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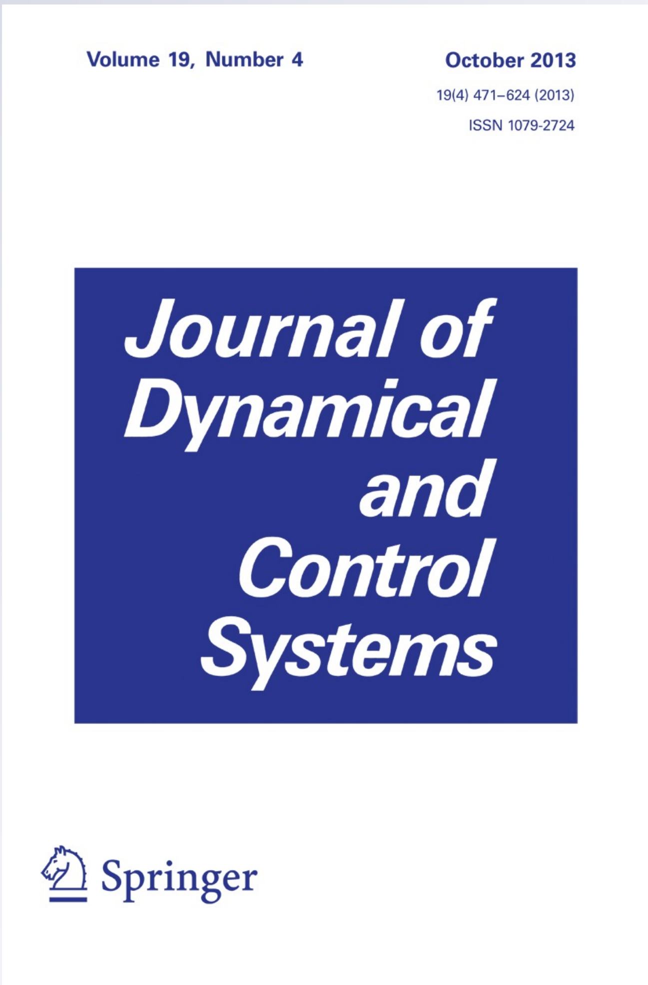

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Exact Controllability of the Distributed System, Governed by String Equation with Memory

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Abstract We will consider the exact controllability of the distributed system, governed by string equation with memory. It will be proved that this mechanical system can be driven to an equilibrium point in a finite time, the absolute value of the distributed control function being bounded. In this case, the memory kernel is a linear combination of exponentials.

Keywords Controllability to rest · String equation with memory · Bounded control

Mathematics Subject Classifications (2010) 93C20 · 35L53

1 Introduction

In this article, we will consider the problem of exact controllability of a system, governed by the integro-differential equation

$$\Theta_{tt}(t,x) - K(0)\Theta_{xx}(t,x) - \int_0^t K'(t-s)\Theta_{xx}(s,x)ds = u(t,x), \quad x \in (0;\pi), \quad t > 0. \tag{1}$$

$$\Theta|_{t=0} = \varphi_0(x), \quad \Theta_t|_{t=0} = \varphi_1(x), \tag{2}$$

$$\Theta|_{x=0} = 0, \quad \Theta|_{x=\pi} = 0. \tag{3}$$

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Here,

$$K(t) = \sum_{j=1}^{N} \frac{c_j}{\gamma_j} e^{-\gamma_j t},$$

where c_j and γ_j are given positive constant numbers, $u(t, x)$ is a control supported (in x) on an interval $(0, \pi)$ and $|u(t, x)| \le M$, $M > 0$ is a given constant number. The goal of the control is to drive this mechanical system to rest in a finite time. We say that a system is controllable to rest when for every initial conditions φ_0, φ_1, we can find a control u such that the corresponding solution $\Theta(t, x)$ of problem (1)–(3) and its first derivative of t $\Theta_t(t, x)$ hit zero at $t = T$.

Similar problems for membranes and plates were studied earlier in monograph [1]. It was proved that the vibrations of these mechanical systems can be driven to rest by applying bounded (in absolute value) and volume-distributed control functions. The existence of a bounded (in absolute value) boundary control that drives a string to rest was proved in [2]. In this case, the so-called moment problem was effectively applied. An overview of the results concerning the boundary controllability of distributed systems can be found in [3]. Problems of controllability of systems similar to Eq. 1 were considered in [4]. A condition under which a solution to the heat equation with memory cannot be driven to rest in a finite time was formulated. This condition is there are roots of some analytic functions of a complex variable in the domain of holomorphism.

