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On the Problem of Precise Control of the System Obeying the Delay String Equation

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Abstract—Consideration was given to the problem of precise control of a system obeying the equation of string with integral "memory." This system was proved to be reducible to the quiescent state in a finite time with the use of a distributed action bound in magnitude. The time to stop oscillations was also estimated.

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1. INTRODUCTION

The present paper considers precise control of a system obeying the following equation:

$$\Theta_{tt}(t,x) - K(0)\Theta_{xx}(t,x) - \int\limits_0^t K'(t-s)\Theta_{xx}(s,x)ds = u(t,x), \quad x \in (0;\pi), \quad t > 0, \tag{1}$$

$$\Theta|_{t=0} = \varphi_0(x), \quad \Theta_t|_{t=0} = \varphi_1(x), \tag{2}$$

$$\Theta|_{x=0} = 0, \quad \Theta|_{x=\pi} = 0. \tag{3}$$

At that,

$$K(t) = \frac{c_1}{\gamma_1}e^{-\gamma_1 t} + \frac{c_2}{\gamma_2}e^{-\gamma_2 t},$$

where $c_{1,2}$ and $\gamma_{1,2}$ are positive constants, $u(t,x)$ is the control distributed in the variable x over the interval $(0,\pi)$, and $|u(t,x)| \leqslant M$, $M > 0$. The control is aimed to drive the system to the quiescent state in a finite time. The system is said to be reducible to the complete quiescent state if for any initial conditions one can determine a control such that the corresponding problem solution and its derivative with respect to t vanish in a finite time.

Similar problems were considered previously in the monograph [1] for the two-dimensional membranes and plates. It was proved that the oscillations of such systems can be stopped in a finite time with the use of a control action distributed over the entire surface of the plant under consideration. The problem of optimization of the boundary control of string oscillations was first considered in [2]. In this case, the method of moments was used to advantage. The results of numerous authors concerning the boundary controllability of membranes and plates with the use of various boundary conditions were compiled in the review paper [3]. The issues of controllability of system similar to (1) were considered in [4] where a condition was given under which the heat-conductivity equation with integral "memory" cannot be reduced to the complete quiescent state in a finite time. This condition lies in the presence of roots of some complex-analytical function in its holomorphy domain.

We are going to prove that the complete controllability of the system obeying Eq. (1) is lost if the carrier of the function $u(t,x)$ in x is concentrated as in [4] over the interval $[a,b]$ contained completely in $[0,\pi]$. We represent Eq. (1) as

$$\frac{\partial}{\partial t}\left(\Theta_t(t,x)-\int\limits_0^t K(t-s)\Theta_{xx}(s,x)ds-\int\limits_0^t u(s,x)ds\right)=0.$$

The function $\Theta(t,x)$ is, obviously, the solution of Eq. (1) if and only if it is the solution of the equation

$$\Theta_t(t,x)-\int\limits_0^t K(t-s)\Theta_{xx}(s,x)ds-\int\limits_0^t u(s,x)ds=f(x),$$

where $f(x)$ is an arbitrary function. By substituting $t=0$ in the above equation and using the second initial condition of (2), we get

$$f(x)=\varphi_1(x).$$

Let now $\varphi_1(x)\equiv 0$. By assuming that

$$P(t,x)=\int\limits_0^t u(s,x)ds,$$

we reduce (1)–(3) to the problem

$$\Theta_t(t,x)-\int\limits_0^t K(t-s)\Theta_{xx}(s,x)ds=P(t,x),\quad x\in(0;\pi),\quad t>0, \tag{1*}$$

$$\Theta|_{t=0}=\varphi_0(x), \tag{2*}$$

$$\Theta|_{x=0}=0,\quad \Theta|_{x=\pi}=0. \tag{3*}$$

We notice that the carrier of the function $P(t,x)$ belongs to the carrier of the function $u(t,x)$. Consequently, if $supp\{u(t,x)\}$ is contained in $[a,b]$ which in turn is completely contained in $[0,\pi]$, then this is also true for $supp\{P(t,x)\}$. Namely the same problem was considered in [4]. If the function $K(t)$ is the sum of two exponents, the condition for no complete controllability is knowingly satisfied. Then, there is an initial function φ_0 such that the solution of problem (1*)–(3*) cannot be reduced to the complete quiescent state by any control action $P(t,x)$ from the corresponding functional class.

