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SOME PROBLEMS OF DISTRIBUTED AND BOUNDARY CONTROL FOR SYSTEMS WITH INTEGRAL AFTEREFFECT

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Abstract. We consider the problem of exact control for a system described by an equation with integral "memory." It is shown that, under certain conditions, this system can be brought to rest in finite time by distributed control bounded in absolute value, and, in a special one-dimensional case, by control applied to an end-point of the interval. We consider different types of kernels in the integral term of the equation and describe some relationships between problems of controllability of some hyperbolic and parabolic systems.

1. Introduction

Consider the following problem of distributed control:

$$\theta_t(t,x) - \int\limits_0^t K(t-s)\Delta\theta(s,x)\,ds = u(t,x), \quad t>0, \quad x\in\Omega, \tag{1}$$

$$\theta|_{t=0} = \xi(x), \tag{2}$$

$$\theta|_{\partial\Omega} = 0. \tag{3}$$

Here and in what follows, $\Omega \subset \mathbb{R}^n$ is a bounded domain with an infinitely smooth boundary; $K(t)$ is a twice continuously differentiable function $K(0) = p > 0$; $u(t,x)$ is the control distributed (with respect to x) over Ω. Without loss of generality, we can ssume that $p = 1$.

For the sake of brevity, we use the symbols $\theta(t)$ and $u(t)$ instead of $\theta(t,x)$ and $u(t,x)$, respectively, implying thereby that $\theta(t)$ and $u(t)$ are functions of t with values in certain Banach spaces.

The problem is to bring the mechanical system under consideration to a state of rest over a finite time interval by means of the control, which is bounded in absolute value.

System (1)–(3) is said to be *controllable*, if for any initial value ξ we can find an instant $T > 0$ and control $u(t)$ such that $u(t) \equiv 0$ for $t > T$; $|u(t,x)| \leq \varepsilon$, $t > 0$, $x \in \Omega$; and $\theta^u(t) \equiv 0$ for $t > T$, where $\theta^u(t)$ is the corresponding solution of problem (1)–(3).

Integro-differential equations with nonlocal terms of convolution type occur in various applications, such as mechanics of heterogeneous media, theory of viscoelasticity, thermal physics, kinetic theory of gases, etc. The authors' interest in integro-differential operators of this type originates in studies of heterogeneous media, of which several should be mentioned here. Thus, the monographs [1–3] are aimed at developing strict mathematical methods for the construction of effective characteristics of strongly nonhomogeneous media and the investigation of closeness of the solutions and eigenvalues of the boundary value problems for such media to the corresponding effective (homogenized) characteristics. Spectral problems for the effective (homogenized) models of continua consisting of several phases with distinct rheological characteristics (elastic material — viscous fluid) are studied in [4]. In [5], the method of two-scale convergence was applied for the first time to spectral problems for strongly inhomogeneous media and the convergence of their spectra to that of the limit problem was examined.

A convolution kernel often happens to be represented by a finite or infinite sum of decaying exponents. In the theory of heterogeneous media, for instance, there is a strict mathematical justification of the fact that the effective model of a two-phase medium consisting of an elastic material and a viscous fluid is

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described by a system with such convolution kernels. The viscosity of the fluid being small or finite determines, in the effective model, the absence or the presence of a term with third-order derivatives modeling the Kelvin–Voigt friction [1]. In models of viscoelastic materials, experimental values obtained for relaxation or creep kernels are often approximated by sums of exponents. In thermal physics, heat conductivity laws with integral aftereffects have been studied in many publications, for instance, [6]. In some cases, integral aftereffects in heat conductivity laws give rise to a thermal front moving with a finite velocity. This fact is important, because it allows one to overcome the paradoxical situation of heat propagation velocity being infinite for the heat equation.

The present paper contains a brief review of the works on the existence and the uniqueness of solutions of the said systems, as well as the controllability of such systems. Nonlocal terms of convolution type in equations and systems bring about some interesting qualitative effects, which are absent in the case of equations and systems containing only differential terms. Systems of this type exhibit some properties of parabolic systems describing dissipation phenomena, as well as hyperbolic systems associated with vibrations and wave propagation. In spectral problems for such equations and systems, the spectrum consists of two parts: real and complex, the first corresponding to pure dissipation of energy, and the second pertaining to vibration processes [7]. For obtaining solutions of such problems, there is an analogue of the Fourier method [8].

As a rule, systems of the said type are uncontrollable in the following sense: there are initial states from which it is impossible to bring the system to a given state (for instance, complete rest) by applying control to a part of the domain or its boundary. In this connection, we should recall the results of [9] for the vibrating string equation and the heat equation: by applying control at one end of a string or a warm rod, the system can be brought to rest or zero temperature. These results have been extended to multi-dimensional cases in [10].

On the other hand, if control is applied to the entire domain, then the presence of integral terms of convolution type has positive effect: the time of bringing a system to a desired state is much shorter than that for the system without the nonlocal terms. In order to prove this property, one can use the spectral method proposed in [11] for the construction of control, which has been done in [12] for systems with nonlocal terms of convolution type. Initially, the said loss of complete controllability was noticed in [15] with regard to systems with a single spatial variable. The results of this paper indicate that systems with the property of complete controllability, in the case of control applied to a part of the domain or its boundary, are an exception among integro-differential systems; in most cases, complete controllability is absent. On can say that completely controllable systems form a set of "zero measure" among integro-differential equations with nonlocal terms of convolution type.

Similar problems for equations without integral terms, which describe two-dimensional membranes and plates, were previously studied in [11], where it was shown that vibrations of such systems can be stopped on a finite time interval by applying control distributed over the whole surface of the object under consideration. In monograph [9], optimization of boundary control of string vibrations was addressed for the first time and, in this case, the method of moments was effectively applied. The review [10] describes the results obtained by many authors regarding boundary controllability of plates and membranes by means of boundary conditions of various types. Some problems of controllability for systems similar to (1) were addressed in [15], where a condition was found ensuring that the solution of the heat equation with integral "memory" cannot be brought to rest on a finite time interval. This condition amounts to the existence of roots of a certain complex-analytic function in its domain of holomorphy.

Let us consider some simple but important examples.

