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Exact Bounded Boundary Controllability to Rest for the Two-Dimensional Wave Equation

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Abstract
The problem of the exact bounded control of oscillations of the two-dimensional membrane is considered. Control force is applied to the boundary of the membrane, which is located in a domain on a plane. The goal of the control is to drive the system to rest in finite time.

Keywords Controllability to rest · Wave equation · Boundary control · Bounded control

Mathematics Subject Classification 35L05 · 35L20 · 35B37

1 Introduction

The problem of exact boundary controllability of oscillations of a plane membrane is considered. Control force has a restriction on its absolute value. We will prove that the plane membrane can be driven to rest in finite time. Exact mathematical definitions will be provided. It should be noted that the given method for the proof in this article can be used in the case of any other dimension, but here the two-dimensional case is provided for a clear and simple presentation.

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The problem of full stabilization in finite time in case of the distributed control is described in the article [1]. This reference also contains the upper estimate for optimal control time.

Previously, the question of the control of oscillations of a plane membrane by means of boundary forces is considered by many authors (i. g. overviews of Russell [2], Lions [3] and the monograph [4], as well as the literature provided there). Some results of these investigations were developed in [5]. The well-known work [6] considers the equation for the vibration of a string. In this work, it was proved that if the control is applied to the end of the string, then the system can be driven to rest. The author used the so-called moment method. A relevant paper is [7] where the L_2-norm of the steering control for large times is studied (but in the simpler case of Dirichlet controls).

In the monograph [4], the problem of exact zero-controllability of a membrane is considered, the existence of the boundary control is proven and the time estimate is given which is required for driving to rest. Here authors, while studying the problem in various formulations, often reject the requirement of optimality of the control and solve the problem of controllability but the method used in this case, nevertheless, gives an optimal, in some sense, control. What is more, problems with restrictions of the force's absolute value are not considered, explicit forms for control functions are not found, and only theorems of existence are proven.

The statement of the problem in the article essentially differs from the one in [2,4], because the value of a control force on the boundary has to satisfy the condition: $|u(t,x)| \leq \varepsilon$. Note, here the aim is to find not an optimal control, but the admissible (satisfying this condition) control.

Moreover the article [8] should be mentioned as it has a result which is similar to the present one. In [8] a membrane is considered, which has one part of the boundary fixed and there is a control on the another part. This control function is defined by the Neumann condition and is bounded. Two parts of the boundary have some important geometrical restrictions. The aim of the control process is to achieve the system's state such as the shift and the velocity are equal to zero.

2 The Statement of the Problem

Let us consider the initial-boundary value problem for the two-dimensional wave equation:

$$w_{tt}(t,x) - \Delta w(t,x) = 0, \quad (t,x) \in Q_T = (0,T) \times \Omega, \tag{1}$$

$$w|_{t=0} = \varphi(x), \quad w_t|_{t=0} = \psi(x), \quad x \in \Omega, \tag{2}$$

$$\frac{\partial w}{\partial \nu} = u(t,x), \quad (t,x) \in \Sigma, \tag{3}$$

where $\Omega \subset R^2$ is a bounded, star-shaped relatively some circular disk domain with an infinitely smooth boundary, ν—the outer normal to the boundary of the domain Ω, Σ is a lateral surface of a cylinder Q_T. Initial data $\varphi(x)$ and $\psi(x)$ are given and will be chosen in suitable Hilbert spaces, $u(t, x)$ is a control function defined on the boundary $\Gamma = \partial\Omega$.

Let $\varepsilon > 0$ be an given arbitrary number. Let us impose the constraint on the control function:

$$|u(t, x)| \le \varepsilon, \tag{4}$$

The problem is to construct a control $u(t, x)$ satisfying inequality (4) such that the corresponding solution $w(t, x)$ to the initial-boundary value problem (1)–(3) and its derivative with respect to t become $(C, 0)$ at some time T, i.e.,

$$w(T, x) = C, \ \ w_t(T, x) = 0, \tag{5}$$

for all $x \in \Omega$. In this case, C is some constant. If we obtained a control $u(t, x)$ such that conditions (5) are achieved, then the system (1)–(3) is called *controllable to rest*.

Note that the constant C in this case is not arbitrary, but depends on the choice of the initial data. The nature of this dependence will be indicated below.

The following theorem is the main result of this article.

Theorem 1 *Let $\varphi(x) \in H^6(\Omega)$ and $\psi(x) \in H^5(\Omega)$ such that*

$$\begin{aligned} &\frac{\partial\varphi(x)}{\partial\nu} = \Delta\varphi(x) = \frac{\partial\Delta\varphi(x)}{\partial\nu} = \Delta^2\varphi(x) = \frac{\partial\Delta^2\varphi(x)}{\partial\nu} = 0, \ \ x \in \Gamma, \\ &\psi(x) = \frac{\partial\psi(x)}{\partial\nu} = \Delta\psi(x) = \frac{\partial\Delta\psi(x)}{\partial\nu} = \Delta^2\psi(x) = 0, \ \ x \in \Gamma. \end{aligned} \tag{6}$$

Then the system (1)–(3) is controllable to rest.