Let us prove now that system (1) is uncontrollable if $u(t, x)$ is supported (in x), as well as in [4], on an interval $[a, b]$ which is properly contained in $[0, \pi]$. It is clear that Eq. 1 can be written in the following form:

$$\frac{\partial}{\partial t}\left(\Theta_t(t, x) - \int_0^t K(t-s)\Theta_{xx}(s, x)ds - \int_0^t u(s, x)ds\right) = 0.$$

Obviously, function $\Theta(t, x)$ is a solution of Eq. 1 if and only if this function is a solution of the following equation:

$$\Theta_t(t, x) - \int_0^t K(t-s)\Theta_{xx}(s, x)ds - \int_0^t u(s, x)ds = f(x), \tag{4}$$

where $f(x)$ is an arbitrary function. Let t in Eq. 4 be equal to zero; then, we obtain

$$f(x) = \varphi_1(x).$$

Let $\varphi_1(x) \equiv 0$. We introduce

$$P(t, x) = \int_0^t u(s, x)ds.$$

Thus, problem (1)–(3) reduce to the problem

$$\Theta_t(t, x) - \int_0^t K(t-s)\Theta_{xx}(s, x)ds = P(t, x), \quad x \in (0; \pi), \quad t > 0. \tag{5}$$

$$\Theta|_{t=0} = \varphi_0(x), \tag{6}$$

$$\Theta|_{x=0} = 0, \quad \Theta|_{x=\pi} = 0. \tag{7}$$

Note that the support of $P(t, x)$ is the subset of $\text{supp}\{u(t, x)\}$. Consequently, if $\text{supp}\{u(t, x)\} \subset [a, b]$ that in its turn is properly contained in $[0, \pi]$, then the same is true for $\text{supp}\{P(t, x)\}$. It is the problem considered in [4]. If $K(t)$ is a linear combination of two exponentials, then systems (5)–(7) are uncontrollable to rest. It means that there is an initial condition φ_0 such that for any control $P(t, x)$, where $P(t, x)$ belongs to the corresponding space, the solution of Eqs. 5–7 cannot be driven to rest.

Using arguments similar to those discussed above, it can be proved that system (1)–(3) is uncontrollable to rest if $K(t)$ is a linear combination of N exponentials, where $N \geq 2$.

2 Preliminaries

Let $A := -\frac{\partial^2}{\partial x^2}$ be an operator acting on a Sobolev space $H := H^2(0, \pi)$ with boundary condition (3).

We denote by $W^2_{2,\gamma}(R_+, A)$ the linear space of functions $f : R_+ = (0, +\infty) \to H$ equipped with the norm

$$\|\Theta\|_{W^2_{2,\gamma}(R_+,A)} = \left(\int\limits_0^{+\infty} e^{-2\gamma t} \left(\left\|\Theta^{(2)}(t)\right\|^2_H + \|A\Theta(t)\|^2_H \right) dt \right)^{\frac{1}{2}}, \quad \gamma \geq 0.$$

For more details about $W^2_{2,\gamma}(R_+, A)$, see chapter 1 of the monograph [5].

Definition 2.1 A function $\Theta(t, x)$ is called a strong solution of problem (1)–(3) if for some $\gamma \geq 0$, this function belongs to the space $W^2_{2,\gamma}(R_+, A)$, satisfies Eq. 1 almost everywhere (in t) on the positive semiaxis R_+ and satisfies the initial condition (2).

Let us denote the function of a complex variable λ by

$$l_n(\lambda) := \lambda^2 + n^2 \lambda \hat{K}(\lambda),$$

where

$$\hat{K}(\lambda) = \sum_{k=1}^{N} \frac{c_k}{\gamma_k(\lambda + \gamma_k)}.$$

Now, we formulate two theorems (see [6]) which are devoted to correct solvability of the initial boundary value problem (1)–(3).

Theorem 2.2 *Let the function $\Theta(t, x) \in W^2_{2,\gamma}(R_+, A)$, $\gamma > 0$, be a strong solution of problem (1)–(3) with $u(t, x) \equiv 0$, $t \in R_+$. Then, for any $t \in R_+$, the following representation is true:*

$$\Theta(t, x) = \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{(\varphi_{1n} + \lambda_n^+ \varphi_{0n}) e^{\lambda_n^+ t} \sin nx}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{(\varphi_{1n} + \lambda_n^- \varphi_{0n}) e^{\lambda_n^- t} \sin nx}{l_n^{(1)}(\lambda_n^-)}$$
$$+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left(\sum_{k=1}^{N-1} \frac{(\varphi_{1n} - q_{k,n} \varphi_{0n}) e^{-q_{k,n} t}}{l_n^{(1)}(-q_{k,n})} \right) \sin nx, \tag{8}$$

where $-q_{k,n}$ *is real zeros of the function* $l_n(\lambda)$ $(q_{k,n} > 0)$, $\lambda_n^{\pm}$ *is a pair of complex conjugate zeros of* $l_n(\lambda)$ *and series* (8) *converges in the norm of the space* H.

Theorem 2.3 *Suppose that* $u(t,x) \in C([0,T],H)$ *for any* $T > 0$; $\Theta(t,x) \in W^2_{2,\gamma}(R_+, A)$ *is a strong solution of problem* (1)–(3) *for some* $\gamma > 0$, $\varphi_0 = \varphi_1 = 0$. *Then, for any* $t \in R_+$, *the following representation holds:*

$$\Theta(t,x) = \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\omega_n(t,\lambda_n^+)\sin nx + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\omega_n(t,\lambda_n^-)\sin nx + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N-1}\omega_n(t,-q_{k,n})\right)\sin nx, \tag{9}$$

where

$$\omega_n(t,\lambda) = \frac{\int_0^t u_n(s)e^{\lambda(t-s)}ds}{l_n^{(1)}(\lambda)}, \quad u_n(t) = \frac{2}{\pi}\int_0^{\pi} u(t,x)\sin nx dx$$

and series (9) *converges in the norm of the space* H.

