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CONTROL OF SYSTEMS WITH DISTRIBUTED PARAMETERS

On a Boundary Controllability Problem for a System Governed by the Two-Dimensional Wave Equation

I. V. Romanov[a,*] **and A. S. Shamaev**[b,c,**]

[a] *National Research University Higher School of Economics, Moscow, 109028 Russia*
[b] *Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russia*
[c] *Moscow State University, Moscow, Russia*
**e-mail: romm1@list.ru*
***e-mail: sham@rambler.ru*

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Abstract—The boundary controllability of oscillations of a plane membrane is studied. The magnitude of the control is bounded. The controllability problem of driving the membrane to rest is considered. The method of proof proposed in this paper can be applied to any dimension but only the two-dimensional case is considered for simplicity.

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INTRODUCTION

The problem of driving a two dimensional membrane to rest using a bounded control force applied on the membrane boundary in a finite amount of time is solved. In the statement of the control problem, the initial data (displacement and velocity) are assumed to be smooth and satisfy certain boundary conditions. The force boundary control is determined by the inhomogeneous Neumann condition. The smoothness conditions imposed on the initial data in this paper are significantly weaker than the conditions used in [1].

The solution of the problem is divided into two phases. In the first phase, the solution is stabilized in a small neighborhood of the state of rest using the friction applied on the domain boundary. By choosing a small friction coefficient, we ensure that the control is small. We use the results obtained in [2–4] concerning the stabilization of membrane energy using boundary conditions of a special form. In the second phase of control, the membrane oscillations are completely suppressed. Here an important role is played by the method used to extend the initial data to a bounded domain and the consideration of a special initial-boundary value problem for the two-dimensional wave equation in this domain. In this case, the control is the normal derivative of the solution to this initial-boundary value problem on the boundary of the domain occupied by the membrane. Note that the control in this phase is actually determined by the method used to continue the initial data to the bounded domain mentioned above. In this construct, the key role is played by the time reversibility of the classical wave equation. Such a control was used in works by many authors in the 1970–1990s. In this case, the absolute value of the control force is bounded because the solution to the original problem was in the first phase driven to a small neighborhood in the norm of a Sobolev space. The rigorous mathematical definitions will be given below.

Note that the method of proof presented in this paper can be used for any dimension, but we consider only the two-dimensional case for the clearness and simplicity of presentation.

The controllability to rest in finite time for the case of distributed control was proved in [5]. In that paper, an upper bound on the optimal control time was also obtained.

Earlier, the control of membrane vibrations using boundary forces was studied by many authors (e.g., see reviews [2, 6] and references therein). In [7], the problem of damping the oscillations of a bounded string using boundary control is studied. It is proved that the oscillations of the string can be completely damped in a finite amount of time using bounded control, and the time needed for complete damping is estimated. In [8], the optimal control of systems with distributed parameters and a finite number of degrees of freedom is considered, and optimality conditions that are similar to Pontryagin's maximum principle are formulated. However, these conditions do not always give a constructive way for building the

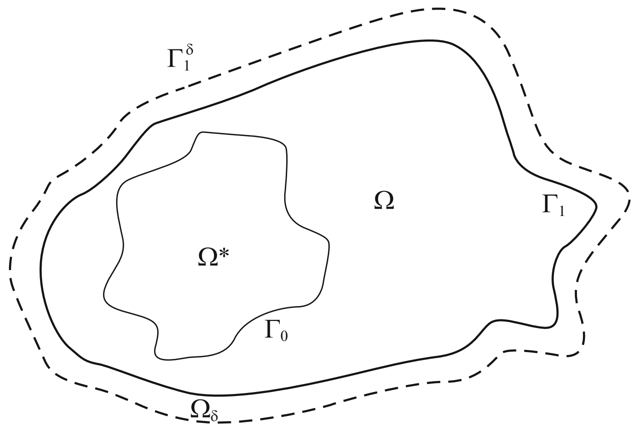

Fig. 1.

optimal control. In [6], the problem of driving an oscillating membrane to rest is studied, the existence of the corresponding boundary control is proved, and the time needed to stop the oscillations is estimated. In many problem statements, the authors drop the optimality conditions and consider only the controllability problem, which significantly simplifies the analysis. In [6], only existence theorems are proved without imposing constraints on the absolute value of the control forces, and no explicit expressions for the controls are obtained.