By reasoning as above, one can prove that controllability of problem (1)–(3) is impossible if $K(t)$ is the sum of N exponential functions where $N\geqslant 2$.

Now, we consider the case where the kernel $K(t)$ is given by

$$K(t)=\frac{c}{\gamma}e^{-\gamma t}.$$

Then,

$$\Theta_{tt}(t,x)-\frac{c}{\gamma}\Theta_{xx}(t,x)+c\int\limits_0^t e^{-\gamma(t-s)}\Theta_{xx}(s,x)ds=u(t,x),\quad x\in(0;\pi),\quad t>0.$$

This equation is representable as

$$\Theta_t(t,x) - \frac{c}{\gamma}\int\limits_0^t e^{-\gamma(t-s)}\Theta_{xx}(s,x)ds - \int\limits_0^t u(s,x)ds = \varphi_1(x).$$

By eliminating the integral term from the two last equations, we establish that

$$\Theta_{tt}(t,x) - \frac{c}{\gamma}\Theta_{xx}(t,x) + \gamma\Theta_t(t,x) = U(t,x),$$

where

$$U(t,x) = u(t,x) + \gamma\int\limits_0^t u(s,x)ds + \gamma\varphi_1(x).$$

In the case of $K(t) = \frac{c}{\gamma}e^{-\gamma t}$, problem (1)–(3) is equivalent to that of control of oscillations of a string with external friction. Using the results and methods of [3], one can readily demonstrate controllability of the system obeying the equations of oscillations of a string with friction. As the result, in the case where the kernel $K(t)$ has the form of one exponential function, system (1)–(3) is controllable as well.

2. BASIC DEFINITIONS AND AUXILIARY STATEMENTS

Let $A := -\frac{\partial^2}{\partial x^2}$ be the differential operator defined over the elements of the Sobolev space $H := H^2(0,\pi)$ with the boundary conditions (3). We denote by $W^2_{2,\gamma}(R_+, A)$ the space of functions on the semiaxis $R_+ = (0, +\infty)$ having values in H and endowed with the norm

$$\|\Theta\|_{W^2_{2,\gamma}(R_+,A)} = \left(\int\limits_0^{+\infty} e^{-2\gamma t}\left(\left\|\Theta^{(2)}(t)\right\|_H^2 + \|A\Theta(t)\|_H^2\right)dt\right)^{\frac{1}{2}}, \quad \gamma \geqslant 0.$$

For more details about the space $W^2_{2,\gamma}(R_+, A)$ see Ch. I of the monograph [5].

Definition. The function Θ is called the strong solution of problem (1)–(3) if for some $\gamma \geqslant 0$ it belongs to the space $W^2_{2,\gamma}(R_+, A)$ and satisfies almost everywhere on the semiaxis R_+ Eq. (1), as well as the initial conditions (2).

We define the function of complex variable λ

$$l_n(\lambda) := \lambda^2 + n^2\lambda\hat{K}(\lambda),$$

where

$$\hat{K}(\lambda) = \frac{c_1}{\gamma_1(\lambda+\gamma_1)} + \frac{c_2}{\gamma_2(\lambda+\gamma_2)}$$

and give two theorems (see [6]) devoted to representation of the solution of problem (1)–(3) as series.