The following lemma should be stated:

Lemma 2.4 *For any natural number n, the equality holds*

$$\frac{1}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{l_n^{(1)}(\lambda_n^-)} + \sum_{k=1}^{N-1}\frac{1}{l_n^{(1)}(-q_{k,n})} = 0.$$

Proof We shall deal with the solution of problem (1)–(3) in the case of $\varphi_0 = \varphi_1 = 0$. According to Theorem 2.3, this solution has the form Eq. 9, with $u(t,x)$ being arbitrary and satisfying theorem conditions. Taking the partial derivative of $\Theta(t,x)$ with respect to t, we obtain

$$\begin{aligned}\frac{\partial\Theta(t,x)}{\partial t} &= \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\left(\frac{1}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{l_n^{(1)}(\lambda_n^-)} + \sum_{k=1}^{N-1}\frac{1}{l_n^{(1)}(-q_{k,n})}\right)u_n(t)\sin nx \\ &+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\lambda_n^+\omega_n(t,\lambda_n^+)\sin nx + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\lambda_n^-\omega_n(t,\lambda_n^-)\sin nx \\ &+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{N-1}(-q_{k,n})\omega_n(t,-q_{k,n})\right)\sin nx.\end{aligned} \tag{10}$$

Since $\Theta_t(t,x)|_{t=0} = 0$, then for any natural number n from Eq. 10 arises

$$\left(\frac{1}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{l_n^{(1)}(\lambda_n^-)} + \sum_{k=1}^{N-1} \frac{1}{l_n^{(1)}(-q_{k,n})}\right) u_n(0) = 0. \tag{11}$$

By virtue of the fact that $u(t,x)$ is arbitrary, it is chosen in such a way that all its Fourier coefficients $u_n(t)$ with respect to $t = 0$ are non-zero. Thus, dividing Eq. 11 by $u_n(0)$, we obtain the required statement. Lemma is proved. □

3 The Main Results

It is the following theorem which presents the main result of the article:

Theorem 3.1 *Let $\varphi_0 \in C^3[0,\pi]$ and $\varphi_0(0) = \varphi_0(\pi) = \varphi_0''(0) = \varphi_0''(\pi) = 0$, $\varphi_1 \in C^3[0,\pi]$ and $\varphi_1(0) = \varphi_1(\pi) = \varphi_1''(0) = \varphi_1''(\pi) = 0$; $M > 0$ is a certain constant. Then, there are a time point $T > 0$ and a control $u(t,x) \in C([0,T], H)$ depending on the value of M, such that the solution Θ of problem (1)–(3) satisfies the equalities*

$$\Theta(T,x) = \Theta_t'(T,x) = 0, \quad \forall x \in (0,\pi) \tag{12}$$

and the restriction

$$|u(t,x)| \le M, \quad \forall t \in (0,T],\ x \in (0,\pi).$$

Proof Let $u(t,x)$ be the function satisfying the theorem conditions and T is some instant of time. According to Theorems 2.2 and 2.3, the solution of problem (1)–(3) could be represented as Eqs. 8 and 9. Hence, we obtain

$$\begin{aligned}\Theta(t,x) &= \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(\varphi_{1n}+\lambda_n^+\varphi_{0n})e^{\lambda_n^+ t}\sin nx}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(\varphi_{1n}+\lambda_n^-\varphi_{0n})e^{\lambda_n^- t}\sin nx}{l_n^{(1)}(\lambda_n^-)}\\
&+\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\sum_{k=1}^{N-1}\left(\frac{(\varphi_{1n}-q_{k,n}\varphi_{0n})e^{-q_{k,n}t}}{l_n^{(1)}(-q_{k,n})}\right)\sin nx + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\int_0^t u_n(s)e^{\lambda_n^+(t-s)}ds}{l_n^{(1)}(\lambda_n^+)}\sin nx\\
&+\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\int_0^t u_n(s)e^{\lambda_n^-(t-s)}ds}{l_n^{(1)}(\lambda_n^-)}\sin nx + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\sum_{k=1}^{N-1}\left(\frac{\int_0^t u_n(s)e^{-q_{k,n}(t-s)}ds}{l_n^{(1)}(-q_{k,n})}\right)\sin nx.\end{aligned} \tag{13}$$

