ε^{-1}, which can be easily seen from the estimate (3.21).

We note that, in the case $\varphi_1 \neq 0$ and $u(t, x) \equiv 0$, the solution $\theta(t, x)$ does not converge (as $t \to +\infty$) to zero, but to the following function (the limit state of the system):

$$\frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\varphi_{1n}}{l_n^{(1)}(-q_{0,n})} \psi_n(x).$$

Acknowledgments

The work is supported by the Russian Science Foundation (project No. 21-11-00151).

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Submitted on March 15, 2022