(1) $K(t) \equiv 1$. Then (1) is obviously reduced to the wave equation by differentiating in t.
(2) $K(t) \equiv e^{-\gamma t}$, $\gamma > 0$. In this case, (1) reduces to the wave equation with friction against the external medium. This equation is also referred to as "telegraph equation."
(3) Let us replace the integral term in equation (1) by the expression $K(t) * \Delta\theta(t, x)$, where the convolution kernel $K(t)$ can be a distribution, in general. Let $K(t) = \delta(t)$ be the Dirac delta-function. Thus (1) becomes the classical heat equation.

(4) $K(t) \equiv \sum_{j=1}^{N} c_j e^{-\gamma_j t}$, $c_j, \gamma_j > 0$. Equations with such kernels occur in problems of thermal physics, viscoelasticity, and mechanics of heterogeneous media.

(5) $K(t) \equiv t^{-\gamma}$, where $0 < \gamma < 1$ or $-\infty < \gamma < 0$. These are the so-called kernels of the Abelian type, which occur, for instance, in the theory of creep. In general, such kernels do not belong to the class C^2.

Obviously, systems with kernels from examples (1)–(3) can be brought to complete rest by an arbitrarily small (in absolute value) distributed control. The case of example (4) is more difficult. Complete bounded controllability of a system whose kernel is a sum of finitely many decaying exponents is established in [12] and [13] for one-dimensional and multi-dimensional domains, respectively.

In what follows, we are going to use well-known definitions and notions. By PW_+ we denote the linear space of Laplace transforms of all square-summable functions on $(0, +\infty)$ with a compact support in $[0, +\infty)$. It is well known that $\varphi(\lambda) \in \mathrm{PW}_+$, if and only if it is an entire function satisfying the following conditions:

(1) there are positive constants C and T such that $|\varphi(\lambda)| \le Ce^{T|\lambda|}$. Note that C and T depend on $\varphi(\lambda)$;

(2) $\sup\limits_{x \ge 0} \int_{-\infty}^{+\infty} |\varphi(x + iy)|^2 \, dy < +\infty$.

By PW_+^T we denote the space of the Laplace transforms of all $f(t) \in L_2(0, T)$ extended by zero for $t > T$.

Let $A := \Delta$ be the Laplace operator with the domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. Let $\{\varphi_k(x)\}_{k=1}^{+\infty}$ be an orthonormal system of eigenfunctions of the Dirichlet problem for the Laplace operator A in the domain Ω, with the corresponding system of eigenvalues $\{\lambda_k^2\}_{k=1}^{+\infty}$, so that $\Delta\varphi_k + \lambda_k^2 \varphi_k = 0$.

Let l_α be the space of all sequences $\{c_k\}_{k=1}^{+\infty}$ for which the series

$$\sum_{k=1}^{+\infty} |c_k|^2 \lambda_k^{2\alpha}$$

is convergent. We introduce the space

$$D(A^\alpha) = \left\{ f(x) = \sum_{k=1}^{+\infty} f_k \varphi_k(x) \colon \{f_k\}_{k=1}^{+\infty} \in l_\alpha \right\}.$$

2. Definition of a Solution. Existence and Uniqueness Theorems

Following [16], we define a solution of problem (1)–(3) with the help of an integral equation. To this end, consider the square root of the Laplace operator, $B = i(-\Delta)^{1/2}$, with the domain $H_0^1(\Omega)$. Consider the semi-group e^{Bt}, $t \in \mathbb{R}$, generated by B. As in [16], we introduce the "cosine operator"

$$R_+(t) = \frac{1}{2}(e^{Bt} + e^{-Bt}).$$

Let $u \in L_2\big(0, T; L_2(\Omega)\big)$, where $T > 0$ is arbitrary, and let $\xi \in L_2(\Omega)$.

Definition 1 (see [16]). A function

$$\theta(t) \in C\big([0, T]; L_2(\Omega)\big)$$

is called a *solution* of problem (1)–(3) if it satisfies the following integral equation:

$$\theta(t) = R_+(t)\xi + \int_0^t R_+(t-s)u(s)\, ds + \int_0^t L(t-s)\theta(s)\, ds, \tag{4}$$

where the operator $L(t)$ (with t fixed) maps $z \in L_2(\Omega)$ to

$$L(t)z = R_+(t)K'(0)z - K'(t)z + \int\limits_0^t R_+(t-\zeta)K''(\zeta)z\,d\zeta.$$

As we know (see [16]), under the above assumptions on the right-hand side and the initial value, a solution of problem (1)–(3), as defined above, exists and is unique in the said class.

3. Bounded Distribulted Control

The system of eigenfunctions $\{\varphi_k(x)\}_{k=1}^{+\infty}$ forms a basis in $L_2(\Omega)$. It is well known that the action of the semi-group e^{At} on a function $\psi \in D(A)$ is defined by

$$e^{At}\psi(x) = \sum_{k=1}^{+\infty} \psi_k e^{-\lambda_k^2 t}\varphi_k(x),$$

where ψ_k are the coefficients in the Fourier expansion of $\psi(x)$. Then, the action of the semigroup e^{Bt} on $\psi \in H_0^1(\Omega)$ is expressed by

$$e^{Bt}\psi(x) = \sum_{k=1}^{+\infty} \psi_k e^{i\lambda_k t}\varphi_k(x),$$

which immediately implies that

$$R_+(t)\psi(x) = \sum_{k=1}^{+\infty} \psi_k \cos\lambda_k t\varphi_k(x).$$

We seek a solution of problem (1)–(3) as an expansion into series with respect to the above eigenfunctions:

$$\theta(t,x) = \sum_{k=1}^{+\infty} \theta_k(t)\varphi_k(x). \tag{5}$$

It is assumed that the solution is defined for all $t \in [0,+\infty)$, while the control u is also defined for all t and identically vanishes on the set $\{t\colon t > T\}$.

