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# Some Problems of Controllability of Distributed Systems Governed by Integrodifferential Equations ⋆

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**Abstract:** The article is devoted to the problems of controllability of some systems with memory. These systems are described by partial integrodifferential equations. Null controllability and controllability to rest of the solutions are considered. In contrast to partial differential equations, these terms are not identical to each other.



*Keywords:* Controllability to rest, null controllability, noncontrollability, Gurtin-Pipkin equation, systems with memory

## 1. INTRODUCTION

Integrodifferential equations with integral terms in a form of convolution often arise in various applications such as mechanics of heterogeneous medium, the theory of viscoelasticity, thermophysics, gas kinetic theory. In some cases the kernel of convolution is a sum (finite of infinite) of damped exponential functions. For example, theory of heterogeneous medium proves that a model, which describes two-phase medium (elastic medium and viscous fluid), is represented by the integrodifferential system with kernels of convolution in a form of finite and infinite sum of damped exponents. The laws of heat conducting with integral memory are studied in different researches (see, for example, Gurtin, Pipkin (1968)). Sometimes the integral memory in heat conducting may lead to appearance of a heat front, which moves with a finite velocity. This property is important as the propagation velocity of classical heat equation is infinite.

The questions of solvability and asymptotic behavior of solutions for equations of this type were investigated for in Dafermos (1970), Desh, Miller (1987). In Munoz Rivera, Naso, Vegni (2003), it was proved that the energy for some dissipative system decays polynomially when the memory kernel decays exponentially. The problem of null controllability is considered in Pandolfi (2005).

This article is devoted to the problems of boundary and distributed controllability. The aim of it is null controllability and (or) controllability to rest of considered systems. Note, that in general these terms are different for systems with memory because if a solution of this system achieves a null state then, generally speaking, it leaves this state in the future. The article also includes some numerical results illustrating the obstacles which may arise when the system with memory is controlled.

We do not focus on theorems of solvability of initial boundary value problems, here we consider the qualitative aspects of control theory of systems with memory.

## 2. CONTROLLABILITY TO REST FOR SOME DISTRIBUTED SYSTEM WITH MEMORY

Let us consider the problem of distributed controllability in the form:

$$\theta_t(t,x) - \int_0^t K(t-s)\theta_{xx}(s,x)ds = u(t,x), \qquad (1)$$

$$t>0, \quad x\in(0,\pi).$$

$$\theta|_{t=0} = \xi(x), \qquad (2)$$

$$\theta(t,0)=\theta(t,\pi)=0, \qquad (3)$$

where

$$K(t)=\sum_{j=1}^{N} c_j e^{-\gamma_j t}, \quad c_j,\gamma_j>0,\ j=1,...,N, \qquad (4)$$

and $u(t,x)$ is a control function, distributed (with respect to $x$) on the interval $(0,\pi)$.

For brevity we write $\theta(t)$ and $u(t)$ instead of $\theta(t,x)$ and $u(t,x)$ respectively. It means that $\theta(t)$ and $u(t)$ are functions of variable $t$, values of which are elements of some Banach space.

We consider an operator of the second derivative with the minus sign: $A=-\frac{d^2}{dx^2}$ and its domain

$$Dom(A) = H^2(0,\pi)\cap H_0^1(0,\pi).$$

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Let us expand formally the solution $\theta(t)$,the control $u(t)$ and the initial condition $\xi$ into a series of eigenfunctions of Dirichlet problem for the second derivative operator:

$$\theta(t,x)=\sum_{n=1}^{+\infty}\theta_n(t)\varphi_n(x),\ u(t,x)=\sum_{n=1}^{+\infty}u_n(t)\varphi_n(x),$$

$$\xi(x)=\sum_{n=1}^{+\infty}\xi_n\varphi_n(x),$$

where $\varphi_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx)$.

