

Lectures on relativistic gravity and cosmology.

Lectures 5-6

Alexei A. Starobinsky

Landau Institute for Theoretical Physics RAS

Faculty of Physics, National Research University
"Higher School of Economics"

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Light motion in the Schwarzschild metric

The Kerr metric

The matter metric energy-momentum tensor

The Einstein equations with matter

Some physical numbers and black holes

$$R_{\odot} = 6.963 \cdot 10^{10} \text{ cm}, \quad M_{\odot} = 1.989 \cdot 10^{33} \text{ g}, \quad r_{g,\odot} \approx 3 \text{ km}$$

$$1 \text{ au} = 1.495978707 \cdot 10^{13} \text{ cm}$$

$$1 \text{ pc} = 3.086 \cdot 10^{18} \text{ cm} = 206265 \text{ au} \approx 3.26 \text{ ly}$$

Two types of black holes.

1) Stellar mass black holes $M = (3 - 150) M_{\odot}$

a) in X-ray double stellar systems,

b) in BH merger events seen by GW antennas.

2) Supermassive black holes in galaxy centres and quasars

$$M = (10^6 - 6 \cdot 10^{10}) M_{\odot}.$$

Black hole in the center of our Galaxy (Milky Way)

Sagittarius A* (Sgr A*).

Distance from us ≈ 8 kpc.

$M \approx 4.3 \cdot 10^6 M_{\odot}$, $r_g \approx 0.09$ au .

Determined by observation of star motion around it for more than 20 years (R. Genzel and A. Ghez). Stars with the closest approach:

1) S4714, $T = 12$ y, $R_{pc} \approx 13$ au, $e = 0.985$;

2) S2, $T = 16$ y, $R_{pc} \approx 120$ au, $e = 0.884$, $\delta\phi = 16'$ per orbital period.

Light motion in the Schwarzschild metric

$$g^{ik} \frac{\partial \psi}{\partial x^i} \frac{\partial \psi}{\partial x^k} = 0$$

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

For motion in the equatorial plane:

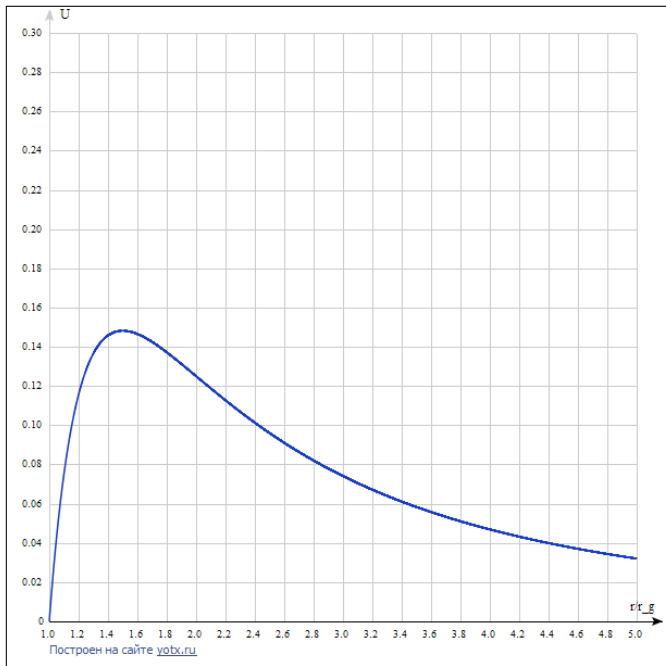
$$\psi = -\omega t + L\phi + \psi_r(r), \quad \psi_r = \int \frac{dr}{1 - \frac{r_g}{r}} \sqrt{\omega^2 - L^2 U(r)}$$

$$U(r) = \left(1 - \frac{r_g}{r}\right) \frac{1}{r^2}$$

$L = b\omega$ where b is the impact parameter. The light trajectory

$$\frac{\partial \psi}{\partial L} = \text{const}, \quad \phi = \int \frac{dr}{r^2 \sqrt{b^{-2} - U(r)}}$$

For $r_g = 0$, $r = \frac{b}{\cos \phi}$.



