Lectures on relativistic gravity and cosmology.

Lectures 19-20

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Inflation

The inflationary hypothesis (I follow the way how it was introduced in my JETP Lett.1979 and PLB 1980 papers):

Some part of the world which includes all its presently observable part was as much symmetric as possible during some period in the past - both with respect to the geometrical background and to the state of all quantum fields (no particles).

Non-universal (due to the specific initial condition) explanation of the cosmological arrow to time - chaos, or entropy (in some not well defined sense) can only grow after inflation.

Still this state is an intermediate attractor for a set of pre-inflationary initial conditions with a non-zero measure. Also it is not a unique one, there exists a class of such states leading to the same observable predictions.

Successive realization of this idea is based on the two more detailed and independent assumptions.

1. Existence of a metastable quasi-de Sitter stage in our remote part which preceded the hot Big Bang. During it, the expansion of the Universe was accelerated and close to the exponential one, $|\dot{H}| \ll H^2$.

2. The origin of all inhomogeneities in the present Universe is the effect of gravitational creation of pairs of particles - antiparticles and field fluctuations during inflation from the adiabatic vacuum (no-particle) state for Fourier modes covering all observable range of scales (and possibly somewhat beyond).

Existing analogies in other areas of physics.

1. The present dark energy, though the required degree of metastability for the primordial dark energy is much more than is proved for the present one (more than 60 e-folds vs. ~ 3).

2. Creation of electrons and positrons in an external electric field.

Outcome of inflation

In the super-Hubble regime ($k \ll aH$) in the coordinate representation in the synchronous gauge with some additional conditions fixing it completely:

$$ds^2 = dt^2 - a^2(t)(\delta_{lm} + h_{lm})dx^l dx^m, \quad l, m = 1, 2, 3$$

$$h_{lm} = 2\xi(\mathbf{r})\delta_{lm} + \sum_{a=1}^{2} g^{(a)}(\mathbf{r})\, e_{lm}^{(a)}$$

$$e_l^{l(a)} = 0, \;\; g^{(a)}_{\;\;,l}\, e_m^{l(a)} = 0, \;\; e_{lm}^{(a)}\, e^{lm(a)} = 1$$

$\xi = -\mathcal{R}$ describes primordial scalar perturbations, g – primordial tensor perturbations (gravitational waves (GW)). The most important quantities:

$$P_\xi(k), \;\; \frac{d \ln P_\xi(k)}{d \ln k} \equiv n_s(k) - 1, \;\; r(k) \equiv \frac{P_g}{P_\xi}$$

Both $|n_s - 1|$ and r are small during slow-roll inflation.

Existence of constant (quasi-isotropic) modes

For FLRW models filled by ideal fluids, it was known already to Lifshitz (1946). For a wide class of modified scalar-tensor gravity theories, it was proved in A. A. Starobinsky, S. Tsujikawa and J. Yokoyama, Nucl. Phys. B 610, 383 (2001). However, their existence is much more general. From the mathematical point of view, constant modes appear simply due to the existence of non-degenerate solutions of the same gravity models in the isotropic and spatially flat FLRW space-time. By construction, these solutions always have 3 non-physical (gauge) arbitrary constants of integration due to the possibility of arbitrary and independent rescaling of all spatial coordinates. Making these constants slightly inhomogeneous converts them to the leading terms of physical constant modes (one scalar and two tensor ones). Moreover, these constants (now functions of spatial coordinates) need not be small, they can be arbitrarily large: $a^2(t)\delta_l^m \rightarrow a^2(t)c_l^m(\mathbf{r})$.

In fact, metric perturbations $h_{\alpha\beta}$ are quantum (operators in the Heisenberg representation) and remain quantum up to the present time. But, after omitting of a very small part, decaying with time, they become commuting and, thus, equivalent to classical (c-number) stochastic quantities with the Gaussian statistics (up to small terms quadratic in $\xi,\ g$).