Let us explain the meaning of the smoothness conditions for the initial data and conditions (6). The proof of Theorem 1 consists of two steps. The first step stabilizes the considered solution and its first derivative with respect to t in a small neighborhood of equilibrium $(C, 0)$ in the norm of $C^4(\overline{\Omega}) \times C^3(\overline{\Omega})$, and the second step allows to drive to rest the system in this small vicinity. The first stage of the proof (stabilization of the solution) is connected with the introduction of friction at the boundary of the domain. This friction creates dissipation of energy, which, in turn, leads to stabilization. This friction is control. In this case, the constraint (4) will be achieved due to the sufficient “smallness” of this friction and this “smallness” is caused by variation of a certain coefficient. Conditions (6) are required in order to the problem statement remained correct for any selected coefficient of friction.

3 The First Step of the Control

Here, we state the task to stabilize a pair $(w(t, x), w_t(t, x))$ in an arbitrarily small vicinity of $(C, 0)$ in the norm of the space $\mathcal{C}^4(\overline{\Omega}) = C^4(\overline{\Omega}) \times C^3(\overline{\Omega})$ where $w(t, x)$ is the solution to the system (1)–(3) and $w_t(t, x)$ is its first derivative with respect to t. What is more the control function should satisfy the restriction (4).

At the first step we state the problem to stabilize (w, w_t) in the small enough neighborhood of $(C, 0)$ in the norm of $\mathcal{H}^6(\Omega) = H^6(\Omega) \times H^5(\Omega)$. In this case, we have the restriction (4) for the control function.

For this purpose, we use [9,10]. In these articles authors consider a friction on Γ which is defined by $w_t(t, x)$. More exactly they consider the initial-boundary value problem (1)–(2) with a new boundary condition:

$$\frac{\partial w(t,x)}{\partial \nu} = -k\frac{\partial w(t,x)}{\partial t}, \quad x \in \Gamma, \tag{7}$$

where $k > 0$ is a friction coefficient. Let us illuminate shortly the questions of solvability of this problem.

Let us denote

$$H = L_2(\Omega), \quad V = H^1(\Omega),$$

Let us define in the space $V \times H$ an unbounded operator

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

with the domain

$$D(\mathfrak{A}) = \{(w_1, w_2) \in H^2(\Omega) \times H^1(\Omega) : \frac{\partial w_1}{\partial \nu} = -kw_2, \ x \in \Gamma\}.$$

It is a well-known fact that the norm in the space $D(\mathfrak{A})$ can be represent in the following form:

$$\|(w_1, w_2)\|_{D(\mathfrak{A})} = \|(w_1, w_2)\|_{V\times H} + \|\mathfrak{A}(w_1, w_2)\|_{V\times H}. \tag{8}$$

Let us consider the following system of differential equations:

$$\bar{w}_t = \mathfrak{A}\bar{w}, \tag{9}$$

where $\bar{w} = (w_1, w_2)$.

It is known (see [9,10]) that an operator $\mathfrak{A}$ is a generator of strongly continuous semigroup of linear bounded operators.

It is a well-known fact that if initial data (φ, ψ) is an element of $D(\mathfrak{A}^k)$, $k = 0, 1, 2, \ldots$, then we have:

$$(w_1(t), w_2(t)) \in C\left([0, T]; D(\mathfrak{A}^k)\right).$$

We note that in our case we have (φ, ψ) as an element of $D(\mathfrak{A}^5)$.

Let $(\varphi, \psi) \in V \times H$. It is proved (see [9,10]) that for the energy of the system, we have:

$$E(t) \to 0, \ t \to +\infty. \tag{10}$$

where

$$E(t) = \int\limits_{\Omega} \left\{ w_{1,x_1}^2(t,x) + w_{1,x_2}^2(t,x) + w_2^2(t,x) \right\} \mathrm{d}x$$

is an energy of the system.

We introduce:

$$C = \frac{1}{|\Gamma|} \int\limits_{\Gamma} \varphi(x) \mathrm{d}\Gamma + \frac{1}{k|\Gamma|} \int\limits_{\Omega} \psi(x) \mathrm{d}x, \tag{11}$$

where $|\Gamma|$ is a length of Γ. Let $w(t,x) = v(t,x) + C$ and consider a new initial-boundary value problem for $v(t,x)$ (analogous to 1, 2, 7):

$$\bar{v}_t = \mathfrak{A}\bar{v}, \tag{12}$$

$$(v_1, v_2)|_{t=0} = (\varphi(x) - C, \psi(x)), \quad x \in \Omega, \tag{13}$$

where $\bar{v} = (v_1, v_2)$. Obviously in this case $v_1 = v$ and $v_2 = v_t$.

