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CONTROL IN SYSTEMS
WITH DISTRIBUTED PARAMETERS

Investigation of Controllability for Some Dynamic Systems with Distributed Parameters Described by Integrodifferential Equations

I. V. Romanov[a,b]

[a] *National Research University Higher School of Economics, Moscow, 101000 Russia*
[b] *Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, 119526 Russia*
e-mail: romm1@list.ru

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Abstract—The problem of distributed controllability for the Gurtin–Pipkin equation with a kernel represented by some series of decreasing exponential functions is considered, while certain conditions are imposed on the coefficients and exponents. It is proved that this system cannot be brought to rest even if the control action is applied to the entire region.

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INTRODUCTION

This study is devoted to the problems of the distributed control of oscillations of a system described by the Gurtin–Pipkin equation. This equation contains a convolutional (variable in time) term, often referred to as memory. For the first time this equation appears in the article [1]. The question is raised about the possibility of bringing such systems to a state of rest. Note that, generally speaking, this concept for systems with memory is not equivalent to bringing the system to the zero state. As will be made clear below, controllability at rest is not always possible for such models, even if the control action is applied to the entire area occupied by the mechanical system. In the course of the proof, we will not observe due rigor in the part concerning the solvability of the initial boundary value problems, but will pay more attention to the qualitative side of the controllability issue.

We consider an important class of kernels that has the form of a series of a countably series of decreasing exponential functions. We will also mention the abelian-type kernel (a kernel with a singularity). These kernels are used in various models of mechanics, in particular, to describe some oscillatory processes.

1. THE PROBLEM OF IRREDUCIBILITY TO A STATE OF REST OF A SYSTEM DESCRIBED BY THE GURTIN–PIPKIN EQUATION AND THE KERNEL IN THE FORM OF A SERIES OF DECREASING EXPONENTIAL FUNCTIONS

We consider the initial boundary value problem

$$\theta_t(t,x) = \int_0^t K_1(t-s)\Delta\theta(s,x)ds + u(t,x), \quad x \in \Omega, \quad t > 0. \tag{1.1}$$

$$\theta|_{t=0} = \varphi(x), \tag{1.2}$$

$$\theta(t,x) = 0, \quad x \in \partial\Omega. \tag{1.3}$$

Hereinafter,

$$K_1(t) = \sum_{j=1}^{+\infty} \frac{c_j}{\gamma_j} e^{-\gamma_j t},$$

where c_j and γ_j are given positive constants such that

$$0 < \gamma_1 < \gamma_2 < \ldots < \gamma_j < \ldots, \quad \gamma_j \to +\infty, \quad j \to +\infty.$$

In this case Ω is a limited singly connected area in R^n with an infinitely smooth border and Δ is the Laplace operator with domain

$$D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega),$$

and $u(t, x)$ is the control function. The existence and uniqueness of a solution to problem (1.1)–(1.3) under the additional conditions imposed on the kernel $K_1(t)$ and the right-hand side of u were proved in [2].

For equations similar to (1.1) and in a number of particular cases (for example, [3–5]), it is possible to prove that the oscillations of the system can be completely stopped in a finite time if the control is applied to the entire domain Ω. In [6], it is proved that a mechanical system described by the Gurtin–Pipkin equation for two-dimensional domains and a wide class of continuous kernels cannot be brought to rest by the control being applied only to a subdomain whose closure is contained in Ω. Note that $K_1(t)$ is an example of such a kernel.

Now let the kernel have the form

$$K_2(t) = \frac{1}{\Gamma(1-\sigma)} t^{-\sigma}, \quad \sigma \in (0,1), \tag{1.4}$$

where $\Gamma(\lambda)$ is the gamma function. This is the Abel kernel. In [7], for a one-dimensional heat equation with integral memory and kernel (1.4), it is proved that the given system is uncontrollable at rest if the control is applied to one end of the segment and the other end is fixed. Note that controllability at rest in this case is unattainable even if we control the entire region [8].

Hereinafter, for the kernel $K_1(t)$, we require the following condition to be satisfied:

$$\sum_{j=1}^{+\infty} \frac{c_j}{\gamma_j} < +\infty. \tag{1.5}$$

Equation (1.1) is the Gurtin–Pipkin integrodifferential equation that describes the process of heat propagation in media with memory and the process of sound propagation in viscoelastic media; it also arises in averaging problems in perforated media (Darcy's law). Note that in some models the derivative $K_1'(t)$ from the kernel has a singularity at $t = 0$, i.e.,

$$\sum_{j=1}^{+\infty} c_j = +\infty. \tag{1.6}$$

A detailed description of models for conditions (1.5) and (1.6), as well as physical laws, can be found in the article [2].

