Lectures on relativistic gravity and cosmology.

Lectures 1-2

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Moscow, 02.09.2023

Special Theory of Relativity

Special relativity kinematics

Special relativity dynamics

Special relativity electrodynamics

The principle of relativity and the maximal velocity

Inertial reference frames (IRF): they exist and move evenly and rectilinearly with respect to each other.

The principle of relativity

All laws of nature are the same in all inertial reference frames.

Hypothesis: there exists a maximal velocity. According to the principle of relativity, it should be the same in all inertial systems. A natural candidate: the light velocity in vacuum:

$$c = 2.99792458 \cdot 10^{10}\,\mathrm{cm/c}$$

The principle of relativity + the maximal velocity = the Special Theory of Relativity (SR).

Classification of areas in physics

Three independent fundamental dimensional physical constants: $c, \ \hbar, \ G$. All other physical constants can be made dimensionless.

c - relativistic physics.
$\hbar$ - quantum physics.
G - gravitational physics.
$c, \ \hbar$ - relativistic quantum physics.
$c, \ G$ - relativistic gravitational physics.
$c, \ \hbar, \ G$ - relativistic quantum gravitational physics.

Interval

The space-time interval s_{12} between two events:

$$s_{12}^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2.$$

Its differential form:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = g_{ik}dx^i dx^k,\ i,k = 0,1,2,3,\ x^0 = ct$$

g_{ik} - the metric tensor, $g_{il}g^{kl} = \delta_i^k$.
The first conventional choice of sign: space-time signature $(+ - - -)$.
The Einstein rule: $a_i b^i \equiv g_{ik} a^i b^k \equiv \sum_{i=0}^{3} a_i b^i$.

For light propagation: $s_{12} = ds^2 = 0$ in all IFR.

For non-zero intervals: $ds^2 = a(|\mathbf{V}|)\mathbf{ds'^2}$ due to homogeneity and isotropy of space and homogeneity of time.
Let K, K_1, K_2 - three IRF, and $\mathbf{V_1}$ and $\mathbf{V_2}$ - relative velocities of K_1 and K_2 with respect to K. Then

$$ds^2 = a(V_1)ds_1^2 = a(V_2)ds_2^2$$

On the other hand,

$$ds_1^2 = a(V_{12})ds_2^2$$

where V_{12} is the modulus of velocity of K_2 with respect to K_1. Thus,

$$\frac{a(V_1)}{a(V_2)} = a(V_{12}) \to a(V) \equiv const = 1$$

$$ds^2 = ds'^2, \quad s = s'$$

Light cone

$s_{12}^2 > 0$ - a time-like interval, inside the light cone of the event 1 (future light cone for $t_2 > t_1$, past light cone for $t_2 < t_1$). It is possible to find IRF where both events 1 and 2 occur in the same place. In this IRF, the time difference between these events (called the proper time difference) is $\tau = s_{12}/c$.

$s_{12}^2 < 0$ - a space-like interval, outside the light cone of the event 1. No absolute simultaneity. Depending on the choice of an IRF, the event 2 can be in the past, in the future or simultaneous to the event 1.

The Lorentz transformation

The Lorentz transformation:

$$x = \gamma(V)(x' + Vt'),\ y = y',\ z = z',\ t = \gamma(V)\left(t' + \frac{V}{c^2}x'\right)$$

where the Lorentz factor is

$$\gamma(V) = \left(1 - \frac{V^2}{c^2}\right)^{-1/2}$$

The Lorentz group $O(1,3)$.

Compare with the Galilean transformation:

$$x = x' + Vt',\ y = y',\ z = z',\ t = t'$$

4-velocity and 4-acceleration

4-velocity:

$$u^i = \frac{dx^i}{ds}, \quad u_i u^i = 1, \ u^i = (\gamma(v), \ \gamma(v)\,\mathbf{v})$$

$$ds = c d\tau = c\gamma^{-1}(v) dt$$

3-velocity transformation (IRF K' is moving with the velocity V relative to IRF K along the x-axis):

$$v_x = \frac{v'_x + V}{1 + \frac{v'_x V}{c^2}}, \quad v_y = \frac{v'_y}{\gamma(V)\left(1 + \frac{v'_x V}{c^2}\right)}, \quad v_z = \frac{v'_z}{\gamma(V)\left(1 + \frac{v'_x V}{c^2}\right)}$$

where $\mathbf{v} = \frac{d\mathbf{r}}{dt}, \ \mathbf{v}' = \frac{d\mathbf{r}'}{dt}$.

