Lectures on relativistic gravity and cosmology.

Lectures 7-8

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Number of equations and their structure

$$R_{ik} - \frac{1}{2} g_{ik} R = g_{ik}\Lambda + 8\pi G T_{ik} \ .$$

10 equations.

1. Due to the Bianchi identity $R^k_{i;k} = 0$, covariant differentiation with summation gives $T^k_{i;k} = 0$.
The equations for gravitational field already contain dynamical equations for matter in themselves.
2. No second time derivatives in $(0-0)$ and $(0-\alpha)$ equations $(\alpha = 1, 2, 3)$.
3. No second time derivatives of g_{00} and $g_{0\alpha}$ in all equations.

Number of 'physically different' variables in g_{ik}: $10 - 4 = 6$.
The synchronous reference system: $g_{00} = 1, \ g_{0\alpha} = 0$.
For an ideal barotropic fluid or gas: $T_{ik} = (\rho + p)\, u_i u_k - p g_{ik}$,
4 more variables: $\rho, \ u^i$. The equations of state $p = p(\rho)$ have to be added separately.

Total active gravitational mass

For the spherically symmetric case ($\Lambda = 0$)

$$R_0^0 - \frac{1}{2}R = -e^{-\lambda}\left(-\frac{\lambda'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = 8\pi G T_0^0$$

$$\lambda = -\ln\left(1 - \frac{8\pi G}{r}\int_0^r T_0^0 r^2\, dr\right) > 0$$

Outside the body ($T_0^0 = 0$ for $r > r_0$), $\lambda = -\ln\left(1 - \frac{2GM}{r}\right)$.
Thus,

$$M = 4\pi \int_0^{r_0} T_0^0 r^2\, dr$$

Difference between $4\pi r^2\, dr$ and $dV = 4\pi r^2 e^{\lambda/2} dr$ - the gravitational mass defect.

Gravitational field far away from bodies

$$g_{ik} = \eta_{ik} + h_{ik}, \;\; \psi_i^k = h_i^k - \frac{1}{2}\delta_i^k h, \;\; h = h_l^l$$

The Lorentz gauge: $\frac{\partial \psi_i^k}{\partial x^k} = 0$. Then

$$R_{ik} = -\frac{1}{2}\eta^{ik}\frac{\partial^2 h_{ik}}{\partial x^i \partial x^k} = -\frac{1}{2}\Box h_{ik}, \;\; \Box = \frac{\partial^2}{\partial t^2} - \triangle$$

.

The static case, order r^{-1}.

$$h_{00}^{(1)} = -\frac{r_g}{r}, \;\; h_{\alpha\beta}^{(1)} = -\frac{r_g}{r}\,\delta_{\alpha\beta}, \;\; h_{0\alpha}^{(1)} = 0$$

They are determined from the comparison with the Schwarzschild solution re-written in terms of the Cartesian coordinates x, y, z. Let $r = \rho\left(1 + \frac{r_g}{4\rho}\right)^2$ and $\rho^2 = x^2 + y^2 + z^2$.

$$ds^2 = \frac{\left(1 - \frac{r_g}{4\rho}\right)^2}{\left(1 + \frac{r_g}{4\rho}\right)^2} dt^2 - \left(1 + \frac{r_g}{4\rho}\right)^4 (dx^2 + dy^2 + dz^2)$$

As a result:

$$ds^2 = (1 + 2\varphi)\, dt^2 - (1 - 2\varphi)(dx^2 + dy^2 + dz^2),$$

where φ is the Newtonian gravitational potential.

In the next order r^{-2} ($\propto c^{-3}$ in usual units)

$$h_{00}^{(2)} = \frac{1}{2}\left(\frac{r_g}{r}\right)^2, \;\; h_{\alpha\beta}^{(2)} = \frac{3}{8}\left(\frac{r_g}{r}\right)^2 \delta_{\alpha\beta}, \;\; h_{0\alpha}^{(2)} = -2GL_{\alpha\beta}\frac{n_\beta}{r^2}$$

where $L_{\alpha\beta}$ is the total angular momentum 4-tensor.