Theorem 1. *Let $u(t,x) = 0$ for $t \in R_+$, and the function $\Theta(t,x) \in W^2_{2,\gamma}(R_+, A)$, $\gamma > 0$ be strong solution of problem (1)–(3). Then, for any $t \in R_+$ the solution $\Theta(t,x)$ of problem (1)–(3) is representable as the sum of the series*

$$\begin{aligned}\Theta(t,x) &= \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(\varphi_{1n} + \lambda_n^+\varphi_{0n})e^{\lambda_n^+ t}\sin nx}{l_n^{(1)}(\lambda_n^+)} \\ &+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(\varphi_{1n} + \lambda_n^-\varphi_{0n})e^{\lambda_n^- t}\sin nx}{l_n^{(1)}(\lambda_n^-)} \\ &+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(\varphi_{1n} - q_n\varphi_{0n})e^{-q_n t}\sin nx}{l_n^{(1)}(-q_n)}, \end{aligned} \tag{4}$$

converging in norm of the space H, where $-q_n$ are the real zeros of the meromorphic function $l_n(\lambda)$ ($q_n > 0$), $\lambda_n^{\pm}$ is a pair of the complex-conjugate zeros, and $l_n^{(1)}(\lambda)$ is the derivative of the function $l_n(\lambda)$.

Theorem 2. *Let $u(t,x) \in C([0,T],H)$ for any $T > 0$, the function $\Theta(t,x) \in W_{2,\gamma}^2(R_+, A)$ be strong solution of problem (1)–(3) for some $\gamma > 0$, and $\varphi_0 = \varphi_1 = 0$. Then, for any $t \in R_+$ the solution $\Theta(t,x)$ of problem (1)–(3) is representable as the sum of the series*

$$\begin{aligned}\Theta(t,x) &= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \omega_n(t,\lambda_n^+) \sin nx \\ &+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \omega_n(t,\lambda_n^-) \sin nx \\ &+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \omega_n(t,-q_n) \sin nx,\end{aligned} \tag{5}$$

converging in norm of the space H, where

$$\omega_n(t,\lambda) = \frac{\int\limits_0^t u_n(s) e^{\lambda(t-s)} ds}{l_n^{(1)}(\lambda)}$$

and $u_n(t)$ is the nth coefficient at the expansion of the function $u(t,x)$ in the Fourier series with respect to sines.

Theorems 1 and 2 are used in what follows. Now, we formulate and prove an auxiliary statement.

Lemma. *The equality*

$$\frac{1}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{l_n^{(1)}(\lambda_n^-)} + \frac{1}{l_n^{(1)}(-q_n)} = 0$$

is satisfied for any natural index n.

Proof. Let us consider the solution of problem (1)–(3) in the case where $\varphi_0 = \varphi_1 = 0$. Under Theorem 2, this solution has the form (5), the function $u(t,x)$ being arbitrary and satisfying the theorem conditions. We determine the partial derivative of $\Theta(t,x)$ with respect to the variable t:

$$\begin{aligned}\frac{\partial \Theta(t,x)}{\partial t} &= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left(\frac{1}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{l_n^{(1)}(\lambda_n^-)} + \frac{1}{l_n^{(1)}(-q_n)} \right) u_n(t) \sin nx \\ &+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \lambda_n^+ \omega_n(t,\lambda_n^+) \sin nx + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \lambda_n^- \omega_n(t,\lambda_n^-) \sin nx \\ &+ \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} (-q_n) \omega_n(t,-q_n) \sin nx.\end{aligned} \tag{6}$$

Since $\Theta_t(t,x)|_{t=0} = 0$, with the use of (6) we establish for any natural index n that

$$\left(\frac{1}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{l_n^{(1)}(\lambda_n^-)} + \frac{1}{l_n^{(1)}(-q_n)} \right) u_n(0) = 0. \tag{7}$$

In virtue of arbitrariness of the function $u(t,x)$, we select it so as to have all its Fourier coefficient $u_n(t)$ other than zero for $t = 0$. Then, the division by $u_n(0)$ in equality (7) provides what we desired and proves the lemma.

3. MAIN RESULTS

The following theorem represents the main result of the present paper.

