ISSN 1064-5624, Doklady Mathematics, 2016, Vol. 94, No. 2, pp. 607–610. © Pleiades Publishing, Ltd., 2016.
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CONTROL THEORY

Exact Bounded Boundary Controllability of Vibrations of a Two-Dimensional Membrane

I. V. Romanov[a, b]* and A. S. Shamaev[c, d]**

Presented by Academician of the RAS F.L. Chernous'ko May 10, 2016

Received May 13, 2016

Abstract—The boundary control of vibrations of a plane membrane is considered. A constraint is imposed on the absolute value of the control function. The goal of the control is to drive the membrane to rest. The proof technique used in this paper can be applied to a membrane of any dimension, but the two-dimensional case is considered for simplicity and illustrative purposes.

DOI: 10.1134/S1064562416050057

The possibility of full stabilization in a finite time in the case of distributed control was proved in [1]. Additionally, an upper estimate for the optimal control was given in [1].

The boundary controllability of vibrations of a plane membrane has been considered by numerous authors (see, e.g., review articles by Russell [2] and Lions [3] and references therein). The problem of stabilizing vibrations of a restricted string by means of boundary control was considered in [4]. Specifically, it was proved that the string can be driven to rest in a finite time when the absolute value of the control function is restricted, and the time required for full stabilization was estimated. In [5] optimal control problems for distributed parameter systems were considered and optimality conditions similar to those for systems with a finite number of degrees of freedom were stated. However, this approach often fails to provide a constructive method for finding an optimal control. In [3] the problem of stabilizing membrane vibrations was considered, the existence of a boundary control was proved, and the time required for driving the membrane to rest was estimated. In many formulations of problems, the authors rejected the requirement of control optimality and considered only the controllability problem, which considerably simplified the study. Moreover, no constraint was imposed on the absolute value of control functions and no explicit expressions for control functions were found, but only existence theorems were proved.

The statement of the problem considered in this paper differs substantially from that considered in [2, 3], since the value of the control function on the boundary has to satisfy the condition $|u(t,x)| \le \varepsilon$. Note also that we search not for an optimal control, but rather for an admissible one (i.e., satisfying initial constraints).

Consider the following initial–boundary value problem for the vibration equation of a two-dimensional membrane:

$$w_{tt}(t,x) - \Delta w(t,x) = 0, \quad (t,x) \in Q_T = (0,T)\times\Omega, \quad (1)$$

$$w|_{t=0} = \varphi(x), \quad w_t|_{t=0} = \psi(x), \quad x \in \Omega, \quad (2)$$

$$\frac{\partial w}{\partial \nu} = u(t,x), \quad (t,x) \in \Sigma, \quad (3)$$

where Ω is a two-dimensional domain with a smooth boundary, ν is the outward normal to the boundary of Ω, Σ is the lateral surface of the cylinder Q_T, the initial data $\varphi(x)$ and $\psi(x)$ are assumed to be given and are chosen in what follows from suitable Sobolev spaces, and $u(t,x)$ is a control function defined on the boundary of Ω.

Let $\varepsilon > 0$ be an arbitrary given number. On the control function, we impose the constraint

$$|u(t,x)| \le \varepsilon. \quad (4)$$

[a] National Research University Higher School of Economics, Myasnitskaya ul. 20, Moscow, 101000 Russia
[b] Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Profsoyuznaya ul. 65, Moscow, 117997 Russia
[c] Ishlinskii Institute for Problems in Mechanics, Russian Academy of Sciences, pr. Vernadskogo 101-1, Moscow, 119526 Russia
[d] Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119991 Russia
*e-mail: romm1@list.ru,
**e-mail: sham@rambler.ru

The problem is to construct a control $u(t, x)$ satisfying inequality (4) such that the corresponding solution $w(t, x)$ of problem (1)–(3) and its derivative with respect to t vanish at some time T, i.e.,

$$w(T,x) = 0, \quad w_t(T,x) = 0 \tag{5}$$

for all $x \in \Omega$. If we can obtain a control $u(t, x)$ such that conditions (5) are satisfied, then system (1)–(3) is called controllable.

The following theorem is the main result of this paper.

Theorem 1. *Let* $\varphi(x) \in H^7(\Omega)$ *and* $\psi(x) \in H^6(\Omega)$ *be such that*

$$\frac{\partial \varphi(x)}{\partial \nu} = 0, \quad \psi(x) = 0, \quad x \in \partial\Omega. \tag{6}$$

Then there exists a time T and a control $u(t, x)$ *satisfying constraint* (4) *such that system* (1)–(3) *is controllable.*

The proof of Theorem 1 consists of two steps. At the first step, the considered solution and its first derivative with respect to t are stabilized in a sufficiently small neighborhood of zero in the norm of $C^5(\bar{\Omega}) \times C^4(\bar{\Omega})$. At the second step, the system in this small neighborhood is driven to rest.

Below, we describe the basic ideas behind the proof of this result.