In particular:

$$\hat{\xi}_k = \xi_k\, i(\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}}) + \mathcal{O}\left((\hat{a}_{\mathbf{k}} - \hat{a}^\dagger_{\mathbf{k}})^2\right) + ... + \mathcal{O}(10^{-100})(\hat{a}_{\mathbf{k}} + \hat{a}^\dagger_{\mathbf{k}}) + ,,,$$

The last term is time dependent, it is affected by physical decoherence and may become larger, but not as large as the second term.

Remaining quantum coherence: deterministic correlation between $\mathbf{k}$ and $-\mathbf{k}$ modes - shows itself in the appearance of acoustic oscillations.

Physical scales related to inflation

"Naive" estimate where I use the reduced Planck mass $\tilde{M}_{Pl} = (8\pi G)^{-1}$.

I. Curvature scale

$$H \sim \sqrt{P_\xi}\tilde{M}_{Pl} \sim 10^{14}\mathrm{GeV}$$

II. Inflaton mass scale

$$|m_{infl}| \sim H\sqrt{|1-n_s|} \sim 10^{13}\mathrm{GeV}$$

New range of mass scales significantly less than the GUT scale.

The simplest one-parametric inflationary models

1. The $R + R^2$ model (A. A. Starobinsky, Phys. Lett. B 91, 99 (1980)):

$$\mathcal{L} = \frac{f(R)}{16\pi G}, \quad f(R) = R + \frac{R^2}{6M^2}$$

$$M = 2.6 \times 10^{-6} \left(\frac{55}{N}\right) G^{-1/2} \approx 3.1 \times 10^{13}\,\mathrm{GeV}$$

$$n_s - 1 = -\frac{2}{N} \approx -0.036, \;\; r = \frac{12}{N^2} = 3(n_s - 1)^2 \approx 0.004, \;\; n_t = -\frac{r}{8}$$

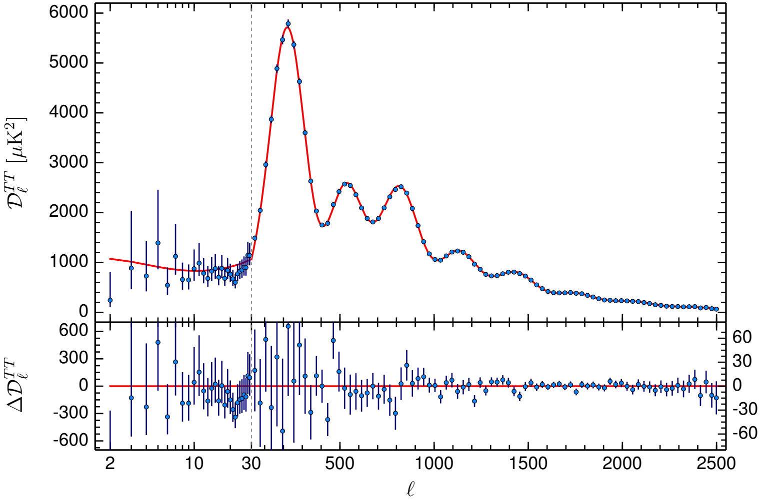
$$N(k) = \ln \frac{a_0 T_\gamma}{k} - \mathcal{O}(10), \quad H_{dS}(N = 55) = 1.4 \times 10^{14}\,\mathrm{GeV}$$

2. The same prediction from the scalar field model with $V(\phi) = \frac{\lambda\phi^4}{4}$ at large ϕ and strong non-minimal coupling to gravity $\xi R\phi^2$ with $\xi < 0, \; |\xi| \gg 1$ (Spokoiny 1984), including the Higgs inflationary model (Bezrukov & Shaposhnikov 2008).