Using Friedrichs' (Poincare) inequality (see [11]), we have

$$\int\limits_{\Omega} v^2(t,x) \mathrm{d}x \le C_3 \left\{ \int\limits_{\Omega} \left(\left(\frac{\partial v}{\partial x_1} \right)^2 + \left(\frac{\partial v}{\partial x_2} \right)^2 \right) \mathrm{d}x + \left(\int\limits_{\Gamma} v(t,x) \mathrm{d}\Gamma \right)^2 \right\}.$$

$$\int\limits_{\Gamma} v(t,x) \mathrm{d}\Gamma = -\frac{1}{k} \int\limits_0^t \int\limits_{\Gamma} \frac{\partial v(s,x)}{\partial \nu} \mathrm{d}\Gamma \mathrm{d}s + \int\limits_{\Gamma} \varphi(x) \mathrm{d}\Gamma - |\Gamma| C$$

$$= -\frac{1}{k} \int\limits_0^t \int\limits_{\Omega} \Delta v(s,x) \mathrm{d}x \mathrm{d}s + \int\limits_{\Gamma} \varphi(x) \mathrm{d}\Gamma - |\Gamma| C$$

$$= -\frac{1}{k} \int\limits_0^t \int\limits_{\Omega} v_{ss}(s,x) \mathrm{d}x \mathrm{d}s + \int\limits_{\Gamma} \varphi(x) \mathrm{d}\Gamma - |\Gamma| C$$

$$= -\frac{1}{k} \int\limits_{\Omega} v_t(t,x) \mathrm{d}x + \frac{1}{k} \int\limits_{\Omega} \psi(x) \mathrm{d}x + \int\limits_{\Gamma} \varphi(x) \mathrm{d}\Gamma - |\Gamma| C = -\frac{1}{k} \int\limits_{\Omega} v_t(t,x) \mathrm{d}x.$$

From the last estimations, we obtain

$$\| w(t,\cdot) - C \|_{L_2(\Omega)} \to 0, \quad t \to +\infty. \tag{14}$$

Let $(\varphi - C, \psi)$ be an element of $D(\mathfrak{A})$ and $(v_1(t), v_2(t))$ is a corresponding (to these initial data) solution. We consider now the following Cauchy problem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathfrak{A}\bar{v}(t) = \mathfrak{A}^2 \bar{v}(t), \;\; \mathfrak{A}\bar{v}(0) = \mathfrak{A}(\varphi - C, \psi).$$

We note that

$$\mathfrak{A}(v_1(t), v_2(t)) = (v_2(t), \Delta v_1(t)). \tag{15}$$

Then from (10) and (15), we obtain

$$\int\limits_{\Omega} \left\{ v_{2,x_1}^2(t) + v_{2,x_2}^2(t) + (\Delta v_1(t))^2 \right\} \mathrm{d}x \to 0, \;\; t \to +\infty. \tag{16}$$

Combining (8), (10), (14) and (16), we have:

$$\|(v_1(t), v_2(t))\|_{D(\mathfrak{A})} \to 0, \;\; t \to +\infty. \tag{17}$$

Let initial condition be an element of $D(\mathfrak{A})$ then for the corresponding solution, we can obtain (using the theory of elliptic boundary value problems (see, for example, [12] or [13])) the following estimate:

$$\|v_1(t)\|_{H^2(\Omega)} \le N_1 \left(\|\Delta v_1(t)\|_{L_2(\Omega)} + k\|v_2(t)\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|v_1(t)\|_{L_2(\Omega)} \right), \tag{18}$$

where N_1 does not depend on (v_1, v_2). Using (10), (16) and the last estimate one can easily prove that $v_1(t)$ tends to zero when $t \to +\infty$ in the norm of $H^2(\Omega)$.

Consider the space $D(\mathfrak{A}^2)$. Using the theory of elliptic boundary value problems, we can describe this space effectively:

$$D(\mathfrak{A}^2) = \{(v_1, v_2) \in H^3(\Omega) \times H^2(\Omega) : \frac{\partial v_1}{\partial \nu} = -kw_2, \; \frac{\partial v_2}{\partial \nu} = -k\Delta v_1, \; x \in \Gamma\}.$$

Let $(v_1(t), v_2(t))$ be the solution to (1, 2), (7) then $(v_1(t), v_2(t))$ is an element of $C\left([0, T]; D(\mathfrak{A}^2)\right)$. We have

$$\mathfrak{A}^2(v_1, v_2) = (\Delta v_1, \Delta v_2). \tag{19}$$

It follows from (19) that

$$\int\limits_{\Omega} \left\{ (\Delta v_{1,x_1}(t))^2 + (\Delta v_{1,x_2}(t))^2 + (\Delta v_2(t))^2 \right\} \mathrm{d}x \to 0, \;\; t \to +\infty. \tag{20}$$

