

Lectures on relativistic gravity and cosmology.

Lectures 3-4

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Non-inertial reference systems

The equivalence principle

The Riemann tensor

The vacuum Einstein equations with a cosmological constant

The Schwarzschild metric

Non-inertial reference frames (NIRF)

General transformation of coordinates, 4-vectors and 4-tensors:

$$x^i = x^i(\tilde{x}^k), \quad dx^i = \frac{\partial x^i}{\partial \tilde{x}^k} d\tilde{x}^k, \quad A^i = \frac{\partial x^i}{\partial \tilde{x}^k} \tilde{A}^k, \quad A_i = \frac{\partial \tilde{x}^k}{\partial x^i} \tilde{A}_k$$

$$A^{ik} = \frac{\partial x^i}{\partial \tilde{x}^l} \frac{\partial x^k}{\partial \tilde{x}^m} \tilde{A}_{lm}, \quad A_{ik} = \frac{\partial \tilde{x}^l}{\partial x^i} \frac{\partial \tilde{x}^m}{\partial x^k} \tilde{A}_{lm}, \quad A^i{}_k = \frac{\partial x^i}{\partial \tilde{x}^l} \frac{\partial \tilde{x}^m}{\partial x^k} \tilde{A}^l{}_m$$

δ_k^i - 4-tensor. The metric tensor: $ds^2 = g_{ik} dx^i dx^k$. Properties:
 $g_{ik} = g_{ki}$, $g_{il} g^{kl} = \delta_k^i$, $g \equiv \text{Det}(g_{ik}) < 0$, $A^i = g^{ik} A_k$, $A_i = g_{ik} A^k$.

Transformation of the 4-volume differential from the Minkowski metric in the Cartesian coordinates:

$$J = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)} = \frac{1}{\sqrt{-g}}, \quad d\tilde{\Omega} \rightarrow \frac{1}{J} d\Omega = \sqrt{-g} d\Omega$$

An important particular case: transformation to an uniformly rotating reference system (tilted coordinates - in non-rotating RF):

$$\tilde{x} = x \cos \Omega t - y \sin \Omega t, \quad \tilde{y} = x \sin \Omega t + y \cos \Omega t, \quad \tilde{z} = z, \quad \tilde{t} = t$$

$$ds^2 = [c^2 - \Omega^2(x^2 + y^2)] dt^2 - dx^2 - dy^2 - dz^2 \\ + 2\Omega y dt dx - 2\Omega x dt dy$$

Light cylinder: $x^2 + y^2 = \frac{c^2}{\Omega^2}$. The reference system may not be realized by rigid physical bodies beyond it.

Covariant partial derivative

Semicolon - covariant partial derivative.

$\frac{\partial a}{\partial x^i} \equiv a_{;i} = a_{,i}$, but $A^i_{,k}$ is not a 4-tensor and $dA^i = A^i(x^k + dx^k) - A^i(x^k)$ is not a 4-vector. To determine the difference of two 4-vectors, they should be first placed into one point of space-time by using the **vector parallel transport** operation. The change of a 4-vector A^i after its parallel transport from x^l to $x^l + dx^l$ is given by:

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l$$

where Γ^i_{kl} are the Christoffel symbols (or, the affine connection). By considering the transport of the scalar product, $\delta(A_i B^i) = 0$, we get $\delta A_i = \Gamma^k_{il} A_k dx^l$.

The total covariant change in A^i and A_i :

$$DA^i = dA^i - \delta A^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma_{kl}^i A^k \right) dx^l$$

$$DA_i = dA_i - \delta A_i = \left(\frac{\partial A_i}{\partial x^l} - \Gamma_{il}^k A_k \right) dx^l$$

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma_{kl}^i A^k, \quad A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma_{il}^k A_k$$

The **assumptions**:

1) $g_{ik;l} = 0$ - no **nonmetricity**;

2) $\Gamma_{kl}^i = \Gamma_{lk}^i$ - no **torsion**.

Then the Christoffel symbols coincide with the Levi-Civita connection. Let us define $\Gamma_{i,kl} = g_{im} \Gamma_{kl}^m$ (here comma does not mean derivative).