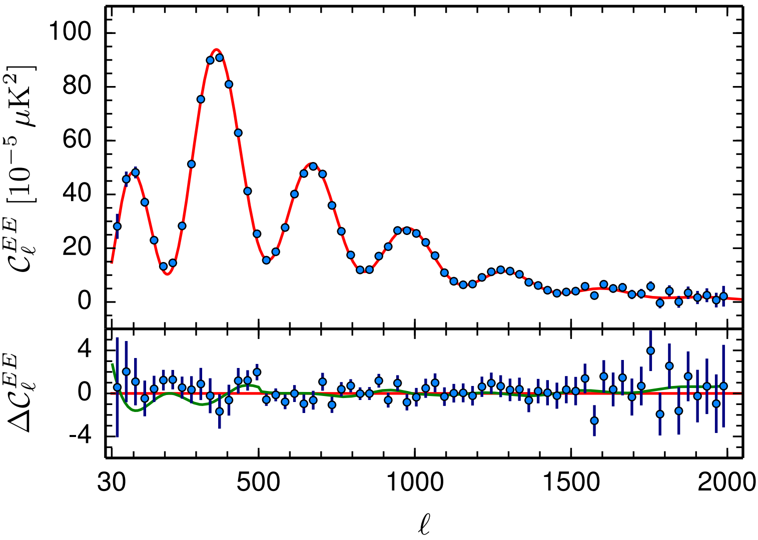
CMB temperature anisotropy multipoles

Red curve - prediction of the simplest one-parametric models.

$\Delta T(\theta,\phi) = \sum_{lm} a_{lm} Y_{lm}(\theta,\phi),\ \ D_l^{TT} = \frac{l(l+1)}{2\pi(2l+1)} \sum_m |a_{lm}|^2.$



CMB E-mode polarization multipoles



The simplest purely geometrical inflationary model

$$\mathcal{L} = \frac{R}{16\pi G} + \frac{N^2}{288\pi^2 P_\xi(k)} R^2 + (\text{small rad. corr.})$$
$$= \frac{R}{16\pi G} + 5.1 \times 10^8 \, R^2 + (\text{small rad. corr.})$$

The quantum effect of creation of particles and field fluctuations works twice in this model:
a) at super-Hubble scales during inflation, to generate space-time metric fluctuations;
b) at small scales after inflation, to provide scalaron decay into pairs of matter particles and antiparticles (AS, 1980, 1981).
Weak dependence of the time t_r when the radiation dominated stage begins:

$$N(k) \approx N_H + \ln \frac{a_0 H_0}{k} - \frac{1}{3} \ln \frac{M_{\rm Pl}}{M} - \frac{1}{6} \ln (M_{\rm Pl} t_r)$$

where $N_H = \ln \frac{k_B T_\gamma}{\hbar H_0} \approx 67.2$ (note that $(1 - n_s) N_H \sim 2$).

Evolution of the $R + R^2$ model

1. During inflation ($t < 0,\ |Mt| \gg 1,\ H \gg M$) - slow-roll:

$$H = \frac{M^2}{6}|t|, \quad |\dot{H}| \ll H^2.$$

2. After inflation ($t > 0,\ Mt \gg 1,\ H \ll M$) - dust-like behaviour driven by massive scalarons

$$a(t) \propto t^{2/3}\left(1 + \frac{2}{3Mt}\sin M(t - t_1)\right).$$

Transition to radiation dominated stage occurs after the scalaron decay into pairs of particles and antiparticles of all quantum fields and their thermalization.

The most effective decay channel: into minimally coupled scalars with $m \ll M$. Then the formula

$$\frac{1}{\sqrt{-g}}\frac{d}{dt}(\sqrt{-g}\,n_s) = \frac{R^2}{576\pi}$$

(Ya. B. Zeldovich and A. A. Starobinsky, JETP Lett. 26, 252 (1977)) can be used for simplicity, but the full integral-differential system of equations for the Bogoliubov α_k, β_k coefficients and the average EMT was in fact solved in AS (1981). Scalaron decay into graviton pairs is suppressed (A. A. Starobinsky, JETP Lett. 34, 438 (1981)).