Then, substituting (5) into (4) and expanding $u(t,x)$ into series with respect to the eigenfunctions, we obtain

$$\begin{aligned}\theta_k(t) = \xi_k\cos\lambda_k t + \int\limits_0^t K'(0)\theta_k(s)\cos\lambda_k(t-s)\,ds - \int\limits_0^t K'(t-s)\theta_k(s)\,ds \\ + \int\limits_0^t \theta_k(s)\int\limits_0^{t-s} K''(\zeta)\cos\lambda_k(t-s-\zeta)d\zeta\,ds + \int\limits_0^{t-s} u_k(s)\cos\lambda_k(t-s)\,ds,\end{aligned} \tag{6}$$

where $\{u_k(t)\}_{k=1}^{+\infty}$ and $\{\xi_k\}_{k=1}^{+\infty}$ are the Fourier coefficients of control $u(t,x)$ and the initial value $\xi(x)$, respectively. Integrating by parts (twice) in the fourth term on the right-hand side of (6) (thereby getting rid of the second derivative $K''(\zeta)$), we obtain

$$\theta_k(t) = \xi_k\cos\lambda_k t - \lambda_k^2\int\limits_0^t \theta_k(s)\int\limits_0^{t-s} K(\zeta)\cos\lambda_k(t-s-\zeta)d\zeta\,ds + \lambda_k\int\limits_0^t u_k(s)\sin\lambda_k(t-s)\,ds. \tag{7}$$

Taking the Laplace transforms of both sides of equation (7) and expressing $\hat{\theta}_k(\lambda)$, we get

$$\hat{\theta}_k(\lambda) = \frac{\hat{u}_k(\lambda) + \xi_k}{\lambda + \lambda_k^2\hat{K}(\lambda)}. \tag{8}$$

First, consider the simplest case when the function $\lambda+\lambda_k^2\hat{K}(\lambda)$ of the complex variable λ, for each k, has only a single simple root μ_k, i.e.,

$$\lambda+\lambda_k^2\hat{K}(\lambda)=(\lambda-\mu_k)L_k(\lambda),$$

where $L_k(\mu_k)\neq 0$ and $\operatorname{Re}\mu_k\leq 0$. Then, in order for $\hat{\theta}_k(\lambda)$ to be holomorphic, we must have

$$\hat{u}_k(\mu_k)=-\xi_k. \tag{9}$$

Relation (9) can be written in integral form as follows:

$$\int\limits_0^T u_k(t)e^{-\mu_k t}dt=-\xi_k. \tag{10}$$

Theorem 1. *Let $\xi\in D(A^\alpha)$, where $\alpha>n/2$, and let the function $L_k(\lambda)$ be such that*

$$F(\lambda)\frac{1}{L_k(\lambda)}\in \mathrm{PW}_+^T$$

for any $F(\lambda)\in\mathrm{PW}_+^T$, where T is an arbitrary number greater than some $T_0>0$. Then there is an instant $T=T'$ and control $u(t,x)\in C([0,T']\times\Omega)$ such that $u(t,x)$ identically vanishes for $t>T'$, satisfies the inequality $|u(t,x)|\leq\varepsilon$ for any $t>0$, $x\in\Omega$, and the support (in t) of $\theta^u(t)$ belongs to $[0,T']$.

Proof. Let

$$u_k(t)=-\frac{\xi_k}{T}e^{\mu_k t},\quad t\in[0,T]. \tag{11}$$

Obviously, this function is a solution of the simplest integral equation (10). For $u_k(t)$ of the form (11), all $\hat{\theta}_k(\lambda)$ are entire functions. It remains to show that they belong to PW_+. We have

$$\hat{u}_k(\lambda)=\frac{\xi_k}{T}\frac{1-e^{T(\mu_k-\lambda)}}{\mu_k-\lambda}.$$

Clearly, $\hat{u}_k(\lambda)\in\mathrm{PW}_+^T$ by construction. Let us show that the function

$$\frac{\hat{u}_k(\lambda)+\xi_k}{\lambda-\mu_k}$$

is also an element of PW_+^T. To this end, consider the following auxiliary differential equation and boundary condition:

$$\dot{y}(t)-\mu_k y(t)=u_k(t),\quad y(0)=\xi_k.$$

The solution of this simple Cauchy problem obviously has the form

$$y(t)=\xi_k e^{\mu_k t}+\int\limits_0^t u_k(s)e^{\mu_k(t-s)}\,ds.$$

For $u_k(t)$ given by (11), we clearly have $y(T)=0$. Therefore, $\hat{y}(\lambda)\in\mathrm{PW}_+^T$, and it is easy to see that $\hat{y}(\lambda)$ is equal to $(\hat{u}_k(\lambda)+\xi_k)/(\lambda-\mu_k)$. Actually, any control $u_k(t)\in L_2(0,T)$ realizing a solution of the integral equation (10) is such that the corresponding function $(\hat{u}_k(\lambda)+\xi_k)/(\lambda-\mu_k)$ belongs to PW_+^T. Our choice of control in the form (11) is due merely to the requirement that the desired control should be bounded in absolute value. Now, using the assumptions of the theorem, we find that

$$\frac{\hat{u}_k(\lambda)+\xi_k}{\lambda-\mu_k}\,\frac{1}{L_k(\lambda)}\in PW_+^T.$$

Taking into account the inequalities $\operatorname{Re}\mu_k\leq 0$ for all k and the explicit form (11) of the Fourier coefficients of $u(t,x)$, we obtain

$$|u_k(t)|\leq\frac{1}{T}|\xi_k|. \tag{12}$$

Further, using the Cauchy inequality, we get

$$|u(t,x)|^2 \le \frac{1}{T^2}\sum_{k=1}^{+\infty}\lambda_k^{-2\alpha}\varphi_k^2(x)\sum_{k=1}^{+\infty}\lambda_k^{2\alpha}|\xi_k|^2.$$

It is known (see [17]) that the series $\sum \lambda_k^{-2\alpha}\varphi_k^2(x)$ is uniformly bounded if $\alpha > n/2$. Moreover,

$$\sum_{k=1}^{+\infty}\lambda_k^{2\alpha}|\xi_k|^2 = \int_\Omega \big(A^\alpha\xi(x)\big)^2\,dx.$$

Our assumptions on ξ in the statement of this theorem obviously ensure that the series for $u(t,x)$ is convergent. Therefore, the last estimate for u shows that for $T = T'$ sufficiently large, the control function can be made arbitrarily small in absolute value. The theorem is proved. □

Now, consider a more difficult case of the function $\lambda + \lambda_k^2\hat{K}(\lambda)$ having precisely two distinct simple roots $\mu_{1,k}$ and $\mu_{2,k}$ with nonpositive real parts.