1. THE FIRST STEP OF THE CONTROL

Let us apply the results of [2], where friction is introduced on the boundary of Ω (or on its portion). The friction is given by the first derivative of the solution $w(t, x)$ with respect to t, i.e., the initial–boundary value problem (1), (2) is considered with boundary condition

$$\frac{\partial w(t,x)}{\partial \nu} = -k\frac{\partial w(t,x)}{\partial t}, \quad (t,x) \in (0,+\infty) \times \partial\Omega, \tag{7}$$

where $k > 0$ is the friction coefficient.

Consider the energy of system (1), (2), (7):

$$E(t) = \int_{\Omega} (w_{x_1}^2(t,x) + w_{x_2}^2(t,x) + w_t^2(t,x))dx.$$

It was proved in [2] that friction on the boundary leads to energy dissipation in the system, namely,

$$\frac{dE(t)}{dt} \le 0, \quad t \ge 0. \tag{8}$$

Moreover, it is true that

$$E(t) \le \frac{C(\varphi,\psi)}{1+t}, \quad t \ge 0, \tag{9}$$

where $C(\varphi, \psi)$ is a constant depending on the initial data of the system and the friction coefficient k.

Consider auxiliary system (1), (2), (7) (system with friction on the boundary). By using estimate (9), we can prove that the pair

$$(w(t,x), w_t(t,x))$$

can be made arbitrarily small in the norm of $C^5(\bar{\Omega}) \times C^4(\bar{\Omega})$ by choosing a sufficiently large $t > 0$.

Let $w_0(t, x)$ be the solution of (1), (2), (7) corresponding to initial data $(\varphi(x), \psi(x))$. Substituting w_0 only into the right-hand side of condition (7), we obtain system (1)–(3) with a known boundary condition:

$$u^{(1)}(t,x) = -k\frac{\partial w_0}{\partial t}, \ \ (t, x) \in (0; T_1] \times \partial\Omega.$$

It is obvious (by the uniqueness theorem) that the solution $w(t, x)$ of system (1)–(3) is such that the pair $(w(t, x), w_t(t, x))$ can also be made arbitrarily small in the norm of $C^5(\bar{\Omega}) \times C^4(\bar{\Omega})$ by choosing a sufficiently large $t > 0$.

Using inequality (8), we can prove that

$$\left|\frac{\partial w_0(t,x)}{\partial t}\right| \le C_1(\varphi,\psi), \quad (t,x) \in (0,+\infty) \times \Omega,$$

where $C_1(\varphi, \psi)$ is independent of k. Therefore, control constraint (4) can be satisfied by choosing a sufficiently small k.

2. THE SECOND STEP OF THE CONTROL

The pair of functions $w = w(T_1, x)$ and $\dot{w} = w_t(T_1, x)$ (T_1 is the control time at the first step) is regarded as new initial data for problem (1)–(3). Recall that, to according to the above argument, these new initial conditions are small enough in the norm of $C^5(\bar{\Omega}) \times C^4(\bar{\Omega})$. Let us apply the control method consisting in extending initial data to an unbounded domain. This method has been applied in many studies (see, e.g., [2] and references therein).

Let Ω_δ be a δ-neighborhood of Ω. Consider an extension operator E. It is a linear continuous operator from the space $C^5(\bar{\Omega}) \times C^4(\bar{\Omega})$ to $C^5(\bar{\Omega}_\delta) \times C^4(\bar{\Omega}_\delta)$ such that the support of the extended pair ($w_0^e(x)$, $w_1^e(x)$) its derivatives of 5th and 4th orders (respectively) inclusive belongs to $\bar{\Omega}_{\frac{\delta}{2}}$. Note that, outside Ω_δ, the functions can be extended by zero to the whole plane. In a more general case, E was constructed in [6].

Let $(w_0(x), w_1(x))$ be an arbitrary element of the space

$$\mathscr{C}^5(\bar{\Omega}) = C^5(\bar{\Omega}) \times C^4(\bar{\Omega}).$$

Consider the following Cauchy problem in the plane:

$$w_{tt}(t,x) - \Delta w(t,x) = 0,\quad (t,x) \in Q = (0,+\infty)\times R^2, \tag{10}$$

$$w|_{t=0} = w_0^e(x),\quad w_t|_{t=0} = w_1^e(x),\quad x\in R^2, \tag{11}$$

where $(w_0^e(x),\, w_1^e(x)) = E(w_0(x),\, w_1(x)) \in C^5(\bar{\Omega}_\delta)\times C^4(\bar{\Omega}_\delta)$. Then the solution of this Cauchy problem and its first time derivative at a sufficiently large $t = T_2$ are such that $(w^s(t, x),\, w_t^s(t, x)) \in \mathscr{C}^5(\bar{\Omega})$ (since the support of the initial data is compact).