For this channel of the scalaron decay:

$$N(k) \approx N_H + \ln\frac{a_0 H_0}{k} - \frac{5}{6}\ln\frac{M_{\rm Pl}}{M}$$

Possible microscopic origins of this phenomenological model.

1. Follow the purely geometrical approach and consider it as the specific case of the fourth order gravity in 4D

$$\mathcal{L} = \frac{R}{16\pi G} + AR^2 + BC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + (\text{small rad. corr.})$$

for which $A \gg 1,\ A \gg |B|$. Approximate scale (dilaton) invariance and absence of ghosts in the curvature regime $A^{-2} \ll (RR)/M_P^4 \ll B^{-2}$.

One-loop quantum-gravitational corrections are small (their imaginary parts are just the predicted spectra of scalar and tensor perturbations), non-local and qualitatively have the same structure modulo logarithmic dependence on curvature.

2. Another, completely different way:

consider the $R + R^2$ model as an approximate description of GR + a non-minimally coupled scalar field with a large negative coupling ξ ($\xi_{conf} = \frac{1}{6}$) in the gravity sector::

$$\mathcal{L} = \frac{R}{16\pi G} - \frac{\xi R\phi^2}{2} + \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi), \quad \xi < 0, \quad |\xi| \gg 1 \ .$$

Geometrization of the scalar:

for a generic family of solutions during inflation and even for some period of non-linear scalar field oscillations after it, the scalar kinetic term can be neglected, so

$$\xi R\phi = -V'(\phi) + \mathcal{O}(|\xi|^{-1}) \ .$$

No conformal transformation, we remain in the the physical (Jordan) frame!

These solutions are the same as for $f(R)$ gravity with

$$\mathcal{L} = \frac{f(R)}{16\pi G}, \; f(R) = R - \frac{\xi R \phi^2(R)}{2} - V(\phi(R)).$$

For $V(\phi) = \frac{\lambda(\phi^2 - \phi_0^2)^2}{4}$, this just produces $f(R) = \frac{1}{16\pi G}\left(R + \frac{R^2}{6M^2}\right)$ with $M^2 = \lambda/24\pi\xi^2 G$ and $\phi^2 = |\xi|R/\lambda$.

The same theorem is valid for a multi-component scalar field.

More generally, R^2 inflation (with an arbitrary n_s, r) serves as an intermediate dynamical attractor for a large class of scalar-tensor gravity models.

Inflation in GR

Inflation in GR with a minimally coupled scalar field with some potential.

In the absence of spatial curvature and other matter:

$$H^2 = \frac{\kappa^2}{3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right)$$

$$\dot{H} = -\frac{\kappa^2}{2}\dot{\phi}^2$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

where $\kappa^2 = 8\pi G$ ($\hbar = c = 1$).

Reduction to the first order equation

It can be reduced to the first order Hamilton-Jacobi-like equation for $H(\phi)$. From the equation for $\dot{H}$, $\frac{dH}{d\phi} = -\frac{\kappa^2}{2}\dot{\phi}$. Inserting this into the equation for H^2, we get

$$\frac{2}{3\kappa^2}\left(\frac{dH}{d\phi}\right)^2 = H^2 - \frac{\kappa^2}{3}V(\phi)$$

Time dependence is determined using the relation

$$t = -\frac{\kappa^2}{2}\int \left(\frac{dH}{d\phi}\right)^{-1} d\phi$$

However, during oscillations of ϕ, $H(\phi)$ acquires non-analytic behaviour of the type $const + \mathcal{O}(|\phi - \phi_1|^{3/2})$ at the points where $\dot{\phi} = 0$, and then the correct matching with another solution is needed.

Inflationary slow-roll dynamics

Slow-roll occurs if: $|\ddot{\phi}| \ll H|\dot{\phi}|, \;\; \dot{\phi}^2 \ll V$, and then $|\dot{H}| \ll H^2$.