The Tolman formula for the total mass

Another expression for the total mass in the stationary case - the Tolman formula.

$$R_0^0 = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{-g} g^{i0} \Gamma^\alpha_{0i} \right)$$

Integrating over the 3-space, using the Gauss theorem and the formula for a metric far away from bodies at the remote 2-sphere, we get:

$$\int R_0^0 \sqrt{-g}\, dV = 2\pi r_g$$

$$M = \int (T_0^0 - T_1^1 - T_2^2 - T_3^3) \sqrt{-g}\, dV$$

Weak gravitational waves (GW) in flat space-time

In the Lorentz gauge $\psi^k_{i,k} = 0$,

$$\Box h^k_i = 0.$$

In flat space-time, GW propagate with the velocity of light. Recently confirmed by the observation of the GW170817 + GRB 170817A event - merging of two neutron stars at the distance 40 ± 10 Mpc. X-ray signal: 1.7 s after the peak of the GR one.

$$|v_{GW} - 1| < 3 \times 10^{-15}$$

The Lorentz gauge still does not fix the reference system unambiguously. The remaining freedom of coordinate transformations: $\tilde{x}^i = x^i + \xi^i$, $\Box \xi^i = 0$.

Consider a plane gravitational wave moving right along the $x^1 = x$ axis. ψ_i^k are functions of $t - x$. From the Lorentz gauge condition: $\dot{\psi}_i^1 = \dot{\psi}_i^0$.

$$\psi_1^1 = \psi_1^0, \ \psi_2^1 = \psi_2^0, \ \psi_3^1 = \psi_3^0, \ \psi_0^1 = \psi_0^0.$$

Additional transformation: $\tilde{x}^i = x^i + \xi^i(t - x)$. Used to make

$$\psi_1^0 = \psi_2^0 = \psi_3^0 = \psi_2^2 + \psi_3^3 = 0.$$

Then $\psi_1^1 = \psi_2^1 = \psi_3^1 = \psi_0^0 = \psi_i^i = 0$ and $\psi_i^k = h_i^k$. Remaining non-zero components are $h_{22} = -h_{33}$ and h_{23}. Thus, GW are transverse, traceless and have two polarization states. They have two degrees of freedom, and initial conditions for vacuum gravity at a space-like Cauchy hypersurface are given by 4 arbitrary functions of spatial coordinates. In the presence of a barotropic matter, 4 more arbitrary functions (energy density and spatial velocity) appear.

Weak high frequency GW in curved vacuum space-time

$$g_{ik} = g_{ik}^{(0)} + h_{ik}$$

Weak: $|h_i^k| \ll 1$.
High frequency: $\omega \gg L_R^{-1} = (R_{iklm}^{(0)} R^{iklm(0)})^{1/4}$ where $R_{iklm}^{(0)}$ is the background Riemann tensor, so that $|R_i^{k(1)}| \gg |R_i^{k(0)}|$.

$$\Gamma_{kl}^{i(1)} = \frac{1}{2}(h^i_{l;k} + h^i_{k;l} - h_{kl}{}^{;i}),$$

$$R_{ik}^{(1)} = \frac{1}{2}(h^l_{i;k;l} + h^l_{k;i;l} - h_{ik}{}^{;l}{}_{;l} - h_{;i;k})\,.$$

Imposing the generally covariant Lorentz gauge $\psi^k_{i;k} = 0$ and using the condition $\omega L_R \gg 1$, we get

$$\psi_{ik}{}^{;l}{}_{;l} = 0\,.$$

The Isaacson EMT of GW

The remaining freedom of coordinate transformations: $\tilde{x}^i = x^i + \xi^i$, $\xi^{i;k}{}_{;k} = 0$. Can be used to make $h \equiv h^i_i = 0$, then $h^k_{i;k} = 0$. After that the remaining admissible transformations should satisfy $\xi^i_{;i} = 0$.

Let us average the space-time over scales much more than ω^{-1} but much less than L_R. Then

$$< R^{(2)}_{ik} > \approx -\frac{1}{4} < h^n_{m,i} h^m_{n,k} >$$

.

$$R^{(0)}_{ik} \approx - < R^{(2)}_{ik} > = 8\pi G T^{(GW)}_{ik},$$

$$T^{(GW)}_{ik} = \frac{1}{32\pi G} < h^n_{m,i} h^m_{n,k} > .$$

For a plane GW in flat space-time considered previously,

$$T^{01(GW)} = \frac{1}{16\pi G}\left[\dot{h}^2_{23} + \frac{1}{4}(\dot{h}_{22} - \dot{h}_{33})^2\right].$$