In this case, (8) takes the form

$$\hat{\theta}_k(\lambda) = \frac{\hat{u}_k(\lambda)+\xi_k}{(\lambda-\mu_{1,k})(\lambda-\mu_{2,k})L_k(\lambda)}. \tag{13}$$

In order for the function $\hat{\theta}_k$ to be holomorphic, it is necessary that

$$\hat{u}_k(\mu_{1,k}) = -\xi_k, \quad \hat{u}_k(\mu_{2,k}) = -\xi_k. \tag{14}$$

Relations (14) can be written in integral form,

$$\int_0^T u_k(t)e^{-\mu_{1,k}t}\,dt = -\xi_k, \quad \int_0^T u_k(t)e^{-\mu_{2,k}t}\,dt = -\xi_k. \tag{15}$$

Theorem 2. *Let $|\mu_{1,k}-\mu_{2,k}| \ge \delta > 0$ for any k, $\xi \in D(A^\alpha)$ with $\alpha > n/2$, and let the function $L_k(\lambda)$ satisfy the condition*

$$\frac{F(\lambda)}{\lambda-a}\,\frac{1}{L_k(\lambda)} \in \mathrm{PW}_+^T$$

for any $F(\lambda) \in \mathrm{PW}_+^T$ such that a is its root and T is an arbitrary number greater than some $T_0 > 0$. Then there exist an instant $T = T'$ and control $u(t,x) \in C([0,T']\times\Omega)$ such that $u(t,x)$ identically vanishes for $t > T'$ and satisfies the inequality $|u(t,x)| \le \varepsilon$ for any $t > 0$, $x \in \Omega$, and the support of $\theta^u(t)$ belongs to $[0,T']$.

Proof. A solution of system (15) can be constructed in the form

$$u_k(t) = C_{1,k}e^{\mu_{1,k}t} + C_{2,k}e^{\mu_{2,k}t}. \tag{16}$$

It is not difficult to show that this choice of control ensures that the function $u(t,x)$ is bounded in absolute value. For this purpose, it suffices to substitute (16) into (15) and use Cramer's rule to find a solution of the resulting system with respect to the unknown $C_{1,k}$ and $C_{2,k}$. Simple estimates for the determinants yield

$$|u_k(t)| \le \frac{TM|\xi_k|}{T^2-4/\delta^2},$$

where $M > 0$ is a constant independent of k.

Now, let us show that

$$\frac{\hat{u}_k(\lambda)+\xi_k}{(\lambda-\mu_{1,k})(\lambda-\mu_{2,k})} \in \mathrm{PW}_+^T.$$

To this end, consider the following auxiliary system of differential equations with initial conditions:

$$\dot{y}_1(t) - \mu_{1,k} y_1(t) = u_k(t), \quad y_1(0) = \xi_k,$$
$$\dot{y}_2(t) - \mu_{2,k} y_2(t) = u_k(t), \quad y_2(0) = \xi_k.$$

Obviously, the solution of this simple system has the form

$$y_1(t) = \xi_k e^{\mu_{1,k} t} + \int_0^t u_k(s) e^{\mu_{1,k}(t-s)}\, ds,$$

$$y_2(t) = \xi_k e^{\mu_{2,k} t} + \int_0^t u_k(s) e^{\mu_{2,k}(t-s)}\, ds.$$

For $u_k(t)$ defined by (16), we obviously have $y_1(T) = 0$ and $y_2(T) = 0$. Therefore, just as in the case of a single zero, we find that the functions

$$\frac{\hat{u}_k(\lambda) + \xi_k}{\lambda - \mu_{1,k}}, \quad \frac{\hat{u}_k(\lambda) + \xi_k}{\lambda - \mu_{2,k}}$$

belong to PW_+^T. Now, let us prove a simple lemma.

Lemma 1 (an analogue of Levinson's theorem [19])**.** *Let $\hat{f}(\lambda)$ belong to PW_+^T and $\hat{f}(\mu) = 0$, $\mu \in \mathbb{C}$. Then*

$$\frac{\hat{f}(\lambda)}{\lambda - \mu}$$

also belongs to PW_+^T.

Proof. Let $f(t)$ be the inverse image of the Laplace transform $\hat{f}(\lambda)$. The support of $f(t)$ lies on the interval $[0, T]$. Consider the convolution of $f(t)$ with $e^{\mu t}$,

$$g(t) = \int_0^{+\infty} f(s) e^{\mu(t-s)}\, ds = \int_0^T f(s) e^{\mu(t-s)}\, ds.$$

Note that the inverse image of $\hat{f}(\lambda)/(\lambda - \mu)$ coincides with $g(t)$. For $t > T$, we have

$$g(t) = e^{\mu t} \int_0^T f(s) e^{-\mu s}\, ds = e^{\mu t} \hat{f}(\mu) = 0,$$

and therefore, the support of $g(t)$ belongs to $[0, T]$. The lemma is proved. $\square$

By Lemma 1, we have

$$\frac{\hat{u}_k(\lambda) + \xi_k}{(\lambda - \mu_{1,k})(\lambda - \mu_{2,k})} \in \mathrm{PW}_+^T,$$

since $\mu_{2,k}$ is a null-point of the function $(\hat{u}_k(\lambda) - \xi_k)/(\lambda - \mu_{1,k})$. Now, using the assumptions of the theorem, we find that

$$\frac{\hat{u}_k(\lambda) + \xi_k}{(\lambda - \mu_{1,k})(\lambda - \mu_{2,k})} \frac{1}{L_k(\lambda)} \in PW_+^T.$$

The theorem is proved. $\square$

To illustrate how Theorem 2 can be applied, consider the kernel from example (2), i.e., $K(t) = e^{-\gamma t}$, $\gamma > 0$. Then, $\hat{K}(\lambda) = 1/(\lambda + \gamma)$ and

$$\lambda + \lambda_k^2 \hat{K}(\lambda) = \lambda + \lambda_k^2 \frac{1}{(\lambda + \gamma)} = (\lambda^2 + \gamma\lambda + \lambda_k^2) \frac{1}{(\lambda + \gamma)}.$$

Therefore, in this case, $L_k(\lambda) = 1/(\lambda + \gamma)$ does not depend on k. Since the sequence of eigenvalues λ_k^2 tends to infinity as $k \to +\infty$, it follows that, for all large enough k, both roots $\mu_{1,k}$ and $\mu_{2,k}$ of the quadratic equation $\lambda^2 + \gamma\lambda + \lambda_k^2 = 0$ are complex with negative real parts. Let us verify the conditions of the theorem. Consider an arbitrary function $F(\lambda) \in PW_+^T$ such that $F(a) = 0$ for some $a \in \mathbb{C}$. Then

$$(\lambda + \gamma)\frac{F(\lambda)}{\lambda - a} = F(\lambda) + \frac{a + \gamma}{\lambda - a}F(\lambda).$$

Hence we obtain the desired result.