Light deflection at large distances

Let $r \gg r_g, b \gg r_g$.

Let us change the radial variable $r \rightarrow r + \frac{r_g}{2}$ and expand the radial part of the eikonal in powers of r_g/r .

$$\psi_r = \omega \int dr \sqrt{1 + \frac{2r_g}{r} - \frac{b^2}{r^2}} = \psi_r^0 + \omega r_g \int \frac{dr}{\sqrt{r^2 - b^2}}$$

The total change of the eikonal when the light beam moves from $r = R$ to $r = b$ and back to $r = R$ is

$$\Delta\psi_r = \Delta\psi_r^0 + 2\omega r_g \int_b^R \frac{dr}{\sqrt{r^2 - b^2}}$$

The light deflection angle is

$$\Delta\phi = -\frac{\partial\Delta\psi_r}{\partial L} = -\frac{\partial\Delta\psi_r^0}{\partial L} + \frac{2r_g R}{b\sqrt{R^2 - b^2}}$$

In the limit $R \rightarrow \infty$, $\Delta\phi = \pi + \delta\phi$, where $\delta\phi = \frac{2r_g}{b}$.
For Sun ($b = R_\odot$), $\delta\phi = 1.75''$. Tested with the 10^{-5} accuracy.

Photon sphere and black hole shadow

The unstable photon orbit: $U'(r) = 0 \rightarrow r = r_1 = \frac{3}{2}r_g$. The corresponding impact parameter

$$b_{crit} = U(r_1)^{-1/2} = \frac{3\sqrt{3}}{2} r_g$$

This impact parameter gives the answer to the two problems:

- 1) the absorption cross-section of ultrarelativistic particles by a black hole $\sigma_{abs} = \pi b_{crit}^2$,
- 2) the radius of the black hole shadow $r_{sh} = b_{crit}$.

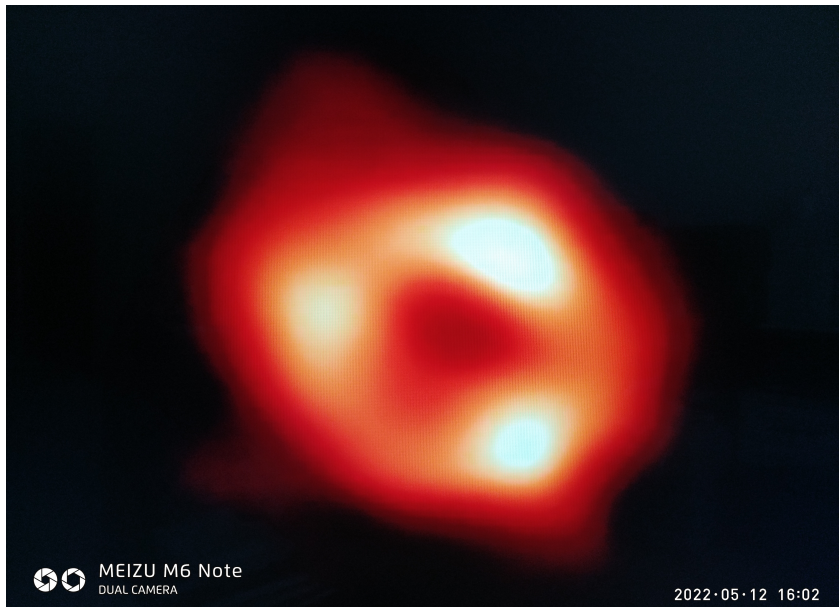
ArXiv:1906.11238: observation of the shadow of the black hole in the M87 galaxy with $M = (6.5 \pm 0.7) \cdot 10^9 M_\odot$, $r_g \approx 1.9 \cdot 10^{15}$ cm ≈ 130 au and distance from the Sun $R \approx 17$ Mpc: $2\theta_{sh} = 42 \pm 3$ micro-as ($\approx 2 \cdot 10^{-10}$).