Then the given problem falls into a countable number of problems:

$$\dot{\theta}_n(t)=-n^2\int_0^t K(t-s)\theta_n(s)ds+u_n(t),\quad t>0,\quad (5)$$

$$\theta_n(0)=\xi_n.$$

We integrate the left and right parts of equation (5) within limits from null to $t$. Hence we get the Volterra equation of second type:

$$\theta_n(t)=-n^2\int_0^t Q(t-s)\theta_n(s)ds+\xi_n+f_n(t),\quad t>0,\ (6)$$

where

$$Q(s)=\int_0^s K(\tau)d\tau,$$

$$f_n(t)=\int_0^t u_n(\tau)d\tau.$$

According to Ivanov (2013) we define the space $\mathcal{H}_s$ as a domain of operator $A^{\frac{s}{2}}$, where $A$ is mentioned second derivative operator, which is taken with a sign minus and has a domain $Dom(A)$. In this case parameter $s$ is a real number. It is known that space $\mathcal{H}_s$ can be effectively described by the following construction. Let us consider the space of series $l_s$, defined as:

$$l_s=\left\{\{a_n\}:\sum_{n=1}^{+\infty}|a_n|^2n^{2s}<+\infty\right\}.$$

Hence the equality takes place:

$$\mathcal{H}_s=\left\{w(x)=\sum_{n=1}^{+\infty}w_n\varphi_n(x):\{w_n\}\in l_s\right\}.$$

Now, as in Ivanov (2013), we define $\mathfrak{H}_{s,\varepsilon}$ as a space of functions

$$w(t,x)=\sum_{n=1}^{+\infty}w_n(t)\varphi_n(x),$$

supplied by the norm

$$\|w\|^2_{\mathfrak{H}_{s,\varepsilon}}=\sum_{n=1}^{+\infty}n^{2s}\|e^{-2\varepsilon t}w_n(t)\|^2_{L_2(0,+\infty)},$$

where $\varepsilon>0$ is some number.

The function $\theta(t,x)=\sum\theta_n(t)\varphi_n(x)$ from $\mathfrak{H}_{s,\varepsilon}$ is called *a solution of problem* (1)—(3) if all functions $\theta_n$ satisfy the Volterra equation (6).

The article Ivanov (2013) proves that for any right part $u(t)\in L_2(0,+\infty;\mathcal{H}_s)$, the initial condition $\xi\in\mathcal{H}_s$ and $\varepsilon>0$ the unique solution of problem (1)—(3) exists.

Let us define the term of controllability of system (1)—(3) to rest. We consider some initial perturbation $\xi\in L_2(0,\pi)$.

*Definition 1.* We say that system (1)—(3) is controllable to rest for this $\xi$ if there are a time instant $T>0$ and function $u(t)\in C([0,T];L_2(0,\pi))$ such that $u(t)\equiv 0$ at $t>T$, for which the Fourier coefficients $\theta_n(t)$ of considered solution $\theta(t)$ are equal to zero identically at a set $t>T$.

Let $PW_+$ denote the linear space of the Laplace transforms of elements of $L_2(0,+\infty)$ such that they are equal to zero on the set $\{t:t>T\}$ for some $T>0$. It is a well-known fact that $\varphi(\lambda)\in PW_+$ if and only if it is an entire function, such that

1) there are real numbers $C$ and $T$ such that $|\varphi(\lambda)|\leqslant Ce^{T|\lambda|}$. Note that $C$ and $T$ depend on $\varphi(\lambda)$.

2) $\sup\limits_{x\geqslant 0}\int_{-\infty}^{+\infty}|\varphi(x+iy)|^2dy<+\infty$.

Let us define space $PW_+^T$, which consists of the Laplace transform of all elements $f(t)\in L_2(0,T)$, extended by zero when $t>T$.