Theorem 3. *Let* $\varphi_0 \in C^3[0,\pi]$ *and* $\varphi_0(0) = \varphi_0(\pi) = \varphi_0''(0) = \varphi_0''(\pi) = 0$, $\varphi_1 \in C^3[0,\pi]$ *and* $\varphi_1(0) = \varphi_1(\pi) = \varphi_1''(0) = \varphi_1''(\pi) = 0$, *and* $M > 0$ *be an arbitrary constant number. Then, there exists a time instant* $T > 0$ *and control* $u(t,x) \in C([0,T], H)$ *depending on the choice of the constant* M *such that the equalities*

$$\Theta(T,x) = \Theta_t'(T,x) = 0 \tag{8}$$

are satisfied for the corresponding solution of problem (1)–(3) for any $x \in (0,\pi)$, *as well as the constraint*

$$|u(t,x)| \leqslant M$$

for any $t \in (0,T]$ *and* $x \in (0,\pi)$.

Proof. Let $u(t,x)$ be some function satisfying the theorem conditions, and T be some time instant. It follows from Theorems 1 and 2 that the solution of problem (1)–(3) is representable as series (4) and (5). Thus, we get

$$\begin{aligned}
\Theta(t,x) &= \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(\varphi_{1n}+\lambda_n^+\varphi_{0n})e^{\lambda_n^+ t}\sin nx}{l_n^{(1)}(\lambda_n^+)} + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(\varphi_{1n}+\lambda_n^-\varphi_{0n})e^{\lambda_n^- t}\sin nx}{l_n^{(1)}(\lambda_n^-)} \\
&+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(\varphi_{1n}-q_n\varphi_{0n})e^{-q_n t}\sin nx}{l_n^{(1)}(-q_n)} + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\int_0^t u_n(s)e^{\lambda_n^+(t-s)}ds}{l_n^{(1)}(\lambda_n^+)}\sin nx \\
&+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\int_0^t u_n(s)e^{\lambda_n^-(t-s)}ds}{l_n^{(1)}(\lambda_n^-)}\sin nx + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\int_0^t u_n(s)e^{-q_n(t-s)}ds}{l_n^{(1)}(-q_n)}\sin nx.
\end{aligned} \tag{9}$$

Now,

$$\begin{aligned}
\frac{\partial\Theta(t,x)}{\partial t} &= \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\lambda_n^+(\varphi_{1n}+\lambda_n^+\varphi_{0n})e^{\lambda_n^+ t}\sin nx}{l_n^{(1)}(\lambda_n^+)} \\
&+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\lambda_n^-(\varphi_{1n}+\lambda_n^-\varphi_{0n})e^{\lambda_n^- t}\sin nx}{l_n^{(1)}(\lambda_n^-)} + \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(-q_n)(\varphi_{1n}-q_n\varphi_{0n})e^{-q_n t}\sin nx}{l_n^{(1)}(-q_n)} \\
&+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\left(\frac{1}{l_n^{(1)}(\lambda_n^+)}+\frac{1}{l_n^{(1)}(\lambda_n^-)}+\frac{1}{l_n^{(1)}(-q_n)}\right)u_n(t)\sin nx \\
&+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\lambda_n^+\int_0^t u_n(s)e^{\lambda_n^+(t-s)}ds}{l_n^{(1)}(\lambda_n^+)}\sin nx \\
&+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{\lambda_n^-\int_0^t u_n(s)e^{\lambda_n^-(t-s)}ds}{l_n^{(1)}(\lambda_n^-)}\sin nx \\
&+ \frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{(-q_n)\int_0^t u_n(s)e^{-q_n(t-s)}ds}{l_n^{(1)}(-q_n)}\sin nx.
\end{aligned} \tag{10}$$