We will consider the system to be manageable at rest, if for all the initial conditions φ there is a control $u(t, x)$ and point in time $T > 0$ such that $u(t, x)$ is identically zero for $t > T$ and the corresponding solution $\theta(t, x, u)$ of problem (1.1)–(1.3) is also identically zero for $t > T$. Note that for systems with memory, the concepts of controllability in the state of rest and controllability in the zero state are not identical. In many cases, the solution, having reached the zero value at some point in time can then exit from this value.

For equations of form (1.1) and various types of kernels, we can pose, for example, the following control problems: bring to rest a moving compact or for the entire domain Ω by controlling a fixed subdomain. It is also possible to pose a problem of boundary control, as is done, for example, in [6, 7]. This article will show that for the kernel $K_1(t)$ there is no controllability at rest for system (1.1)–(1.3) even if the control is performed over the entire domain. Questions remain about the possibility of bring some other types of kernels to a state of rest. For similar equations, it is sometimes possible to achieve controllability at rest if the control is applied to a compact moving according to a certain law (there is no controllability beyond a fixed subdomain). For example, the problem of controllability at rest for the one-dimensional equation of the oscillation of a string with memory was considered in [9]. In this case, the kernel in the integral term of the equation is identically equal to unity, and the control is concentrated on a subsegment (part of the string) that moves at a constant speed.

Let us show that in the control problem (1.1)–(1.3) for the kernel $K_1(t)$ it is (generally speaking) impossible to bring the system to rest. We consider the control

$$u(t) \in L_2((0,T);\, L_2(\Omega)),$$

continued by zero at $t > T$. Next, we prove that there exists such an initial condition $\varphi(x)$ at which the motion of the system cannot be stopped. More precisely, there is an initial condition $\varphi(x)$, such that for each control function $u(t,x)$, which is identically equal to zero for $t > T$ for some $T > 0$, and the corresponding solution cannot be identically equal to zero outside the bounded segment (with respect to the variable t).

Definition. The following number is called the convergence index of a sequence of complex numbers $\{z_k\}$,

$$\tau = \inf\left\{\alpha > 0 : \sum_{k=1}^{+\infty} \frac{1}{|z_k|^{\alpha}} < +\infty\right\}.$$

Theorem. *Let us assume that condition* (1.5) *is also satisfied for the sequence of exponents* $\{\gamma_k\}$ *of the kernel* $K_1(t)$, $\tau > 1$. *Then controllability in the state of rest for the system* (1.1)–(1.3) *does not take place*.

Proof. We consider the orthonormal system of eigenfunctions $\{\psi_n\}$ and eigenvalues $-a_n^2$ ($n = 1,2,\ldots$) of the operator Δ with respect to the boundary condition (1.3). Let

$$\varphi(x) = \xi_1\psi_1(x),$$

where $\xi_1 \neq 0$. Let us decompose the solution $\theta(t,x)$ and control action $u(t,x)$ into Fourier series in the mentioned system of eigenfunctions (this is the basis in $L_2(\Omega)$). The result is a countable system of integrodifferential equations:

$$\dot{\theta}_n(t) = -a_n^2\int_0^t K_1(t-s)\theta_n(s)ds + u_n(t), \quad t > 0, \quad n = 1,2,\ldots \tag{1.7}$$

Obviously, due to the choice of φ, only the first equation in system (1.7) has a nonzero initial condition.

Let us make the Laplace transform of both parts in equality (1.7) for $n = 1$:

$$(\lambda + a_1^2\hat{K}_1(\lambda))\hat{\theta}_1(\lambda) = \xi_1 + \hat{u}_1(\lambda). \tag{1.8}$$

Recall the definition of space PW_+ as a linear space of images of the Laplace transforms from elements from $L_2(0,+\infty)$ such that they are zero on the set $\{t : t > T\}$ for some $T > 0$. It is known that $f(\lambda) \in PW_+$ if and only if it is an entire function such that

(1) there are real numbers C and T, such that $|f(\lambda)| \leq Ce^{T|\lambda|}$ (note that C and T depend on $f(\lambda)$);

(2) $\sup\limits_{x\geq 0}\int_{-\infty}^{+\infty}|f(x+iy)|^2\,dy < +\infty$.