We make the Laplace transform of equation (5):

$$\lambda\hat{\theta}_n(\lambda)-\xi_n=-n^2\hat{K}(\lambda)\hat{\theta}_n(\lambda)+\hat{u}_n,$$

where

$$\hat{\theta}_n(\lambda)=\int_0^{+\infty}e^{-\lambda t}\theta_n(t)dt.$$

Hence,

$$\hat{\theta}_n(\lambda)=\frac{\xi_n+\hat{u}_n(\lambda)}{\lambda+n^2\hat{K}(\lambda)};\quad \hat{K}(\lambda)=\sum_{j=1}^{N}\frac{c_j}{\lambda+\gamma_j}.$$

We obtain:

$$\lambda+n^2\hat{K}(\lambda)=\lambda+n^2\sum_{j=1}^{N}\frac{c_j}{\lambda+\gamma_j},$$

and reduce to a common denominator: $(\lambda+\gamma_1)...(\lambda+\gamma_N)$. In formula for $\hat{\theta}_n(\lambda)$ this denominator is multiplied by numerator of fraction

$$\frac{\xi_n+\hat{u}_n(\lambda)}{\lambda+n^2\hat{K}(\lambda)},$$

Obviously, that equation $\lambda+n^2\hat{K}(\lambda)=0$ has $N+1$ roots. It is proved (see Ivanov, Sheronova (2010)), that these roots are different and have negative real parts. We denote them as $\mu_{1,n},\mu_{2,n},...,\mu_{N+1,n}$.

*Theorem 1.* Suppose that for some initial perturbation $\xi$ and time instant $T$ there is a control function $u(t)\in C([0,T];L_2(0,\pi))$ such that $u(t)\equiv 0$ at $t>T$ and satisfies the following conditions:

$$\hat{u}_n(\mu_{i,n})=-\xi_n,\ i=1,2,...,N+1,\quad (7)$$

for all natural $n$. Then system (1)—(3) is controllable to rest for this initial condition $\xi$.

**Proof.**

Note, that conditions (7) for each $n$ is the moments problem:

$$\int_0^T u_n(s)e^{-\mu_{i,n}s}ds = -\xi_n,\ i = 1, 2, ..., N+1.$$

We set $\hat{\theta}_n(\lambda)$ in a form

$$\hat{\theta}_n(\lambda) = \frac{(\xi_n + \hat{u}_n(\lambda))(\lambda+\gamma_1)...(\lambda+\gamma_N)}{(\lambda-\mu_{1,n})(\lambda-\mu_{2,n})\cdots(\lambda-\mu_{N+1,n})}. \quad (8)$$

The article Romanov, Shamaev (2016) proves, that functions

$$\frac{\hat{u}_n(\lambda)+\xi_k}{\lambda-\mu_{i,n}},\ i = 1, 2, ..., N+1,$$

are in $PW_+^T$ for any natural $n$. There is the following lemma

*Lemma 2.* (The analogue of Levinson theorem) Let function $\hat{f}(\lambda)$ be in space $PW_+^T$, moreover $\hat{f}(\mu) = 0$, $\mu \in \mathbb{C}$. Then

$$\frac{\hat{f}(\lambda)}{\lambda-\mu}$$

is also in $PW_+^T$.

We set the notation:

$$U_1(\lambda) = \frac{\hat{u}_k(\lambda)+\xi_n}{\lambda-\mu_{1,k}}.$$

There is an obvious equality:

$$(\lambda+\gamma_1)\frac{U_1(\lambda)}{\lambda-\mu_{2,k}} = U_1(\lambda) + \frac{\mu_{2,k}+\gamma_1}{\lambda-\mu_{2,k}}U_1(\lambda).$$

Lemma 2 and the last equality imply that function

$$U_2(\lambda) = (\lambda+\gamma_1)\frac{U_1(\lambda)}{\lambda-\mu_{2,k}}$$

is in space $PW_+^T$, because for example function $U_1$ has got a null $\mu_{2,n}$. The following equality is right for function $U_2$:

$$(\lambda+\gamma_2)\frac{U_2(\lambda)}{\lambda-\mu_{3,k}} = U_2(\lambda) + \frac{\mu_{3,k}+\gamma_2}{\lambda-\mu_{3,k}}U_2(\lambda).$$

Analogously we obtain that $\hat{\theta}_n(\lambda)$ is also in space $PW_+^T$.