Necessary conditions: $|V'| \ll \kappa V, \;\; |V''| \ll \kappa^2 V$. Then

$$H^2 \approx \frac{\kappa^2 V}{3}, \; \dot{\phi} \approx -\frac{V'}{3H}, \; N(t) \equiv \ln \frac{a(t_f)}{a} \approx \kappa^2 \int_{\phi_f}^{\phi} \frac{V}{V'}\, d\phi$$

where the index f denotes the end of inflation.

First obtained in A. A. Starobinsky, Sov. Astron. Lett. 4, 82 (1978) in the $V = \frac{m^2\phi^2}{2}$ case and for a bouncing model.

Kinematic origin of scalar perturbations

Local duration of inflation in terms of $N_{tot} = \ln\left(\frac{a(t_f)}{a(t_i)}\right)$ is different in different points of space: $N_{tot} = N_{tot}(\mathbf{r})$. Here the index i denotes the beginning of inflation. Then

$$\xi(\mathbf{r}) = \Delta N_{tot}(\mathbf{r}) = \frac{\delta N_{tot}(\phi)}{\delta\phi}\,\delta\phi(\mathbf{r})\,.$$

Correct generalization to the non-linear case: the space-time metric after the end of inflation at super-Hubble scales

$$ds^2 = dt^2 - a^2(t)e^{2N_{tot}(\mathbf{r})}(dx^2 + dy^2 + dz^2)\,.$$

First derived in A. A. Starobinsky, Phys. Lett. B 117, 175 (1982) in the case of single-field inflation.

Simplified proof assuming $H \approx H_0 = const$ during inflation

Due to appearance of inhomogeneity caused by quantum fluctuations, in the super-Hubble slow-roll regime

$\phi(t) \to \phi(t - t_f(\mathbf{r}))\,,\ a(t) \to a(t - t_f(\mathbf{r}))\,.$ Then

$ds^2 = dt^2 - e^{2H_0 t} dl^2 = dt^2 - e^{2H_0 t_f(\mathbf{r})} e^{2H_0(t - t_f(\mathbf{r}))} dl^2 =$
$dt^2 - e^{2H_0 t_f(\mathbf{r})} a^2(t - t_f(\mathbf{r})) dl^2$

where $a(t)$ is the exact background solution for all times and $dl^2 = \delta_{lm} dx^i dx^m$. For large t after inflation,

$$ds^2 = dt^2 - e^{2H_0 t_f(\mathbf{r})} a^2(t) dl^2\,.$$

For small $H_0 \delta t_f(\mathbf{r})$, $\xi = H_0 \delta t_f(\mathbf{r}) = \delta N_{tot} = -\frac{H_0 \delta\phi(\mathbf{r})}{\dot\phi}\,.$

Visualizing small differences in the number of e-folds

Duration of inflation was finite inside our past light cone. In terms of e-folds, difference in its total duration in different points of space can be seen by the naked eye from a smoothed CMB temperature anisotropy map.

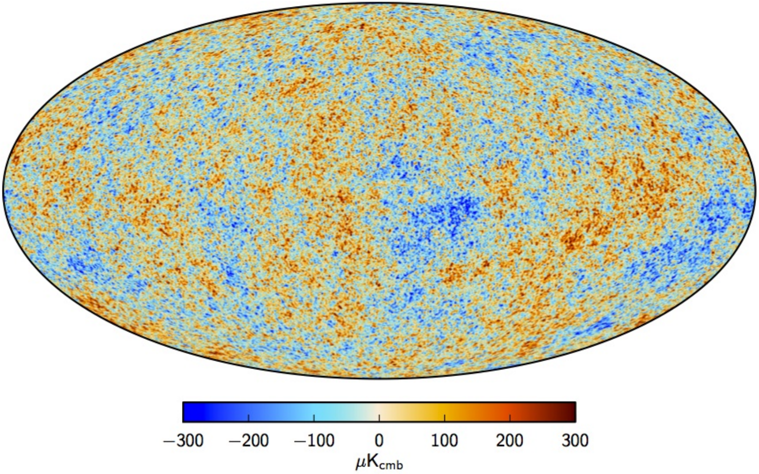
For $\ell \lesssim 50$, neglecting the Silk and Doppler effects, as well as the ISW effect due the presence of dark energy,