Therefore,

$$
\begin{aligned}
\frac{\partial \Theta(t,x)}{\partial t} &= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\lambda_n^+(\varphi_{1n} + \lambda_n^+ \varphi_{0n}) e^{\lambda_n^+ t} \sin nx}{l_n^{(1)}(\lambda_n^+)} \\
&\quad + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\lambda_n^-(\varphi_{1n} + \lambda_n^- \varphi_{0n}) e^{\lambda_n^- t} \sin nx}{l_n^{(1)}(\lambda_n^-)} \\
&\quad + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \sum_{k=1}^{N-1} \left(\frac{(-q_{k,n})(\varphi_{1n} - q_{k,n}\varphi_{0n}) e^{-q_{k,n} t}}{l_n^{(1)}(-q_{k,n})} \right) \sin nx \\
&\quad + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left(\frac{1}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{l_n^{(1)}(\lambda_n^-)} + \frac{1}{l_n^{(1)}(-q_n)} \right) u_n(t) \sin nx \\
&\quad + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\lambda_n^+ \int_0^t u_n(s) e^{\lambda_n^+(t-s)} ds}{l_n^{(1)}(\lambda_n^+)} \sin nx + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{\lambda_n^- \int_0^t u_n(s) e^{\lambda_n^-(t-s)} ds}{l_n^{(1)}(\lambda_n^-)} \sin nx \\
&\quad + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \sum_{k=1}^{N-1} \left(\frac{(-q_{k,n}) \int_0^t u_n(s) e^{-q_{k,n}(t-s)} ds}{l_n^{(1)}(-q_{k,n})} \right) \sin nx. \qquad (14)
\end{aligned}
$$

Note that the fourth summand in Eq. 14 is equal to zero, by Lemma 2.4. Using condition (12) and formulas (13) and (14), we obtain

$$
\begin{aligned}
&-\left(\frac{(\varphi_{1n} + \lambda_n^+ \varphi_{0n}) e^{\lambda_n^+ T}}{l_n^{(1)}(\lambda_n^+)} + \frac{(\varphi_{1n} + \lambda_n^- \varphi_{0n}) e^{\lambda_n^- T}}{l_n^{(1)}(\lambda_n^-)} + \sum_{k=1}^{N-1} \frac{(\varphi_{1n} - q_{k,n}\varphi_{0n}) e^{-q_{k,n} T}}{l_n^{(1)}(-q_{k,n})} \right) \\
&= \frac{\int_0^T u_n(s) e^{\lambda_n^+(T-s)} ds}{l_n^{(1)}(\lambda_n^+)} + \frac{\int_0^T u_n(s) e^{\lambda_n^-(T-s)} ds}{l_n^{(1)}(\lambda_n^-)} \\
&\quad + \sum_{k=1}^{N-1} \frac{\int_0^T u_n(s) e^{-q_{k,n}(T-s)} ds}{l_n^{(1)}(-q_{k,n})}, \qquad n = 1, 2, \ldots, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\lambda_n^+(\varphi_{1n} + \lambda_n^+ \varphi_{0n}) e^{\lambda_n^+ T}}{l_n^{(1)}(\lambda_n^+)} - \frac{\lambda_n^-(\varphi_{1n} + \lambda_n^- \varphi_{0n}) e^{\lambda_n^- T}}{l_n^{(1)}(\lambda_n^-)} \\
&\quad - \sum_{k=1}^{N-1} \frac{(-q_{k,n})(\varphi_{1n} - q_{k,n}\varphi_{0n}) e^{-q_{k,n} T}}{l_n^{(1)}(-q_{k,n})} \\
&= \frac{\lambda_n^+ \int_0^T u_n(s) e^{\lambda_n^+(T-s)} ds}{l_n^{(1)}(\lambda_n^+)} + \frac{\lambda_n^- \int_0^T u_n(s) e^{\lambda_n^-(T-s)} ds}{l_n^{(1)}(\lambda_n^-)} \\
&\quad + \sum_{k=1}^{N-1} \frac{(-q_{k,n}) \int_0^T u_n(s) e^{-q_{k,n}(T-s)} ds}{l_n^{(1)}(-q_{k,n})}, \qquad n = 1, 2, \ldots \quad . \qquad (16)
\end{aligned}
$$

We introduce

$$a_n = -(\varphi_{1n} + \lambda_n^+ \varphi_{0n}), \quad \bar{a}_n = -(\varphi_{1n} + \lambda_n^- \varphi_{0n}),$$

$$b_{k,n} = -(\varphi_{1n} + (-q_{k,n})\varphi_{0n}), \quad k = 1, 2, ..., N-1.$$

Let us set equal coefficients preceding

$$\frac{1}{l_n^{(1)}(\lambda_n^+)}, \quad \frac{1}{l_n^{(1)}(\lambda_n^-)}, \quad \frac{1}{l_n^{(1)}(-q_{k,n})}, \quad k = 1, 2, ..., N-1.$$

in the right and left parts of Eqs. 15 and 16. Thus, a new moment problem occurs:

$$\int_0^T u_n(s) e^{\lambda_n^+(T-s)} ds = a_n e^{\lambda_n^+ T}, \quad \int_0^T u_n(s) e^{\lambda_n^-(T-s)} ds = \bar{a}_n e^{\lambda_n^- T}, \quad n = 1, 2, ...,$$

$$\int_0^T u_n(s) e^{-q_{k,n}(T-s)} ds = b_{k,n} e^{-q_{k,n} T}, \quad k = 1, 2, ..., N-1, \quad n = 1, 2, ... \quad . \tag{17}$$