The statement of the problem considered in this paper is significantly different from the statement presented in [2, 6] because the magnitude of the control force on the boundary of the domain must satisfy the condition $|u(t,x)| \le \varepsilon$. Note that we do not seek the optimal control but rather a feasible (i.e., satisfying the given constraints) control.

1. STATEMENT OF THE PROBLEM

Let Ω be a bounded domain in R^2 with an infinitely smooth boundary, ν be the unit outward normal to the boundary of Ω, and Σ be the lateral surface of the cylinder $Q_T = (0,T)\times\Omega$. Let the boundary of Ω consist of two parts Γ_0 and Γ_1, i.e.,

$$\partial\Omega = \Gamma_0 \cup \Gamma_1.$$

We also assume that

$$\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \varnothing.$$

Therefore, Γ_0 must be the boundary of a bounded domain Ω^* such that $\Omega \cap \Omega^* = \varnothing$ (see Fig. 1).

Introduce the notation

$$\Sigma_i = (0,T)\times\Gamma_i, \quad i = 0,1.$$

Consider the initial-boundary value problem for the equation of membrane oscillations

$$w_{tt}(t,x) - \Delta w(t,x) = 0, \quad (t,x)\in Q_T, \tag{1.1}$$

$$w\big|_{t=0} = \varphi(x), \quad w_t\big|_{t=0} = \psi(x), \quad x\in\Omega, \tag{1.2}$$

$$w(t,x) = 0, \quad (t,x)\in\Sigma_0, \tag{1.3}$$

$$\frac{\partial w}{\partial \nu} = u(t,x), \quad (t,x)\in\Sigma_1. \tag{1.4}$$

Let $\varepsilon > 0$ be an arbitrary number. The problem is to construct a control u satisfying the inequality

$$|u(t,x)| \le \varepsilon \tag{1.5}$$

such that the corresponding solution w and its derivative with respect to t vanish at a certain time T, i.e., $w(T, x) = 0$ and $w_t(T, x) = 0$ for all $x \in \Omega$. If this problem has a solution, then system (1.1)–(1.3) is said to be *controllable*.

Consider the space

$$\mathcal{H}_0^3(\Omega) = \{(w_1, w_2) \in H^3(\Omega) \times H^2(\Omega) : w_1(x) = w_2(x) = \Delta w_1 = 0,\ x \in \Gamma_0\}.$$

The following theorem is the main result of this paper.

Theorem. *Let the boundary* Ω *additionally satisfy the following condition: there exists a point* $x_0 \in R^2$ *such that*

(1) $(x - x_0) \cdot \nu \leq 0,\ x \in \Gamma_0,$

(2) $(x - x_0) \cdot \nu \geq \beta > 0,\ x \in \Gamma_1$; *moreover, let* $(\varphi(x), \psi(x)) \in \mathcal{H}_0^3(\Omega)$ *and*

$$\frac{\partial \varphi}{\partial \nu} = \psi = \frac{\partial \psi}{\partial \nu} = \Delta\varphi = 0, \qquad x \in \Gamma_1. \tag{1.6}$$

Then system (1.1)–(1.3) is controllable.

The proof of this theorem consists of two phases. In the first phase, the solution and its time derivative are driven to a small neighborhood of zero in the norm of $\mathcal{H}_0^3(\Omega)$; in the second phase, the system is driven to rest from this neighborhood.

Note that the initial data in the control problem must be sufficiently smooth. This is explained by the fact that the proof uses Sobolev's embedding theorem for replacing Sobolev's norm by the norm in the space of smooth functions. In this case, the order of smoothness is decreased.

2. FIRST PHASE OF CONTROL

In the first phase, we drive the solution to system (1.1)–(1.4) and its first derivative with respect to t to an arbitrarily small neighborhood of zero in the norm of the space $\mathcal{H}_0^3(\Omega)$. The control used for this purpose must satisfy constraint (1.5).

For this purpose, we use the results obtained in [3, 4]. The idea underlying these results is that friction determined by the derivative of $w(t, x)$ with respect to t is introduced on Γ_1; i.e., the initial-boundary value problem (1.1)–(1.3) subject to the boundary condition

$$\frac{\partial w(t,x)}{\partial \nu} = -k\frac{\partial w(t,x)}{\partial t}, \qquad x \in \Gamma_1, \tag{2.1}$$

where $k > 0$ is the friction coefficient, is considered. Let us briefly discuss the solvability issues of this initial-boundary value problem and the regularization of its solution.