$$\frac{\Delta T(\theta,\phi)}{T_\gamma} = -\frac{1}{5}\xi(r_{LSS},\theta,\phi) = -\frac{1}{5}\Delta N_{tot}(r_{LSS},\theta,\phi)\,,$$

where $r_{LSS} = \eta_0 - \eta$.
For $\frac{\Delta T}{T} \sim 10^{-5}$, $\Delta N \sim 5\times 10^{-5}$, and for $H \sim 10^{14}\,\mathrm{GeV}$, $\Delta t \sim 5 t_{Pl}$!

CMB temperature anisotropy

Planck-2015: P. A. R. Ade et al., arXiv:1502.01589



Quantum generation of perturbations during inflation

Quantization with the adiabatic vacuum initial condition (in the tensor case, omitting the polarization tensor):

$$\hat{\psi} = (2\pi)^{-3/2} \int \left[\hat{a}_{\mathbf{k}}\, \psi_k(\eta)\, e^{-i\mathbf{kr}} + \hat{a}^{\dagger}_{\mathbf{k}}\, \psi^*_k\, e^{i\mathbf{kr}} \right] d^3k$$

where ψ stands for ξ, g^a correspondingly and ψ_k satisfies the equation

$$\frac{1}{f}(f\psi_k)'' + \left(k^2 - \frac{f''}{f} \right) \psi_k = 0, \quad \eta = \int \frac{dt}{a(t)}$$

For GW: $f = a$, for scalar perturbations in scalar field driven inflation in GR: $f = \frac{a\dot{\phi}}{H}$ where, in turn, the background scalar field satisfies the equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0$$

How the two basic hypothesis of the inflationary paradigm work.

I. Inflationary background: $t = \infty$ corresponds to $\eta = 0$ and $H(\eta) \equiv \frac{a'}{a^2}$ is bounded and slowly decreasing in this limit, so that $\frac{f''}{f} \sim \frac{2}{\eta^2}$. Then

$$\eta \to -0 : \quad \psi_k(\eta) \to \psi(k) = const, \quad P_(k) = \frac{k^3|\psi^2(k)|}{2\pi^2}$$

II. Adiabatic vacuum initial condition:

$$\eta \to -\infty : \quad \psi_k(\eta) = \frac{e^{-ik\eta}}{f\sqrt{2k}}$$

Combining both conditions and neglecting the mass term:

$$\psi_k(\eta) \approx \frac{e^{-ik\eta}}{f\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right), \; a(\eta) \approx \frac{1}{H(\eta)|\eta|}$$

Spectral predictions of the one-field inflationary scenario in GR

Scalar (adiabatic) perturbations:

$$P_\xi(k) = \frac{H_k^4}{4\pi^2\dot{\phi}^2} = \frac{GH_k^4}{\pi|\dot{H}|_k} = \frac{128\pi G^3 V_k^3}{3V_k'^2}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$. Through this relation, the number of e-folds from the end of inflation back in time $N(t)$ transforms to $N(k) = \ln \frac{k_f}{k}$ where $k_f = a(t_f)H(t_f)$.

The spectral slope

$$n_s(k) - 1 \equiv \frac{d \ln P_\xi(k)}{d \ln k} = \frac{1}{\kappa^2}\left(2\frac{V_k''}{V_k} - 3\left(\frac{V_k'}{V_k}\right)^2\right)$$

is small by modulus – confirmed by observations!

