Loop homology of moment-angle complexes and quasitoric manifolds

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Loop homology with field coefficients

X (a simply connected topological space with a basepoint) $\rightsquigarrow$
its loop space ΩX (a connected H-space) $\rightsquigarrow$
$H_*(\Omega X;\Bbbk)$ (a connected graded Hopf algebra).

Milnor–Moore theorem

$H_*(\Omega X;\mathbb{Q})\cong U(L)$, where $L=\pi_*(\Omega X)\otimes\mathbb{Q}\cong\pi_{*-1}(X)\otimes\mathbb{Q}$ considered as a graded Lie algebra with the Whitehead bracket.

How to compute $H_*(\Omega X;\Bbbk)$? Adams' cobar construction:
$H_*(\Omega X;\Bbbk)\cong\mathcal{E}xt_{C^*(X;\Bbbk)}(\Bbbk,\Bbbk)$, Adams–Hilton models...
If X is formal: $H_*(\Omega X;\Bbbk)=\mathrm{Ext}_{H^*(X;\Bbbk)}(\Bbbk,\Bbbk)$.

- Typical task is to define $H_*(\Omega X;\Bbbk)$ by generators and relations (minimally, if possible).

Quasitoric manifolds

Standard action of the compact n-dimensional torus: $T^n \curvearrowright \mathbb{C}^n \simeq \mathbb{R}^{2n}$.
The orbit space is $\mathbb{R}^{2n}/T^n = \mathbb{R}^n_{\geq 0}$.
It follows: if $T^n \curvearrowright M^{2n}$ is locally standard, then M/T is a manifold with corners.

Definition (Davis, Januszkiewicz'91)

Smooth manifold M^{2n} with a locally standard action of T^n is quasitoric if $M/T \cong P$ is a simple convex n-dimensional polytope.

Examples:

- $T^n \curvearrowright \mathbb{C}\mathrm{P}^n$ is quasitoric since $\mathbb{C}\mathrm{P}^n/T^n = \Delta^n$.
- All smooth projective toric varieties$/\mathbb{C}$ are quasitoric.
- $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ is quasitoric, but not toric; $P = [0,1]^2$.

Moment-angle complexes

Let $\mathcal{K}$ be an abstract simplicial complex on vertex set $[m] = \{1, \ldots, m\}$.
Define the moment-angle complex $\mathcal{Z}_\mathcal{K}$ as

$$\mathcal{Z}_\mathcal{K} := \bigcup_{J \in \mathcal{K}} (D^2, S^1)^J \subset (D^2)^m, \quad \text{where} \quad (D^2, S^1)^J := \prod_{j \in J} D^2 \times \prod_{j \in [m] \setminus J} S^1.$$

The action $\mathbb{T}^m \curvearrowright (D^2)^m$ restricts to $\mathbb{T}^m \curvearrowright \mathcal{Z}_\mathcal{K}$.

Proposition (Buchstaber, Panov'99)

If $|\mathcal{K}| \simeq S^{n-1}$, then $\mathcal{Z}_\mathcal{K}$ is a $(m+n)$-dimensional topological manifold.

Combinatorial description of quasitoric manifolds

Consider the following combinatorial data $(\mathcal{K}, A)$:

- $\mathcal{K}$ – a polytopal triangulation of S^{n-1} (i.e., $\mathcal{K} \cong \partial P^*$ for a simple n-dimensional polytope P);
- $A : \mathbb{Z}^m \to \mathbb{Z}^n$ – a map of lattices, such that the collection $\{A(e_j),\ j \in J\}$ forms a basis of $\mathbb{Z}^n$ for any $(n-1)$-dimensional simplex $J \in \mathcal{K}$.

Theorem (Davis, Januszkiewicz + Buchstaber, Panov)

Under these conditions,

1. $\mathcal{Z}_\mathcal{K}$ is smoothed canonically;
2. $\mathrm{Ker}(\exp A : \mathbb{T}^m \to \mathbb{T}^n)$ is a $(m-n)$-dimensional torus that acts freely on $\mathcal{Z}_\mathcal{K}$;
3. $M^{2n} = \mathcal{Z}_\mathcal{K} / T^{m-n}$ is a quasitoric manifold over P;
4. All quasitoric manifolds arise this way.