4-acceleration:

$$w^i = \frac{du^i}{ds} = \frac{d^2x^i}{ds^2}, \quad w^i u_i = 0$$

Action

Action for a free massive particle:

$$S = -mc \int_a^b ds$$

The coefficient is chosen in such a way that in the non-relativistic limit $v \ll c$, this expression would give the correct expression for the Lagrangian equal to the particle kinetic energy (up to a constant):

$$S = \int_{t_1}^{t_2} L dt = -mc \int_{t_1}^{t_2} \frac{dt}{\gamma(v)}, \; L \approx = -mc^2 + \frac{mv^2}{2}$$

Equations of motion

The principle of least action (more accurately, the principle of stationary action): $\delta S = 0$.
Taking into account that $ds = \sqrt{dx_i dx^i}$, we get:

$$\delta S = -mc \int_a^b \frac{dx_i \delta dx^i}{ds} = -mc \int_a^b u_i d\delta x^i =$$

$$-mcu_i \delta x^i|_a^b + mc \int_a^b \delta x^i \frac{du_i}{ds} ds$$

Equations of motion follow from the variation of action with respect to arbitrary virtual trajectories with the fixed end-points $(\delta x^i)_a = (\delta x^i)_b = 0$:

$$\frac{du^i}{ds} = 0$$

4-momentum

4-momentum is defined as the coordinate derivative of the action estimated on real trajectories with the fixed initial point only $(\delta x^i)_a = 0$:

$$p_i = -\frac{\partial S}{\partial x^i} = mcu_i = \left(\frac{\mathcal{E}}{c},\ -\mathbf{p}\right)$$

$$\mathcal{E} = m\gamma(v)c^2,\ \mathbf{p} = m\gamma(v)\mathbf{v} = \frac{\mathcal{E}\mathbf{v}}{c^2}$$

$$p^i p_i = m^2 c^2 \to \mathcal{E}^2 = p^2 c^2 + m^2 c^4$$

In the non-relativistic limit: $\mathcal{E} = mc^2 + \frac{mv^2}{2}$. For the $v = 0$, the correct form of the famous Einstein formula follows:

$$\mathcal{E} = mc^2$$

The massless limit: photons, gravitons.

$$\mathcal{E} = |\mathbf{p}|c = \hbar\omega = h\nu$$

Relativistic Hamilton-Jacobi equation

$$\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x^i} = g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} = m^2c^2$$

Transition to the non-relativistic Hamilton-Jacobi equation: introduce the new action $S' = S + mc^2 t$. For $c \to \infty$:

$$\frac{\partial S'}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial S'}{\partial x}\right)^2 + \left(\frac{\partial S'}{\partial y}\right)^2 + \left(\frac{\partial S'}{\partial z}\right)^2\right] = 0$$

The massless limit: light in the geometric optics approximation. The eikonal equation:

$$\frac{\partial \psi}{\partial x_i}\frac{\partial \psi}{\partial x^i} = 0, \quad k_i = \frac{p_i}{\hbar} = -\frac{\partial \psi}{\partial x^i}$$

Particle interactions

$$(\sum_n \mathcal{E}_n)^2 - (\sum_n \mathbf{p}_n)^2 c^2 = \mathrm{inv}$$

A proton with an energy $\mathcal{E} \gg m_p c^2$ is moving through thermal radiation with the characteristic photon energy $\varepsilon_\gamma \sim kT_\gamma \ll \mathcal{E}$. For which energy the process $p + \gamma \to p + \pi_0$ $(n + \pi_+)$ becomes possible ($T_\gamma = 2.725$ K)?

$$(\mathcal{E} + \varepsilon_\gamma)^2 - (pc - \varepsilon_\gamma)^2 = (m_p + m_\pi)^2 c^4$$

$$\mathcal{E} = \frac{2m_p m_\pi + m_\pi^2}{4\varepsilon_\gamma} c^4 \sim 10^{20}\mathrm{eV}$$

The Greisen-Zatsepin-Kuzmin effect (1966) resulting in the cutoff (GZK cutoff) in the spectrum of high-energy cosmic rays - observed (2007)!