Tensor perturbations (A. A. Starobinsky, JETP Lett. 30, 682 (1979)):

$$P_g(k) = \frac{16GH_k^2}{\pi}; \quad n_g(k) \equiv \frac{d\ln P_g(k)}{d\ln k} = -\frac{1}{\kappa^2}\left(\frac{V'_k}{V_k}\right)^2$$

The consistency relation:

$$r(k) \equiv \frac{P_g}{P_\xi} = \frac{16|\dot{H}_k|}{H_k^2} = 8|n_g(k)|$$

Tensor perturbations are always suppressed by at least the factor $\sim 8/N(k)$ compared to scalar ones. For the present Hubble scale, $N(k_H) = (50 - 60)$. Typically, $|n_g| \leq |n_s - 1|$, so $r \leq 8(1 - n_s) \sim 0.3$ – confirmed by observations!

The most recent upper limit on r

G. Galloni et al., JCAP 04 (2023) 062; arXiv:2208.00188:

$$r_{0.01} < 0.028 \text{ at the 95\% CL.}$$

For comparison, in the chaotic inflationary model $V(\varphi) \propto |\varphi|^n$, $r = \frac{4n}{N}$, $1 - n_s = \frac{n+2}{2N}$. The r upper bound gives $n < 0.5$ for $N_{0.01} = (55 - 60)$, but then $1 - n_s \leq 0.022$. Thus, this model is disfavoured by observational data.

The target prediction for r in the 3 simplest (one-parametric) inflationary models having $n_s - 1 = -\frac{2}{N}$ (the $R + R^2$, Higgs and combined Higgs-R^2 models) is

$$r = \frac{12}{N^2} = 3(n_s - 1)^2 \approx 0.004$$

Inverse reconstruction of inflationary models in GR

In the slow-roll approximation:

$$\frac{V^3}{V'^2} = CP_\xi(k(t(\phi))), \quad C = \frac{12\pi^2}{\kappa^6}$$

Changing variables for ϕ to $N(\phi)$ and integrating, we get:

$$\frac{1}{V(N)} = -\frac{\kappa^4}{12\pi^2}\int \frac{dN}{P_\xi(N)}$$

$$\kappa\phi = \int dN\sqrt{\frac{d\ln V}{dN}}$$

Here, $N \gg 1$ stands both for $\ln(k_f/k)$ at the present time and the number of e-folds back in time from the end of inflation. First derived in H. M. Hodges and G. R. Blumenthal, Phys. Rev. D 42, 3329 (1990).

The two-parameter family of isospectral slow-roll inflationary models, but the second parameter shifts the field ϕ only.

Minimal "scale-free" reconstruction

Minimal inflationary model reconstruction avoiding introduction of any new physical scale both during and after inflation and producing the best fit to the Planck data.

Assumption: the numerical coincidence between $2/N_H \sim 0.04$ and $1 - n_s$ is not accidental but happens for all $1 \ll N \lesssim 60$: $P_\xi = P_0 N^2$. Then:

$$V = V_0 \frac{N}{N + N_0} = V_0 \tanh^2 \frac{\kappa\phi}{2\sqrt{N_0}}$$

$$r = \frac{8N_0}{N(N + N_0)}$$

$r \sim 0.003$ for $N_0 \sim 1$. From the upper limit on r:

$$N_0 < \frac{0.028N^2}{8 - 0.028N}$$

$N_0 < 13$ for $N = 55$.

Inflation in $f(R)$ gravity

Purely geometrical realization of inflation.

The simplest model of modified gravity (geometrical primordial dark energy) considered as a phenomenological macroscopic theory in the fully non-linear and non-perturbative regime.

$$S = \frac{1}{16\pi G}\int f(R)\sqrt{-g}\, d^4x + S_m$$

$$f(R) = R + F(R), \quad R \equiv R^{\mu}_{\mu}$$

Here $f''(R)$ is not identically zero. Usual matter described by the action S_m is minimally coupled to gravity.

Vacuum one-loop corrections depending on R only (not on its derivatives) are assumed to be included into $f(R)$. The normalization point: at laboratory values of R where the scalaron mass (see below) $m_s \approx \mathrm{const}$. Metric variation is assumed everywhere. Palatini variation leads to a different theory with a different number of degrees of freedom.