Relation of the Levi-Civita connection to metric

From $g_{ik;l} = 0$:

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,il} + \Gamma_{i,kl}, \quad \frac{\partial g_{il}}{\partial x^k} = \Gamma_{i,kl} + \Gamma_{l,ik}, \quad -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,ki} - \Gamma_{k,il}$$

$$\Gamma_{i,kl} = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i} \right)$$

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

Useful relations:

$$dg = g g^{ik} dg_{ik} = -g g_{ik} dg^{ik}, \quad \Gamma^k_{ik} = \frac{\partial \ln \sqrt{-g}}{\partial x^i}$$

$$A^i_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial(\sqrt{-g} A^i)}{\partial x^i}, \quad \phi^i_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left(\sqrt{-g} g^{ik} \frac{\partial \phi}{\partial x^k} \right)$$

The equivalence principle

The gravitational constant (the 2018 CODATA recommended value) $G = 6.67430(15) \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}$.

Relativistic effects in gravity becomes important when $\frac{|\Delta\varphi|}{c^2}$ is not small compared to unity where φ is the Newtonian gravitational potential. Further in my lectures: $c = 1$.

Two forms of the equivalence principle.

1. The weak equivalence principle (WEP): $m_i = m_{g,pass}$.

Tested with the accuracy $3 \cdot 10^{-15}$ by now (the MICROSCOPE mission, arXiv:2209.15488).

2. The strong equivalence principle (SEP): $m_{g,pass} = m_{g,act}$.

Tested with the accuracy 3×10^{-6} by now (in the triple stellar system PSR J0337+1715 : a neutron star (radio pulsar) and two white dwarfs, arXiv:1807.02059).

Action and equations of motion for a massive particle

WEP: locally motion in the gravitational field is indistinguishable from that in a NIRF. But now we don't assume that the interval ds can be globally transformed to its form in the Minkowski space-time. Locally, it is always possible. Moreover, locally all the Christoffel symbols can be made zero using the local coordinate transformation $\tilde{x}^i = x^i + \frac{1}{2}(\Gamma^i_{kl})_0 x^k x^l$ - the local IRF.

The action has the same form as in Special Relativity:

$$S = -m \int_a^b ds$$

Equations of motion:

$$\frac{Du^i}{ds} \equiv \frac{du^i}{ds} + \Gamma^i_{kl} u^k u^l = 0$$

The Hamilton-Jacobi equation has the same form, too:

$$g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} = m^2$$

The limit $m = 0$ gives the eikonal equation in GR for the motion of light in the geometric optics approximation:

$$g^{ik} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^k} = 0$$

Transition to the non-relativistic Newton gravity:

$$L = -m + \frac{mv^2}{2} - m\varphi \rightarrow ds = \left(1 - \frac{v^2}{2} + \varphi\right) dt \rightarrow$$

$$\rightarrow ds^2 = (1 + 2\varphi)dt^2 - d\mathbf{r}^2 \rightarrow g_{00} = 1 + 2\varphi$$

The Riemann tensor

The new effect appearing in curved space-time: a 4-vector changes after its parallel transport along a closed loop

$$\Delta A_k = \frac{1}{2} R^i{}_{klm} A_i \Delta f^{lm}.$$

An equivalent way to introduce the Riemann tensor (the second conventional choice of sign):

$$A_{i;k;l} - A_{i;l;k} = A_m R^m{}_{ikl}$$

$$R^i{}_{klm} = \frac{\partial \Gamma^i{}_{km}}{\partial x^l} - \frac{\partial \Gamma^i{}_{kl}}{\partial x^m} + \Gamma^i{}_{nl} \Gamma^n{}_{km} - \Gamma^i{}_{nm} \Gamma^n{}_{kl}$$

Properties of the Riemann tensor:

1. $R_{iklm} = -R_{kilm} = -R_{ikml} = R_{lmik}$.
2. $R_{iklm} + R_{imkl} + R_{ilmk} = 0$.
3. $R^n{}_{ikl;m} + R^n{}_{imk;l} + R^n{}_{ilm;k} = 0$ - the Bianchi identity.