Davis–Januszkiewics spaces

The Borel construction: for any action $G \curvearrowright X$, the diagonal action $G \curvearrowright EG \times X$ is free. This gives a homotopy fibration

$$G \to X \to X/\!/G, \quad X/\!/G := (X \times EG)/G.$$

$\mathrm{DJ}_{\mathcal{K}} := \mathcal{Z}_{\mathcal{K}}/\!/\mathbb{T}^m = M/\!/T^n$ is the Davis–Januszkiewicz space.

$$\begin{array}{ccccc} \mathcal{Z}_{\mathcal{K}} & & \xrightarrow{/T^{m-n}} & & M \\ & \searrow^{/\mathbb{T}^m} & & \swarrow_{/T^n} & \\ & & P & & \end{array} \qquad \begin{array}{ccccc} \mathcal{Z}_{\mathcal{K}} & & \xrightarrow{/\!/T^{m-n}} & & M \\ & \searrow^{/\!/\mathbb{T}^m} & & \swarrow_{/\!/T^n} & \\ & & \mathrm{DJ}_{\mathcal{K}}. & & \end{array}$$

Theorem (Buchstaber, Panov'99)

$\mathrm{DJ}_{\mathcal{K}} \simeq (\mathbb{C}\mathrm{P}^\infty)^{\mathcal{K}}$, where $X^{\mathcal{K}} := \bigcup_{J \in \mathcal{K}} \prod_{j \in J} X \subset X^m$.

Two split fibrations

$\mathrm{DJ}_\mathcal{K} = \mathcal{Z}_\mathcal{K} /\!/ \mathbb{T}^m = M /\!/ T^n$, hence we have the fibrations

$$\mathbb{T}^m \to \mathcal{Z}_\mathcal{K} \to \mathrm{DJ}_\mathcal{K}, \quad T^n \to M \to \mathrm{DJ}_\mathcal{K},$$

$$\Omega\mathcal{Z}_\mathcal{K} \to \Omega\mathrm{DJ}_\mathcal{K} \xrightarrow{p} \mathbb{T}^m, \quad \Omega M \to \Omega\mathrm{DJ}_\mathcal{K} \xrightarrow{p'} T^n.$$

Observation (Panov, Ray'08)

The maps p and p' admit homotopy sections ($s : \mathbb{T}^m \to \Omega\mathrm{DJ}_\mathcal{K}$, $p \circ s \sim 1$). (Similar to $\Omega(X \vee Y) \to \Omega(X \times Y)$.)

Theorem (Panov, Ray)

$H_*(\Omega\mathrm{DJ}_\mathcal{K}; \Bbbk) \cong \mathrm{Ext}^*_{\Bbbk[\mathcal{K}]}(\Bbbk, \Bbbk)$, where

$$\Bbbk[\mathcal{K}] = \Bbbk[v_1, \dots, v_m] / \Big(\prod_{j \in J} v_j = 0, \quad \forall J \notin \mathcal{K} \Big)$$

is the Stanley–Reiner ring.

Loop homology as subalgebras

Using the homotopy sections, we obtain the Hopf algebra extensions

$$1 \to H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{k}) \to \mathrm{E}(\mathbb{k}[\mathcal{K}]) \to \Lambda[u_1,\dots,u_m] \to 0,$$

$$1 \to H_*(\Omega M;\mathbb{k}) \to \mathrm{E}(\mathbb{k}[\mathcal{K}]) \to \Lambda[\theta_1,\dots,\theta_n] \to 0.$$

In particular: $H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{k}) \subset \mathrm{E}(\mathbb{k}[\mathcal{K}])$, and

$$\mathrm{E}(\mathbb{k}[\mathcal{K}]) \simeq H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{k}) \otimes \Lambda[u_1,\dots,u_m]$$

as left $H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{k})$-modules (not as algebras!) Similarly,

$$\mathrm{E}(\mathbb{k}[\mathcal{K}]) \simeq H_*(\Omega M;\mathbb{k}) \otimes \Lambda[\theta_1,\dots,\theta_n].$$

Two-step problem:

1. Describe the algebra $\mathrm{E}(\mathbb{k}[\mathcal{K}]) = \mathrm{Ext}^*_{\mathbb{k}[\mathcal{K}]}(\mathbb{k},\mathbb{k})$;
2. Knowing $\mathrm{E}(\mathbb{k}[\mathcal{K}])$, describe its subalgebras $H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{k})$ and $H_*(\Omega M;\mathbb{k})$.

The algebra $\mathrm{E}(\Bbbk[\mathcal{K}])$ in known cases

Simplicial complex $\mathcal{K}$ is flag if the following holds: any set of pairwise connected vertices is a simplex.

The flag case (Fröberg'75)

If $\mathcal{K}$ is flag, then $\Bbbk[\mathcal{K}]$ is a Koszul algebra. Hence

$$\mathrm{E}(\Bbbk[\mathcal{K}]) = T(u_1,\dots,u_m)/(u_i^2=0;\ u_iu_j+u_ju_i=0,\ \{i,j\}\in\mathcal{K}).$$

The "almost flag" case (V.)

Suppose that $\mathcal{K}$ is obtained from a flag complex by removing maximal faces $I_1,\dots,I_r$, and $|I_j|\geq 3$ for all j. Then

$$\mathrm{E}(\Bbbk[\mathcal{K}]) = \frac{T(u_1,\dots,u_m,w_{I_1},\dots,w_{I_r})}{(u_i^2=0;\ u_iu_j+u_ju_i=0,\ \{i,j\}\in\mathcal{K};\ u_iw_{I_j}-w_{I_j}u_i=0,\ i\in I_j)}$$

For general $\mathcal{K}$, not much is known about $\mathrm{E}(\Bbbk[\mathcal{K}])$.