Theory: $\theta_{sh} = 2r_{sh}/R \approx 40$ micro-as.

Shadow of the black hole in the M87 galaxy



Shadow of the black hole in Sgr A*



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The Kerr metric

The stationary solution of the Einstein equations in vacuum.

$$ds^2 = \left(1 - \frac{r_g r}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{r_g a^2 r}{\rho^2} \sin^2 \theta\right) \sin^2 \theta d\phi^2 + \frac{2r_g a r}{\rho^2} \sin^2 \theta dt d\phi ,$$

$$\Delta = r^2 - r_g r + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta, \quad \sqrt{-g} = \rho^2 \sin \theta ,$$

$$r_g = 2GM, \quad L = Ma, \quad 0 \leq a \leq GM$$

The space-time becomes flat for $M = 0$.

The event horizon: $g^{11} = 0$, $r_+ = GM + \sqrt{G^2 M^2 - a^2}$.

The outer surface of the ergosphere:

$$g_{00} = 0, \quad r_e = GM + \sqrt{G^2 M^2 - a^2 \cos^2 \theta}.$$

Geodesic particles inside the ergosphere should corotate with the black hole.

$$g^{ik} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^k} = \frac{1}{\Delta} \left(r^2 + a^2 + \frac{r_g a^2 r}{\rho^2} \sin^2 \theta \right) \left(\frac{\partial S}{\partial t} \right)^2 - \frac{\Delta}{\rho^2} \left(\frac{\partial S}{\partial r} \right)^2 -$$

$$- \frac{1}{\rho^2} \left(\frac{\partial S}{\partial \theta} \right)^2 - \frac{1}{\Delta \sin^2 \theta} \left(1 - \frac{r_g r}{\rho^2} \right) \left(\frac{\partial S}{\partial \phi} \right)^2 + \frac{2r_g a r}{\rho^2 \Delta} \frac{\partial S}{\partial t} \frac{\partial S}{\partial \phi}$$

. Separation of variables in the Hamilton-Jacobi and eikonal equations for an arbitrary motion.

The Christodoulou-Ruffini formula:

$$M^2 = M_{ir}^2 + \frac{L^2}{4M_{ir}^2}$$

where $M_{ir}^2 = \frac{Mr_+}{2G}$, $S_{hor} = 16\pi G^2 M_{ir}^2$. M_{ir} cannot decrease with time in classical General Relativity (the Second Law of the Black Hole Thermodynamics). For $a = a_{max} = GM$, $M_{ir} = \frac{M}{\sqrt{2}}$.

Infinitesimal transformation of the metric tensor

Let us find how the metric tensor changes locally under infinitesimal transformations of coordinates $\tilde{x}^i = x^i + \xi^i$.

$$\begin{aligned}\tilde{g}^{ik}(\tilde{x}^l) &= g^{lm}(x^l) \frac{\partial \tilde{x}^i}{\partial x^l} \frac{\partial \tilde{x}^k}{\partial x^m} = g^{lm} \left(\delta_l^i + \frac{\partial \xi^i}{\partial x^l} \right) \left(\delta_m^k + \frac{\partial \xi^k}{\partial x^m} \right) \approx \\ &\approx g^{ik}(x^l) + g^{im} \frac{\partial \xi^k}{\partial x^m} + g^{kl} \frac{\partial \xi^i}{\partial x^l} .\end{aligned}$$

. On the other hand,

$$\begin{aligned}\tilde{g}^{ik}(\tilde{x}^l) &= \tilde{g}^{ik}(x^l + \xi^l) = \tilde{g}^{ik}(x^l) + \xi^l \frac{\partial \tilde{g}^{ik}}{\partial x^l} , \\ \tilde{g}^{ik}(x^l) &= g^{ik}(x^l) - \xi^l \frac{\partial g^{ik}}{\partial x^l} + g^{il} \frac{\partial \xi^k}{\partial x^l} + g^{kl} \frac{\partial \xi^i}{\partial x^l} = g^{ik}(x^l) + \xi^{i;k} + \xi^{k;i} , \\ \tilde{g}_{ik} &= g^{ik} - \xi_{i;k} - \xi_{k;i} .\end{aligned}$$