Obviously, if moment problem (17) is solvable, then moment problems (15) and (16) are solvable as well. Elimination of common factors in both parts (Eq. 17) allows to represent this system as follows:

$$\int_0^T u_n(s) e^{-\lambda_n^+ s} ds = a_n, \quad \int_0^T u_n(s) e^{-\lambda_n^- s} ds = \bar{a}_n, \quad n = 1, 2, ...,$$

$$\int_0^T u_n(s) e^{q_{k,n} s} ds = b_{k,n}, \quad k = 1, 2, ..., N-1, \quad n = 1, 2, ... \quad . \tag{18}$$

Due to the fact that the complex numbers λ_n^+ and λ_n^-, and a_n and $\bar{a}_n$ are conjugate, the first and the second equations in problem (18) are equivalent. This means that if the function $u_n(s)$ is a solution of the first equation, then it is a solution of the second one. Therefore, it is possible to eliminate one of the equations in Eq. 18, for instance, the second one (actually the countable number of equations is excluded). Thus, the moment problem can be rewritten as follows:

$$\int_0^T u_n(s) e^{-\lambda_n^+ s} ds = a_n, \quad \int_0^T u_n(s) e^{q_{k,n} s} ds = b_{k,n},$$

$$k = 1, 2, ..., N-1, \quad n = 1, 2... \quad . \tag{19}$$

Let us replace $-\lambda_n^+ = \lambda_n$ in Eq. 19. Notice that $Re\lambda_n > 0$ and $q_{k,n} > 0$ (see [6]). Finally, we obtain the system of N moments for each natural number n:

$$\int_0^T u_n(s) e^{\lambda_n s} ds = a_n, \quad \int_0^T u_n(s) e^{q_{k,n} s} ds = b_{k,n}, \quad k = 1, 2, ..., N-1. \tag{20}$$

The solution of Eq. 20 is sought in the following form:

$$u_n(s) = C_{0,n}e^{\lambda_n s} + \sum_{k=1}^{N-1} C_{k,n}e^{q_{k,n}s}, \quad n = 1, 2, ... \quad . \tag{21}$$

Set $C_{0,n}$ and $C_{k,n}$ as some unknown constants. Substituting Eq. 21 in Eq. 20, we get the system of N algebraic equations for each natural number n:

$$C_{0,n}\int_0^T e^{2\lambda_n s}ds + \sum_{k=1}^{N-1} C_{k,n}\int_0^T e^{(\lambda_n+q_{k,n})s}ds = a_n,$$

$$C_{0,n}\int_0^T e^{(\lambda_n+q_{k,n})s}ds + \sum_{k=1}^{N-1} C_{k,n}\int_0^T e^{2q_{k,n}s}ds = b_{k,n}, \quad k = 1, 2, ..., N-1. \tag{22}$$

Let us find the determinant Δ_n of problem (22).

$$\Delta_n = \begin{vmatrix} \int_0^T e^{2\lambda_n s}ds & \int_0^T e^{(\lambda_n+q_{1,n})s}ds & \dots & \int_0^T e^{(\lambda_n+q_{N-1,n})s}ds \\ \int_0^T e^{(q_{1,n}+\lambda_n)s}ds & \int_0^T e^{2q_{1,n}s}ds & \dots & \int_0^T e^{(q_{1,n}+q_{N-1,n})s}ds \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^T e^{(q_{N-1,n}+\lambda_n)s}ds & \int_0^T e^{(q_{N-1,n}+q_{1,n})s}ds & \dots & \int_0^T e^{2q_{N-1,n}s}ds \end{vmatrix}.$$

As far as

$$\int_0^T e^{(q_{i,n}+q_{j,n})s}ds = \frac{1}{q_{i,n}+q_{j,n}}e^{(q_{i,n}+q_{j,n})T} - \frac{1}{q_{i,n}+q_{j,n}}, \tag{23}$$

then using equality (23) and the well-known property of determinants

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_{i1}+c_{i1} & b_{i2}+c_{i2} & \dots & b_{in}+c_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \dots & b_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ c_{i1} & c_{i2} & \dots & c_{in} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}, \tag{24}$$

we obtain

$$\Delta_n = \begin{vmatrix} \frac{e^{2\lambda_n T}}{2\lambda_n} & \frac{e^{(\lambda_n+q_{1,n})T}}{\lambda_n+q_{1,n}} & \cdots & \frac{e^{(\lambda_n+q_{N-1,n})T}}{\lambda_n+q_{N-1,n}} \\ \frac{e^{(q_{1,n}+\lambda_n)T}}{q_{1,n}+\lambda_n} & \frac{e^{2q_{1,n}T}}{2q_{1,n}} & \cdots & \frac{e^{(q_{1,n}+q_{N-1,n})T}}{q_{1,n}+q_{N-1,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{e^{(q_{N-1,n}+\lambda_n)T}}{q_{N-1,n}+\lambda_n} & \frac{e^{(q_{N-1,n}+q_{1,n})T}}{q_{N-1,n}+q_{1,n}} & \cdots & \frac{e^{2q_{N-1,n}T}}{2q_{N-1,n}} \end{vmatrix} + \beta_n(T), \tag{25}$$

where $\beta_n(T)$ is the sum of all other determinants, which are the result of N-fold application of property (24) to each row of the determinant Δ_n.