Introduce the notation

$$H = L_2(\Omega), \quad V = H^1_{\Gamma_0}(\Omega),$$

where

$$H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) : v(x) = 0,\ x \in \Gamma_0\}.$$

In the space $V \times H$, we define the unbounded operator

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

with the domain

$$D(\mathfrak{A}) = \left\{(w_1, w_2) \in H^2(\Omega) \times H^1(\Omega) : w_1(x) = w_2(x) = 0,\ x \in \Gamma_0;\ \frac{\partial w_1}{\partial \nu} = -kw_2,\ x \in \Gamma_1\right\}.$$

It is known that the norm in the space $D(\mathfrak{A})$ squared can be determined by

$$\|(w_1, w_2)\|^2_{D(\mathfrak{A})} = \|(w_1, w_2)\|^2_{V\times H} + \|\mathfrak{A}(w_1, w_2)\|^2_{V\times H}. \tag{2.2}$$

Consider the system of differential equations

$$\overline{w}_t = \mathfrak{A}\overline{w}, \tag{2.3}$$

where $\overline{w} = (w_1, w_2)$.

It is known (e.g., see [4] and work [9] cited therein) that $\mathfrak{A}$ is the generating operator of the contraction semigroup $e^{t\mathfrak{A}}$, i.e. of the semigroup for which

$$\|e^{t\mathfrak{A}}\| \le 1.$$

Note that $\mathfrak{A}$ is a dissipative operator. Indeed, for any $\overline{v} \in D(\mathfrak{A})$, we have

$$(\mathfrak{A}\overline{v}, \overline{v})_{V\times H} = ((v_2, \Delta v_1), (v_1, v_2))_{V\times H} = \int_\Omega \frac{\partial v_2}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}\frac{\partial v_1}{\partial x_2} dx + (\Delta v_1, v_2)_H$$
$$= \int_{\Gamma_1} \frac{\partial v_1}{\partial \nu} v_2 d\Gamma = -k\int_{\Gamma_1} v_2^2 d\Gamma \le 0,$$

which implies dissipativity.

It is known from the theory of continuous semigroups that, if the pair of initial data (φ, ψ) is an element in the space $D(\mathfrak{A}^k)$ $(k = 0, 1, 2, \ldots)$, then the corresponding solution to system (2.3) satisfies the relation

$$(w_1(t), w_2(t)) \in C([0, T]; D(\mathfrak{A}^k)).$$

Assume that $(\varphi, \psi) \in V \times H$. It was proved (see [3] or [4] but for weaker conditions on the domain boundary) that the energy of the system satisfies the inequality

$$E(t) \le Me^{-2\gamma t}E(0), \quad t \ge 0, \tag{2.4}$$

where

$$E(t) = \int_\Omega \{w_{1,x_1}^2(t,x) + w_{1,x_2}^2(t,x) + w_2^2(t,x)\}dx$$

is the energy of the system and the positive constants M and γ are independent of the initial data.

Let $(\varphi, \psi) \in D(\mathfrak{A})$ and $(w_1(t), w_2(t))$ be the solution corresponding to this initial data. Apply the operator $\mathfrak{A}$ to Eq. (2.3) and the initial data (1.2). Then we obtain

$$\frac{d}{dt}\mathfrak{A}\overline{w}(t) = \mathfrak{A}^2\overline{w}(t), \quad \mathfrak{A}\overline{w}(0) = \mathfrak{A}(\varphi, \psi).$$

Note that

$$\mathfrak{A}(w_1(t), w_2(t)) = (w_2(t), \Delta w_1(t)). \tag{2.5}$$

Then, (2.4) and (2.5) imply

$$\int_\Omega \{w_{2,x_1}^2(t) + w_{2,x_2}^2(t) + (\Delta w_1(t))^2\}dx \le Me^{-2\gamma t}\int_\Omega \{\psi_{x_1}^2 + \psi_{x_2}^2 + (\Delta\varphi)^2\}dx. \tag{2.6}$$

