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CONTROL IN SYSTEMS WITH DISTRIBUTED PARAMETERS

Suppression of Oscillations of Thin Plate by Bounded Control Acting to the Boundary

I. V. Romanov[a],* and A. S. Shamaev[b],**

[a] National Research University, Higher School of Economics, Moscow, 101000 Russia

[b] Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, 119991 Russia

*e-mail: romm1@list.ru

**e-mail: sham@rambler.ru

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Abstract—The problem of the exact bounded control of transverse vibrations of a thin plate is considered. Control actions are applied to the boundary of the plate, which fills a certain bounded domain on the plane. The purpose of the control is to completely stop oscillations in a finite time period.

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INTRODUCTION

The paper presents the problem of the boundary control of vibrations of a two-dimensional plate. At the same time, the control actions impose restrictions on the maximum absolute value. Consider the possibility of driving the plate to rest. The exact mathematical definitions will be given below.

The possibility of a complete stop in a finite time in the case of distributed control is proved in the monograph [1]. An upper estimate is also given there for the optimal control time.

The question of controlling the vibrations of plates and membranes using boundary forces has been studied by many authors (for example, [2] and the literature cited there). In [3], the problem of stopping vibrations of a limited string using boundary control is considered, it is proved that it is possible to completely stop string vibrations in a finite time period while limiting the absolute value of the control action, and an estimate is given of the time required to completely stop vibrations. In [4], the optimal control problems for systems with distributed parameters are investigated and the optimality conditions that are similar to Pontryagin's maximum principle for systems with a finite number of degrees of freedom are formulated. Moreover, these conditions do not always lead to a constructive method of constructing the optimal control. In the review paper [2], the problem of completely stopping the membrane's motion is considered, the existence of such a boundary control is proved, and the time required to completely stop the oscillations is estimated. Here, in many problem statements, the authors abandon the requirements of the optimal control and study only the problem of controllability, which greatly facilitates the study; the paper does not consider problems with a restriction on the absolute value of the control forces, and also does not give explicit expressions for the control actions; it only proves the existence theorems.

Bounded control (applied to the boundary) for membranes and plates was described, for example, in [5–7].

In addition to achieving complete rest, for the distributed oscillatory systems there, is the so-called problem of stabilizing the solution. This task consists in setting a feedback control on the boundary of the region that "stabilizes" the solution; i.e., the system's energy tends to zero when time t tends to infinity. For example, in [8], the problem of stabilizing the membrane's energy by the friction introduced at the boundary is considered. More precisely, the boundary of the region occupied by the membrane consists of two parts: Γ_0 and Γ_1, satisfying some additional geometric conditions. At Γ_0 the Dirichlet condition is introduced; i.e., this part of the boundary is rigidly fixed, and a boundary condition of the form is introduced on Γ_1

$$\frac{\partial w}{\partial \nu} = -k\frac{\partial w}{\partial t},$$

where ν is the external unit normal to Γ_1, $k > 0$. The friction set in this way leads to the dissipation of the energy of the system and, consequently, to the stabilization of its oscillations. Since part of the boundary is fixed, the energy of the system coincides with the square of the norm of the direct product of spaces: $H^1 \times L_2$. Therefore, as $t \to +\infty$, the solution of the problem and its first derivative with respect to t (speed) tend to zero according to the norms of spaces H^1 and L_2, respectively. Note that in the stated formulation, the initial data of the problem should be chosen sufficiently smooth and satisfying the matching conditions.

A similar formulation was also considered for the problem of the boundary stabilization of the transverse vibrations of a thin plate [9]. This problem will be described in detail below.

In general, methods of boundary stabilization are quite effective, since they allow driving the system's vibrations over a finite time into an arbitrarily small vicinity of zero, which in practice, as a rule, is equivalent to driving it to rest. However, these methods have a drawback. The time spent on stabilization may be longer than in precision control tasks. For example, methods are known for a plate that make it possible to drive the system's vibrations to rest in an arbitrarily short time period.

1. DESCRIPTION OF THE MAIN METHODS: UNRESOLVED ISSUES

Among the large number of different approaches in the control of distributed oscillatory systems (membranes, plates), three main methods can be distinguished: the method of moments, the method of continuing the solution in an unbounded domain, and the Hilbert uniqueness method.

The method of moments proposed by A.G. Butkovsky is effective for one-dimensional problems (such as string and rod problems). This method consists in decomposing the original problem into a countable number of harmonic oscillator control problems. Unfortunately, this method is of little use for regions of dimensions greater than unity.

