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Igor Romanov & Alexey Shamaev

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Noncontrollability to Rest of the Two-Dimensional Distributed System Governed by the Integrodifferential Equation

Igor Romanov[1] · **Alexey Shamaev**[2,3]

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Abstract In this paper, we examine the controllability problem of a distributed system governed by the two-dimensional Gurtin–Pipkin equation. We consider a system with compactly supported distributed control and show that if the memory kernel is a twice continuously differentiable function, such that its Laplace transformation has at least one root, then the system cannot be driven to equilibrium in finite time.

Keywords Lack of controllability to rest · Equation with memory · Distributed control · Moment problems

Mathematics Subject Classification 45K05

1 Introduction

Integrodifferential equations with nonlocal terms of the convolution type often arise in applications such as mechanics of heterogeneous media, the theory of viscoelasticity, thermal physics, and kinetic theory of gases.

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✉ Igor Romanov
romm1@list.ru

[1] National Research University Higher School of Economics, 20 Myasnitskaya Ulitsa, Moscow, Russia 101000

[2] Institute for Problems in Mechanics RAS, 101 Prosp. Vernadskogo, Block 1, Moscow, Russia 119526

[3] Lomonosov Moscow State University, GSP-1, Leninskie Gory, Moscow, Russia 119991

For example, it was rigorously proved that in the case of a heterogeneous two-phase medium, consisting of viscous fluid and elastic inclusions, the effective equation is integrodifferential, and the corresponding convolution kernel is a finite or infinite sums of decreasing exponential functions.

If the viscosity of a liquid is small (big), the effective equation contains (does not contain) third-order terms corresponding to Kelvin–Voigt friction; we refer to [1] for more details.

In the theory of viscoelasticity, it is a common practice to approximate relaxation kernels by the sum of exponents.

In thermal physics, the law of heat conduction with integral memory is the subject of many research papers, in particular [2].

The presence of integral memory in the law of heat conduction may lead to the appearance of a thermal front, which moves at a finite speed. This makes an important difference with the heat equation, whose solution propagates at infinite speed.

In this paper, we outline of the results of the existence and uniqueness of solutions to these systems and consider the problem of controllability.

2 Problem Statement

We consider the problem of noncontrollability of a system governed by the integrodifferential equation

$$\theta_t(t,x,y) - \int_0^t K(t-s)\Delta\theta(s,x,y)\mathrm{d}s = u(t,x,y),$$

$$t>0, \quad (x,y)\in\Omega, \tag{1}$$

$$\theta|_{t=0} = \xi(x,y), \tag{2}$$

$$\theta|_{\partial\Omega} = 0. \tag{3}$$

Hereinafter, $\Omega \subset \mathbb{R}^2$ is a bounded domain, $K(t)$ is an arbitrary twice continuously differentiable function such that $K(0)=\mu>0$, and $u(t,x,y)$ is a control supported (in x, y) on Ω. The kernel $K(t)$ can be represented, for example, as a sum of decreasing exponential functions, i.e.,

$$K(t) = \sum_{j=1}^{N} c_j \mathrm{e}^{-\gamma_j t},$$

where c_j and γ_j are given positive constants.

For brevity, we write $\theta(t)$ and $u(t)$ instead of $\theta(t,x,y)$ and $u(t,x,y)$, respectively. This also means that $\theta(t)$ and $u(t)$ are functions of t with values in some suitable space.

The goal of the control is to drive this mechanical system to rest in finite time. We say that the system in (1)–(3) is *controllable to rest* if for every initial condition ξ, it is possible find a control $u(t)$ and a time $T>0$ such that $u(t)$ is equal to zero for any $t > T$ and the corresponding solution $\theta(t, u)$ of problem (1)–(3) equals zero for any $t > T$ too. Conversely, the system is *uncontrollable to rest* if there is an initial condition ξ such that for every control $u(t)$ ($u(t)$ is in the suitable class of functions), which equals zero identically outside some finite segment $[0, T]$, the corresponding solution does not equal zero identically outside any finite segment (in t).

In this article, we prove that the system governed by a two-dimensional Gurtin–Pipkin equation is uncontrollable to rest if the distributed control is supported on the subdomain, which is properly contained in an arbitrary bounded domain with a smooth boundary. This result is a generalization of an analogous theorem in [3] devoted to a similar one-dimensional problem. The method used in this paper can also be applied to the case where the dimension of Ω is greater than two; this will be discussed in greater detail in Sect. 6.