Using (17) and (25), we obtain:

$$\|(v_1(t), v_2(t))\|_{D(\mathfrak{A}^2)} \to 0, \;\; t \to +\infty. \tag{21}$$

The theory of elliptic boundary value problems gives us the following estimates:

$$\|v_1(t)\|_{H^3(\Omega)} \le N_2 \left(\|\Delta v_1(t)\|_{H^1(\Omega)} + k\|v_2(t)\|_{H^{\frac{3}{2}}(\Gamma_1)} + \|v_1(t)\|_{L_2(\Omega)} \right). \tag{22}$$

$$\|v_2(t)\|_{H^2(\Omega)} \le N_3 \left(\|\Delta v_2(t)\|_{L_2(\Omega)} + k\|\Delta v_1(t)\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|v_2(t)\|_{L_2(\Omega)} \right) \tag{23}$$

Using the last estimates one can easily prove that $v_1(t)$ tends to zero when $t \to +\infty$ in the norm of $H^3(\Omega)$.

Let us have a look at one more step in detail. Consider the space $D(\mathfrak{A}^3)$. We have: $\mathfrak{A}^3(v_1, v_2) = (\Delta v_2, \Delta^2 v_1)$. Hence, we obtain two equations: $\Delta v_2 = f_1$, $\Delta^2 v_1 = f_2$, where $(f_1, f_2) \in H^1 \times L_2$, and three boundary conditions at Γ:

$$a) \; \frac{\partial v_1}{\partial \nu} = -k v_2, \; b) \; \frac{\partial v_2}{\partial \nu} = -k \Delta v_1, \; c) \; \frac{\partial \Delta v_1}{\partial \nu} = -k \Delta v_2. \tag{24}$$

Let us make a substitution $\Delta v_1 = h$, then the equation $\Delta^2 v_1 = f_2$ with the boundary condition (c) has the form $\Delta h = f_2$, $\frac{\partial h}{\partial \nu} = -k \Delta v_2$. Hence, the following estimation takes place

$$\|h\|_{H^2(\Omega)} \leq N_4 \left(\|\Delta h\|_{L_2(\Omega)} + k \|\Delta v_2\|_{H^{\frac{1}{2}}(\Gamma)} + \|h\|_{L_2(\Omega)} \right).$$

Then, $\Delta v_1 \in H^{\frac{3}{2}}(\Gamma)$ at the boundary of a domain. So from the equation $\Delta v_2 = f_1$ and the boundary condition (b) the following estimation is derived:

$$\|v_2\|_{H^3(\Omega)} \leq N_5 \left(\|\Delta v_2\|_{H^1(\Omega)} + k \|\Delta v_1\|_{H^{\frac{3}{2}}(\Gamma)} + \|v_2\|_{L_2(\Omega)} \right). \tag{25}$$

Then, we get the equation $\Delta v_1 = f_3 \in H^2(\Omega)$ with the boundary condition (a). Using the previous estimation, we obtain:

$$\begin{aligned} \|v_1\|_{H^4(\Omega)} \leq N_6 \Bigg(& \|\Delta^2 v_1\|_{L_2(\Omega)} + k \|v_2\|_{H^{\frac{5}{2}}(\Gamma)} + \\ & k \|\Delta v_2\|_{H^{\frac{3}{2}}(\Gamma)} + \|v_1\|_{L_2(\Omega)} \Bigg). \end{aligned} \tag{26}$$

Continuing in the analogous way, we can prove that $\|v_1(t)\|_{H^6(\Omega)}$ and $\|v_2(t)\|_{H^5(\Omega)}$ tend to zero when $t \to +\infty$. It means that

$$\|w(t) - C\|_{H^6(\Omega)}, \; \|w_t(t)\|_{H^5(\Omega)} \to 0 \quad t \to +\infty.$$

We solve the problem (1), (2), (7) with the given initial conditions, then this solution is substituted to the *only* right part of the equality (7), and we obtain the boundary condition (3) for the initial-boundary value problem (1)–(3). In other words, we make the control function of the problem (1)–(3) be equal to

$$u^{(1)}(t, x) = -k \frac{\partial w_0(t, x)}{\partial t},$$

where w_0 is a solution to the problem (1), (2), (7).

Therefore, it is proved (here we use Sobolev embedding theorem) that controlling for a long time, we can make the values

$$\|w(T_1,\cdot)\|_{C^4(\overline{\Omega})}, \quad \|w_t(T_1,\cdot)\|_{C^3(\overline{\Omega})}$$

arbitrarily close to $(C, 0)$ at some time $t = T_1$.