For two-dimensional regions, the last two methods are much more effective. D.L. Russell proposed continuing the solution of the problem in an unbounded domain. The core of the method lies in the fact that, instead of the original control problem, it considers an initial boundary value problem (with zero Dirichlet boundary conditions) in an unbounded region consisting of a plane without a stellar region. Further, the initial data is extended to this unbounded domain so that the solution corresponding to the new (extended) initial data (together with the speed) comes to the zero state in the initial bounded domain at some point in time. Then, the control is defined as the restriction of the solution of the problem in an unbounded domain on the boundary of the membrane. In this case, the main problem is to determine how to extend the initial conditions. To build this continuation, an important physical property plays a decisive role: in an unbounded region on the chosen compact set, the wave oscillations exponentially stabilize over time to zero if the initial perturbation is chosen to be compact.

The Hilbert uniqueness method was proposed by J.L. Lyons and is based on the operator approach. The problem is reduced to proving the invertibility of some linear operator. Using this method, many authors have studied the controllability issues for a part of the boundary for membranes and plates. Moreover, some important conditions were imposed on the geometry of the boundaries.

Note that in this study we will also use (with significant changes) Russell's method.

The method of moments was applied by Butkovsky to prove bounded controllability; i.e., the control action applied to one end of the string should be bounded in absolute value. In the methods of Russell and Lyons, only the controllability task was set; restrictions on the module of the control function were not imposed. Note that this kind of restriction significantly complicates the task.

In this paper, we investigate the possibility of driving transverse vibrations of a thin plate to rest in the case when the boundary control actions are limited in absolute value. In this case, significant limitations will be imposed on the geometry of the boundary of the region filled with the plate. In addition, some initial conditions will also be imposed on the initial data of the problem, namely, the conditions of smoothness and matching. Controllability issues related to the relaxation of these restrictions remain open. For example, in the presented study, the boundary of the region occupied by the plate should consist of two parts. In other words, a plate with a hole is considered (see details below). It remains unclear whether the oscillations can be driven to rest (by means of the bounded boundary impact) if there is no hole and the region is simply connected. The problem also arises of reducing the degree of smoothness of the initial data; in this study, sufficiently strong smoothness conditions are imposed on the initial perturbation. However, these are challenges for future research studies.

2. FORMULATION OF THE CONTROL PROBLEM

Let Ω be a bounded domain on plane R^2 with an infinitely smooth boundary Γ consisting of two connected parts: Γ_0 and Γ_1, i.e., $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\nu = (\nu_1, \nu_2)$ is the external unit normal to the boundary of region Ω (we consider ν to be defined at every point Γ). Suppose that additionally the following condition is satisfied:

$$\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \varnothing.$$

Let us suppose that Γ_0 should also be the boundary of some bounded domain Ω^* such that $\Omega \cap \Omega^* = \varnothing$ (Fig. 1).

Consider the initial boundary value problem for the equation of transverse vibrations of thin plates:

$$w_{tt}(t,x) + \Delta^2 w(t,x) = 0, \quad (t,x) \in Q_T = (0,T) \times \Omega, \tag{2.1}$$

$$w|_{t=0} = \varphi(x), \quad w_t|_{t=0} = \psi(x), \quad x \in \Omega, \tag{2.2}$$

$$w = \frac{\partial w}{\partial \nu} = 0, \quad (t,x) \in (0,T) \times \Gamma_0, \tag{2.3}$$

$$\Delta w + (1-\mu) B_1 w = u_1(t,x), \quad \frac{\partial \Delta w}{\partial \nu} + (1-\mu)\frac{\partial B_2 w}{\partial \tau} = u_2(t,x), \quad (t,x) \in (0,T) \times \Gamma_1, \tag{2.4}$$

where μ is the Poisson constant ($0 < \mu < 1/2$), $\tau = (-\nu_2, \nu_1)$ is the tangent vector, and B_1 and B_2 are the boundary operators defined by the formulas

$$B_1 w = 2\nu_1\nu_2 \frac{\partial^2 w}{\partial x_1 \partial x_2} - \nu_1^2 \frac{\partial^2 w}{\partial x_2^2} - \nu_2^2 \frac{\partial^2 w}{\partial x_1^2},$$

$$B_2 w = (\nu_1^2 - \nu_2^2)\frac{\partial^2 w}{\partial x_1 \partial x_2} + \nu_1\nu_2 \left(\frac{\partial^2 w}{\partial x_2^2} - \frac{\partial^2 w}{\partial x_1^2}\right).$$

Hereinafter, we assume that at boundary Γ the inequalities are satisfied:

$$x \cdot \nu = x_1\nu_1 + x_2\nu_2 \le 0 \quad \text{at} \quad \Gamma_0,$$

$$x \cdot \nu = x_1\nu_1 + x_2\nu_2 \ge 0 \quad \text{at} \quad \Gamma_1.$$

Let $\varepsilon > 0$ be an arbitrary number. The task is to build such control actions u_1 and u_2 satisfying the inequalities

$$|u_i(t,x)| \le \varepsilon, \quad i = 1,2, \tag{2.5}$$

that the corresponding solution w and its derivative with respect to t vanish at some point in time T, i.e., $w(T,x) = 0$ and $w_t(T,x) = 0$ for all $x \in \Omega$. Zero bias and zero velocity will be called the *state of rest* of the system in question.