3 Literature Review

The presence of nonlocal terms of the convolution type in the equations and systems leads to a number of interesting qualitative effects that are not observed in differential equations and systems of equations. For instance, systems of this type exhibit properties of both parabolic and hyperbolic equations. In spectral problems for such equations and systems, the spectrum is composed of real and complex parts. The former part corresponds to the energy dissipation in the heat equation, and the later corresponds to vibrations. Such equations can be solved using a method similar to the Fourier method.

Moreover, systems of this type are usually uncontrollable to rest, if boundary control or control, which is distributed on part of the domain, is applied. Here, we recall the well-known work [4] that considers the equation for the vibration of a string. In this work, it was proved that if the control is applied to the end of the string, then the system can be driven to rest. The author used the so-called moment method. These results were later generalized to the multidimensional case in [5].

If the control distributed on the whole domain is used, then the integral terms of the convolution type facilitate the process of control. In this case, the control time is significantly reduced. It should be noted that the spectral method proposed in [6] can be successfully adapted to the case of systems with nonlocal terms of the convolution type (we refer to [7] for further details).

The uncontrollability mentioned above was justified in [3] for one-dimensional systems similar to (1).

In most cases, the property of controllability to rest is not observed. For example, in [3], it was proved that a solution to the heat equation with memory cannot be driven to rest in finite time if some auxiliary analytic function has roots. This result is valid for both boundary and distributed controls. Moreover, the case of distributed control can be reduced to the case of boundary control. In our paper, we obtain similar results for the case of two-dimensional domains.

The work in [8] is also notable because the boundary noncontrollability was justified for the heat equation with memory.

Positive results on controllability of a one-dimensional wave equation with memory were obtained in [7]. It was shown that this equation can be driven to rest by applying a bounded distributed control. In this case, the kernel of the integral term in the equation is the sum of N decreasing exponential functions.

Problems similar to (1)–(3) for integrodifferential equations have been widely studied in existing literature. Equation (1) was originally derived in [2]. The questions of solvability and asymptotic behavior of solutions for equations of this type were investigated for in [9,10]. In [11], it was proved that the energy for some dissipative system decays polynomially when the memory kernel decays exponentially.

Problems related to the solvability of system (1)–(3) were considered in [12]. It was proved that a solution belongs to some Sobolev space on the semi-axis (in t) if the kernel $K(t)$ is the sum of exponential functions, each of which tends to zero as $t \to +\infty$.

Interesting explicit formulas for the solution to (1)–(3) were obtained in [13] under the assumption that the kernel $K(t)$ is also the sum of decreasing exponential functions. It follows from these formulas that solutions tend to zero when $t \to +\infty$. All of these works assumed that the kernels of integral terms in the studied equations are nonincreasing functions.

4 Preliminaries

Let $A := \Delta$ be an operator such that

$$\mathrm{dom} A = H^2(\Omega) \cap H_0^1(\Omega),$$

where $\Omega \subset R^2$ is a bounded domain with boundary of class C^2. We now consider the control function $u(t) \in L_2^{\mathrm{loc}}([0, +\infty[; L_2(\Omega))$ and initial condition $\xi \in L_2(\Omega)$.

Definition 4.1 The function

$$\theta(t) \in H_{loc}^1([0, +\infty[; L_2(\Omega)) \cap L_2^{\mathrm{loc}}([0, +\infty[; H^2(\Omega) \cap H_0^1(\Omega))$$

is the solution to problem (1)–(3) if $\theta(t)$ satisfies (1):

$$\theta_t(t) - \int_0^t K(t-s)\Delta\theta(s)\mathrm{d}s = u(t)$$

and initial condition (2): $\theta(0) = \xi$.

Note that in virtue of smoothness of $K(t)$ problem (1)–(3) is solvable, if we impose an additional conditions of smoothness on ξ and $u(t)$ (see [14]).

Let PW_+ denote the linear space of the Laplace transforms of elements of $L_2([0, +\infty[)$ such that they are equal to zero on the set $\{t : t > T\}$ for some $T > 0$. It is a well-known fact that $\varphi(\lambda) \in PW_+$ if and only if it is an entire function, such that

1. there are real numbers C and T such that $|\varphi(\lambda)| \leqslant C\mathrm{e}^{T|\lambda|}$. Note that C and T depend on $\varphi(\lambda)$.
2. $\sup\limits_{x\geqslant 0} \int\limits_{-\infty}^{+\infty} |\varphi(x+iy)|^2 \mathrm{d}y < +\infty$.