Now let us show that the boundary control function $u(t, x)$ can be sufficiently small, i.e., we may satisfy the restriction (4). It is known that

$$\max_{t\in[0,+\infty)} E(t) = E(0) = \int\limits_{\Omega} \left(\varphi_{x_1}^2(x) + \varphi_{x_2}^2(x) + \psi^2(x)\right) \mathrm{d}x.$$

Then using S. L. Sobolev theorems of injections and (23), we have:

$$\begin{aligned}
\|w_2(t)\|_{C(\overline{\Omega})} &\le C_1\|w_2(t)\|_{H^2(\Omega)} \le C_2\|\Delta w_2(t)\|_{L_2(\Omega)} + \\
&\quad +kC_2\|\Delta w_1(t)\|_{H^{\frac{1}{2}}(\Gamma_1)} + C_2\|w_2(t)\|_{L_2(\Omega)} \\
&\le C_2\|\Delta w_2(t)\|_{L_2(\Omega)} + kC_3\|\Delta w_1(t)\|_{H^1(\Omega)} + C_2\|w_2(t)\|_{L_2(\Omega)} \\
&\le C_2\sqrt{\int\limits_{\Omega} \left\{(\Delta w_{1,x_1}(t))^2 + (\Delta w_{1,x_2}(t))^2 + (\Delta w_2(t))^2\right\} \mathrm{d}x} \\
&\quad + kC_3\sqrt{\int\limits_{\Omega} \left\{w_{2,x_1}^2(t) + w_{2,x_2}^2(t) + (\Delta w_1(t))^2\right\} \mathrm{d}x} \\
&\quad + kC_3\sqrt{\int\limits_{\Omega} \left\{(\Delta w_{1,x_1}(t))^2 + (\Delta w_{1,x_2}(t))^2 + (\Delta w_2(t))^2\right\} \mathrm{d}x} \\
&\quad + C_2\sqrt{\int\limits_{\Omega} \left\{w_{1,x_1}^2(t) + w_{1,x_2}^2(t) + w_2^2(t)\right\} \mathrm{d}x} \\
&\le C_2\sqrt{\int\limits_{\Omega} \left\{(\Delta\varphi_{x_1})^2 + (\Delta\varphi_{x_2})^2 + (\Delta\psi)^2\right\} \mathrm{d}x} \\
&\quad + kC_3\sqrt{\int\limits_{\Omega} \left\{\psi_{x_1}^2 + \psi_{x_2}^2 + (\Delta\varphi)^2\right\} \mathrm{d}x} \\
&\quad + kC_3\sqrt{\int\limits_{\Omega} \left\{(\Delta\varphi_{x_1})^2 + (\Delta\varphi_{x_2})^2 + (\Delta\psi)^2\right\} \mathrm{d}x} + C_2\sqrt{\int\limits_{\Omega} \left\{\varphi_{x_1}^2 + \varphi_{x_2}^2 + \psi^2\right\} \mathrm{d}x}.
\end{aligned}$$

Thus, $\|w_2(t)\|_{C(\overline{\Omega})}$ is uniformly bounded for any t because k is near zero.

By virtue of (6), initial conditions (φ, ψ) will be the element of $D(\mathfrak{A}^2)$ for any k. If the coefficient k is small enough, we achieve condition (4).

4 The Second Step of the Control

The reasoning in this first step will present D. L. Russell's method of proving controllability. The main difference is that estimates for solutions are not constructed in Sobolev spaces, but in spaces of smooth functions. Since at the first stage the solution was made small enough in the norm of the space of smooth functions, then at the second stage, as we will see, the solution will remain "small" on the closure of the domain. Therefore, the control constraint (4) remains satisfied.

Now, we have a task to drive the system to rest. A pair of functions

$$(w|_{t=0} = w(T_1, x), w_t|_{t=0} = w_t(T_1, x))$$

is considered to be new initial data for the problem (1)–(3). Bearing in mind that according to the fact proven above these initial conditions are sufficiently close to $(C, 0)$ in the norm of the space $\mathcal{C}^4(\overline{\Omega})$. We shift now the solution w (first step of the control) on the value C, i.e., we change w on $w + C$ and consider the pair

$$(w|_{t=0} = w(T_1, x), w_t|_{t=0} = w_t(T_1, x))$$

that is sufficiently close to (0, 0).