By using conditions (8) and taking into consideration that the fourth addend in (10) vanishes in virtue of the lemma, we establish from (9) and (10) that

$$-\left(\frac{(\varphi_{1n}+\lambda_n^+\varphi_{0n})e^{\lambda_n^+T}}{l_n^{(1)}(\lambda_n^+)}+\frac{(\varphi_{1n}+\lambda_n^-\varphi_{0n})e^{\lambda_n^-T}}{l_n^{(1)}(\lambda_n^-)}+\frac{(\varphi_{1n}-q_n\varphi_{0n})e^{-q_nT}}{l_n^{(1)}(-q_n)}\right)$$
$$=\frac{\int\limits_0^T u_n(s)e^{\lambda_n^+(T-s)}ds}{l_n^{(1)}(\lambda_n^+)}+\frac{\int\limits_0^T u_n(s)e^{\lambda_n^-(T-s)}ds}{l_n^{(1)}(\lambda_n^-)}$$
$$+\frac{\int\limits_0^T u_n(s)e^{-q_n(T-s)}ds}{l_n^{(1)}(-q_n)},\quad n=1,2\ldots, \tag{11}$$

$$-\left(\frac{\lambda_n^+(\varphi_{1n}+\lambda_n^+\varphi_{0n})e^{\lambda_n^+T}}{l_n^{(1)}(\lambda_n^+)}+\frac{\lambda_n^-(\varphi_{1n}v+\lambda_n^-\varphi_{0n})e^{\lambda_n^-T}}{l_n^{(1)}(\lambda_n^-)}+\frac{(-q_n)(\varphi_{1n}-q_n\varphi_{0n})e^{-q_nT}}{l_n^{(1)}(-q_n)}\right)$$
$$=\frac{\lambda_n^+\int\limits_0^T u_n(s)e^{\lambda_n^+(T-s)}ds}{l_n^{(1)}(\lambda_n^+)}+\frac{\lambda_n^-\int\limits_0^T u_n(s)e^{\lambda_n^-(T-s)}ds}{l_n^{(1)}(\lambda_n^-)}$$
$$+\frac{(-q_n)\int\limits_0^T u_n(s)e^{-q_n(T-s)}ds}{l_n^{(1)}(-q_n)},\quad n=1,2\ldots. \tag{12}$$

We denote

$$a_n=-(\varphi_{1n}+\lambda_n^+\varphi_{0n}),\quad \bar a_n=-(\varphi_{1n}+\lambda_n^-\varphi_{0n}),\quad b_n=-(\varphi_{1n}+(-q_n)\varphi_{0n}).$$

To satisfy equalities (11) and (12), in the left and right sides we equate the coefficients at the numbers

$$\frac{1}{l_n^{(1)}(\lambda_n^+)},\quad \frac{1}{l_n^{(1)}(\lambda_n^-)},\quad \frac{1}{l_n^{(1)}(-q_n)}$$

and obtain a new system of moments

$$\int\limits_0^T u_n(s)e^{\lambda_n^+(T-s)}ds=a_ne^{\lambda_n^+T},\quad \int\limits_0^T u_n(s)e^{\lambda_n^-(T-s)}ds=\bar a_ne^{\lambda_n^-T},\quad n=1,2\ldots,$$
$$\int\limits_0^T u_n(s)e^{-q_n(T-s)}ds=b_ne^{-q_nT},\quad n=1,2\ldots. \tag{13}$$

We notice that the solvability of the system of moments (11), (12) follows from the solvability of the system of moments (13). We reduce the identical factors in both sides of equalities (13), and, as the result, the system of moments (13) is given by

$$\int\limits_0^T u_n(s)e^{-\lambda_n^+s}ds=a_n,\quad \int\limits_0^T u_n(s)e^{-\lambda_n^-s}ds=\bar a_n,\quad n=1,2\ldots,$$
$$\int\limits_0^T u_n(s)e^{q_ns}ds=b_n,\quad n=1,2\ldots. \tag{14}$$

In virtue of conjugacy of the complex numbers λ_n^+ and λ_n^-, as well as the numbers a_n and $\bar{a}_n$, the first and second equations in system (14) are equivalent, that is, if the function $u_n(s)$ is the solution of the first equation, then it also satisfies the second equation and vice versa. This enables us to eliminate one of the equations—second, for example,—from system (14). In fact, eliminated is a countable number of equations. Therefore, the system of moments is given by

$$\int_0^T u_n(s)e^{-\lambda_n^+ s}ds = a_n, \quad \int_0^T u_n(s)e^{q_n s}ds = b_n, \quad n = 1, 2 \ldots. \tag{15}$$