5 Main Results

Now, we consider the auxiliary boundary value problem

$$\theta_t(t,x,y) - \int\limits_0^t K(t-s)\Delta\theta(s,x,y)\mathrm{d}s = 0,$$

$$t > 0, \quad (x,y) \in \Omega_0 = \left\{(x,y) : x^2 + y^2 < R^2\right\}, \tag{4}$$

$$\theta|_{t=0} = \xi(x,y), \tag{5}$$

$$\theta|_{\partial\Omega_0} = v(t,x,y), \quad (x,y) \in \partial\Omega_0. \tag{6}$$

Consider the control function $v(t) \in L_2^{\mathrm{loc}}([0,+\infty[; L_2(\partial\Omega_0))$ and initial condition $\xi \in L_2(\Omega_0)$.

We formally multiply (in the sense of the inner product in $L_2(\Omega_0)$) both parts of (4) by the function φ such that $\varphi \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$. Hereinafter, ν is a normal vector to the domain boundary $\partial\Omega_0$. Next, using Green's formula, we formally replace the operator A from $\theta(t)$ with φ. Then, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\theta(t),\varphi\rangle - \int\limits_0^t K(t-s)\left(\langle\theta(s),\Delta\varphi\rangle - \int\limits_{\partial\Omega_0} v(s)\frac{\partial\varphi}{\partial\nu}\mathrm{d}\sigma\right)\mathrm{d}s = 0, \tag{7}$$

where $\langle\cdot,\cdot\rangle$ is the inner product in $L_2(\Omega_0)$.

We say that the function $\theta(t) \in H_{\mathrm{loc}}^1([0,+\infty[; L_2(\Omega_0))$ is *the solution* to problem (4)–(6) if the equality in (7) holds for every $\varphi \in \mathrm{dom}\Delta$ and $\theta(0) = \xi$.

The orthonormalized system of eigenvectors of A are the functions $\varphi_{nm}(x,y)$, which in polar coordinates ($x = r\cos\alpha$, $y = r\sin\alpha$) have the form

$$\tilde{\varphi}_{nm}(r,\alpha) = \frac{J_m\left(\mu_n^m \frac{r}{R}\right)\mathrm{e}^{im\alpha}}{\sqrt{\pi}RJ_m'(\mu_n^m)}, \quad m = 0,1,2,\ldots, \quad n = 1,2,\ldots,$$

where J_m are Bessel functions and μ_n^m are positive roots of J_m. It is a well-known fact that this system is a basis for $L_2(\Omega_0)$. We substitute $\varphi = \varphi_{nm}$ in (7). Then, using the notation $\theta_{nm}(t) = \langle\theta(t),\varphi_{nm}\rangle$, we obtain

$$\frac{\mathrm{d}\theta_{nm}(t)}{\mathrm{d}t} + \lambda_{nm}^2 \int_0^t K(t-s)\theta_{nm}(s)\mathrm{d}\sigma\,\mathrm{d}s = -\int_0^t K(t-s)\left(\int_{\partial\Omega_0} v(s)\frac{\partial\varphi_{nm}}{\partial\nu}\right)\mathrm{d}\sigma\,\mathrm{d}s, \tag{8}$$

where λ_{nm} are the corresponding eigenvalues. Using polar coordinates yields

$$\lambda_{nm}^2 = \frac{(\mu_n^m)^2}{R^2}.$$

Next, we take the Laplace transformation of both parts of (8) and express $\hat{\theta}_{nm}(\lambda)$ as

$$\hat{\theta}_{nm}(\lambda) = \frac{-\hat{K}(\lambda)\int_{\partial\Omega_0} \hat{v}(\lambda)\frac{\partial\varphi_{nm}}{\partial\nu}\mathrm{d}\sigma + \xi_{nm}}{\lambda + \lambda_{nm}^2\hat{K}(\lambda)}, \tag{9}$$

where $\hat{K}(\lambda)$ is the Laplace transform of the kernel $K(t)$:

$$\hat{K}(\lambda) = \int_0^{+\infty} K(t)\mathrm{e}^{-\lambda t}\mathrm{d}t.$$

Lemma 5.1 *In problem* (4)–(6)*, if* $\hat{K}(\lambda)$ *has at least one root* λ_0 *in the domain of holomorphism (we require that this domain exists), then controllability to rest is impossible; that is, there exists an initial condition* $\xi \in L_2(\Omega_0)$*, such that for every control* $v \in L_2^{\mathrm{loc}}([0, +\infty[; L_2(\Omega_0))$*, which is equal to zero on the set* $\{t : t > T\}$ *for some* $T > 0$*, the corresponding solution cannot be equal to zero identically outside any finite segment (in t).*

Proof By using polar coordinates in the integral of equality (9), equality (9) takes the form

$$\hat{\theta}_{nm}(\lambda) = \frac{-\mu_n^m\hat{K}(\lambda)\int_0^{2\pi} \hat{v}_0(\lambda,\alpha)\mathrm{e}^{im\alpha}\mathrm{d}\alpha + \xi_{nm}}{\sqrt{\pi}R\left(\lambda + \lambda_{nm}^2\hat{K}(\lambda)\right)}, \tag{10}$$

where $\hat{v}_0(\lambda,\alpha) := \hat{v}(\lambda, R\cos\alpha, R\sin\alpha)$.

The system of functions $\{\mathrm{e}^{im\alpha}\}_{m\in Z}$ is an orthogonal basis in $L_2(0, 2\pi)$. Thus, we can expand

$$\hat{v}_0(\lambda,\alpha) = \sum_{j=-\infty}^{+\infty} \hat{v}_{0,j}(\lambda)\mathrm{e}^{ij\alpha},$$

where

$$\hat{v}_{0,j}(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-ij\alpha}\hat{v}_0(\lambda,\alpha)\mathrm{d}\alpha.$$

Hence, we obtain

$$\hat{\theta}_{nm}(\lambda) = \frac{-\mu_n^m \hat{K}(\lambda)2\pi\,\hat{v}_{0,-m}(\lambda) + \xi_{nm}}{\sqrt{\pi}R(\lambda + \lambda_{nm}^2 \hat{K}(\lambda))}. \tag{11}$$

Note that if the system is controllable to rest, then $\theta_{nm}(t)$ and $v_{0,-m}(t)$ are equal to zero identically outside some finite segment on $[0, +\infty)$. Thus $\hat{\theta}_{nm}(\lambda)$ and $\hat{v}_{0,-m}(\lambda)$ are in PW_+.

If it follows from the definition of PW_+ that $\hat{\theta}_{nm}(\lambda)$ is an entire function, then it cannot have singularities at the roots of the denominator $\lambda + \lambda_{nm}^2\hat{K}(\lambda)$.

If $\lambda_0 = 0$ is a root of the equation $\hat{K}(\lambda) = 0$, then the equality (11) generally cannot be satisfied for the values $\hat{\theta}_{nm}(\lambda)$, which correspond to functions from PW_+. Hence, in this case controllability to rest is impossible.

Let $\lambda_0 = 0$ be not a root of the equation $\hat{K}(\lambda) = 0$. The control function $\hat{v}_{0,-m}(\lambda)$ has to satisfy the following equalities:

$$\hat{v}_{0,-m}(\lambda) = -\frac{1}{2\pi}\frac{\lambda_{nm}^2\xi_{nm}}{\mu_n^m\lambda}, \tag{12}$$

when $\lambda \neq 0$ is a root of the equation $\lambda + \lambda_{nm}^2\hat{K}(\lambda) = 0$. Since $\lambda_{nm}^2 = \left(\mu_n^m\right)^2/R^2$, then (12) can be rewritten as

$$\hat{v}_{0,-m}(\lambda) = -\frac{1}{2\pi}\frac{\mu_n^m\xi_{nm}}{R^2\lambda}. \tag{13}$$

Note that the equality in (13) can be presented in the following form:

$$\int_0^T v_{0,-m}(t)\mathrm{e}^{-\lambda t}\mathrm{d}t = -\frac{1}{2\pi}\frac{\mu_n^m\xi_{nm}}{R^2\lambda}.$$

The latter equality is the so-called moment problem.