The case of $\lambda + \lambda_k^2 \hat{K}(\lambda)$ having precisely N mutually distinct complex roots is more difficult. This happens, for instance, if $K(t)$ is a sum of finitely many decaying exponential functions (example (4)). The structure of the set of zeroes and their asymptotic behavior are thoroughly examined in [7,20]. In the case of N zeroes, one has to solve countably many finite-dimensional problems for N-dimensional moments. Then, the control function can be constructed in a form similar to (16), but the sum will contain N terms. Some methods for finding solutions of such systems for moments, in a similar case, have been described in [12, 13].

4. Boundary Control

In this section, we consider a one-dimensional problem of boundary control concentrated on one end of an interval. In [15], a sufficient condition was obtained for a system to be noncontrollable by one end. As a consequence, it is impossible to bring the system to rest by applying control on an arbitrary subinterval. In [14], this result was generalized for a two-dimensional case. More precisely, it was shown that if the control function $u(t,x)$ in system (1)–(3) identically vanishes outside a domain D such that $\bar{D} \subset \Omega$ and $\hat{K}(\lambda)$ has at least one root λ_0, then the system is noncontrollable. This means that there is an initial value ξ such that for any control $u(t)$ identically vanishing for $t > T$, the corresponding solution $\theta^u(t)$ does not identically vanish outside any finite interval (with respect to t). Systems noncontrollable by one end (and therefore, by a subinterval) include those whose kernels $K(t)$ can be represented as sums of two or more decaying exponents. In this connection, it should be said that the condition that $\hat{K}(\lambda)$ has no zeroes is insufficient for the controllability of a system (for instance, the Abelian kernel with $\gamma \in (0,1)$; see [15]).

Consider the following one-dimensional problem of boundary control:

$$\theta_t(t,x) - \int_0^t K(t-s)\theta_{xx}(s,x)\,ds = 0, \quad t > 0, \quad x \in (0,\pi). \tag{17}$$

$$\theta|_{t=0} = \xi(x), \tag{18}$$

$$\theta(t,0) = v(t), \quad \theta(t,\pi) = 0. \tag{19}$$

The aim is to find the control $v(t) \in L_2(0,+\infty)$ applied to the left end-point of the segment $[0,\pi]$ such that $v(t)$ identically vanishes on the set $\{t\colon t > T\}$ for some $T > 0$ and ensures that the corresponding solution $\theta^v(t)$ also identically vanishes on $\{t\colon t > T\}$.

Recall the well-known definition of the so-called Dirichlet mapping, denoted by D. In a domain Ω, consider the boundary value problem for the Laplace equation with a nonzero Dirichlet condition for the trace on the boundary Γ. If this trace is an element of $L_2(\Gamma)$, then the solution of this problem exists and is an element of $H^{1/2}(\Omega)$. Thus, we have a mapping from $L_2(\Gamma)$ to $H^{1/2}(\Omega)$. In the one-dimensional case, say $\Omega = (0,\pi)$, the Laplace operator coincides with double differentiation, while the trace on the boundary is just the "value" of the solution at the end-points of the interval. Taking zero as the boundary value at the right end-point and an arbitrary v at the left, we obviously have

$$Dv = \left(1 - \frac{x}{\pi}\right)v.$$

Let $v \in L_2(0,T)$ with an arbitrary $T > 0$, and $\xi \in L_2(0,\pi)$.

Definition 2 (see [16]). A function

$$\theta(t) \in C\big([0,T]; L_2(0,\pi)\big)$$

is called a *solution* of problem (17)–(19), if $\theta(t)$ satisfies the integral equation

$$\theta(t) = R_+(t)\xi - B\int_0^t R_-(t-s)Dv(s)\,ds - \int_0^t L(t-s)Dv(s)\,ds + \int_0^t L(t-s)\theta(s)\,ds, \tag{20}$$

where

$$R_-(t) = \frac{1}{2}(e^{Bt} - e^{-Bt}),$$

the operators $R_+(t)$ and $L(t)$ are defined in Sec. 2.

It is known that under the above assumptions about the boundary and the initial conditions, a solution of problem (17)–(19) exists and is unique in the specified class.

Definition 3. A system of functions $\{e^{\mu_j t}\}_{j=1}^{+\infty}$, $\mu_j \in \mathbb{C}$, $j = 1, 2, \ldots$, is called *minimal* in $L_2(0,T)$ if none of these functions can be approximated, with any given accuracy in the norm of this space, by linear combinations of the rest.

Among such systems, an important role is played by the so-called uniformly minimal systems. A system $\{e^{\mu_j t}\}_{j=1}^{+\infty}$ is called *uniformly minimal* (see [19]) in $L_2(0,T)$ if

$$\operatorname{dist}\big(e^{\mu_j t}, \overline{\operatorname{span}}\{e^{\mu_i t}\}_{i\neq j}\big) \geq \alpha\|e^{\mu_j t}\|_{L_2(0,T)}.$$

It is well known that the eigenfunctions of the Dirichlet problem for the operator

$$A = -\frac{\partial^2}{\partial x^2}$$

have the form

$$\varphi_k(x) = \sqrt{\frac{2}{\pi}}\sin kx.$$

Since these eigenfunctions form a basis in $L_2(0,\pi)$, we seek a solution of problem (17)–(19) in the form of a series:

$$\theta(t,x) = \sum_{k=1}^{+\infty}\theta_k(t)\varphi_k(x). \tag{21}$$

Substituting (21) into (20), we get

$$\begin{aligned}
\theta_k(t) &= \xi_k\cos kt + \int_0^t K'(0)\theta_k(s)\cos k(t-s)\,ds - \int_0^t K'(t-s)\theta_k(s)\,ds \\
&+ \int_0^t \theta_k(s)\int_0^{t-s} K''(\zeta)\cos k(t-s-\zeta)d\zeta\,ds + \sqrt{\frac{2}{\pi}}\int_0^t v(s)\sin k(t-s)\,ds \\
&- \sqrt{\frac{2}{\pi}}\frac{1}{k}\int_0^t K'(0)v(s)\cos k(t-s)\,ds + \sqrt{\frac{2}{\pi}}\frac{1}{k}\int_0^t K'(t-s)v(s)\,ds \\
&- \sqrt{\frac{2}{\pi}}\frac{1}{k}\int_0^t v(s)\int_0^{t-s} K''(\zeta)\cos k(t-s-\zeta)\,d\zeta\,ds.
\end{aligned} \tag{22}$$

Passing to the Laplace transforms of both sides of (22) and expressing $\hat{\theta}_k(\lambda)$, we find that

$$\hat{\theta}_k(\lambda) = \frac{\sqrt{\pi}\xi_k + \sqrt{2}k\hat{K}(\lambda)\hat{v}(\lambda)}{\sqrt{\pi}\big(\lambda + k^2\hat{K}(\lambda)\big)}. \tag{23}$$

If $\lambda = 0$ is a root of the equation $\hat{K}(\lambda) = 0$, then (23) cannot hold, in general, for the quantities $\hat{\theta}_k(\lambda)$ corresponding to functions of class PW_+, and therefore, system (17)–(19) cannot be brought to rest.