We perform in (15) the replacement $-\lambda_n^+ = \lambda_n$ and notice that $Re\lambda_n > 0$ and $q_n > 0$ (see [6, 7]). Finally, we get the countable system of moment pairs

$$\int_0^T u_n(s)e^{\lambda_n s}ds = a_n, \quad \int_0^T u_n(s)e^{q_n s}ds = b_n, \quad n = 1, 2 \ldots. \tag{16}$$

Solution of system (16) is sought in the form

$$u_n(s) = C_{1,n}e^{\lambda_n s} + C_{2,n}e^{q_n s}, \quad n = 1, 2, \ldots, \tag{17}$$

where $C_{1,n}$ and $C_{2,n}$ are some unknown constants. By substituting (17) in (16), we get a countable system of pairs of algebraic equations

$$\begin{gathered} C_{1,n}\int_0^T e^{2\lambda_n s}ds + C_{2,n}\int_0^T e^{(\lambda_n+q_n)s}ds = a_n, \quad n = 1, 2 \ldots, \\ C_{1,n}\int_0^T e^{(\lambda_n+q_n)s}ds + C_{2,n}\int_0^T e^{2q_n s}ds = b_n, \quad n = 1, 2 \ldots. \end{gathered} \tag{18}$$

Let us find the determinant Δ_n of system (18). Since

$$\int_0^T e^{2\lambda_n s}ds = \frac{1}{2\lambda_n}e^{2\lambda_n T} - \frac{1}{2\lambda_n}, \tag{19}$$

we establish with the use of equality (19) that

$$\begin{aligned} \Delta_n &= \frac{1}{4\lambda_n q_n}\left(e^{2\lambda_n T} - 1\right)\left(e^{2q_n T} - 1\right) - \frac{1}{(\lambda_n + q_n)^2}\left(e^{(\lambda_n+q_n)T} - 1\right)^2 \\ &= \frac{1}{4\lambda_n q_n}\left(e^{2(\lambda_n+q_n)T} - e^{2q_n T} - e^{2\lambda_n T} + 1\right) \\ &\quad - \frac{1}{(\lambda_n + q_n)^2}\left(e^{2(\lambda_n+q_n)T} - 2e^{(\lambda_n+q_n)T} + 1\right) \\ &= e^{2(\lambda_n+q_n)T}\left(\frac{1}{4\lambda_n q_n} - \frac{1}{(\lambda_n + q_n)^2}\right) + \left(\frac{1}{4\lambda_n q_n} - \frac{1}{(\lambda_n + q_n)^2}\right) \\ &\quad - \frac{1}{4\lambda_n q_n}\left(e^{2q_n T} + e^{2\lambda_n T}\right) + \frac{2}{(\lambda_n + q_n)^2}e^{(\lambda_n+q_n)T} \\ &= \frac{e^{2(\lambda_n+q_n)T}}{4\lambda_n q_n}\Bigg[1 - e^{-2\lambda_n T} - e^{-2q_n T} + e^{-2(\lambda_n+q_n)T} \\ &\quad - \frac{4\lambda_n q_n}{(\lambda_n + q_n)^2} - \frac{4\lambda_n q_n}{(\lambda_n + q_n)^2}e^{-2(\lambda_n+q_n)T} + \frac{8\lambda_n q_n}{(\lambda_n + q_n)^2}e^{-(\lambda_n+q_n)T}\Bigg]. \end{aligned} \tag{20}$$

We note that the sequence of magnitudes of the complex roots $\{|\lambda_n|\}$ tends to $+\infty$ for $n \to +\infty$, and the sequence of the real numbers $\{q_n\}$ tends to some positive number q (see [6, 7]). Therefore, in virtue of the asymptotic behavior of the numbers λ_n and q_n, there exists a time instant T such that all determinants Δ_n are other than zero for any natural index n.