Let us factor out $e^{\lambda_n T}$ from the first row of the determinant in the right side of Eq. 25 and then take out the same factor from the first column; now, similar action can be made for the second row and column with the factor $e^{q_{1,n}T}$ and so on.

Thus, we get

$$\Delta_n = e^{2\lambda_n T} \prod_{j=1}^{N-1} e^{2q_{j,n}T} \begin{vmatrix} \frac{1}{2\lambda_n} & \frac{1}{\lambda_n+q_{1,n}} & \cdots & \frac{1}{\lambda_n+q_{N-1,n}} \\ \frac{1}{q_{1,n}+\lambda_n} & \frac{1}{2q_{1,n}} & \cdots & \frac{1}{q_{1,n}+q_{N-1,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{q_{N-1,n}+\lambda_n} & \frac{1}{q_{N-1,n}+q_{1,n}} & \cdots & \frac{1}{2q_{N-1,n}} \end{vmatrix} + \beta_n(T). \tag{26}$$

Denote

$$\bar{\Delta}_n = \begin{vmatrix} \frac{1}{2\lambda_n} & \frac{1}{\lambda_n+q_{1,n}} & \cdots & \frac{1}{\lambda_n+q_{N-1,n}} \\ \frac{1}{q_{1,n}+\lambda_n} & \frac{1}{2q_{1,n}} & \cdots & \frac{1}{q_{1,n}+q_{N-1,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{q_{N-1,n}+\lambda_n} & \frac{1}{q_{N-1,n}+q_{1,n}} & \cdots & \frac{1}{2q_{N-1,n}} \end{vmatrix}.$$

Then,

$$\Delta_n = e^{2\lambda_n T} \prod_{j=1}^{N-1} e^{2q_{j,n}T} \left(\bar{\Delta}_n + e^{-2\lambda_n T} \prod_{j=1}^{N-1} e^{-2q_{j,n}T} \beta_n(T) \right).$$

In virtue of the definition of $\beta_n(T)$, the following relation is true:

$$\left| e^{-2\lambda_n T} \prod_{j=1}^{N-1} e^{-2q_{j,n}T} \beta_n(T) \right| \to 0, \quad \text{as } T \to +\infty.$$

Let us decompose the determinant $\bar{\Delta}_n$ by the first row.

$$\bar{\Delta}_n = \frac{1}{2\lambda_n} \begin{vmatrix} \frac{1}{2q_{1,n}} & \frac{1}{q_{1,n}+q_{2,n}} & \cdots & \frac{1}{q_{1,n}+q_{N-1,n}} \\ \frac{1}{q_{2,n}+q_{1,n}} & \frac{1}{2q_{2,n}} & \cdots & \frac{1}{q_{2,n}+q_{N-1,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{q_{N-1,n}+q_{1,n}} & \frac{1}{q_{N-1,n}+q_{2,n}} & \cdots & \frac{1}{2q_{N-1,n}} \end{vmatrix} + \Lambda_n, \tag{27}$$

where Λ_n is the sum of all other determinants, occurred after the decomposition. Notice that there is λ_n^2 in all summands in the denominator after the expansion of these determinants, i.e. $\Lambda_n \sim \frac{1}{\lambda_n^2}$.

Let us make the following notation:

$$P_n = \begin{vmatrix} \frac{1}{2q_{1,n}} & \frac{1}{q_{1,n}+q_{2,n}} & \cdots & \frac{1}{q_{1,n}+q_{N-1,n}} \\ \frac{1}{q_{2,n}+q_{1,n}} & \frac{1}{2q_{2,n}} & \cdots & \frac{1}{q_{2,n}+q_{N-1,n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{q_{N-1,n}+q_{1,n}} & \frac{1}{q_{N-1,n}+q_{2,n}} & \cdots & \frac{1}{2q_{N-1,n}} \end{vmatrix}.$$

P_n is the Cauchy determinant. It is a well-known fact that

$$P_n = \frac{\prod\limits_{N-1\geq i>j\geq 1} (q_{i,n} - q_{j,n})^2}{\prod\limits_{i,j=1}^{N-1} (q_{i,n} + q_{j,n})}.$$

As far as $q_{i,n}$, $i = 1, 2, ..., N-1$ is a pairwise different for any n (see [6, 7]); then, P_n is non-zero.