We now record the index m. Let m, for example, be equal to 1 and n changes from 1 to $+\infty$. We obtain the subsystem of equalities for values of the function $\hat{v}_{0,-1}(\lambda)$ at such points λ that $\lambda + \lambda_{n1}^2\hat{K}(\lambda) = 0$:

$$\hat{v}_{0,-1}(\lambda) = -\frac{1}{2\pi}\frac{\mu_n^1\xi_{n1}}{R^2\lambda}., \quad n = 1, 2, \ldots. \tag{14}$$

Note that $\hat{K}(\lambda)$ has a root $\lambda_0 \neq 0$ (if $K(t)$ is a series of decreasing exponentials, then $\hat{K}(\lambda)$ has a countable number of roots, see [15]). Applying the methods used in

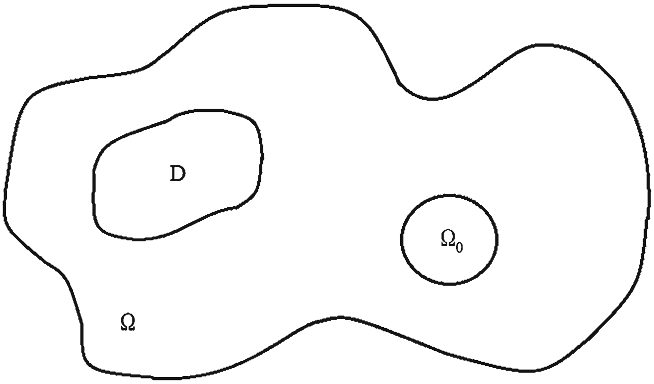

Fig. 1 The illustration to the proof of Theorem 5.1

[3] (in which Rouche's theorem was used), we can prove that there exists a sequence $\{\lambda_n \neq 0\}$ of zeros of

$$\lambda + \frac{(\mu_n^1)^2}{R^2}\hat{K}(\lambda).$$

It is important that this sequence converges to a nonzero complex number. Let us choose $\xi_{2j+1,1} = 0$; hence, $\hat{v}_{0,-1}(\lambda_{2n+1}) = 0$. As the sequence of zeros converges and $\hat{v}_{0,-1}(\lambda)$ is an entire function, then $\hat{v}_{0,-1}(\lambda) \equiv 0$. Then for any n, all $\xi_{2n,1}$ must be zero. However, we can always take some of them to be nonzero numbers. Thus, there exists an initial condition ξ such that for any control function $v(t)$, controllability to rest is impossible. □

We now consider problem (1)–(3). Let D be an arbitrary bounded domain such that $\overline{D} \subset \Omega$. We define $\widetilde{L}_2(D)$ as the space of all elements from $L_2(D)$ extended by zero to the set $\Omega \setminus D$. The following theorem is the main result of this article.

Theorem 5.1 *If the control function $u(t)$ in (1) is an element of*

$$L_2^{\text{loc}}([0, +\infty[; \widetilde{L}_2(D))$$

and $\hat{K}(\lambda)$ has at least one root λ_0 in the domain of holomorphism, then controllability to rest is impossible; that is, there exists an initial condition $\xi \in L_2(\Omega)$ such that for every control $u(t)$, which is equal to zero on the set $\{t : t > T\}$ for some $T > 0$, the corresponding solution cannot be equal to zero identically outside any finite segment (in t).

Proof Since $\overline{D}$ is a closed and bounded subset of Ω, then we can consider a circle Ω_0, which is properly contained in Ω and does not intersect D (see Fig. 1).

Let $\theta(t)$ be a corresponding solution to (1)–(3); note that we require the existence of this solution.

We restrict the solution $\theta(t)$ to Ω_0 and consider a new initial boundary value problem on Ω_0. We first prove that this restriction is correct. By means of the definition 4.1 we obviously have:

$$\theta(t) \in H^1_{\text{loc}}([0, +\infty[; L_2(\Omega_0)) \cap L_2^{\text{loc}}([0, +\infty[; H^2(\Omega_0)). \tag{15}$$

We also can easily establish that for $\theta(t)$ the integral equality (7) holds for any $\varphi \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$ and $v(s)$ is a "restriction" of $\theta(s)$, $s \in (0, t)$, on the boundary of Ω_0.