We establish the determinant $\Delta_{1,n}$:

$$\Delta_{1,n} = a_n \int\limits_0^T e^{(\lambda_n+q_n)s} ds - b_n \int\limits_0^T e^{2q_n s} ds$$
$$= \frac{a_n}{\lambda_n + q_n} e^{(\lambda_n+q_n)T} - \frac{a_n}{\lambda_n+q_n} - \frac{b_n}{2q_n} e^{2q_n T} + \frac{b_n}{2q_n}$$

and similarly find $\Delta_{2,n}$:

$$\Delta_{2,n} = b_n \int\limits_0^T e^{2\lambda_n s} ds - a_n \int\limits_0^T e^{(\lambda_n+q_n)s} ds$$
$$= \frac{b_n}{2\lambda_n} e^{2\lambda_n T} - \frac{b_n}{2\lambda_n} - \frac{a_n}{\lambda_n+q_n} e^{(\lambda_n+q_n)T} + \frac{a_n}{\lambda_n+q_n}.$$

By using the Cramer formulas, we obtain

$$C_{1,n} = \frac{\Delta_{1,n}}{\Delta_n}, \quad C_{2,n} = \frac{\Delta_{2,n}}{\Delta_n}.$$

Consequently, the solution of system (16) at the time instant t is given by

$$u_n(t) = 4\lambda_n q_n \frac{\frac{a_n}{\lambda_n+q_n} e^{(\lambda_n+q_n)T} - \frac{a_n}{\lambda_n+q_n} - \frac{b_n}{2q_n} e^{2q_n T} + \frac{b_n}{2q_n}}{e^{2(\lambda_n+q_n)T}\left(1 - e^{-2\lambda_n T} - e^{-2q_n T} + e^{-2(\lambda_n+q_n)T} - \alpha_n(\lambda_n, q_n, T)\right)} e^{\lambda_n t}$$
$$+ 4\lambda_n q_n \frac{\frac{b_n}{2\lambda_n} e^{2\lambda_n T} - \frac{b_n}{2\lambda_n} - \frac{a_n}{\lambda_n+q_n} e^{(\lambda_n+q_n)T} + \frac{a_n}{\lambda_n+q_n}}{e^{2(\lambda_n+q_n)T}\left(1 - e^{-2\lambda_n T} - e^{-2q_n T} + e^{-2(\lambda_n+q_n)T} - \alpha_n(\lambda_n, q_n, T)\right)} e^{q_n t},$$

where

$$\alpha_n(\lambda_n, q_n, T) = \frac{4\lambda_n q_n}{(\lambda_n+q_n)^2} + \frac{4\lambda_n q_n}{(\lambda_n+q_n)^2} e^{-2(\lambda_n+q_n)T} - \frac{8\lambda_n q_n}{(\lambda_n+q_n)^2} e^{-(\lambda_n+q_n)T}.$$

Let $\lambda_n = \mu_n - i\nu_n$. It was proved in [6, 7] that $\mu_n, \nu_n > 0$ for any natural index n. Let us estimate the magnitude of the functions $u_n(t)$ for any natural n. It is given by