The following theorem is the main result of this paper.

Theorem. *Let functions* $\varphi(x) \in H^6(\Omega)$ *and* $\psi(x) \in H^4(\Omega)$ *be such that they are zero near boundary* Γ (*i.e., they are finite in* Ω). *Then there exist moment T and control actions* $u_1(t,x)$ *and* $u_2(t,x)$ *satisfying constraint* (2.5) *such that system* (2.1)–(2.4) *is controllable to rest.*

The proof of the theorem is divided into two stages and will be carried out in Sections 3 and 4.

Comment. The conditions imposed on the boundary of region Ω are important for proving a decrease in the energy of an auxiliary system (see below). We will use this fact substantially in the future.

3. FIRST STAGE OF CONTROL

At the first stage of control, we consider the problem of stabilizing to zero a solution with respect to the norm of a certain Sobolev space. For this we will use the results of the monograph [9]. We describe an auxiliary problem, namely, Eq. (2.1), initial conditions (2.2), and new boundary conditions:

$$w = \frac{\partial w}{\partial \nu} = 0, \quad (t,x) \in (0,T) \times \Gamma_0, \tag{3.1}$$

$$\Delta w + (1-\mu) B_1 w = 0, \quad \frac{\partial \Delta w}{\partial \nu} + (1-\mu)\frac{\partial B_2 w}{\partial \tau} = \lambda (x\cdot\nu)\frac{\partial w}{\partial t}, \quad (t,x) \in (0,T)\times\Gamma_1, \tag{3.2}$$

where λ is a fixed positive number.

We define the "energy" of system (2.1), (2.2), (3.1), and (3.2):

$$E(t) = \frac{1}{2}\int_\Omega \{w_t^2 + w_{x_1x_1}^2 + w_{x_2x_2}^2 + 2\mu w_{x_1x_1} w_{x_2x_2} + 2(1-\mu) w_{x_1x_2}^2\} dx. \tag{3.3}$$

We introduce the following notation:

$$H = L_2(\Omega), \quad W = H^2_{\Gamma_0}(\Omega),$$

where

$$H^2_{\Gamma_0}(\Omega) = \left\{ v \in H^2(\Omega) \colon v(x) = \frac{\partial v(x)}{\partial \nu} = 0,\ x \in \Gamma_0 \right\}.$$

We define the bilinear forms:

$$a(w,v) = \int_\Omega \{ w_{x_1x_1} v_{x_1x_1} + w_{x_2x_2} v_{x_2x_2} + \mu w_{x_1x_1} v_{x_2x_2} + \mu w_{x_2x_2} v_{x_1x_1} + 2(1-\mu) w_{x_1x_2} v_{x_1x_2} \} dx,$$

$$i(w,v) = \int_\Omega wv dx, \quad b(w,v) = \int_{\Gamma_1} \lambda (x\cdot\nu) wv d\Gamma.$$

Note that $a(w,w)$ is the square of the norm of space $H^2_{\Gamma_0}(\Omega)$ [10], which is equivalent to the usual norm of the Sobolev space $H^2(\Omega)$ (for elements from $H^2_{\Gamma_0}(\Omega)$)

Let be W' be conjugate to space W. Using these forms, we define the linear continuous operators $A \in \mathcal{L}(W, W')$ and $B \in \mathcal{L}(W, W')$:

$$\langle Aw, \hat{w}\rangle = a(w,\hat{w}), \quad \langle Bw, \hat{w}\rangle = b(w,\hat{w}), \quad w, \hat{w} \in W.$$

Also let I be the identity operator acting from H on itself.

Multiply Eq. (1) scalarly on the left and right by the function $v \in W$ and formally, "replace" the corresponding derivatives (using the boundary conditions), we obtain the integral identity

$$\frac{d}{dt}\{i(w_t, v) + b(w,v)\} + a(w,v) = 0. \tag{3.4}$$

Using the definition of operators A, B, and I, the integral identity (3.4) can be rewritten in the form of the system of equations

$$\overline{w}_t = \mathfrak{A}\overline{w}, \tag{3.5}$$

where $\overline{w} = (w_1, w_2)$ and

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ -A & -B \end{pmatrix};$$

moreover,

$$D(\mathfrak{A}) = \{(w_1, w_2) \in W \times W \colon Aw_1 + Bw_2 \in H\}.$$

It is known that operator $\mathfrak{A}$ generates a continuous semigroup, while the norm in space $D(\mathfrak{A})$ can be set as follows [9]:

$$\|(w_1, w_2)\|_{D(\mathfrak{A})} = \|(w_1, w_2)\|_{W\times H} + \|\mathfrak{A}(w_1, w_2)\|_{W\times H}. \tag{3.6}$$