*Remark 1.* If for example initial perturbation $\xi$ has only finite number of nonzero Fourier coefficients, i. e. it can be represented in a form $\xi(x) = \sum_{n=1}^N \xi_n\varphi_n$, then conditions of Theorem 1 are satisfied. It is true, because roots $\mu_{i,n}$ are different for each $n$ and corresponding moments problem (7) is solvable. Here we have to solve finite number of finite dimensional moments problems. In the case if initial condition has infinite number of nonzero coefficients then countable number of finite dimensional moments problems of $N+1$ dimension have to be solved. Methods for such moments systems and similar problems are considered in Romanov, Shamaev (2013) and Romanov, Shamaev (2015).

## 3. PROBLEM OF BOUNDARY NONCONTROLLABILITY OF A ROD WITH MEMORY

We consider the following problem of boundary noncontrollability:

$$\theta_t(t,x) + \int_0^t G(t-s)\theta_{xxxx}(s,x)ds = 0, \quad (9)$$

$$t > 0, \quad x \in (0,\pi).$$

$$\theta|_{t=0} = \xi(x), \quad (10)$$

$$\theta(t,0) = \theta(t,\pi) = 0,\ \theta_{xx}|_{x=0} = v(t),\ \theta_{xx}|_{x=\pi} = 0. \quad (11)$$

If we differentiate equation (9) with respect to $t$, we obtain the equation of oscillation of a rod with additional integral term ("memory").

Consider the control function $v(t) \in L_2^{loc}(0,+\infty)$ and the initial condition $\xi \in L_2(0,\pi)$. Let

$$W = \{\varphi \in H^4(0,\pi) : \varphi(0) = \varphi(\pi) = \\ = \varphi_{xx}(0) = \varphi_{xx}(\pi) = 0\}.$$

We formally multiply (in the sense of the inner product in $L_2(0,\pi)$) both parts of (9) by the function $\varphi \in W$. Next, using integrating by parts, we formally replace the operator $\frac{d^4}{dx^4}$ from $\theta(t)$ to $\varphi$. Then, we obtain

$$\frac{d}{dt}\langle\theta(t),\varphi\rangle + \int_0^t G(t-s)\left(\langle\theta(s),\varphi_{xxxx}\rangle\right.$$

$$\left.+\varphi_{xxx}(0)v(s)\right)ds = 0, \quad (12)$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(0,\pi)$.

We say that the function $\theta(t) \in H^1_{loc}([0,+\infty); L_2(0,\pi))$ is *the solution* to problem (9)–(11) if the equality in (12) holds for every $\varphi \in W$ and $\theta(0) = \xi$. Formally we do not know if the solution of problem (9)—(11) exists relatively to the given definition. The proof of results bellow is constructed on the assumption that the solution exists. More exactly, we consider only those initial conditions $\xi$ and control functions $v(t)$ for which there is the solution of (9)—(11).

We think that the given definition of solution is "natural" in some sense. Also we suppose that it is possible to prove the theorem of existence under additional conditions of smoothness of kernel $G(t)$, the control $v(t)$ and the initial condition $\xi$.

We set in (12) $\varphi = \varphi_n$, which is defined above. Then we obtain:

$$\dot{\theta}_n(t) + \int_0^t G(t-s)\left(n^4\theta_n(s) - n^3\sqrt{\frac{2}{\pi}}v(s)\right)ds = 0. \quad (13)$$

Next, we take the Laplace transformation of both parts of (13) and express $\hat{\theta}_n(\lambda)$ as

$$\hat{\theta}_n(\lambda) = \frac{\xi_n + n^3\sqrt{\frac{2}{\pi}}\hat{G}(\lambda)\hat{v}(\lambda)}{\lambda + n^4\hat{G}(\lambda)}, \quad (14)$$

where $\hat{G}(\lambda)$ is the Laplace transform of the kernel $G(t)$.