Assume that system (1.1)–(1.3) is controllable in a state of rest, then the functions $\hat{\theta}_1(\lambda)$ and $\hat{u}_1(\lambda)$ are elements of space PW_+. Therefore, these are entire functions of the exponential type.

We now consider the roots of the equation

$$\lambda + a_1^2\hat{K}_1(\lambda) = 0. \tag{1.9}$$

It is proved in [2] that Eq. (1.9) has (among other things) a countable number of real roots $\{\lambda_k\}$, for which the following inequalities hold:

$$-\gamma_{k+1} < \lambda_k < -\gamma_k, \quad k = 1,2,\ldots \tag{1.10}$$

It follows from (1.10) and the conditions of the theorem that the convergence index τ for a sequence of roots $\{\lambda_k\}$ is greater than one.

By definition, an entire function $f(\lambda)$ has a finite order of growth if there is a number $\mu > 0$ such that

$$\max_{|\lambda|=r}|f(\lambda)| < e^{r^{\mu}}, \quad r \geq r_0(\mu). \tag{1.11}$$

At the same time, the order of growth ρ of the entire function $f(\lambda)$ is called the lower bound those $\mu > 0$ for which (1.11) is true.

Obviously, for an entire function of the exponential type, the order of growth ρ is 1. It is well known in complex analysis that the index of convergence of a sequence of zeros of an entire function does not exceed its order of growth ($\tau \leq \rho$). It follows from (1.8) that the sequence $\{\lambda_k\}$ is a null function $\xi_1 + \hat{u}_1(\lambda)$. However, this is an entire function of the exponential type; hence, its growth order is 1 and the convergence index of the sequence of its zeros does not exceed 1. It was established above that for $\{\lambda_k\}$ number τ is greater than one. The established contradiction proves the theorem.

CONCLUSIONS

The instability of controllability for a kernel consisting of a finite number of exponentials. If the kernel in Eq. (1.1) consists of only a finite number of decreasing exponential functions, then, using the methods of [3, 4], we can prove that the considered mechanical system can be brought to rest in finite time if the control is applied to the entire region. Therefore, from the proved theorem, we can obtain an important corollary about the instability of the controllability of this system. Note that this instability is related to the addition of a small disturbance to the new kernel; i.e., according to the proved theorem, controllability is lost if this disturbance is the remainder of the series $K_1(t)$.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

REFERENCES

1. M. E. Gurtin and A. C. Pipkin, "Theory of heat conduction with finite wave speed," Arch. Ration. Mech. Anal., No. 31, 113–126 (1968).
2. V. V. Vlasov, N. A. Rautian, and A. S. Shamaev, "Spectral analysis and correct solvability of abstract integro-differential equations arising in thermophysics and acoustics," Sovrem. Mat. Fundam. Napravl. **39**, 36–65 (2011).
3. I. Romanov and A. Shamaev, "Exact controllability of the distributed system, governed by string equation with memory," J. Dyn. Control Syst. **18**, 611–623 (2013).
4. I. Romanov and A. Shamaev, "Exact controllability of the distributed system governed by the wave equation with memory," arXiv: 1503.04461.
5. I. Romanov and A. Romanova, "Some problems of controllability of distributed systems governed by integro-differential equations," IFAC-Papers OnLine **51**, 132–137 (2018).
6. I. Romanov and A. Shamaev, "Non-controllability to rest of the two-dimensional distributed system governed by the integrodifferential equation," J. Optimiz. Theory Appl. **170**, 772–782 (2016).
7. S. Ivanov and L. Pandofi, "Heat equations with memory: Lack of controllability to rest," J. Math. Anal. Appl. **355**, 1–11 (2009).
8. A. V. Romanova and I. V. Romanov, "On the problems of controllability and uncontrollability for some mechanical systems described by the equations of vibrations of plates and beams with integral memory," IOP Conf. Ser.: Mater. Sci. Eng. **1083**, 012041-1–9 (2021).
9. U. Biccaria and U. Micu, "Null-controllability properties of the wave equation with a second order memory term," J. Differ. Equat. **267**, 1376–1422 (2019).