Moreover, operator $\mathfrak{A}$ is the generating operator of the semigroup $e^{t\mathfrak{A}}$ of contractions, i.e., such a semigroup for which

$$\|e^{t\mathfrak{A}}\| \le 1.$$

According to the theory of continuous semigroups, if a pair of the initial data (φ, ψ) is an element of space $D(\mathfrak{A}^k)$, $k = 0,1,2,\ldots$, then for the corresponding solution of system (2.3) the inclusion

$$(w_1(t), w_2(t)) \in C([0,T]; D(\mathfrak{A}^k)).$$

The space $D(\mathfrak{A})$ can be described effectively using the theory of elliptic boundary value problems. For this, we consider a boundary value problem (relatively unknown w_1) in the following form:

$$(w_1, w_2) \in W \times W\colon Aw_1 + Bw_2 = f \in H. \tag{3.7}$$

Formally, using Green's formula, we can prove that (3.7) is a variational formulation of the problem [9]:

$$\Delta^2 w_1 = f \in H, \tag{3.8}$$

$$w_1 = \frac{\partial w_1}{\partial \nu} = 0, \quad x \in \Gamma_0, \tag{3.9}$$

$$\Delta w_1 + (1-\mu) B_1 w_1 = 0, \quad \frac{\partial \Delta w_1}{\partial \nu} + (1-\mu)\frac{\partial B_2 w_1}{\partial \tau} = \lambda (x \cdot \nu) w_2, \quad x \in \Gamma_1, \tag{3.10}$$

where $w_2 \in W$. Problem (3.8)–(3.10) is a regular elliptic problem in the sense of the definition given in [11]. In [9], it was shown that solution w_1 of problems (3.8)–(3.10) belongs to the space $H^4(\Omega)$. Therefore, Eq. (3.8) and the boundary conditions (3.9) and (3.10) are satisfied in the classical sense.

Summing up all the above, we obtain that the space $D(\mathfrak{A})$ consists of all pairs $(w_1, w_2) \in H^4(\Omega) \times W$ satisfying the boundary conditions (3.9) and (3.10), and the operator $\mathfrak{A}$ over elements $D(\mathfrak{A})$ can be represented as

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ -\Delta^2 & 0 \end{pmatrix}.$$

We now consider the linear operator $\mathcal{B}$ acting from space $H^4(\Omega)$ on space $L_2(\Omega) \times H^{\frac{3}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ by the rule

$$w \mapsto \left(\Delta^2 w, \alpha(x) w + \beta(x)(\Delta w + (1-\mu) B_1 w), \alpha(x)\frac{\partial w}{\partial \nu} + \beta(x)\left(\frac{\partial \Delta w}{\partial \nu} + (1-\mu)\frac{\partial B_2 w}{\partial \tau}\right)\right),$$

where $\alpha(x) \equiv 1$ if $x \in \Gamma_0$ and $\alpha(x) \equiv 0$ if $x \in \Gamma_1$; and $\beta(x) \equiv 0$ if $x \in \Gamma_0$ and $\beta(x) \equiv 1$ if $x \in \Gamma_1$. As mentioned above, (3.8)–(3.10) is a regular elliptic problem in terms of [11, Ch. 2]. Then $\mathcal{B}$ is a Fredholm operator. Therefore, we fall into the field of application of the theory of solvability of elliptic boundary value problems considered in [12].

Let us suppose that $(\varphi, \psi) \in W \times H$. It is proved that the inequality is true for the energy of the system [9]

$$E(t) \le M e^{-2\gamma t} E(0), \quad t \ge 0, \tag{3.11}$$

where the positive constants M and γ are independent of the initial data.

Let $(\varphi, \psi) \in D(\mathfrak{A})$ and $(w_1(t), w_2(t))$ be the solution corresponding to these initial data. We act on Eq. (3.5) and the initial conditions (2.2) by operator $\mathfrak{A}$. Therefore, we obtain

$$\frac{d}{dt}\mathfrak{A}\overline{w}(t) = \mathfrak{A}^2 \overline{w}(t), \quad \mathfrak{A}\overline{w}(0) = \mathfrak{A}(\varphi, \psi).$$

Note that

$$\mathfrak{A}(w_1(t), w_2(t)) = (w_2(t), -\Delta^2 w_1(t)). \tag{3.12}$$

Then from (3.11) and (3.12) we have

$$\frac{1}{2}\int_\Omega \{w_{2,x_1x_1}^2 + w_{2,x_2x_2}^2 + 2\mu w_{2,x_1x_1} w_{2,x_2x_2} + 2(1-\mu) w_{2,x_1x_2}^2 + (\Delta^2 w_1)^2\}dx$$
$$\le \frac{1}{2} M e^{-2\gamma t}\int_\Omega \{\psi_{x_1x_1}^2 + \psi_{x_2x_2}^2 + 2\mu\psi_{x_1x_1}\psi_{x_2x_2} + 2(1-\mu)\psi_{x_1x_2}^2 + (\Delta^2\varphi)^2\}dx. \quad (3.13)$$