In order that $\hat{\theta}_k \in \mathrm{PW}_+$, it is necessary (but not sufficient) that

$$\int_0^T v(t)e^{-\lambda t}\,dt = \frac{\sqrt{\pi}k\xi_k}{\sqrt{2}\lambda}, \tag{24}$$

for $\lambda \neq 0$ satisfying the equation $\lambda + k^2\hat{K}(\lambda) = 0$. If $\hat{K}(\lambda)$ has at least one zero, then the problem for the moments (24) has no solutions, in general (see [15]), and therefore, the system is noncontrollable.

Let $\Lambda = \{\mu_j\}$ be the set of all roots of the equation $\lambda + k^2\hat{K}(\lambda) = 0$ for each k, and let $\mu_j \neq 0$ for any j. Here $j \in J$, where J is a set of indexes. Note that the set $\{\mu_j\}$ coincides with the spectrum of the original problem, i.e., the set of all λ for which there is a nontrivial solution of the equation $\lambda\hat{\theta} + \hat{K}(\lambda)\hat{\theta}_{xx} = 0$ with the corresponding homogeneous boundary conditions.

Denote by $n_\Lambda(t)$ the number of points of the spectrum Λ belonging to the circle of radius t with center at the origin. We introduce a parameter a^* describing the "density" of the spectrum:

$$a^* = \overline{\lim_{r\to+\infty}}\, \frac{\pi}{r}\int_0^r \frac{n_\Lambda(t)}{t}\,dt.$$

In examples (4) and (5) (for $\gamma \in (-\infty, 0)$), we have $a^* = \infty$ and the corresponding systems are noncontrollable, because of the presence of accumulation points in the spectrum and the "slow" convergence of $|\mu_j|$ to infinity (i.e., $\sum_j 1/|\mu_j|^\sigma = +\infty$ for some $\sigma > 1$). On the other hand, in example (5) for $0 < \gamma < 1$, the points of the spectrum tend to infinity at "a sufficiently fast" rate and $a^* = 0$. This fact indicates that control might be possible within an arbitrarily short time interval, but it is shown in [15] that a system with such a kernel is noncontrollable. For the heat equation (example (3)), the quantities $|\mu_j|$ tend to infinity as j^2, the spectrum Λ is "rarefied," and $a^* = 0$, which, as in the previous case, suggests the possibility of control over an arbitrarily short time. This fact has indeed been established in [18] for a more general two-dimensional equation, and it seems that those methods are applicable in the one-dimensional case. In examples (1) and (2), the parameter a^* is finite and it can also be shown that $a^* = T^*$, where T^* is the time of the optimal speed-in-action. Moreover, in these cases, one can find explicit expressions for the optimal control yielding the solution of the speed-in-action problem.

Our next statement follows from (24).

Theorem 3. *Suppose that the speed-in-action problem for system* (17)–(19) *has a solution. Then the time of the optimal speed-in-action T^* is not less than a^*.*

Proof. Suppose that for the control time we have $T < a^*$. Let us show that in this case the system cannot be brought to rest. As shown in [19, p. 131], if $\{\lambda_j\}$ is a system of complex values, $0 \notin \{\lambda_j\}$, and

$$\overline{\lim_{r\to+\infty}}\left(\int_0^r \frac{n_{\{\lambda_j\}}(t)}{t}\,dt - \frac{T}{\pi}r - \frac{1}{2}\log r\right) > -\infty,$$

then the system of exponents $\{e^{i\lambda_j t}\}$ is non-minimal in $L_2(-T/2, T/2)$.

Since $e^{-\mu_j t} = e^{i(i\mu_j)t}$ and $n_\Lambda(t) \equiv n_{\{i\mu_j\}}(t)$, the above statement ensures that the system $\{e^{-\mu_j t}\}$ is non-minimal in $L_2(-T/2, T/2)$. It is not difficult to show that the system is also non-minimal in $L_2(0, T)$.

Let us show that the set $\{e^{-\mu_j t}\}$ is dense in $L_2(0,T)$ for $T < a^*$. We are going to use the following result established in [21]. If $\{\lambda_j\}$ is a system of complex values, $0 \notin \{\lambda_j\}$, and

$$\overline{\lim_{r\to+\infty}} \left(\int\limits_0^r \frac{n_{\{\lambda_j\}}(t)}{t}\, dt - \frac{T}{\pi} r + \frac{1}{2}\log r \right) > -\infty,$$

then the system of exponents $\{e^{i\lambda_j t}\}$ is dense in $L_2(0,T)$. This statement immediately implies that the system $\{e^{-\mu_j t}\}$ is dense in $L_2(0,T)$ for $T < a^*$.