Hence,

$$\begin{aligned} \Delta_n &= e^{2\lambda_n T} \prod_{j=1}^{N-1} e^{2q_{j,n}T} \left(\frac{1}{2\lambda_n} P_n + \Lambda_n + e^{-2\lambda_n T} \prod_{j=1}^{N-1} e^{-2q_{j,n}T} \beta_n(T) \right) \\ &= \frac{1}{2\lambda_n} P_n e^{2\lambda_n T} \prod_{j=1}^{N-1} e^{2q_{j,n}T} \left(1 + \frac{2\lambda_n}{P_n} \Lambda_n + \frac{2\lambda_n}{P_n} e^{-2\lambda_n T} \prod_{j=1}^{N-1} e^{-2q_{j,n}T} \beta_n(T) \right). \end{aligned}$$

Let us denote

$$\bar{\Lambda}_n = \frac{2\lambda_n}{P_n} \Lambda_n, \quad \bar{\beta}_n(T) = \frac{2\lambda_n}{P_n} e^{-2\lambda_n T} \prod_{j=1}^{N-1} e^{-2q_{j,n}T} \beta_n(T).$$

It leads to the following equation:

$$\Delta_n = \frac{1}{2\lambda_n} P_n e^{2\lambda_n T} \prod_{j=1}^{N-1} e^{2q_{j,n}T} \left(1 + \bar{\Lambda}_n + \bar{\beta}_n(T)\right). \tag{28}$$

Notice that $\bar{\Lambda}_n \sim \frac{1}{\lambda_n}$, $\bar{\beta}_n(T) \to 0$ as $T \to +\infty$, the sequence of the modules of complex roots $\{|\lambda_n|\}$, tends to $+\infty$ as $n \to +\infty$, but the sequence of real numbers $\{q_{k,n}\}_{n=1}^{\infty}$ converges to some positive numbers q_k; actually, $q_{k,n} = q_k + o(n^{-2})$ (see [6, 7]). Thus, due to the asymptotic properties of λ_n and $q_{k,n}$, there is T such that all determinants Δ_n are non-zero for any natural index n.

Let us determine $\Delta_{0,n}$ by the following formula:

$$\Delta_{0,n} = \begin{vmatrix} a_n & \int_0^T e^{(\lambda_n+q_{1,n})s}ds & \cdots & \int_0^T e^{(\lambda_n+q_{N-1,n})s}ds \\ b_{1,n} & \int_0^T e^{2q_{1,n}s}ds & \cdots & \int_0^T e^{(q_{1,n}+q_{N-1,n})s}ds \\ \vdots & \vdots & \ddots & \vdots \\ b_{N-1,n} & \int_0^T e^{(q_{N-1,n}+q_{1,n})s}ds & \cdots & \int_0^T e^{2q_{N-1,n}s}ds \end{vmatrix}.$$

Set likewise $\Delta_{k,n}$, where $k = 1, 2, ..., N-1$:

$$\Delta_{k,n} = \begin{vmatrix} \int\limits_0^T e^{2\lambda_n s} ds & \int\limits_0^T e^{(\lambda_n + q_{1,n})s} ds & \dots & a_n & \dots & \int\limits_0^T e^{(\lambda_n + q_{N-1,n})s} ds \\ \int\limits_0^T e^{(q_{1,n} + \lambda_n)s} ds & \int\limits_0^T e^{2q_{1,n} s} ds & \dots & b_{1,n} & \dots & \int\limits_0^T e^{(q_{1,n} + q_{N-1,n})s} ds \\ \vdots & \vdots & & \vdots & & \vdots \\ \int\limits_0^T e^{(q_{N-1,n} + \lambda_n)s} ds & \int\limits_0^T e^{(q_{N-1,n} + q_{1,n})s} ds & \dots & b_{N-1,n} & \dots & \int\limits_0^T e^{2q_{N-1,n} s} ds \end{vmatrix},$$

where the column $\{a_n, b_{1,n}, ..., b_{N-1,n}\}$ takes the kth place.

Applying Cramer's rule, we obtain

$$C_{0,n} = \frac{\Delta_{0,n}}{\Delta_n}, \quad C_{k,n} = \frac{\Delta_{k,n}}{\Delta_n}, \quad k = 1, 2, ..., N-1.$$

Thus, the solution of Eq. 20 at the instant of time t has the following form:

$$u_n(t) = \frac{\Delta_{0,n}}{\Delta_n} e^{\lambda_n t} + \sum_{k=1}^{N-1} \frac{\Delta_{k,n}}{\Delta_n} e^{q_{k,n} t}.$$

Let $\lambda_n = \mu_n - i\nu_n$. In [6], it has been proved that $\mu_n, \nu_n > 0$ for any natural index n. The estimation of modulus of the function $u_n(t)$ for any natural n should be provided. So we have

$$|u_n(t)| \leq \frac{|\Delta_{0,n}|}{|\Delta_n|} e^{\lambda_n T} + \sum_{k=1}^{N-1} \frac{|\Delta_{k,n}|}{|\Delta_n|} e^{q_{k,n} T}. \tag{29}$$