Emission of GW by non-relativistic matter

$$\frac{1}{2}\Box\psi_k^i = -8\pi G\tau_i^k.$$

From the Lorentz gauge $\psi_{i,k}^k = 0$, it follows that $\tau_{i,k}^k = 0$.
Solution in the form of retarded potentials:

$$\psi_i^k = -4G\int(\tau_i^k)_{t-R}\frac{dV}{R}.$$

Assuming that all matter velocities are small compared to the light velocity, we can write:

$$\psi_i^k = -\frac{4G}{R_0}\int(\tau_i^k)_{t-R_0}\,dV.$$

Calculation of the integral.

$$\frac{\partial \tau_{\alpha\gamma}}{\partial x^\gamma} = \frac{\partial \tau_{0\gamma}}{\partial x^0}, \quad \frac{\partial \tau_{0\gamma}}{\partial x^\gamma} = \frac{\partial \tau_{00}}{\partial x^0}.$$

Multiplying the first equation by x^β, integrating over the volume and symmetrizing over α, β, we get

$$\int \tau_{\alpha\beta}\, dV = -\frac{1}{2}\frac{\partial}{\partial x^0}\int (\tau_{\alpha 0} x^\beta + \tau_{\beta 0} x^\alpha)\, dV.$$

Multiplication of the second equation by $x^\alpha x^\beta$ and integration over volume leads to

$$\frac{\partial}{\partial x^0}\int \tau_{00} x^\alpha x^\beta\, dV = -\int (\tau_{\alpha 0} x^\beta + \tau_{\beta 0} x^\alpha)\, dV.$$

Combining these expression, we get

$$\int \tau_{\alpha\beta}\, dV = \frac{1}{2}\left(\frac{\partial}{\partial x^0}\right)^2 \int \tau_{00} x^\alpha x^\beta\, dV.$$

Since $\tau_{00} = \rho$,

$$\psi_{\alpha\beta} = -\frac{2G}{R_0}\left(\frac{\partial}{\partial t}\right)^2 \int \rho x^\alpha x^\beta \, dV.$$

Let us introduce the quadrupole moment of mass distribution:

$$D_{\alpha\beta} = \int \rho \left(3x^\alpha x^\beta - r^2\delta_{\alpha\beta}\right) dV.$$

At large distances from the source and locally, the GW can be considered as a plane one. Along the x^1 axis,

$$h_{23} = -\frac{2G}{3R_0}\ddot{D}_{23}, \;\; h_{22} - h_{33} = -\frac{2G}{3R_0}(\ddot{D}_{22} - \ddot{D}_{33}).$$

The energy flux to the x^1 direction is

$$T^{01} = \frac{G}{36\pi R_0^2}\left[\left(\frac{\dddot{D}_{22} - \dddot{D}_{33}}{2}\right)^2 + \dddot{D}_{23}^2\right].$$

The polarization tensor $e_{\alpha\beta}$. Properties:

$$e_{\alpha\alpha} = 0, \ \ e_{\alpha\beta} n_\beta = 0, \ \ e_{\alpha\beta} e_{\alpha\beta} = 1.$$

The intensity of radiation into the solid angle *do* (restoring *c*):

$$dI = \frac{G}{72\pi c^5} (\dddot{D}_{\alpha\beta} e_{\alpha\beta})^2 \, do.$$

Summing over two polarizations of GW, we get:

$$dI = \frac{G}{36\pi c^5} \left[\frac{1}{4} (\dddot{D}_{\alpha\beta} n_\alpha n_\beta)^2 + \frac{1}{2} \dddot{D}^2_{\alpha\beta} - \dddot{D}_{\alpha\beta} \dddot{D}_{\alpha\gamma} n_\beta n_\gamma \right] do.$$

The total GW radiation flux I (luminosity L in GW):

$$I = -\frac{d\mathcal{E}}{dt} = \frac{G}{45 c^5} \dddot{D}^2_{\alpha\beta} \, .$$

The (hypothetical) limiting luminosity in GR:
$L \sim \frac{c^5}{G} = 3.63 \cdot 10^{59} \, \mathrm{erg/s}$ (compare to $L_\odot = 3.83 \cdot 10^{33} \, \mathrm{erg/s}$).