Combining (2.2), (2.4), and (2.6), we obtain

$$\|(w_1(t), w_2(t))\|_{D(\mathfrak{A})} \le M_1 e^{-\gamma t}\|(\varphi, \psi)\|_{D(\mathfrak{A})}, \quad t \ge 0. \tag{2.7}$$

Let the initial data belong to $D(\mathfrak{A})$. Then, the theory of elliptic boundary value problems (see [10, p. 98]) implies that the corresponding solution satisfies the bound

$$\|w_1(t)\|_{H^2(\Omega)} \le N_1\left(\|\Delta w_1(t)\|_{L_2(\Omega)} + k\|w_2(t)\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|w_1(t)\|_{L_2(\Omega)}\right), \tag{2.8}$$

where the constant N_1 is independent of w_1. This bound, (2.4), and (2.6) imply that $w_1(t)$ tends to zero as $t \to +\infty$ in the norm of $H^2(\Omega)$ because w_1 is fixed on a part of the boundary.

A consequence of these bounds and reasoning is the equivalence of norms in the spaces $D(\mathfrak{A})$ and $H^2 \times H^1$.

Now consider the space $D(\mathfrak{A}^2)$. As before, we use the theory of solvability of elliptic boundary value problems to effectively describe this space by

$$D(\mathfrak{A}^2) = \Big\{(w_1, w_2) \in H^3(\Omega) \times H^2(\Omega) : w_1(x) = w_2(x) = \Delta w_1(x) = 0, x \in \Gamma_0;$$
$$\frac{\partial w_1}{\partial \nu} = -kw_2,\ \frac{\partial w_2}{\partial \nu} = -k\Delta w_1,\ x \in \Gamma_1\Big\}.$$

Let $(w_1(t), w_2(t))$ be the solution to problem (1.1)–(1.3), (2.1); then, it lies in the space $C([0,T]; D(\mathfrak{A}^2))$. We have

$$\mathfrak{A}^2(w_1, w_2) = (\Delta w_1, \Delta w_2). \tag{2.9}$$

Equality (2.9) and [4] imply that

$$\int_\Omega \{(\Delta w_{1,x_1}(t))^2 + (\Delta w_{1,x_2}(t))^2 + (\Delta w_2(t))^2\}dx \le Me^{-2\gamma t}\int_\Omega \{(\Delta\varphi_{x_1})^2 + (\Delta\varphi_{x_2})^2 + (\Delta\psi)^2\}dx. \tag{2.10}$$

By combining (2.7) and (2.10), we obtain

$$\|(w_1(t), w_2(t))\|_{D(\mathfrak{A}^2)} \le M_2 e^{-\gamma t}\|(\varphi, \psi)\|_{D(\mathfrak{A}^2)}, \quad t \ge 0. \tag{2.11}$$

Using the theory of elliptic boundary value problems (see [10, p. 98]), we obtain

$$\|w_1(t)\|_{H^3(\Omega)} \le N_2\left(\|\Delta w_1(t)\|_{H^1(\Omega)} + k\|w_2(t)\|_{H^{\frac{3}{2}}(\Gamma_1)} + \|w_1(t)\|_{L_2(\Omega)}\right), \tag{2.12}$$

$$\|w_2(t)\|_{H^2(\Omega)} \le N_3\left(\|\Delta w_2(t)\|_{L_2(\Omega)} + k\|\Delta w_1(t)\|_{H^{\frac{1}{2}}(\Gamma_1)} + \|w_2(t)\|_{L_2(\Omega)}\right), \tag{2.13}$$

where the constants N_2 and N_3 are independent of (w_1, w_2).

Now (2.6) and (2.10) imply that $\Delta w_1(t)$ tends to zero (as $t \to +\infty$) in the norm of $H^1(\Omega)$. Therefore, by Sobolev's trace theorem, $\Delta w_1(t)$ tends to zero also in the norm of $H^{\frac{1}{2}}(\Gamma_1)$. Then, using (2.4), (2.13), and again (2.10), we conclude that $w_2(t)$ tends to zero (as $t \to +\infty$) in norm in $H^2(\Omega)$. Therefore, bound (2.12) implies that $w_1(t)$ tends to zero as $t \to +\infty$ in norm of $H^3(\Omega)$ because w_1 is fixed on a part of the boundary.

These bounds and reasoning imply that the norms in the spaces $D(\mathfrak{A}^2)$ and $H^3 \times H^2$ are equivalent.