Expanding the determinants $\Delta_{0,n}$, $\Delta_{k,n}$, $k = 1, 2, ..., N-1$, it is clear that the part of the summands consists of the different exponential products. Notice that the exponential product with the largest number of factors in the determinant $\Delta_{0,n}$ has the form

$$e^{2q_{1,n} T} e^{2q_{2,n} T} \dots e^{2q_{N-1,n} T} \quad \text{or} \quad e^{\lambda_n T} e^{2q_{1,n} T} e^{2q_{2,n} T} \dots e^{q_{j,n} T} \dots e^{2q_{N-1,n} T},$$

and in $\Delta_{k,n}$

$$e^{\lambda_n T} e^{2q_{1,n} T} e^{2q_{2,n} T} \dots e^{q_{k,n} T} \dots e^{2q_{N-1,n} T} \quad \text{or}$$

$$e^{2\lambda_n T} e^{2q_{1,n} T} e^{2q_{2,n} T} \dots e^{2q_{k-1,n} T} e^{2q_{k+1,n} T} \dots e^{2q_{N-1,n} T} \quad \text{or}$$

$$e^{2\lambda_n T} e^{2q_{1,n} T} e^{2q_{2,n} T} \dots e^{q_{k,n} T} \dots e^{q_{j,n} T} \dots e^{2q_{N-1,n} T}, \quad k \neq j.$$

Thus, there is at least one exponent with a positive index in the denominator of all summands in the right side of estimation (29). It means that it is possible to make the modulus of the function $u_n(t)$, and hence of the control $u(t)$, be indefinitely small by means of increasing the time control. Using Eq. 28, we obtain

$$|u_n(t)| \le \frac{2|\lambda_n||\Delta_{0,n}|}{|P_n|e^{2\mu_n T}\prod_{j=1}^{N-1} e^{2q_{j,n}T}\left(1-|\bar{\Lambda}_n|-|\bar{\beta}_n(T)|\right)}e^{\lambda_n T}$$

$$+\sum_{k=1}^{N-1}\frac{2|\lambda_n||\Delta_{k,n}|}{|P_n|e^{2\mu_n T}\prod_{j=1}^{N-1} e^{2q_{j,n}T}\left(1-|\bar{\Lambda}_n|-|\bar{\beta}_n(T)|\right)}e^{q_{k,n}T}. \tag{30}$$

Now, it is obvious that

$$|u(t,x)| \le \sum_{n=1}^{\infty}|u_n(t)|. \tag{31}$$

Using Eqs. 29 and 30, let us estimate the time required to stabilize the system, providing that the function $u(t,x)$ satisfies the condition

$$|u(t,x)| \le \varepsilon, \tag{32}$$

where ε is an arbitrary constant.

As far as the sequences of real numbers $\{\mu_n\}$, $\{\nu_n\}$, $\{q_{k,n}\}$ are such that $\mu_n = \mu + o(n^{-2})$, $\nu_n = Dn$ and $q_{k,n} = q_k + o(n^{-2})$, where μ, D and q_k are some positive numbers (see [6, 7]), and moreover the sequences $\{|a_n|\}$, $\{|b_{k,n}|\}$, $\{|\Lambda_n|\}$ tend to zero, then the following estimations take place:

$$|u(t,x)| \le \frac{c}{e^{c_1 T}}\left(\sum_{n=1}^{\infty}|\lambda_n||a_n| + \sum_{n=1}^{\infty}|\lambda_n|\left(\sum_{k=1}^{N-1}|b_{k,n}|\right)\right) \le \frac{c_2}{e^{c_1 T}} \le \varepsilon, \tag{33}$$

where c, c_1 and c_2 are certain constants and T is great enough. Notice that the convergence of the numerical series in Eq. 33 is caused by the fact that the initial conditions belong to the corresponding class of functions (see the theorem statement). Thus, the time T required to drive the system to rest is a solution of the equation

$$\frac{c_2}{e^{c_1 T}} = \varepsilon.$$

Finally, we obtain

$$T = -\frac{1}{c_1}\ln\frac{\varepsilon}{c_2}. \tag{34}$$

Equality (34) shows that in the case of the system with "memory", the time required to stabilize is essentially smaller and then the time for driving to rest the string without the integral delays, if

$$T \sim \frac{c}{\varepsilon}$$

(see [2]). At the same time, controllability of the system with "memory" is lost during the transition from the control distributed on the segment $[a,b]$ to the one distributed on the subsegment. □

There is a coefficient preceding the function $\Theta_{xx}(t, x)$ corresponding to the kernel $K'(t-s)$ in the integral part of Eq. 1. Since this coefficient can be arbitrary, there is no loss of generality in the current form of the equation. In order to perform it, the arbitrary constant should be added to $K(t)$. This constant is read as the exponential function with zero index. It can be verified that if a new kernel $K_1(t)$ has the form

$$K_1(t) = K(t) + C,$$

then the results [6] remain valid, and hence, the results of this article about the controllability still stand. In addition, if $K_1(t) = C_1 e^{-\lambda_1 t} + C_2$, then according to [4] about the lack of controllability to rest, it is proposed that systems (1)–(3) are not controlled in case when the control function $u(t, x)$ is supported on a subsegment.

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