Now we state the theorem which is analogous to results in researches Ivanov, Pandolfi (2009) and Romanov, Shamaev (2016) for similar problems. It considers the case when the operator of the second derivative is in the integral term of the main equation. Note that Romanov, Shamaev (2016) considers two-dimensional Gurtin-Pipkin equation and a control, which is distributed on a subdomain, the boundary of a two-dimensional domain being fixed. The proof of our theorem is constructed with a help of method used in Ivanov, Pandolfi (2009).

*Theorem 3.* In problem (9)—(11), if $\hat{G}(\lambda)$ has at least one root $\lambda_0$ in the domain of holomorphism (we require that this domain exists), then controllability to rest is impossible; that is, there exists an initial condition $\xi \in L_2(0,\pi)$, such that for every control

$$v \in L_2^{loc}(0,+\infty),$$

which is equal to zero on the set $\{t : t > T\}$ for some $T > 0$, the corresponding solution can not be equal to zero identically outside any finite segment (in $t$).

**Proof.** If $\lambda_0 = 0$ is a root of the equation $\hat{G}(\lambda) = 0$, then the equality (14) generally cannot be satisfied for the values $\hat{\theta}_n(\lambda)$, which correspond to functions from $PW_+$. Hence, in this case controllability to rest is impossible.

Let $\lambda_0 = 0$ be not a root of the equation $\hat{G}(\lambda) = 0$. The control function $\hat{v}(\lambda)$ has to satisfy the following equalities:

$$\hat{v}(\lambda) = \sqrt{\frac{\pi}{2}}\frac{n}{\lambda}\xi_n, \qquad (15)$$

when $\lambda \neq 0$ is a root of the equation $\lambda + n^4\hat{G}(\lambda) = 0$.

Note that the equality in (15) can be presented in the following form:

$$\int_0^T v(t)e^{-\lambda t}dt = \sqrt{\frac{\pi}{2}}\frac{n}{\lambda}\xi_n. \qquad (16)$$

The latter equality is the so-called moments problem.

Note that $\hat{G}(\lambda)$ has a root $\lambda_0 \neq 0$ (if $G(t)$ is a series of decreasing exponentials, then $\hat{G}(\lambda)$ has a countable number of roots, see Ivanov, Sheronova (2010)). Applying the methods used in Ivanov, Pandolfi (2009) (in which Rouche's theorem was used), we can prove that there exists a sequence $\{\lambda_n \neq 0\}$ of zeros of

$$\lambda + n^4\hat{G}(\lambda).$$

It is important that this sequence converges to a nonzero complex number. Let us choose $\xi_{2n+1} = 0$; hence, $\hat{v}(\lambda_{2n+1}) = 0$. As the sequence of zeros converges and $\hat{v}(\lambda)$ is an entire function, then $\hat{v}(\lambda) \equiv 0$. Then for any $n$, all $\xi_{2n}$ must be zero. However, we can always take some of them to be nonzero numbers. Thus, there exists an initial condition $\xi$ such that for any control function $v(t)$, controllability to rest is impossible.

*Remark 2.* If kernel $G(t)$ has the form of (4) and $N \geq 2$, then a solution of system (9)—(11) is not controllable to rest.

## 4. THE PROBLEM OF NULL CONTROLLABILITY. NUMERICAL EXPERIMENTS

The results of the section are nonstrict and experimental. Consider the problem (9)—(11). Here, as in Section 2, we concern the kernel in the form (4). Let us formally differentiate equation (9) with respect to $t$. Then we obtain:

$$\theta_{tt}(t,x) + G(0)\theta_{xxxx}(s,x) + \int_0^t G'(t-s)\theta_{xxxx}(s,x)ds = 0. \qquad (17)$$

For this equation we have two initial conditions for $\theta(0)$ and for $\theta_t(0)$. Let

$$\theta|_{t=0} = \xi(x), \;\; \theta_t|_{t=0} = 0. \qquad (18)$$

Then the solution of (17), (18), (11) is the solution of (9)—(11). Note that we can define the solution of (17), (18), (11) in analogous way as we have done it in the previous section. We state a question whether the last system is null controllable by means of boundary control $v(t)$. It means that $\theta|_{t=T} = 0$, $\theta_t|_{t=T} = 0$ at some $T > 0$.