For the given initial conditions, we solve problem (1.1)–(1.3), (2.1), then substitute this solution *only* into the right-hand side of (2.1), and obtain the boundary condition (1.5) for the initial-boundary value problem (1.1)–(1.4). In other words in the first phase we use the control

$$u^{(1)}(t,x) = -k\frac{\partial w_1^0(t,x)}{\partial t}$$

on Γ_1 in problem (1.1)–(1.4). Here w_1^0 is the solution of problem (1.1)–(1.3), (2.1).

Therefore, we have proved that, if the control is applied for sufficiently long time, we can make the quantity

$$\|(w(T_1,\cdot), w_t(T_1,\cdot))\|_{\mathcal{H}_0^3(\Omega)}$$

arbitrarily small at time $t = T_1$. Note that the functions $\varphi(x)$ and $\psi(x)$ must satisfy the conditions

$$\frac{\partial\varphi(x)}{\partial\nu} = -k\psi(x), \quad \frac{\partial\psi}{\partial\nu} = -k\Delta\varphi x \in \Gamma_1. \tag{2.14}$$

Due to (1.6), condition (2.14) is satisfied for any k.

Let us now show that the control $u(t,x)$ can also be made arbitrarily small; i.e., we can satisfy condition (1.5). To this end note that, due to the contraction property of the semigroup generated by the operator $\mathfrak{A}$, it holds that

$$\max_{t\in[0,+\infty)} E(t) = E(0) = \int_\Omega (\varphi_{x_1}^2(x) + \varphi_{x_2}^2(x) + \psi^2(x))dx.$$

Therefore, we have

$$\|e^{t\mathfrak{A}}(\varphi,\psi)\|_{D(\mathfrak{A}^2)} \le \|(\varphi,\psi)\|_{D(\mathfrak{A}^2)}.$$

Thus, using Sobolev's embedding theorems and (2.13), we obtain

$$\begin{aligned}\|w_2(t)\|_{C(\bar{\Omega})} &\le C_1\|w_2(t)\|_{H^2(\Omega)} \le C_2\|\Delta w_2(t)\|_{L_2(\Omega)} + kC_2\|\Delta w_1(t)\|_{H^{\frac{1}{2}}(\Gamma_1)} + C_2\|w_2(t)\|_{L_2(\Omega)}\\ &\le C_2\|w_2(t)\|_{H^1(\Omega)} + C_2\|\Delta w_2(t)\|_{L_2(\Omega)} + kC_3\|\Delta w_1(t)\|_{H^1(\Omega)} + C_2\|w_2(t)\|_{L_2(\Omega)}\\ &\le (C_2 + kC_3)\left\|e^{t\mathfrak{A}}(w_1,w_2)\right\|_{D(\mathfrak{A}^2)} \le (C_2 + kC_3)\|(\varphi,\psi)\|_{D(\mathfrak{A}^2)}.\end{aligned}$$

Since the coefficient k can be chosen arbitrarily small, $\|w_2(t)\|_{C(\bar{\Omega})}$ is bounded due to the last bound. We also note that, due to the boundary conditions (1.6), the initial data (φ,ψ) lie in the space $D(\mathfrak{A}^2)$ for any k. By choosing the friction coefficient k sufficiently small, we conclude that condition (1.5) is fulfilled.

3. SECOND PHASE OF CONTROL

Now we want to drive the system to complete rest. We consider the functions $w|_{t=0} = w(T_1,x)$ and $w_t|_{t=0} = w_t(T_1,x)$ as the new initial data in problem (1.1)–(1.4). Recall that, as has been proved above, these initial data (the pair of functions) are sufficiently small in the norm of the space $\mathcal{H}_0^3(\Omega)$.

Consider the domain Ω_δ that by definition is the δ-neighborhood of the domain Ω without the points of the domain $\bar{\Omega}^*$ (Fig. 1). The domain Ω_δ is constructed such that the exterior contour of its boundary (it is called Γ_1^δ) satisfies condition (2) in the formulation of the theorem. Let ν_δ be the outward normal to the boundary of Ω_δ.

