Approximation of the C_0-semigroup of the heat equation by iterations of high-order Chernoff functions

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Let $(X, \|\cdot\|)$ be any Banach space and $\mathscr{L}(X)$ denotes the set of all bounded linear operators on X. Next we will use the notions of *strongly continuous one-parameter semigroup* (or just C_0*–semigroup*), *contractive semigroup* and *generator of a strongly continuous semigroup*, definitions of which can be found, for example, in the book of Engel and Nagel [1]. In 1968 Paul Chernoff proved the following theorem:

Theorem 1 (Chernoff [2]). *Let X be a Banach space, $F(t)$ be a strongly continuous function from $[0,\infty)$ to the set of linear contraction operators on X, such that $F(0) = I$. Suppose that the closure A of the strong derivative $F'(0)$ is the generator of contractive C_0–semigroup $\{e^{tA}\}_{t\geq 0}$. Then $[F(t/n)]^n$ converges to e^{tA} in the strong operator topology.*

Let us note that this theorem does not contain an estimate of the rate of convergence. In 2022 was published the theorem that provides such estimate under certain conditions:

Theorem 2 (Galkin, Remizov [3]). *Suppose that:*

1) $T > 0$, $M_1 \geq 1$, $w \geq 0$. $(A, D(A))$ is generator of C_0-semigroup $(e^{tA})_{t\geq 0}$ in a Banach space X, such that $\|e^{tA}\| \leq M_1 e^{wt}$ for $t \in [0, T]$.

2) There are a mapping $F\colon (0, T] \to \mathscr{L}(X)$ and constant $M_2 \geq 1$ such that we have $\|(F(t))^k\| \leq M_2 e^{kwt}$ for all $t \in (0, T]$ and all $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$.

3) $m \in \mathbb{N} \cup \{0\}$, $p \in \mathbb{N}$, subspace $\mathcal{D} \subset D(A^{m+p})$ is $(e^{tA})_{t\geq 0}$-invariant.

4) There exist such functions $K_j\colon (0, T] \to [0, +\infty)$, $j = 0, 1, \ldots, m + p$ that for all $t \in (0, T]$ and all $x \in \mathcal{D}$ we have $\Big\|F(t)x - \sum_{k=0}^{m} \frac{t^k A^k x}{k!}\Big\| \leq t^{m+1} \sum_{j=0}^{m+p} K_j(t)\|A^j x\|$.

Then for all $t > 0$, all integer $n \geq t/T$ and all $x \in \mathcal{D}$ we have

$$\|(F(t/n))^n x - e^{tA}x\| \leq \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} C_j(t/n)\|A^j x\|,$$

where $C_{m+1}(t) = K_{m+1}(t)e^{-wt} + M_1/(m + 1)!$ and $C_j(t) = K_j(t)e^{-wt}$ for all $j \neq m + 1$.

The mapping $F\colon (0, T] \to \mathscr{L}(X)$ is called a *Chernoff function of order m for operator*

A iff it satisfies the conditions of theorem 2. Let $UC_b(\mathbb{R})$ be the Banach space of all uniformly continuous bounded functions $f\colon \mathbb{R} \to \mathbb{R}$ with the norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$, and linear operator $L = [f \mapsto f'']$ has domain $D(L) = \{f \in UC_b(\mathbb{R}) \mid f'' \in UC_b(\mathbb{R})\}$. Here we are interesting *how to construct space–shift based Chernoff function S_m of any order m for operator L.* Previously , the following results were known in this direction:

In 2016 Ivan Remizov [4] found Chernoff function of order 1 containing 3 summands:

$$[S_1(t)f](x) = \frac{1}{2}f(x) + \frac{1}{4}f(x + 2\sqrt{t}) + \frac{1}{4}f(x - 2\sqrt{t}) = f(x) + tf''(x) + o(t).$$

In 2019 Alexander Vedenin found Chernoff function of order 2 with 3 summands too:

$$[S_2(t)f](x) = \frac{2}{3}f(x) + \frac{1}{6}f(x + \sqrt{6t}) + \frac{1}{6}f(x - \sqrt{6t}) = f(x) + tf''(x) + \frac{t^2}{2}f^{IV}(x) + o(t^2).$$

In general, the following theorem is true:

Theorem 3. *For any natural m, there is a unique Chernoff function S_m of order m for the operator $L = [f \mapsto f'']$, having the form* $[S_m(t)f](x) = \sum_{i=1}^{m+1} a_i \cdot f(x + b_i t^{s_i})$.

In this case, the following conditions will be met:

1) $s_1 = \ldots = s_{m+1} = 1/2$;

2) the numbers $b_1, \ldots, b_{m+1}$ are different roots of the orthogonal Chebyshev–Hermite polynomials;

3) the numbers $a_1, \ldots, a_{m+1}$ are the Christoffel coefficients corresponding to the quadrature nodes $b_1, \ldots, b_{m+1}$ and can be calculated by the formulas

$$a_i = \frac{2^{m+2}(m+1)!\sqrt{\pi}}{(H'_{m+1}(b_i))^2}, \quad i = 1, \ldots, m+1.$$

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