Combining (3.11), (3.12), and (3.13), we write

$$\|(w_1(t), w_2(t))\|_{D(\mathfrak{A})} \le M_1 e^{-\gamma t}\|(\varphi,\psi)\|_{D(\mathfrak{A})}. \quad (3.14)$$

Because $(w_2(t), -\Delta^2 w_1(t)) \in W \times H$, then, using the theory of elliptic boundary value problems [12, p. 98], we obtain

$$\|w_1\|_{H^4(\Omega)} \le M_2(\|\Delta^2 w_1\|_{L_2(\Omega)} + \lambda M_3\|w_2\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|w_1\|_{L_2(\Omega)})$$
$$\le M_4(\|\Delta^2 w_1\|_{L_2(\Omega)} + \lambda M_5\sqrt{a(w_2,w_2)} + M_6\sqrt{a(w_1,w_1)}).$$

The last estimate obviously implies the equivalence of the norm of space $D(\mathfrak{A})$ and space norms $H^4 \times H^2$ (for items from $D(\mathfrak{A})$).

We now turn to the consideration of space $D(\mathfrak{A}^2)$. As before, using the theory of solvability of elliptic boundary value problems, this space can be effectively described; i.e., $D(\mathfrak{A}^2)$ consists of all pairs

$$(w_1, w_2) \in H^6(\Omega) \times (H^4(\Omega) \cap W)$$

satisfying the boundary conditions (3.9) and (3.10), as well as the boundary conditions

$$\Delta^2 w_1(x) = \frac{\partial \Delta^2 w_1}{\partial \nu} = 0, \quad x \in \Gamma_0, \quad (3.15)$$

$$\Delta w_2 + (1-\mu)B_1 w_2 = 0, \quad \frac{\partial \Delta w_2}{\partial \nu} + (1-\mu)\frac{\partial B_2 w_2}{\partial \tau} = \lambda(x\cdot\nu)(-\Delta^2 w_1), \quad x \in \Gamma_1. \quad (3.16)$$

Let $(w_1(t), w_2(t))$ be the solution to problem (2.1)–(2.3) and (3.2), then it belongs to the space $C([0,T]; D(\mathfrak{A}^2))$. We have

$$\mathfrak{A}^2(w_1, w_2) = (-\Delta^2 w_1, -\Delta^2 w_2). \quad (3.17)$$

From (3.11) and [9] it follows that

$$a(\Delta^2 w_1, \Delta^2 w_1) + \int_\Omega (\Delta^2 w_2)^2 dx \le N e^{-2\gamma t}\left(a(\Delta^2\varphi, \Delta^2\varphi) + \int_\Omega (\Delta^2\psi(x))^2 dx\right). \quad (3.18)$$

Combining (3.11) and (3.18), we obtain

$$\|(w_1(t), w_2(t))\|_{D(\mathfrak{A}^2)} \le N_1 e^{-\gamma t}\|(\varphi,\psi)\|_{D(\mathfrak{A}^2)}. \quad (3.19)$$

Applying the theory of elliptic boundary value problems [12, p. 98], we can write

$$\|w_1(t)\|_{H^6(\Omega)} \le N_2(\|\Delta^2 w_1(t)\|_{H^2(\Omega)} + \lambda N_3\|w_2(t)\|_{H^{\frac{5}{2}}(\Gamma_1)} + \|w_1(t)\|_{L_2(\Omega)}), \quad (3.20)$$

$$\|w_2(t)\|_{H^4(\Omega)} \le N_4(\|\Delta^2 w_2(t)\|_{L_2(\Omega)} + \lambda N_5\|\Delta^2 w_1(t)\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|w_2(t)\|_{L_2(\Omega)}), \quad (3.21)$$

where are the constants N_2, N_3, N_4, and N_5 independent of the selection of (w_1, w_2).

From (3.18) it follows that $\Delta^2 w_1(t)$ tends to zero (for $t \to +\infty$) according to the norm W (or $H^2(\Omega)$). Consequently, by Sobolev's trace theorem, $\Delta^2 w_1(t)$ will tend to zero and at the norm $H^{\frac{1}{2}}(\Gamma_1)$. Then, using also (3.11) and again (3.18), we obtain $w_2(t)$ tends to zero (for $t \to +\infty$) at the norm in $H^4(\Omega)$. Thus, from estimate (3.20) it follows that $w_1(t)$ tends to zero at $t \to +\infty$ at the rate of $H^6(\Omega)$, because w_1 is fixed to parts of the boundary along with its normal derivative.