Define the space

$$\mathcal{H}_0^3(\Omega_\delta) = \{(w,v)\in H^3(\Omega_\delta)\times H^2(\Omega_\delta)\colon w(x) = v(x) = \Delta w(x) = 0,\ x\in\Gamma_0\}.$$

We also consider an arbitrary pair of functions

$$(f(x), g(x))$$

in the space $\mathcal{H}_0^3(\Omega)$. We extend this pair to zero (the linear continuation operator

$$E:\mathcal{H}_0^3(\Omega)\to\mathcal{H}_0^3(\Omega_\delta)$$

exists and is bounded) to the domain Ω_δ while preserving smoothness. The construction of the continuation operator E is well known and is thoroughly described in [11].

Following Russell, we will denote the initial data functions thus continued by $f^e(x)$ and $g^e(x)$.

Consider the initial-boundary value problem for the membrane oscillation equation in the domain Ω_δ:

$$w_{tt}(t,x) - \Delta w(t,x) = 0,\quad (t,x)\in Q = (0,+\infty)\times\Omega_\delta, \tag{3.1}$$

$$w|_{t=0} = f^e(x),\quad w_t|_{t=0} = g^e(x),\quad x\in\Omega_\delta, \tag{3.2}$$

$$w(t,x) = 0,\quad (t,x)\in(0,+\infty)\times\Gamma_0, \tag{3.3}$$

$$\frac{\partial w(t,x)}{\partial\nu_\delta} = -k\frac{\partial w(t,x)}{\partial t},\quad x\in\Gamma_1^\delta. \tag{3.4}$$

Similarly to the preceding section, the solution to problem (3.1)–(3.4) satisfies the bound

$$\|(w(t),w_t(t))\|_{V_\delta\times H_\delta} \le M_4 e^{-\gamma t}\|(f^e,g^e)\|_{V_\delta\times H_\delta},\quad t\ge 0, \tag{3.5}$$

where

$$V_\delta = \{v\in H^1(\Omega_\delta)\colon v(x) = 0,\ x\in\Gamma_0\},\quad H_\delta = L_2(\Omega_\delta).$$

Below, we use the method (in a modified form) described in [2] and applied in boundary control problems for the wave equation.

Let the initial conditions $f(x)$ and $g(x)$ for $x \in \Omega$ be given. We extend these functions to Ω_δ using the linear bounded operator E. Then $(f^e, g^e) = E(f, g)$. Thus, we obtain the initial-boundary value problem (3.1)–(3.4). Let $w^s(t,x)$ be the solution to this problem. For the domain Ω_δ, we consider the operator $\mathfrak{A}_\delta$ that is constructed by complete analogy with the operator $\mathfrak{A}$ for the domain Ω. Then, we have the bound

$$\left\|(w_1^s(t), w_2^s(t))\right\|_{D(\mathfrak{A}_\delta^2)} \le M_5 e^{-\gamma_1 t} \|(f^e, g^e)\|_{D(\mathfrak{A}_\delta^2)}, \quad t \ge 0. \tag{3.6}$$

Due to the equivalence of norms, it holds that

$$\left\|(w_1^s(t), w_2^s(t))\right\|_{\mathcal{H}_0^3(\Omega_\delta)} \le M_6 e^{-\gamma_1 t} \|(f^e, g^e)\|_{\mathcal{H}_0^3(\Omega_\delta)}, \quad t \ge 0. \tag{3.7}$$

Consider a sufficiently large time $t = T_2$ and the constraint on the solution together with its time derivative at T_2 in Ω. It is clear that at $t = T_2$, due to (3.7) and the continuity of the operator E, it holds that

$$\left\|(w_1^s(T_2, \cdot), w_2^s(T_2, \cdot))\right\|_{\mathcal{H}_0^3(\Omega)} \le M_7 e^{-\gamma_1 T_2} \left\|(f, g)\right\|_{\mathcal{H}_0^3(\Omega)}. \tag{3.8}$$

Let, by definition,

$$(w_1^{s,e}(T_2, x), w_2^{s,e}(T_2, x)) = E(w_1^s(T_2, x)\big|_\Omega, w_2^s(T_2, x)\big|_\Omega).$$