Let us expand the solution of (17), (18), (11):

$$\theta(t,x) = \sum_{n=1}^{+\infty}\theta_n(t)\varphi_n(x).$$

Then we obtain equations (for $n = 1, 2, ...$)

$$\ddot{\theta}_n(t) + G(0)\left(n^4\theta_n(t) - n^3\sqrt{\frac{2}{\pi}}v(t)\right) + \int_0^t G'(t-s)\left(n^4\theta_n(s) - n^3\sqrt{\frac{2}{\pi}}v(s)\right)ds = 0 \qquad (19)$$

and initial conditions

$$\theta_n(0) = \xi_n, \;\; \dot{\theta}_n(0) = 0. \qquad (20)$$

Using these initial conditions we can see that equations (13) and (19) are equivalent.

Let us introduce the following notation:

$$w(t) = \sqrt{\frac{2}{\pi}}\int_0^t G(t-s)v(s)ds. \qquad (21)$$

Therefore equations (13) have the form:

$$\dot{\theta}_n(t) + n^4\int_0^t G(t-s)\theta_n(s)ds = n^3 w(t), \quad n = 1, 2, ... \qquad (22)$$

Furthermore, there is the initial condition $\theta_n(0) = \xi_n$ for each equation (22). The equation (22) should be solved for each $n$. For this purpose we make the Laplace transform of the both parts of (22) and obtain

$$\hat{\theta}_n(\lambda) = \frac{\xi_n + n^3\hat{w}(\lambda)}{\lambda + n^4\hat{G}(\lambda)}, \qquad (23)$$

Introduce the formal notation:

$$Q_n(t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{e^{\lambda t}}{\lambda + n^4\hat{G}(\lambda)}d\lambda.$$

Then, formally using Mellin's inverse formula, we get the solution of the Cauchy problem of equation (22) for any $n$:

$$\theta_n(t) = \xi_n Q_n(t) + n^3\int_0^t Q_n(t-s)w(s)ds. \qquad (24)$$

Let us define number $\delta$ in the formula of $Q_n(t)$. It is obvious that the equation

$$\lambda + n^4\hat{G}(\lambda) = 0 \qquad (25)$$

has $N+1$ roots. Using methods from Ivanov, Sheronova (2010) we may show that real parts of roots of the equation for any natural $n$ are negative and there may be only two complex-conjugate roots of one multiplicity. Note there are no complex roots in some cases, but for given kernel $G(t)$ in the form (4) there is always a number $n_1$ such that the equation (25) has mentioned complex roots for any $n > n_1$. Let us consider homogeneous equation (22) (the right part is equal to null). The explicit form of

$$\frac{1}{\lambda + n^4 \hat{G}(\lambda)}$$

and described above structure of roots of the equation (25) imply that each solution of this homogeneous equation tends to zero exponentially when $t \to +\infty$. Hence we may set $\delta$ equal to zero in the formula of $Q_n(t)$.

Then we get:

$$Q_n(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{e^{i\alpha t}}{i\alpha + n^4 G_1(\alpha)} d\alpha,$$

where

$$G_1(\alpha) = \int\limits_0^{+\infty} G(t) e^{-i\alpha t} dt.$$

We use formula (24) in order to drive the solution of problem (17), (18), (11) to the null state at some instant $T > 0$. We have:

$$\int\limits_0^t Q_n(t-s) w(s) ds = \sqrt{\frac{2}{\pi}} Q_n * (G * v) =$$

$$\sqrt{\frac{2}{\pi}} (Q_n * G) * v = \sqrt{\frac{2}{\pi}} \int\limits_0^t R_n(t-s) v(s) ds,$$