Generators and relations of connected $\mathbb{k}$-algebras

Proposition (C.T.C. Wall'60)

Let $S = T(a_1, \ldots, a_N)/(r_1 = \cdots = r_M = 0)$ be a connected graded algebra over a field $\mathbb{k}$, such that the elements a_i, r_j are homogeneous and the collections $\{a_i\}$ and $\{r_j\}$ are minimal under inclusion. Then

$$\bigoplus_{i=1}^{N} \mathbb{k} \cdot a_i \simeq \mathrm{Tor}_1^S(\mathbb{k}, \mathbb{k}), \quad \bigoplus_{j=1}^{M} \mathbb{k} \cdot r_j \simeq \mathrm{Tor}_2^S(\mathbb{k}, \mathbb{k}).$$

It follows: if we compute $\mathrm{Tor}^{H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{k})}(\mathbb{k}, \mathbb{k})$ and $\mathrm{Tor}^{H_*(\Omega M;\mathbb{k})}(\mathbb{k}, \mathbb{k})$, we will know the number and the degrees of generators and relations in minimal presentations of these algebras.

Moreover, we can recover generators and relations from representatives of basic elements in Tor_1 and Tor_2...

Generators and relations of certain subalgebras

Let $S \subset A$ be a subalgebra and V be a vector space such that $A \simeq S \otimes V$ as left S-modules. Then $\operatorname{Tor}^S(\mathbb{k}, \mathbb{k})$ can be computed as follows.

1. Take any "small" resolution

$$\cdots \to A \otimes M_2 \to A \otimes M_1 \to A \otimes M_0 \to \mathbb{k} \to 0$$

   of the left A-module $\mathbb{k}$.

2. Consider it as a free resolution

$$\cdots \to S \otimes (V \otimes M_2) \to S \otimes (V \otimes M_1) \to S \otimes (V \otimes M_0) \to \mathbb{k} \to 0$$

   of the left S-module $\mathbb{k}$.

3. Apply $\mathbb{k} \otimes_S (-)$ and compute the homology:

$$\operatorname{Tor}_i^S(\mathbb{k}, \mathbb{k}) = H_i\Big[V \otimes M_\bullet\Big].$$

Results in the flag case

Theorem (Grbic,Panov,Theriault,Wu'16)

Let $\mathcal{K}$ be a flag simplicial complex. The algebra $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is minimally generated by the following set of $\sum_{J\subset[m]}\dim\widetilde{H}_0(\mathcal{K}_J;\Bbbk)$ elements:

$$[u_{i_1},[u_{i_2},\dots[u_{i_k},u_j]\dots]],\quad J=\{i_1<\dots<i_k\}\sqcup\{j\},\ j\neq\max(J),$$

where j is a maximal vertex in its connected component of $\mathcal{K}_J$.

Theorem

Let $\mathcal{K}$ be a flag simplicial complex. Then the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is minimally defined by $\sum_{J\subset[m]}\dim\widetilde{H}_0(\mathcal{K}_J;\Bbbk)$ generators and $\sum_{J\subset[m]}\dim\widetilde{H}_1(\mathcal{K}_J;\Bbbk)$ relations of degree $|J|$.

Theorem

Let M be a quasitoric mfld such that $\mathcal{K}$ is flag. Then the algebra $H_*(\Omega M;\Bbbk)$ is defined by $h_1(P)=m-n$ generators and $h_2(P)$ relations.

Results (almost flag case)

Suppose that $\mathcal{K}$ is obtained from a flag simplicial complex $\mathcal{K}^f$ by removing some maximal faces $I_1, \dots, I_r$, and $|I_j| \geq 3$ for all j.

Theorem

$H_*(\Omega\mathcal{Z}_{\mathcal{K}};\mathbb{k})$ is the free product of $H_*(\Omega\mathcal{Z}_{\mathcal{K}^f};\mathbb{k})$ and the tensor algebra on $\sum_{j=1}^r 2^{m-|I_j|}$ generators

$$[u_{\ell_1},[u_{\ell_2},\dots,[u_{\ell_p},w_{I_j}]\dots]],\quad L=\{\ell_1<\dots<\ell_p\}\subset[m]\setminus I_j.$$

Theorem

Let M be a quasitoric manifold such that $\mathcal{K}$ satisfies the above conditions. Then the algebra $H_*(\Omega M;\mathbb{k})$ is minimally generated by the following $(m-n)+r$ elements:

1. $(m-n)$ linear combinations of $u_1,\dots,u_m$ that correspond to the basis of $\mathrm{Ker}(A:\mathbb{Z}^m\to\mathbb{Z}^n)$;
2. the elements $w_{I_1},\dots,w_{I_r}$.

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