Moreover,

$$\|(w^s(T_2,\cdot), w_t^s(T_2,\cdot))\|_{\mathscr{C}^5(\bar{\Omega})} \le \frac{M}{T_2}\|(w_0^e, w_1^e)\|_{\mathscr{C}^5(R^2)}, \tag{12}$$

where

$$\mathscr{C}^5(R^2) = C^5(R^2)\times C^4(R^2).$$

We restrict $(w^s(T_2, x)\, w_t^s(T_2, x))$ to Ω and then again apply the operator E, i.e.,

$$(w_0^{s,e}(T_2,x), w_1^{s,e}(T_2,x)) = E(w^s(T_2,x)|_\Omega, w_t^s(T_2,x)|_\Omega).$$

Now consider the inverse Cauchy problem with initial conditions

$$w(t,x)|_{t=T_2} = -w_0^{s,e}(T_2,x),\quad w_t(t,x)|_{t=T_2} = -w_1^{s,e}(T_2,x). \tag{13}$$

Let $w^i(t, x)$ be its solution.

Since Eq. (1) is reversible in time, we have the estimate

$$\|(w^i(0,\cdot), w_t^i(0,\cdot))\|_{\mathscr{C}^5(\bar{\Omega})} \le \frac{M}{T_2}\|(w_0^{s,e}(T_2,x), w_1^{s,e}(T_2,x))\|_{\mathscr{C}^5(R^2)}. \tag{14}$$

Let us restrict $(w^i(0, x),\, w_t^i(0, x))$ to Ω, i.e.,

$$(w^{i,r}(0,x), w_t^{i,r}(0,x)) = (w^i(0,x), w_t^i(0,x))|_\Omega.$$

Therefore, the pair $(w^{i,r}(0, x),\, w_t^{i,r}(0, x))$ is obtained from $(w_0(x),\, w_1(x))$ by applying a linear continuous operator, which is denoted by L. Thus,

$$(w^{i,r}(0,x), w_t^{i,r}(0,x)) = L(w_0(x), w_1(x)),$$
$$L\colon \mathscr{C}^5(\bar{\Omega}) \to \mathscr{C}^5(\bar{\Omega}).$$

It follows from (12) and (14) that $\|L\| < 1$ if T_2 is sufficiently large. Thus, the operator $I + L$ is invertible (I is the identity operator). Therefore, for any initial condition $(w,\, \dot{w}) \in \mathscr{C}^5(\bar{\Omega})$, there is a pair $(w_0(x),\, w_1(x)) \in \mathscr{C}^5(\bar{\Omega})$ such that

$$(w,\dot{w}) = (w_0(x), w_1(x)) + L(w_0(x), w_1(x)). \tag{15}$$

Representation (15) determines the control method used at the second step. Specifically, the pair $(w_0(x),\, w_1(x))$ is extended to the whole plane by applying the extension operator E; then the above-described procedure (the construction of L) is used to construct the pair $(w^i(0, x),\, w_t^i(0, x))$ (the restriction of which to Ω is $L(w_0(x),\, w_1(x))$). Then the extensions of the initial data in the original problem (these data are obtained after applying the first-step control) are specified by the sum

$$(w_0^e(x), w_1^e(x)) + (w^i(0,x), w_t^i(0,x)). \tag{16}$$

Note that $(w^i(0,\, x),\, w_t^i(0,\, x))$ plays the role of a "small" perturbation. By the construction of L, the solution of the Cauchy problem with initial conditions (16) and its derivative with respect to t obviously vanish identically in Ω at $t = T_2$. Then, as a boundary control, we use the value of the outward normal derivative of the solution to the Cauchy problem on the boundary of Ω.

To conclude, the control at the second step is represented in operator form:

$$u^{(2)}(t,x) = \frac{\partial}{\partial \nu_\Omega} K_+^t[(I + (-K_-^{T_2})\mathrm{ERK}_+^{T_2})E(I + R(-K_-^{T_2})\mathrm{ERK}_+^{T_2}E)^{-1}\{w,\dot{w}\}],\quad (t,x)\in(0,T_2]\times\partial\Omega,$$

where E is the extension operator from Ω to Ω_δ, R is the restriction operator from R^2 to Ω, and K_+^t and K_-^t are the solution operators of the Cauchy problem (Poisson's formula) in direct and reverse time. Note that the minus sign preceding $K_-^{T_2}$ is explained by initial conditions (13).

The initial data $(w,\, \dot{w})$ at the second step of the control can be made (due to the first step) arbitrarily small in the norm of $\mathscr{C}^5(\bar{\Omega})$. It is well known that, corresponding to these initial data (after they are extended to R^2 as described above), the solution of the Cauchy problem for every fixed $t > 0$ belongs to the space

$$\mathscr{C}^4(R^2) = C^4(R^2)\times C^3(R^2).$$

By using this smoothness property, the energy conservation law, and the Sobolev embedding theorem, is possible is to prove that the restriction of the solution of the Cauchy problem to $\bar{\Omega}$ (i.e., the solution of the original initial—boundary value problem at the second step) can be made arbitrarily small in the norm of $C^1(\bar{\Omega})$ at every $t \ge 0$ if $(w,\, \dot{w})$ is previously made sufficiently small. As a result, condition (4) is satisfied.

ACKNOWLEDGMENTS

This work was supported by the Russian Science Foundation, project no. 16-11-10343.

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Translated by I. Ruzanova