We can consider the "restriction" of the solution to $\partial\Omega_0$ as the control. Thus, we obtain the boundary control problem on the circle Ω_0. The solution to this problem automatically exists. Using the previous lemma, we have proved that the controllability to rest of this new problem is impossible; in this case, it is impossible to stop oscillations of the origin problem in (1)–(3). □

6 Generalizations and Related Topics

To obtain an analogous result for the Gurtin–Pipkin equation when the dimension of Ω is greater than two, it is necessary to use the orthonormalized system of eigenvectors of A defined in Ω_0, where Ω_0 is a ball. These eigenvectors, for example, in $\mathbb{R}^3$, are constructed by means of spherical harmonics

$$Y_m^l(\alpha, \varphi),\ 0 \le \alpha \le \pi,\ 0 \le \varphi < 2\pi,\ m = 0, 1, \dots,\ l = 0, \pm 1, \pm 2, \dots, \pm m,$$

while we use functions $\mathrm{e}^{im\alpha}, 0 \le \alpha < 2\pi, m = 0, 1, \dots,$ for two-dimensional domains. Using the orthogonal property of spherical harmonics Y_m^l on the unit sphere, we expand $\hat{v}_0(\lambda)$. After this expansion, all steps of the proof remain unchanged (see the proof of Lemma 5.1). We choose $\mathbb{R}^2$ in the present paper as the clearest and most evident way to demonstrate the idea of the proof.

It can likely be proved that if

$$K(t) = \sum_{j=1}^{N} c_j \mathrm{e}^{-\gamma_j t}$$

and if the control function is not supported only on the subdomain, which is properly contained in Ω, then the problem in (1)–(3) is controllable to rest. Note that if $K(t) = C > 0$, then (1) is a classical wave equation, and $\hat{K}(\lambda)$ does not have a null. It is a well-known fact that the system is controllable to rest if we use a control supported on a subdomain. If $K(t) = q\mathrm{e}^{-\gamma t}$ and $q, \gamma > 0$, then (1) can be reduced to

$$\theta_{tt}(t, x, y) - q\Delta\theta(t, x, y) + \gamma\theta_t(t, x, y) = P(t, x, y),$$

where

$$P(t, x, y) = \frac{\mathrm{d}u(t, x, y)}{\mathrm{d}t} + \gamma u(t, x, y).$$

In this case, P is considered to be a new control. The latter equation is a damped wave equation. We now consider the one-dimensional case. In this case, instead of the Laplace operator Δ, we write the second derivative $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$:

$$\theta_{tt}(t,x) - q\theta_{xx}(t,x) + \gamma\theta_t(t,x) = P(t,x). \tag{16}$$

In a later publication, we will prove that oscillations of the string governed by (16), where $P(t,x) \equiv 0$, can be stopped if we apply the control to the end of the string, the second end being fixed. Using this fact, we are optimistic that it can be proved that by means of control $P(t,x)$ contained in a subsegment (in x), the system is also controllable to rest.

Finally, we note a link between stability and controllability for the one-dimensional case. If we consider the equation

$$\theta_{tt} - \alpha\theta_{xx} + q\theta_{xx} * \mathrm{e}^{-\gamma t} = 0,$$

where $*$ is the convolution and $\alpha > 0$, then solutions of this equation are stable if $q \in [0, \alpha\gamma]$ and unstable if $q < 0$ or $q > \alpha\gamma$. Furthermore, if $q = 0$ or $q = \alpha\gamma$, then the system is controllable to rest if we use the boundary control.

7 Optimization and Controllability

Problems of controllability of systems with memory are important and have been considered in existing literature. It is advisable for a system to have controllability to rest, which can be accomplished by selective forcing. This relates to both problems of controllability of heat flow and viscoelastic systems. However, the result of this article shows that there are some cases where controllability to rest of a system is impossible. Therefore, we modify the statement of our task; specifically, our goal is not to drive the system to a given position, but to minimize the residual (in the norm of some functional space) between the desirable result and the obtained system's phase condition. Herein, we proved that the residual is nonzero; however, the actual value of the error and how the control function should be chosen to minimize this error are open research topics. Therefore, these problems serve as possible foundations for our future research.

8 Conclusions

In this article, we proved that a system governed by the two-dimensional Gurtin–Pipkin equation is uncontrollable to rest if the distributed control is supported on the subdomain, which is properly contained in an arbitrary bounded domain with a smooth boundary. In this case, the memory kernel is a twice continuously differentiable function such that its Laplace transformation has at least one root.

Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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