The consequence of these estimates and reasoning is the equivalence of norms in spaces $D(\mathfrak{A}^2)$ and $H^6 \times H^4$ (for elements from $D(\mathfrak{A}^2)$)

From estimate (3.20), it obviously follows that the value

$$\|(w(t,\cdot), w_t(t,\cdot))\|_{D(\mathfrak{A}^2)}$$

tends to zero when $t \to +\infty$.

Let $\varphi(x)$ and $\psi(x)$ satisfy the conditions of the theorem. Using the theory of operator semigroups, we find that in this case there exists a solution to system (2.1)–(2.3) and (3.2) such that

$$w \in C([0,+\infty); H^6(\Omega)), \quad w_t \in C([0,+\infty); H^4(\Omega)).$$

We solve the auxiliary problem (2.1)–(2.3) and (3.2) with the given initial conditions, then this solution is substituted only into the right side of the second equality (3.2). Thus, we obtain the boundary conditions (2.3) and (2.4) for the initial boundary value problem (2.1)–(2.4). In other words, the control actions in problem (2.1)–(2.4) are determined by the formulas

$$u_1^{(1)}(t,x) \equiv 0, \quad u_2^{(1)}(t,x) = \lambda(x\cdot\nu)\frac{\partial w_0}{\partial t}, \quad (t,x) \in \Sigma,$$

where w_0 is the solution to the auxiliary problem (2.1)–(2.3) and (3.2).

Thus, it is proved that by controlling for a sufficiently long time, we can make the value

$$\|(w(t,\cdot), w_t(t,\cdot))\|_{D(\mathfrak{A}^2)}$$

arbitrarily small at some sufficiently large moment in time $t = T_1$. And due to the proved equivalence of the norms, the value

$$\|(w(t,\cdot), w_t(t,\cdot))\|_{\mathcal{H}_0^6(\Omega)}$$

will also be arbitrarily small where

$$\mathcal{H}_0^6(\Omega) = \left\{(w,v) \in H^6(\Omega)\times(H^4(\Omega)\cap W) \colon w(x) = \frac{\partial w}{\partial \nu} = \Delta^2 w = \frac{\partial \Delta^2 w}{\partial \nu} = 0,\ x \in \Gamma_0\right\}.$$

More precisely, due to the equivalence of the norms, the following estimate is correct:

$$\|(w(t,\cdot), w_t(t,\cdot))\|_{\mathcal{H}_0^6(\Omega)} \le N_6 e^{-\gamma t}\|(\varphi,\psi)\|_{\mathcal{H}_0^6(\Omega)}. \tag{3.22}$$

We show now that the boundary control action $u_2^{(1)}(t,x)$ can also be made small enough, i.e., satisfy constraint (2.5). For this, we note that since $e^{t\mathfrak{A}}$ is the semigroup of contractions, it follows that

$$\|w_t(t,\cdot)\|_{H^2(\Omega)} \le M_1^* \|(\varphi,\psi)\|_{D(\mathfrak{A})}, \quad t \ge 0.$$

In the last estimate, the expression on the right-hand side depends only on the initial data and does not depend on coefficient λ. Using the last inequality and Sobolev's theory on embedding, we obtain the boundedness of the module $w_t(t,x)$ on the closure of the cylinder

$$Q_{T_1} = (0,T_1)\times\Omega$$

by the constant depending only on the initial data. Choosing coefficient λ sufficiently close to zero, we obtain that condition (2.5) is satisfied.

4. SECOND STAGE OF CONTROL

We now set the task of driving the described system to complete rest. We will consider functions $w|_{t=0} = w(T_1,x)$ and $w_t|_{t=0} = w_t(T_1,x)$ to be the new initial data in problem (2.1)–(2.4). Recall that, according to what was proved above, these initial conditions (a pair of functions) are sufficiently small in the norm of the space $\mathcal{H}_0^6(\Omega)$.

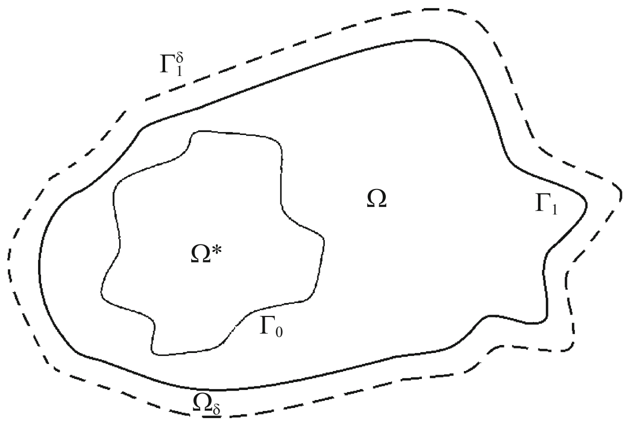

Fig. 1.

Consider the region Ω_δ, which, by definition, is the δ-vicinity of region Ω without points of set $\overline{\Omega^*}$ (Fig. 1). We construct region Ω_δ so that the outer contour of its boundary (we call it Γ_1^δ) satisfies the conditions that were previously imposed on Γ_1. Also let ν_δ be the external unit normal to the boundary of region Ω_δ.