The Ricci tensor and the gravitational field action

The Ricci tensor is defined as (the third conventional choice of sign):

$$R^i_k = R^m_{imk}$$

$$R_{ik} = R_{ki}, \quad R^k_{i;k} = \frac{1}{2} R_{,i}$$

The Ricci scalar (or, scalar curvature): $R \equiv R^i_i$

The action for the gravitational field:

$$S_g = -\frac{1}{16\pi G} \int (R + 2\Lambda) \sqrt{-g} d\Omega$$

The action can be represented in a not generally covariant form containing squares of the first derivatives of the metric tensor using the identity

$$\sqrt{-g} R = \sqrt{-g} g^{ik} (\Gamma^m_{il} \Gamma^l_{km} - \Gamma^l_{ik} \Gamma^m_{lm}) + \frac{\partial(\sqrt{-g} w^i)}{\partial x^i}$$

The vacuum Einstein equations with a cosmological constant

$$\delta S_g / \delta g^{ik} = 0$$

The useful relation $\delta \sqrt{-g} = -\frac{1}{2\sqrt{-g}} \delta g = -\frac{1}{2} \sqrt{-g} g_{ik} \delta g^{ik}$.

$$(-16\pi G) \delta S_g = \int (R_{ik} - \frac{1}{2} g_{ik} (R + 2\Lambda)) \delta g^{ik} \sqrt{-g} d\Omega + \int g^{ik} \delta R_{ik} \sqrt{-g} d\Omega$$

It can be shown that the last term in the integrand is the total derivative:

$$\sqrt{-g} g^{ik} \delta R_{ik} = \frac{\partial}{\partial x^l} (\sqrt{-g} v^l)$$

In the local IRF where all $\Gamma_{kl}^i = 0$, $v^l = g^{ik} \delta \Gamma_{ik}^l - g^{il} \delta \Gamma_{ik}^k$.

The vacuum Einstein equations with a cosmological constant

$$R_{ik} - \frac{1}{2} g_{ik} R = g_{ik} \Lambda$$

The differentiation ($; k$) gives $0 = 0$ due to the Bianchi identity.

Spherically symmetric vacuum gravitational field

The most general spherically symmetric space-time metric:

$$ds^2 = A(t, r)dt^2 + B(t, r)dtdr - C(t, r)dr^2 - D(t, r)(d\theta^2 + \sin^2 \theta d\phi^2)$$

Remaining freedom of coordinate transformations:

$$r = f_1(\tilde{r}, \tilde{t}), \quad t = f_2(\tilde{r}, \tilde{t}).$$

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

$$\Gamma_{11}^1 = \frac{\lambda'}{2}, \quad \Gamma_{10}^0 = \frac{\nu'}{2}, \quad \Gamma_{00}^1 = \frac{\nu'}{2} e^{\nu-\lambda}$$

$$\Gamma_{00}^0 = \frac{\dot{\nu}}{2}, \quad \Gamma_{10}^1 = \frac{\dot{\lambda}}{2}, \quad \Gamma_{11}^0 = \frac{\dot{\lambda}}{2} e^{\lambda-\nu}$$

$$\Gamma_{12}^2 = \Gamma_{13}^3 = \frac{1}{r}, \quad \Gamma_{22}^1 = -re^{-\lambda}, \quad \Gamma_{33}^1 = -re^{-\lambda} \sin^2 \theta$$

$$\Gamma_{23}^3 = \cot \theta, \quad \Gamma_{33}^2 = -\sin \theta \cos \theta$$

$$\Lambda = 0, \quad R_1^1 - \frac{1}{2}R = -e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = 0$$

$$R_0^0 - \frac{1}{2}R = -e^{-\lambda} \left(-\frac{\lambda'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = 0, \quad R_0^1 = -e^{-\lambda} \frac{\dot{\lambda}}{r} = 0$$

From the last equation: $\lambda = \lambda(r)$. The difference of the second and third equations gives $\lambda + \nu = f(t)$. $f(t)$ can be made zero by using the remaining possible transformation of time $t = f_3(\tilde{t})$. Thus, spherically symmetric vacuum gravitational field appears to be static (for the range of r for which it has sense). Then integration of the second equation gives $e^{-\lambda} = e^\nu = 1 - \frac{r_g}{r}$, $r_g = 2GM$.

The Schwarzschild metric describing the outer region of a non-rotating black hole:

$$ds^2 = \left(1 - \frac{r_g}{r} \right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

Space-time remains regular at $r = r_g$.