To simplify our further exposition, we assume, without loss of generality, that for each k, the equation $\lambda + \lambda_k^2 \hat{K}(\lambda) = 0$ has just one root μ_k. Therefore, from (24), we obtain the system of moments

$$\int\limits_0^T v(t) e^{-\mu_k t}\, dt = \frac{\sqrt{\pi} k \xi_k}{\sqrt{2}\mu_k}. \tag{25}$$

As we have shown, the system $\{e^{-\mu_k t}\}$ is dense and non-minimal in $L_2(0,T)$, and thus, there is an element $e^{-\mu_r t}$ without which the system remains dense in this space. Set $\xi_k = 0$ for all $k \neq r$, and $\xi_r = 1$. Then, in view of the density of the system $\{e^{-\mu_k t}\}_{k\neq r}$, we have $v(t) \equiv 0$, which is impossible. The theorem is proved. $\square$

Remark. Suppose that $T^* = a^*$ and the following conditions hold:

(1) the equation $\lambda + k^2 \hat{K}(\lambda) = 0$ has precisely N mutually disjoint complex roots $\{\mu_k^i\}$, $\mu_k^i \neq 0$, $i = 1, \dots, N$, for each k. Recall that instead of $\{\mu_k^i\}$ we also write $\{\mu_j\}$, where $j \in J$ (in our case, J the set of all positive integers);
(2) the corresponding system of exponents $\{e^{-\mu_j t}\}$, supplemented by the identical unity (i.e. e^0), is dense and uniformly minimal on the interval $[0, a^*]$;
(3) all roots $\{\mu_j\}$ satisfy the condition $\|e^{-\mu_j t}\|_{L_2(0,a^*)} \geq \alpha$ with $\alpha > 0$ independent of j.

Then the optimal control has the form

$$v(t) = C f_0(t) + \sum_{j=1}^{+\infty} f_j(t) d_j,$$

where $\{f_j(t)\}_{j=0}^{+\infty}$ is a system biorthogonal to $\{1\} \cup \{e^{-\mu_j t}\}$ on the interval $[0, a^*]$, d_j are certain known numbers (the right-hand sides of system (24)), and C is an unknown constant.

Indeed, the existence of the biorthogonal system and the L_2-convergence of the series for $v(t)$ follow from the fact that the system $\{1\} \cup \{e^{-\mu_j t}\}$ is uniformly minimal on the interval $[0, a^*]$, together with the condition

$$\|e^{-\mu_j t}\|_{L_2(0,a^*)} \geq \alpha.$$

Of course, it is assumed here that the initial values are sufficiently smooth (see Sec. 2), in order to ensure a sufficiently fast decay of the Fourier coefficients ξ_k.

Next, we give an example of a kernel whose Laplace transform has no zeroes, its corresponding system being noncontrollable. Let $K(t) = e^{-2\gamma t}$ (see example (2)) with a sufficiently small γ. The equation with this kernel is equivalent to the telegraph equation. In this case, $\hat{K}(\lambda)$ has no zeroes and Theorem 3 can be applied. Equation (17) takes the form

$$\theta_t(t,x) - \int\limits_0^t e^{-2\gamma(t-s)} \theta_{xx}(s,x)\, ds = 0. \tag{26}$$

Since the eigenfunctions form a basis, we can use the expansion (21), which gives us the countable system of equations

$$\dot{\theta}_k(t) - \int\limits_0^t e^{-2\gamma(t-s)} \left(-k^2\theta_n(s) + \sqrt{\frac{2}{\pi}} kv(s) \right) ds = 0, \quad \theta_k(0) = \xi_k. \tag{27}$$

Differentiating the last equation and eliminating the integral term, we find that the Cauchy problem (27) is equivalent to the problem

$$\ddot{\theta}_k(t) + 2\gamma\dot{\theta}_k(t) + k^2\theta_k(t) = \sqrt{\frac{2}{\pi}} kv(t), \quad \theta_k(0) = \xi_k, \quad \dot{\theta}_k(0) = 0. \tag{28}$$

The solution of the Cauchy problem (28) has the form

$$\theta_k(t) = \xi_k e^{-\gamma t} \cos \alpha_k t + \frac{\gamma}{\alpha_k} \xi_k e^{-\gamma t} \sin \alpha_k t + h_k \int\limits_0^t v(s) e^{-\gamma(t-s)} \sin \alpha_k (t-s)\, ds, \tag{29}$$

where

$$\alpha_k = \sqrt{k^2 - \gamma^2}, \quad h_k = \frac{k}{\alpha_k}\sqrt{\frac{2}{\pi}}.$$

Differentiating (29), we get

$$\begin{aligned}\dot{\theta}_k(t) = -\xi_k \left(\alpha_k + \frac{\gamma^2}{\alpha_k} \right) e^{-\gamma t} \sin \alpha_k t \\ - h_k \gamma \int\limits_0^t v(s) e^{-\gamma(t-s)} \sin \alpha_k (t-s)\, ds + h_k \alpha_k \int\limits_0^t v(s) e^{-\gamma(t-s)} \cos \alpha_k (t-s)\, ds.\end{aligned} \tag{30}$$

The conditions of rest, $\theta_k(T) = 0$ and $\dot{\theta}_k(T) = 0$, yield

$$- \xi_k \left(\alpha_k + \frac{\gamma^2}{\alpha_k} \right) \sin \alpha_k T - h_k \gamma \int\limits_0^T v(s) e^{\gamma s} \sin \alpha_k (T-s)\, ds + h_k \alpha_k \int\limits_0^T v(s) e^{\gamma s} \cos \alpha_k (T-s)\, ds = 0,$$

$$\xi_k \cos \alpha_k T + \frac{\gamma}{\alpha_k} \xi_k \sin \alpha_k T + h_k \int\limits_0^T v(s) e^{\gamma s} \sin \alpha_k (T-s)\, ds = 0.$$

Multiplying the second equation by γ and adding to the first, we obtain the equivalent system

$$\begin{aligned}&\int\limits_0^T v(s) e^{\gamma s} \cos \alpha_k (T-s)\, ds = \frac{\xi_k}{h_k} \sin \alpha_k T, \\ &\int\limits_0^T v(s) e^{\gamma s} \sin \alpha_k (T-s)\, ds = -\frac{\xi_k}{h_k} \cos \alpha_k T - \frac{\gamma}{h_k \alpha_k} \xi_k \sin \alpha_k T.\end{aligned} \tag{31}$$

Taking $T = 2\pi$ in the system for moments (31), changing the variable to $t = 2\pi - s$, and setting $w(t) = v(2\pi - t)$, we obtain

$$\begin{aligned} &\int_0^{2\pi} w(t)e^{-\gamma t}\cos(\alpha_k t)\,dt = e^{-2\pi\gamma}\frac{\xi_k}{h_k}\sin(2\pi\alpha_k), \\ &\int_0^{2\pi} w(t)e^{-\gamma t}\sin(\alpha_k t)\,dt = -e^{-2\pi\gamma}\frac{\xi_k}{h_k}\cos(2\pi\alpha_k) - \frac{\gamma e^{-2\pi\gamma}}{h_k\alpha_k}\xi_k\sin(2\pi\alpha_k). \end{aligned} \tag{32}$$