$$|u_n(t)| \leqslant \frac{4q_n|\lambda_n||a_n|}{|\lambda_n+q_n| e^{q_n T}(1 - \beta_n(T) - |\alpha_n|)} + \frac{4q_n|\lambda_n||a_n|}{|\lambda_n+q_n| e^{(\mu_n+2q_n)T}(1 - \beta_n(T) - |\alpha_n|)}$$
$$+ \frac{4q_n|\lambda_n||b_n|}{|2q_n| e^{\mu_n T}(1 - \beta_n(T) - |\alpha_n|)} + \frac{4q_n|\lambda_n||b_n|}{|2q_n| e^{(\mu_n+2q_n)T}(1 - \beta_n(T) - |\alpha_n|)}$$
$$+ \frac{4q_n|\lambda_n||b_n|}{|2\lambda_n| e^{q_n T}(1 - \beta_n(T) - |\alpha_n|)} + \frac{4q_n|\lambda_n||b_n|}{|2\lambda_n| e^{(2\mu_n+q_n)T}(1 - \beta_n(T) - |\alpha_n|)}$$
$$+ \frac{4q_n|\lambda_n||a_n|}{|\lambda_n+q_n| e^{\mu_n T}(1 - \beta_n(T) - |\alpha_n|)} + \frac{4q_n|\lambda_n||a_n|}{|\lambda_n+q_n| e^{(2\mu_n+q_n)T}(1 - \beta_n(T) - |\alpha_n|)}, \tag{21}$$

where $\beta_n(T) = e^{-2\mu_n T} + e^{-2q_n T} + e^{-2(\mu_n+q_n)T}$. We also estimate the magnitude of the desired control. Obviously,

$$|u(t,x)| \leqslant \sum_{n=1}^{\infty} |u_n(t)|. \tag{22}$$

By means of (21) and (22) we estimate the time required to stop the system, provided that the function $u(t,x)$ is constrained by

$$|u(t,x)| \leqslant \varepsilon, \tag{23}$$

where ε is an arbitrary constant.

Since the sequences of the real numbers $\{\mu_n\}$, $\{\nu_n\}$, $\{q_n\}$ are such that $\mu_n = \mu + o(n^{-2})$, $\nu_n = Dn$ and $q_n = q + o(n^{-2})$, where μ, D, q are some positive constant numbers (see [6, 7]), and the sequences $\{|a_n|\}$, $\{|b_n|\}$, and $\{|\alpha_n|\}$ tend to zero, the following estimates are true:

$$|u(t,x)| \leqslant \frac{c}{e^{c_1 T}} \left(\sum_{n=1}^{\infty} |a_n| + \sum_{n=1}^{\infty} |\lambda_n||b_n| \right) \leqslant \frac{c_2}{e^{c_1 T}} \leqslant \varepsilon, \tag{24}$$

where c, c_1, and c_2 are some constants and T is sufficiently great. We note that the numerical series in estimate (24) converge in virtue of the initial data selected from the corresponding classes of functions (see the formulation of the theorem).

Consequently, the time T of oscillation damping can be established from the equation

$$\frac{c_2}{e^{c_1 T}} = \varepsilon.$$

Finally, we obtain

$$T = -\frac{1}{c_1} \ln \frac{\varepsilon}{c_2}. \tag{25}$$

4. CONCLUSIONS

As can be seen from equality (25), in the case of a system with "memory" the time to stop oscillations is much less than the time that suffices to stop a string without the integral delay where

$$T \sim \frac{c}{\varepsilon}$$

(see [2]). At the same time, in the problem with "memory" its complete controllability is lost at passing from the control distributed over the entire interval $[a,b]$ to a subinterval.

Remark 1. For $N > 2$, there also exists a result on controllability and estimation of the time of driving the solution to the complete quiescent state. The present paper confined itself to the case of $N = 2$ to avoid cumbersome calculations.

Remark 2. In Eq. (1), the function $\Theta_{xx}(t,x)$ is preceded by a coefficient coordinated with the kernel $K'(t-s)$ in the integral term of the equation. Nevertheless, this form of the equation does not restrain generality because this coefficient can be arbitrary. For that, an arbitrary constant must be added to $K(t)$. It can be understood as the number multiplied by an exponential function with the zero exponent. It is possible to verify that if the modified kernel $K_1(t)$ has the form

$$K_1(t) = K(t) + C,$$

then the results of [6] remain valid and, consequently, the controllability results of this work retain their validity. At that, if $K_1(t) = C_1 e^{-\lambda_1 t} + C_2$, then on the basis of the findings of [4] on no controllability one can state that system (1)–(3) is not controllable if the carrier of the function $u(t,x)$ is concentrated on a subinterval.

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