We define the spaces

$$W_\delta = \left\{ v \in H^2(\Omega_\delta) \colon v(x) = \frac{\partial v(x)}{\partial \nu_\delta} = 0,\ x \in \Gamma_0 \right\},$$

$$\mathcal{H}_0^6(\Omega_\delta) = \left\{ (w, v) \in H^6(\Omega_\delta) \times (H^4(\Omega_\delta) \cap W_\delta) \colon w(x) = \frac{\partial w}{\partial \nu_\delta} = \Delta^2 w = \frac{\partial \Delta^2 w}{\partial \nu_\delta} = 0,\ x \in \Gamma_0 \right\}.$$

We also consider an arbitrary pair of functions

$$(f(x), g(x))$$

from the space $\mathcal{H}_0^6(\Omega)$. We extend this pair to the null (the linear extension operator

$$E \colon \mathcal{H}_0^6(\Omega) \to \mathcal{H}_0^6(\Omega_\delta)$$

exists and is bounded) in region Ω_δ while maintaining smoothness. In this case, the pair extended on Ω_δ will be identically equal to zero in the narrow strip adjacent to boundary Γ_1^δ from inside. The construction of the continuation operator E is well known and described in detail in [11].

The functions of the initial data extended in this way, following D.L. Russell, we designate, respectively, $f^e(x)$ and $g^e(x)$.

Let us consider the initial boundary value problem for the equation of plate oscillation in region Ω_δ:

$$w_{tt}(t, x) + \Delta^2 w(t, x) = 0, \quad (t, x) \in Q = (0, +\infty) \times \Omega_\delta, \tag{4.1}$$

$$w|_{t=0} = f^e(x), \quad w_t|_{t=0} = g^e(x), \quad x \in \Omega_\delta, \tag{4.2}$$

$$w(t, x) = \frac{\partial w}{\partial \nu_\delta} = 0, \quad (t, x) \in (0, +\infty) \times \Gamma_0, \tag{4.3}$$

$$\Delta w + (1 - \mu) B_1^\delta w = 0, \quad \frac{\partial \Delta w}{\partial \nu_\delta} + (1 - \mu) \frac{\partial B_2^\delta w}{\partial \tau_\delta} = \lambda (x \cdot \nu) \frac{\partial w}{\partial t}, \quad x \in \Gamma_1^\delta. \tag{4.4}$$

For the solution of problem (4.1)–(4.4) similarly to the previous section, the following estimate is true:

$$\|(w(t), w_t(t))\|_{W_\delta \times H_\delta} \le N e^{-\gamma_1 t} \|(f^e, g^e)\|_{W_\delta \times H_\delta}, \quad t \ge 0, \tag{4.5}$$

where $H_\delta = L_2(\Omega_\delta)$ and $\gamma_1 > 0$.

Next, we use the method (in a modified form) described in [8] and applied in the boundary controllability problems for the wave equation.

Let there be some initial conditions $f(x)$ and $g(x)$, $x \in \Omega$. Let us extend them on Ω_δ using the linear bounded operator E. Then $(f^e, g^e) = E(f, g)$. We obtain the initial boundary value problem (4.1)–(4.4). Let $w^s(t,x)$ be the solution to this problem. For region Ω_δ, consider operator $\mathfrak{A}_\delta$, which is constructed in exactly the same way as operator $\mathfrak{A}$ for region Ω. Then the following estimation is performed:

$$\left\|(w_1^s(t), w_2^s(t))\right\|_{D(\mathfrak{A}_\delta^2)} \le N_1^* e^{-\gamma_1 t} \left\|(f^e, g^e)\right\|_{D(\mathfrak{A}_\delta^2)}. \tag{4.6}$$

Consider some sufficiently large moment in time $t = T_2$ and restriction of the solution and its time derivative at time T_2 to region Ω. Obviously, for $t = T_2$ by (4.6) and the continuity of the operator E, the correct estimate is

$$\left\|(w_1^s(T_2,\cdot), w_2^s(T_2,\cdot))\right\|_{\mathcal{H}_0^6(\Omega)} \le M_7 e^{-\gamma_1 T_2} \left\|(f, g)\right\|_{\mathcal{H}_0^6(\Omega)}. \tag{4.7}$$

By definition let

$$(w_1^{s,e}(T_2,x), w_2^{s,e}(T_2,x)) = E(w_1^s(T_2,x)|_\Omega, w_2^s(T_2,x)|_\Omega).$$