Let us consider the domain Ω_δ, which is δ-vicinity of the domain Ω. Also let take an arbitrary pair $(w_0(x), w_1(x))$ from the space $\mathcal{C}^4(\overline{\Omega})$. Consider an extension operator E. It is a linear continuous operator from the space $\mathcal{C}^4(\overline{\Omega})$ to $\mathcal{C}^4(\overline{\Omega}_\delta)$ such that the support of the extended pair $(w_0^e(x), w_1^e(x))$, and its derivatives of fourth and third orders (respectively) inclusive belongs to $\overline{\Omega}_\delta$. Moreover

$$(w_0^e(x), w_1^e(x)) = (w_0(x), w_1(x)), \text{ if } x \in \overline{\Omega}.$$

Note that, outside $\overline{\Omega}_\delta$, the functions can be extended by zero to the whole plane. In a more general case, E was constructed in [13].

Extended in this way functions are denoted (as above) as $w_0^e(x)$ and $w_1^e(x)$, according to D. L. Russell.

Let us consider the Cauchy problem for the equation of membrane's oscillations on a plane R^2:

$$w_{tt}(t, x) - \Delta w(t, x) = 0, \quad (t, x) \in Q = (0, +\infty) \times R^2, \tag{27}$$

$$w|_{t=0} = w_0^e(x), \quad w_t|_{t=0} = w_1^e(x), \quad x \in R^2. \tag{28}$$

It is known that the solution to the problem (27), (28) has the form (Poisson's formula):

$$w(t, x) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int\limits_{|y-x|<t} \frac{w_0^e(y)\mathrm{d}y}{\sqrt{t^2 - |y-x|^2}}\right) + \frac{1}{2\pi}\int\limits_{|y-x|<t} \frac{w_1^e(y)\mathrm{d}y}{\sqrt{t^2 - |y-x|^2}}. \tag{29}$$

We use the formula (29) for estimating the absolute value of the solution $w(t,x)$ uniformly by the initial data. The absolute value of $w(t,x)$ is estimated in case $x \in \overline{\Omega}_\delta$. We compute the first derivative with respect to t in the right part of (29):

$$w(t,x) = \frac{1}{2\pi t} \int\limits_{|y-x|<t} \frac{w_0^e(y) + (y-x)\cdot \nabla w_0^e(y)}{\sqrt{t^2 - |y-x|^2}} \mathrm{d}y + \frac{1}{2\pi} \int\limits_{|y-x|<t} \frac{w_1^e(y)\mathrm{d}y}{\sqrt{t^2 - |y-x|^2}}. \tag{30}$$

As initial data $(w_0^e(x), w_1^e(x))$ have a compact support then there is large enough time $t^* > 0$ such that for any $t > t^*$ and for any $x \in \overline{\Omega}_\delta$, we obtain

$$w(t,x) = \frac{1}{2\pi t} \int\limits_{\Omega_\delta} \frac{w_0^e(y) + (y-x)\cdot \nabla w_0^e(y)}{\sqrt{t^2 - |y-x|^2}} \mathrm{d}y + \frac{1}{2\pi} \int\limits_{\Omega_\delta} \frac{w_1^e(y)\mathrm{d}y}{\sqrt{t^2 - |y-x|^2}}. \tag{31}$$

Note that we choose t such as $t^2 - |y-x|^2 \geq \alpha > 0$ for any $x, y \in \overline{\Omega}_\delta$.

The following rough evaluation follows from the explicit form of (31):

$$\|w(t,\cdot)\|_{C^4(\overline{\Omega}_\delta)} \leq \frac{C_1}{t} \|w_0^e\|_{C^4(R^2)} + \frac{C_2}{t} \|w_1^e\|_{C^3(R^2)}. \tag{32}$$

Differentiating $w(t,x)$ with respect to t, we obtain the rough estimate in the space of the pair of functions $\mathcal{C}^4(\overline{\Omega}_\delta) = C^4(\overline{\Omega}_\delta) \times C^3(\overline{\Omega}_\delta)$

$$\|(w(t,\cdot), w_t(t,\cdot))\|_{\mathcal{C}^4(\overline{\Omega}_\delta)} \leq \frac{M}{t} \|(w_0^e, w_1^e)\|_{\mathcal{C}^4(R^2)}, \quad t > t^*, \tag{33}$$

where a number M does not depend on initial data.

Further, we use the method described in [2] and applied to problems of the boundary controllability for a wave equation.