Let us formulate the initial-boundary value problem in reverse time (i.e., for $t \le T_2$) for the same equation

$$\frac{d}{dt}(w_1, w_2) = (w_2, \Delta w_1) \tag{3.9}$$

subject to the boundary condition on Γ_1^δ

$$\frac{\partial w_1}{\partial \nu_\delta} = k w_2, \tag{3.10}$$

condition (3.3) on Γ_0, and the initial conditions

$$w_1(t)\big|_{t=T_2} = -w_1^{s,e}(T_2, x), \quad w_2(t)\big|_{t=T_2} = -w_2^{s,e}(T_2, x). \tag{3.11}$$

Let $(w_1^i(t), w_2^i(t))$ be the solution of the initial-boundary value problem (3.3) and (3.9)–(3.11) in reverse time. As before, we have the bound

$$\left\|(w_1^i(0, \cdot), w_2^i(0, \cdot))\right\|_{\mathcal{H}_0^3(\Omega)} \le M_7 e^{-\gamma_1 T_2} \left\|(w_1^s(T_2, x), w_2^s(T_2, x))\right\|_{\mathcal{H}_0^3(\Omega)}. \tag{3.12}$$

Consider the sum of solutions in forward and reverse time restricted to the domain Ω:

$$w(t, x) = w^s(t, x) + w^i(t, x), \quad x \in \Omega. \tag{3.13}$$

This sum satisfies Eq. (1.1), and the restriction of the outward normal derivative of $w(t, x)$ to the lateral surface of the cylinder can be used as the desired boundary control $u(t, x)$.

It is clear that solution (3.13) with the initial conditions

$$w\big|_{t=0} = f^e(x) + w^{i,r}(0, x), \quad w_t\big|_{t=0} = g^e(x) + w_t^{i,r}(0, x), \quad x \in \Omega, \tag{3.14}$$

(the subscript r denotes the restriction to Ω) is identically equal to zero in Ω at the time $t = T_2$ together with its derivative with respect to t. Note that the value of the corresponding solution to the initial-boundary value problem with the initial conditions (3.14) on the boundary of Ω gives the desired control.

The pair $(w_1^{i,r}(0, x), w_2^{i,r}(0, x))$ is obtained from the pair $(f(x), g(x))$ by applying a linear continuous operator (we denote it by L) with the norm less than unity (a consequence of bounds (3.8) and (3.12)). It is clear that the sums on the right-hand sides in (3.14) give all the elements of the space $\mathcal{H}_0^3(\Omega)$. Indeed, (3.14) can be written in the form

$$(I + L)(f(x), g(x)) = (w\big|_{t=0}, w_t\big|_{t=0}), \tag{3.15}$$

where I is the identity operator. Therefore, since $\|L\| < 1$, the operator $I + L$ acting from $\mathcal{H}_0^3(\Omega)$ to itself is invertible.

Now, we represent the control function that should be found (in the second phase of control) in the form

$$u^{(2)}(t,x) = \frac{\partial}{\partial \nu} P[(S_+(t) - S_-(T_2 - t)ERS_+(T_2))\, E(I + L)^{-1}\{w|_{t=0}, w_t|_{t=0}\}], \tag{3.16}$$

$x \in \Gamma_1$, where R is the restriction operator from Ω_δ to Ω, $S_+(t)$ and $S_-(T_2 - t)$ are the resolving operators of the dissipative problem in forward and reverse time, respectively, P is the projection $(a,b) \mapsto a$, and

$$L = -RS_-(T_2)ERS_+(T_2)E.$$

Thus, we have proved that the system with arbitrary smooth initial data can be driven to rest. Now we show that, by choosing sufficiently small initial data, we can drive the system to rest using a small (in absolute value) boundary control.

Let the pair $(w|_{t=0}, w_t|_{t=0})$ be sufficiently small in the norm of the space $\mathcal{H}_0^3(\Omega)$. Formula (3.16) immediately implies the smallness of the control $u^{(2)}(t,x)$ because all the operators involved in this formula are continuous.

Therefore, the restriction of the normal derivative of the solution to the initial-boundary value problem to the boundary of Ω (the Neumann condition in the control problem) can be made smaller (in absolute value) than any given ε if the control time is sufficiently large. Therefore, the desired constraint on the control $u(t,x)$ is satisfied, which completes the proof of the theorem.

CONCLUSIONS

The existence of a boundary control completely damping oscillations of a membrane in a finite amount of time is proved. This control must satisfy a constraint on its absolute value. In the formulation of the main theorem, the initial velocity and the shape of the membrane boundary satisfy certain conditions.

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