Testing the Lorentz invariance

1. Strong interaction: particles with up to $\sim 3 \cdot 10^{20}$ eV in high-energy cosmic rays.
2. Electromagnetic and weak interactions.
arXiv:1807.06504 - photons and neutrino events from the blazar TXS 0506+056 ($\varepsilon_\nu \approx 290$ TeV).
Photons: $\mathcal{E}_3 \gtrsim 10^{20}$ GeV, $\mathcal{E}_4 \gtrsim 10^{11}$ GeV (corrections $\frac{p^3c^3}{\mathcal{E}_3}$ and $\frac{p^4c^4}{\mathcal{E}_4^2}$ in the dispersion relation for $\mathcal{E}^2(p)$).
Neutrinos: $\mathcal{E}_3 \gtrsim 10^{19}$ GeV, $\mathcal{E}_4 \gtrsim 10^{11}$ GeV.
3. Gravitational interaction.
GW170817 + GRB 170817A event. Distance: 40 ± 10 Mpc.
X-ray signal: 1.7 s after the peak of the GR one.

$$\left|\frac{v_{GW}}{c} - 1\right| < 3 \cdot 10^{-15}$$

Electromagnetic interactions

New physical quantity: electric charge e. Quantized, $\pm n$ for free particles (protons, electrons, etc.), $\pm\frac{n}{3}$ for quarks.

The fine structure constant $\alpha = \frac{e^2}{\hbar c}$. The 2018 CODATA recommended value $\alpha^{-1} = 137.035999084(21)$.

No change in the Maxwell equations: they are already invariant under the Lorentz transformation.
The 4-vector potential: $A^i = (\phi, \mathbf{A})$.
The electromagnetic field tensor: $F_{ik} = A_{k,i} - A_{i,k}$.
In the 3D form:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \mathrm{grad}\ \phi, \ \mathbf{H} = \mathrm{rot}\ \mathbf{A}$$

$$E_x = F_{01},\ E_y = F_{02},\ E_z = F_{03},\ H_x = F_{32},\ H_y = F_{13},\ H_z = F_{21}$$

Gauge transformations: $A_k \to A'_k = A_k - \frac{\partial f}{\partial x^k}$.

Action and equations for a charge

Action for a point massive charge:

$$S = -mc\int_a^b ds - \frac{e}{c}\int_a^b A_i dx^i$$

Equations of motion:

$$mc\frac{du^i}{ds} = \frac{e}{c}F^{ik}u_k$$

In the 3D form:

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c}[\mathbf{vH}], \quad \frac{d\mathcal{E}}{dt} = e\mathbf{Ev}$$

where $\mathcal{E}$ and **p** are the same as in the absence of electromagnetic field.

Action and equations for the field

Action or the electromagnetic field:

$$S_{em} = -\frac{1}{16\pi c}\int F_{ik}F^{ik}d\Omega, \quad d\Omega = c\,dt\,dx\,dy\,dz$$

The Maxwell equations (comma - partial derivative):

$$F_{ik,l} + F_{li,k} + F_{kl,i} = 0$$

$$F^{ik}_{\ ,k} = \frac{4\pi}{c}j^i$$

The electric 4-current $j^i = (c\rho, \mathbf{j})$.

Testing the constancy of α.

1. Laboratory measurements (2008):
$\frac{\dot{\alpha}}{\alpha} = (-1.6 \pm 2.3) \times 10^{-17}$ per year.
2. Measurements using remote quasars: $|\frac{\Delta\alpha}{\alpha}| \lesssim 10^{-5}$ for the last 10-12 billion years.