where

$$R_n(\tau) = \int\limits_0^\tau Q_n(\tau - p) G(p) dp.$$

Therefore we obtain:

$$\int\limits_0^T R_n(T-s) v(s) ds = -\frac{1}{n^3} \sqrt{\frac{\pi}{2}} \xi_n Q_n(T),\ n = 1, 2, ..., \tag{26}$$

$$\int\limits_0^T R_n'(T-s) v(s) ds = -\frac{1}{n^3} \sqrt{\frac{\pi}{2}} \xi_n Q_n'(T),\ n = 1, 2, \dots . \tag{27}$$

The system of integral equations (26), (27) is infinite-dimensional moments problem. We do not know whether this problem has a solution. Next we try to construct a function $v(s)$, which is a solution of only $N_1$ first equations of systems (26) and (27) under the assumption that perturbation of such control is "small enough".

Consider the first $N_1$ equations of systems (26) and (27). $v(s)$ is seek in the form:

$$v(s) = \sum_{n=1}^{N_1} D_n R_n(T-s) + \sum_{n=1}^{N_1} L_n R_n'(T-s), \tag{28}$$

where $D_n$ and $L_n$ — unknown constants. We substitute (28) to (26), (27). In order to find $D_n$, $L_n$ we use the first $N_1$ equations of systems (26), (27). As a result we get a system of linear algebraic equations: $A\overline{C} = \overline{b}$. Here $A$ is a matrix:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where $A_i$, i=1,2,3,4, are matrixes such that

$$A_1 = \left\{ \int\limits_0^T R_k(T-s) R_j(T-s) ds \right\}_{k,j=1,2,\dots,N_1},$$

$$A_2 = \left\{ \int\limits_0^T R_k(T-s) R_j'(T-s) ds \right\}_{k,j=1,2,\dots,N_1},$$

$$A_3 = \left\{ \int\limits_0^T R_k'(T-s) R_j(T-s) ds \right\}_{k,j=1,2,\dots,N_1},$$

$$A_4 = \left\{ \int\limits_0^T R_k'(T-s) R_j'(T-s) ds \right\}_{k,j=1,2,\dots,N_1}$$

and

$$\overline{C} = \begin{pmatrix} \overline{D} \\ \overline{L} \end{pmatrix},\ \bar{b} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \end{pmatrix},$$

where

$$\overline{D} = \{D_n\},\ \overline{L} = \{L_n\},$$

$$\bar{b}_1 = \left\{ -\frac{1}{n^3} \sqrt{\frac{\pi}{2}} \xi_n Q_n(T) \right\},$$

$$\bar{b}_2 = \left\{ -\frac{1}{n^3} \sqrt{\frac{\pi}{2}} \xi_n Q_n'(T) \right\}$$

are column vectors with indecies $n = 1, 2, ..., N_1$. If determinant of matrix $A$ is nonzero, then $\overline{C}$ is uniquely determined.

Now let us observe the numerical experiments. For simplicity we use kernel $G(t)$ as a sum of two exponential finctions:

$$G(t) = e^{-t} + e^{-2t}.$$

For numerical experiments we replace the expansion of solution $\theta(t, x)$ into series by the finite sum:

$$\sum_{n=1}^{N^*} \theta_n(t) \varphi_n(x).$$

As the initial condition we use a function $\xi(x) = 10\varphi_1(x)$. Fig. 1 illustrates the solution of the problem without control and Fig. 2 shows the solution with the control function.

The numerical experiments show that constructed type of control, which reduces oscillations of the rod without memory, does not give the same results in case when the equation has memory term. The amplitude of oscillations decreases a little, not essentially. That is why the important question arises what a control function should be in order to minimize the amplitude of oscillations and their velocity if there is an integral term in the equation (time for control is given). It is the aim of further research.