Let us multiply the second equation of system (32) by i and add the result to the first equation. Doing the same with this system, after multiplying the second equation by $-i$, we obtain the equivalent system

$$\begin{aligned} &\int_0^{2\pi} w(t)e^{(-\gamma+i\alpha_k)t}\,dt = q_k, \\ &\int_0^{2\pi} w(t)e^{(-\gamma-i\alpha_k)t}\,dt = \bar{q}_k, \end{aligned} \tag{33}$$

where

$$q_k = e^{-2\pi\gamma}\frac{\xi_k}{h_k}\sin(2\pi\alpha_k) - ie^{-2\pi\gamma}\frac{\xi_k}{h_k}\left(\cos(2\pi\alpha_k) + \frac{\gamma}{\alpha_k}\sin(2\pi\alpha_k)\right).$$

Consider the system of exponents $\{e^{\pm i\alpha_k t}\}_{k=1}^{+\infty}$ (here, i is the imaginary unit) and set $e_{\pm k}(t) = e^{\pm i\alpha_k t}$. Let us supplement this system with the element $e_0 = 1$ and show that the resulting system is uniformly minimal in $L_2(0, 2\pi)$. Note that

$$\sqrt{k^2 - \gamma^2} = k + O\left(\frac{1}{k}\right).$$

We have

$$\alpha_k = k + O\left(\frac{1}{k}\right).$$

It is known that this is a system of equiconvergence (see [19, p. 170]), and therefore, it is minimal. It has also been shown (see [19, p. 137, Theorem 2.15]) that in this case, equiconvergence is equivalent to uniform minimality, since the system is dense in $L_2(0, 2\pi)$ for sufficiently small γ. Then there is a biorthogonal system $\{f_{\pm k}(t)\}_{k=0}^{+\infty}$ (see [19]) such that

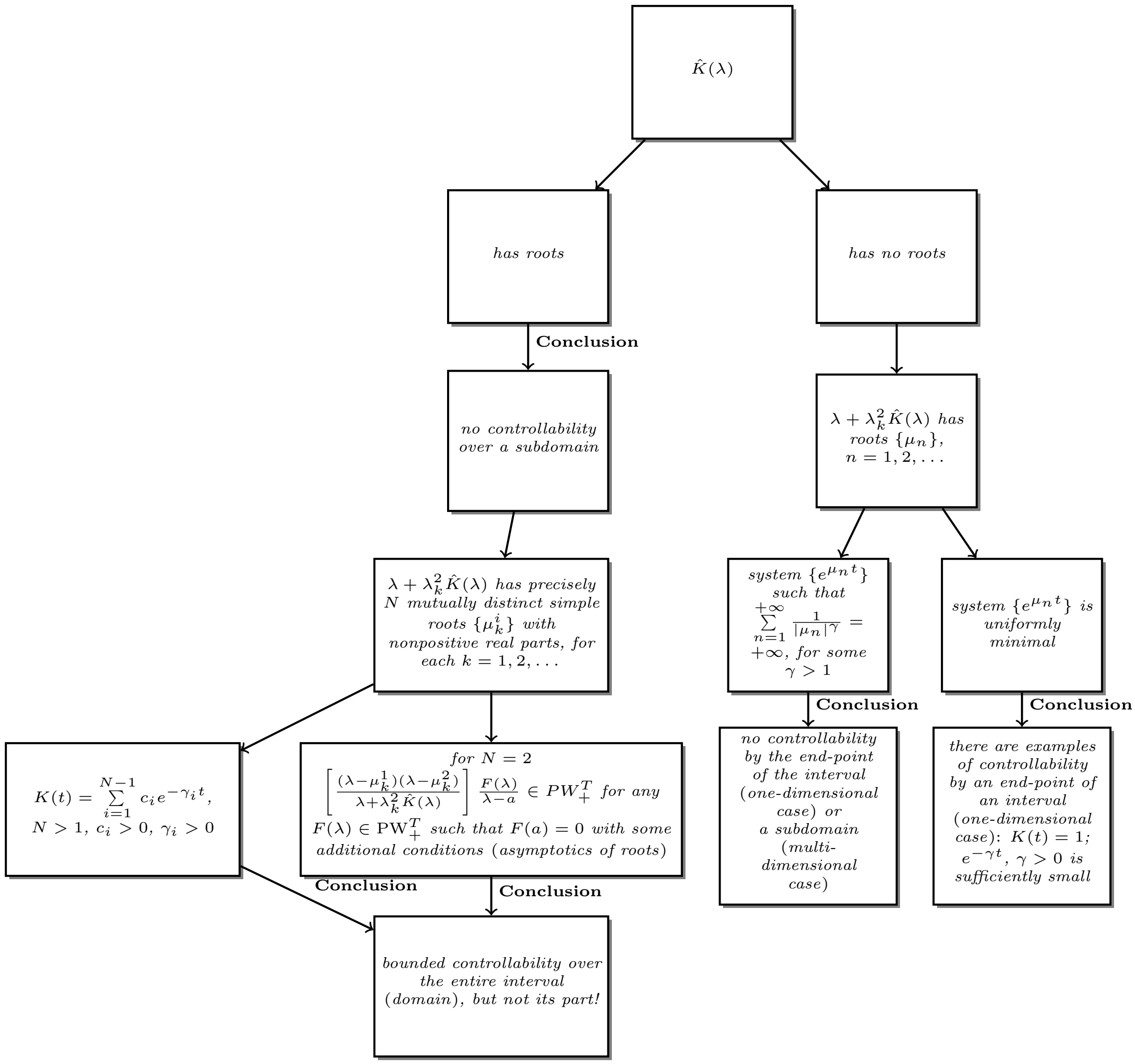
$$\int_0^{2\pi} f_k(t)e_j(t)\,dt = \delta_{kj}, \quad k, j = 0, \pm 1, \pm 2, \ldots,$$

and this biorthogonal system is uniformly bounded in the norm of $L_2(0, 2\pi)$. Let us represent the control function $w(t)$, which is a solution of system (33), in the form of a series with respect to the biorthogonal system:

$$w(t) = e^{\gamma t}\sum_{k=\pm 1, \pm 2, \ldots} f_k(t)q_k, \tag{34}$$

where $q_{-k} = \bar{q}_k$, by definition.

In conclusion, we present a scheme that gives a general idea of the questions considered in this paper.



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