The Killing equations define infinitesimal transformations of coordinates which do not change the space-time metric:

$$\xi^{i;k} + \xi^{k;i} = 0$$

The matter metric energy-momentum tensor

The matter action $S = \int \mathcal{L}_m \sqrt{-g} d\Omega$.

$$\begin{aligned}\delta S &= \int \left(\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g^{ik}} \delta g^{ik} + \frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial(\partial g^{ik}/\partial x^l)} \delta \frac{\partial g^{ik}}{\partial x^l} \right) d\Omega = \\ &= \int \left(\frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l} \frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial(\partial g^{ik}/\partial x^l)} \right) \delta g^{ik} d\Omega .\end{aligned}$$

Definition of the matter metric energy-momentum tensor (EMT)

$$\frac{1}{2} \sqrt{-g} T_{ik} = \frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l} \frac{\partial(\mathcal{L}_m \sqrt{-g})}{\partial(\partial g^{ik}/\partial x^l)} .$$

Then $\delta S = \frac{1}{2} \int T_{ik} \delta g^{ik} \sqrt{-g} d\Omega = -\frac{1}{2} \int T^{ik} \delta g_{ik} \sqrt{-g} d\Omega$.

$$\delta S = \frac{1}{2} \int T_{ik} (\xi^{i;k} + \xi^{k;i}) \sqrt{-g} d\Omega = \int T_{ik} \xi^{i;k} \sqrt{-g} d\Omega$$

Note that δg_{ik} are not independent here, while ξ^i is an arbitrary infinitesimal vector!

Conservation of the matter metric EMT

$$\delta S = \int \left((T_i^k \xi^i)_{;k} - T_{i;k}^k \xi^i \right) \sqrt{-g} d\Omega = - \int T_{i;k}^k \xi^i \sqrt{-g} d\Omega,$$

since $\sqrt{-g} A_{;i}^i = \frac{\partial(\sqrt{-g} A^i)}{\partial x^i}$.

$$\delta S = 0 \rightarrow T_{i;k}^k = 0$$

1. EMT for the electromagnetic field:

$$\mathcal{L}_m = -\frac{1}{16\pi} F_{ik} F_{lm} g^{il} g^{km}$$

$$T_{ik} = \frac{1}{4\pi} \left(-F_{il} F_k{}^l + \frac{1}{4} F_{lm} F^{lm} g_{ik} \right), \quad T \equiv T_i^i = 0.$$

2. EMT for an ideal barotropic fluid (or gas):

$$T_{ik} = (\rho + p) u_i u_k - p g_{ik}.$$

Formally $\mathcal{L}_m = p \equiv \frac{1}{2} \left((\rho + p) u_i u_k g^{ik} - (\rho - p) \right)$.

The Einstein equations with matter and a cosmological constant

$$S = -\frac{1}{16\pi G} \int (R + 2\Lambda) \sqrt{-g} d\Omega + \int \mathcal{L}_m \sqrt{-g} d\Omega .$$

$$\frac{\delta S}{\delta g^{ik}} = 0 \rightarrow R_{ik} - \frac{1}{2} g_{ik} R = g_{ik} \Lambda + 8\pi G T_{ik} , \quad R = -4\Lambda - 8\pi G T .$$

Transition to the Newtonian limit:

$$g_{00} = 1 + 2\varphi, \quad T_0^0 = \rho, \quad \Gamma_{00}^\alpha = \frac{\partial \varphi}{\partial x^\alpha}, \quad R_0^0 = -\frac{R}{2} = \frac{\partial \Gamma_{00}^\alpha}{\partial x^\alpha} = \Delta \varphi ,$$

$$\Delta \varphi = 4\pi G \rho$$