## REFERENCES


C. M. Dafermos. Asymptotic stability in viscoelasticity. *Arch.Ration.Mech.Anal.*, 37:297-308, 1970.

W. Desh, R. K. Miller. Exponential stabilization of Volterra integrodifferential equations in Hilbert space. *J. Differ. Equ.*, 70:366-389, 1987.

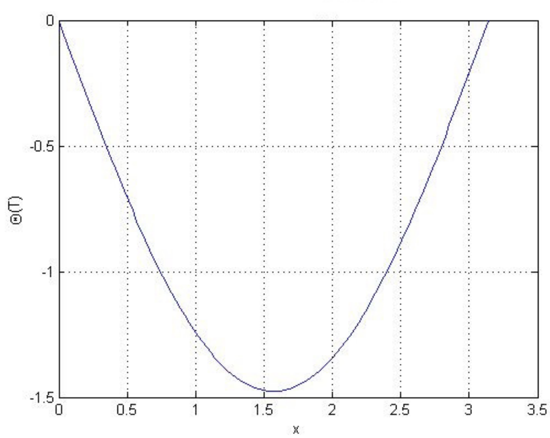


Fig. 1. The solution without control at $t = T$ ($N^* = 20$, $T = 2.5$, $v = 0$)

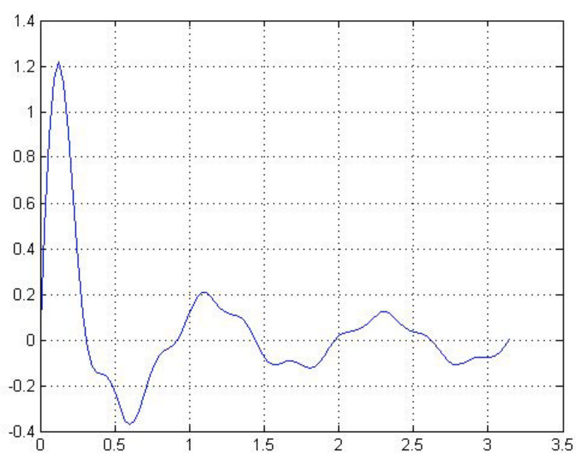


Fig. 2. The solution with control at $t = T$ ($N^* = 20$, $N_1 = 5$, $T = 2.5$, $v \neq 0$)

M. E. Gurtin, A. C. Pipkin. Theory of Heat Conduction with Finite Wave Speed. *Arch. Ration. Mech. Anal.*, 31: 113–126, 1968.

S. Ivanov. Regularity of the Gurtin-Pipkin Equation. *arXiv. doi* 1205.0616v3:12 p, 2013.

S. Ivanov, L. Pandolfi. Heat Equation with Memory: Lack of Controllability to Rest. *Journal of Mathematical Analysis and Applications*, 355 (1):1–11, 2009.

S. Ivanov, T. Sheronova. Spectrum of the Heat Equation with Memory. *arXiv. doi* 0912.1818v3, 10 p, 2010.

J. E. Munoz Rivera, M. G. Naso, F. M. Vegni. Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory. *Journal of Mathematical Analysis and Applications* 286:692-704, 2003.

L. Pandofi. The controllability of the GurtinPipkin equations: a cosine operator approach. *Appl. Math. Optim.* 52:143-165, 2005.

I. Romanov, A. Shamaev. Exact Controllability of the Distributed System Governed by String Equation with Memory. *Journal of Dynamical and Control Systems*, 19(4):611–623, 2013.

I. Romanov, A. Shamaev. Exact Controllability of the Distributed System Governed by the Wave Equation with Memory. *arXiv. doi* 1503.04461v2, 2015.

I. Romanov, A. Shamaev. On Problems of Distributed and Boundary Control of Systems with Integral Memory. *Trudy of Petrovskiy seminar (in Russian)*, 31:135-158, 2016.

I. Romanov, A. Shamaev. Noncontrollability to Rest of the Two-Dimensional Distributed System Governed by the Integrodifferential Equation. *Journal of Optimization Theory and Applications*, 170 (3):772–782, 2016.