Let us consider some initial conditions $w_0(x)$ and $w_1(x)$, $x \in \Omega$. We extend them to R^2 by means of a linear bounded operator E. Then, we obtain $(w_0^e, w_1^e) = E(w_0, w_1)$. And the Cauchy problem (27), (28) arises. Let $w^s(t,x)$ be the solution to this Cauchy problem. Now consider any large enough time $t = T_2$. We get $(w^s(T_2,x), w_t^s(T_2,x)) \in \mathcal{C}^4(\overline{\Omega})$. The restriction of the function $w^s(T_2,x)$ and its derivative on the domain Ω should be considered. It is obvious that in virtue of (33) the following estimate is correct for $t = T_2$

$$\|(w^s(T_2,\cdot), w_t^s(T_2,\cdot))\|_{\mathcal{C}^4(\overline{\Omega})} \leq \frac{M}{T_2} \|(w_0^e, w_1^e)\|_{\mathcal{C}^4(R^2)}. \tag{34}$$

Let by definition $(w_0^{s,e}(T_2,x), w_1^{s,e}(T_2,x)) = E\left(w^s(T_2,x)|_\Omega, w_t^s(T_2,x)|_\Omega\right)$. Now, let us have a look at the inverse Cauchy problem with initial conditions

$$w(t,x)|_{t=T_2} = -w_0^{s,e}(T_2,x) \quad w_t(t,x)|_{t=T_2} = -w_1^{s,e}(T_2,x). \tag{35}$$

Let $w^i(t,x)$ be the solution to the inverse Cauchy problem with conditions (35). In virtue of invertibility of Eq. (1) with respect to t the following estimate takes place:

$$\|(w^i(0,\cdot), w_t^i(0,\cdot))\|_{C^4(\overline{\Omega})} \le \frac{M}{T_2}\|(w_0^{s,e}(T_2,x), w_1^{s,e}(T_2,x))\|_{C^4(R^2)}. \tag{36}$$

Obviously the solution of the Cauchy problem with initial conditions such as

$$w|_{t=0} = w_0^e(x) + w^i(0,x), \quad w_t|_{t=0} = w_1^e(x) + w_t^i(0,x), \quad x \in R^2, \tag{37}$$

identically equals zero in Ω as well as its first derivative with respect to t at the time $t = T_2$. Now let us consider the restriction of the right parts of (37) in the domain Ω. We regard the initial conditions (the restriction of right parts of (37) in the domain Ω) in the problem of boundary controllability:

$$w|_{t=0} = w_0(x) + w^{i,r}(0,x), \quad w_t|_{t=0} = w_1(x) + w_t^{i,r}(0,x), \quad x \in \Omega. \tag{38}$$

Note that it is the value of the corresponding solution to the Cauchy problem in R^2 with the initial conditions (37) to determine the required control function on the boundary of Ω.

A pair $(w^{i,r}(0,x), w_t^{i,r}(0,x))$ is derived from pair $(w_0(x), w_1(x))$ by means of applying a linear continuous operator, let us denote it as L, with the norm less than 1 (consequence from estimates 34 and 36). Obviously the sums in right parts (38) generate all elements of the space $C^4(\overline{\Omega})$. Indeed, (38) can be written as:

$$(I + L)(w_0(x), w_1(x)) = (w|_{t=0}, w_t|_{t=0}), \tag{39}$$

where I is the identical operator. Hence, as $\|L\| < 1$, so the operator $I + L$, which acts from $C^4(\overline{\Omega})$ to itself, is invertible.

Now let us represent the control function (second step) in the following form:

$$u^{(2)}(t,x) = \frac{\partial}{\partial \nu} P K_+^t \Big[\Big(I + (-K_-^{T_2}) E R K_+^{T_2} \Big) \times \\ E \Big(I + R(-K_-^{T_2}) E R K_+^{T_2} E \Big)^{-1} (w|_{t=0}, w_t|_{t=0}) \Big], \; x \in \partial\Omega,$$

where R is a restriction from R^2 to Ω and $K_+^{T_2}$, $K_-^{T_2}$ are resolving operators of the Cauchy problem and P is a projection: $(a,b) \mapsto a$. We write minus before $K_-^{T_2}$ because of (35).

Thus, we have proven that the system with smooth initial conditions can be driven to rest by means of extending them on the full plane. It is the method to extend which determines a program of the boundary control. Let us show now that if the initial conditions have small enough absolute values, we can drive the system to rest by means of a boundary control which has a small absolute value.

We regard that in the problem (1)–(3), the value of the solution $w(t, x)$ and the value of its derivative $w_t(t, x)$ at $t = T_1$ are small enough in norms of spaces $C^4(\overline{\Omega})$ and $C^3(\overline{\Omega})$, respectively.

Let $(w|_{t=0}, w_t|_{t=0})$ be rewritten according to the formula (38). As continuous operator $I + L$ invertible, so according to Banach's theorem an invertible operator is continuous too. Hence, choosing $(w|_{t=0}, w_t|_{t=0})$ sufficiently small, we can make $(w_0(x), w_1(x))$ be sufficiently small as well. Now let consider the sums (38), which determine data $(w|_{t=0}, w_t|_{t=0})$. Extending these sums on the whole plane by the method above, we obtain initial data (37).