We now give the initial boundary value problem in inverse time (i.e., for $t \le T_2$) for the equation (it remains unchanged)

$$\frac{d}{dt}(w_1, w_2) = (w_2, -\Delta^2 w_1) \tag{4.8}$$

with the boundary condition on Γ_1^δ:

$$\Delta w + (1-\mu) B_1^\delta w = 0, \quad \frac{\partial \Delta w}{\partial \nu_\delta} + (1-\mu)\frac{\partial B_2^\delta w}{\partial \tau_\delta} = -\lambda(x \cdot \nu_\delta)\frac{\partial w}{\partial t}, \tag{4.9}$$

condition (4.3) on Γ_0, and the initial conditions

$$w_1(t)|_{t=T_2} = -w_1^{s,e}(T_2,x), \quad w_2(t)|_{t=T_2} = -w_2^{s,e}(T_2,x). \tag{4.10}$$

Let $(w_1^i(t), w_2^i(t))$ be the solution of the initial boundary value problem (4.3), (4.8), (4.9), and (4.10) in inverse time. Similarly to the previous assessment

$$\left\|(w_1^i(0,\cdot), w_2^i(0,\cdot))\right\|_{\mathcal{H}_0^6(\Omega)} \le M_7 e^{-\gamma_1 T_2} \left\|(w_1^s(T_2,x), w_2^s(T_2,x))\right\|_{\mathcal{H}_0^6(\Omega)}. \tag{4.11}$$

Consider the sum of the solutions in forward and reverse time limited to region Ω:

$$w(t,x) = w^s(t,x) + w^i(t,x), \quad x \in \Omega. \tag{4.12}$$

This sum satisfies Eq. (2.1) and the boundary conditions on Γ, which are determined by the value of the function w itself and its derivatives on the boundary of region Ω.

Obviously, solution (4.12) with the initial conditions of the form

$$w|_{t=0} = f^e(x) + w^{i,r}(0,x), \quad w_t|_{t=0} = g^e(x) + w_t^{i,r}(0,x), \quad x \in \Omega, \tag{4.13}$$

(the index r means the restriction on Ω) is identically equal to zero in Ω together with its first derivative with respect to t at time $t = T_2$. Note that the value of the corresponding solution to the initial boundary value problem with the initial conditions (4.13) on the boundary of region Ω determines the desired control.

The pair $(w_1^{i,r}(0,x), w_2^{i,r}(0,x))$ is obtained from the pair $(f(x), g(x))$ using a linear continuous operator, call it L, with a norm less than one if T_2 is sufficiently large (a consequence of estimates (4.7) and (4.11)). Obviously, the sums on the right-hand sides of (4.13) give all the elements of the space $\mathcal{H}_0^6(\Omega)$. Indeed, (4.13) can be written as

$$(I + L)(f(x), g(x)) = (w|_{t=0}, w_t|_{t=0}), \tag{4.14}$$

where I is the identity operator. Therefore, since $\|L\| < 1$, operator $I + L$ acting from $\mathcal{H}_0^6(\Omega)$ on itself is convertible.

Now we present the desired control functions (at the second stage) in the following form:

$$u_1^{(2)}(t,x) = \Delta w_0 + (1-\mu)\, B_1 w_0, \tag{4.15}$$

$$u_2^{(2)}(t,x) = \frac{\partial \Delta w_0}{\partial \nu} + (1-\mu)\frac{\partial B_2 w_0}{\partial \tau}, \tag{4.16}$$

$$w_0 = P[(S_+(t) - S_-(T_2 - t)\, ERS_+(T_2))\, E\,(I + L)^{-1}\{w|_{t=0}, w_t|_{t=0}\}], \tag{4.17}$$

$x \in \Gamma_1$ where R is the constraint operator from Ω_δ in region Ω, $S_+(t)$ and $S_-(T_2 - t)$ are the resolving operators of the dissipative problem in direct and inverse time, respectively, P is projection $(a,b) \mapsto a$, and

$$L = -RS_-(T_2)\, ERS_+(T_2)\, E.$$

Thus, the possibility of driving to rest a system with arbitrary smooth initial data has been proved. We now show that, by choosing the sufficiently small initial data, we can drive the system to rest by a small modulo boundary control.

Let the pair $(w|_{t=0}, w_t|_{t=0})$ be quite small in the norm of space $\mathcal{H}_0^6(\Omega)$. From formulas (4.15)–(4.17) sufficiently small value of control actions

$$u_1^{(2)}(t,x),\ u_2^{(2)}(t,x)$$

follows automatically, since all the operators in the formula (4.17) are continuous. Therefore, applying Sobolev's embedding theorem, it is easy to show that these control actions will be less than the predetermined ε in absolute value if the control time in the first stage is chosen to be sufficiently large. The latter also means that the required restriction on the control functions is fulfilled. The theorem is proved.

CONCLUSIONS

In the present work, the existence of boundary control is proved, which drives the oscillations of a thin plate to rest in a finite time period. At the same time, the control actions themselves are subject to a restriction in absolute value. In the statement of the main theorem, the initial displacement, initial velocity, and geometry of the plate boundary satisfy certain conditions.

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