Bearing in mind that supports of functions $w_0^e(x)$ and $w_1^e(x)$ are in $\overline{\Omega}_\delta$, and supports of their derivatives with respect to all variables (including the third and the second orders, respectively) are located in $\overline{\Omega}_\delta$ too. The solution $w^s(t, x)$ has a compact support which is located in some bounded domain G_t in R^2 at each moment t because of the finite speed of the wave propagation. Let us take a sufficiently large circle D such as $\overline{G}_t \subset D$, $t \in [0, T_2]$. In this case function, $w^s(t, x)$ is thought as a solution of initial boundary value problem at the domain D with the homogeneous Dirichlet condition for $t \in [0, T_2]$. In virtue of the corresponding smoothness of initial conditions we obtain: $w^s(t, x) \in C([0, T_2]; H^4(D))$ and $w_t^s(t, x) \in C([0, T_2]; H^3(D))$. Then, the energy conservation law takes place:

$$\int\limits_D \left\{ \left(w_{x_1}^s(t, x)\right)^2 + \left(w_{x_2}^s(t, x)\right)^2 + \left(w_t^s(t, x)\right)^2 \right\} \mathrm{d}x$$
$$= \int\limits_D \left\{ \left(\frac{\partial w_0^e(x)}{\partial x_1} \right)^2 + \left(\frac{\partial w_0^e(x)}{\partial x_2} \right)^2 + \left(w_1^e(x)\right)^2 \right\} \mathrm{d}x, \quad t \in [0, T_2]. \tag{40}$$

Now differentiating Eq. (27) and initial conditions (28) with respect to variables x_1, x_2, we obtain the estimate

$$\|w^s(t, \cdot)\|'_{H^3(D)} \le \|w_0^e\|_{H^3(\Omega_\delta)} + \|w_1^e\|_{H^2(\Omega_\delta)},$$

where $\| \cdot \|'_{H^3(D)}$ is a seminorm (term

$$\int\limits_D (w^s(t, x))^2 \mathrm{d}x$$

is absent). The last statement is true because derivatives (of the second order in this case) of function $w^s(t, x)$ are identically zero at domain $D \setminus \overline{G}_t$, and hence they are solutions of differentiated initial boundary value problem with the homogeneous Dirichlet condition at the boundary of the domain D.

Then, the seminorm $\| \cdot \|'_{H^3(D)}$ is a norm. Therefore, we obtain

$$\|w^s(t, \cdot)\|_{H^3(D)} \le C_F \|w_0^e\|_{H^3(\Omega_\delta)} + C_F \|w_1^e\|_{H^2(\Omega_\delta)}.$$

Taking into account the last estimate and the Sobolev embedding theorem, we get

$$\|w^s(t,\cdot)\|_{C^1(\overline{\Omega})} \leq C_S\|w_0^e\|_{H^3(\Omega_\delta)} + C_S\|w_1^e\|_{H^2(\Omega_\delta)}. \tag{41}$$

Summing up, it is proven that the solution $w^s(t,x)$ can be made sufficiently small in the norm $C^1(\overline{\Omega})$ for any $t \in [0, T_2]$. The same argument may be applied to the solution of the inverse Cauchy problem with initial conditions $-w_0^{s,e}(T_2,x)$ and $-w_1^{s,e}(T_2,x)$. In this case, it is important that functions $w_0^{s,e}(T_2,x)$ and $w_1^{s,e}(T_2,x)$ in virtue of inequality (33) are "small" in $\mathcal{C}^4$, if $w_0^e(x)$ and $w_1^e(x)$ are "small". Hence, the restriction of the normal derivative of the solution to the Cauchy problem (27), (28) on the boundary of Ω (Neumann condition of the problem of controllability) is less than given ε with respect to absolute value. The latter means that the required restriction (4) on the control function $u(t,x)$ is satisfied.

5 Conclusions

According to the explicit form of C (see formula 11) and the formula for $u^{(1)}(t,x)$, we can conclude that the solution to the problem (1), (2) and (7) at the first stage is stabilized to this constant that tends to infinity when ε tends to zero. That is, the stricter the restriction on control, the farther away from zero is the terminal state of the system. In connection with this fact, a question of transfer of the membrane from rest (with nonzero shift) to another condition arises. This condition has null shift and velocity. The point is if it is possible to transfer during limited time and with the help of the absolute value limited control defined by the Neumann condition and applied to the boundary. This question still remains. However in case of the one-dimensional problem (the string equation), it can be solved easy enough. The string in rest with nonzero shift should be extended to null smoothly on the entire real axis. Then, watching the fluctuations of the string, after some time, we will see how the segment we are interested in will come to a state with zero shift and velocity. Obviously, this effect takes place due to the presence of a back front of fluctuations. The finiteness of control actions located at the ends of considered segment is provided in virtue of a method to extend a string profile to zero on the entire real axis. This extension should be "gentle" enough. Obviously, this technique for a membrane does not give desired result because there is no back front of fluctuations in the two-dimensional case.

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