Background FRW equations in $f(R)$ gravity

$$ds^2 = dt^2 - a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$H \equiv \frac{\dot{a}}{a}\ , \quad R = 6(\dot{H} + 2H^2)$$

The trace equation (4th order)

$$\frac{3}{a^3}\frac{d}{dt}\left(a^3\frac{df'(R)}{dt}\right) - Rf'(R) + 2f(R) = 8\pi G(\rho_m - 3p_m)$$

The 0-0 equation (3d order)

$$3H\frac{df'(R)}{dt} - 3(\dot{H} + H^2)f'(R) + \frac{f(R)}{2} = 8\pi G\rho_m$$

Reduction to the first order equation

In the absence of spatial curvature and $\rho_m = 0$, it is always possible to reduce these equations to a first order one using either the transformation to the Einstein frame and the Hamilton-Jacobi-like equation for a minimally coupled scalar field in a spatially flat FLRW metric, or by directly transforming the 0-0 equation to the equation for $R(H)$:

$$\frac{dR}{dH} = \frac{(R - 6H^2)f'(R) - f(R)}{H(R - 12H^2)f''(R)}$$

See, e.g. H. Motohashi amd A. A. Starobinsky, Eur. Phys. J. C 77, 538 (2017), but in the special case of the $R + R^2$ gravity this was found and used already in the original AS (1980) paper.

Analogues of large-field (chaotic) inflation: $F(R) \approx R^2 A(R)$ for $R \to \infty$ with $A(R)$ being a slowly varying function of R, namely

$$|A'(R)| \ll \frac{A(R)}{R}\ ,\ |A''(R)| \ll \frac{A(R)}{R^2}\ .$$

Analogues of small-field (new) inflation, $R \approx R_1$:

$$F'(R_1) = \frac{2F(R_1)}{R_1}\ ,\ F''(R_1) \approx \frac{2F(R_1)}{R_1^2}\ .$$

Thus, all inflationary models in $f(R)$ gravity are close to the simplest one over some range of R.

Perturbation spectra in slow-roll $f(R)$ inflationary models

Let $f(R) = R^2\,A(R)$. In the slow-roll approximation $|\ddot{R}| \ll H|\dot{R}|$:

$$P_\xi(k) = \frac{\kappa^2 A_k}{64\pi^2 A_k'^2 R_k^2}, \quad P_g(k) = \frac{\kappa^2}{12 A_k \pi^2}, \quad \kappa^2 = 8\pi G$$

$$N(k) = -\frac{3}{2}\int_{R_f}^{R_k} dR\,\frac{A}{A' R^2}$$

where the index k means that the quantity is taken at the moment $t = t_k$ of the Hubble radius crossing during inflation for each spatial Fourier mode $k = a(t_k)H(t_k)$.

Smooth reconstruction of inflation in $f(R)$ gravity

$$f(R) = R^2 A(R)$$

$$A = const - \frac{\kappa^2}{96\pi^2} \int \frac{dN}{P_\xi(N)}$$

$$\ln R = const + \int dN \sqrt{-\frac{2\, d\ln A}{3\, dN}}$$

The additional assumptions that $P_\xi \propto N^\beta$ and that the resulting $f(R)$ can be analytically continued to the region of small R without introducing a new scale, and it has the linear (Einstein) behaviour there, leads to $\beta = 2$ and the $R + R^2$ inflationary model with $r = \frac{12}{N^2} = 3(n_s - 1)^2$ unambiguously.

Conclusions

- At present, cosmology requires the introduction of four fundamental constants to describe observational data, additional to those known from laboratory physics.
- One new fundamental cosmological parameter $n_s - 1$ has been measured recently, but the theory had been able to predict it more than 30 years before the discovery.
- Regarding the present dark energy:
  a) still no statistically significant deviation from an exact cosmological constant;
  b) one constant is sufficient to describe its properties.
- Regarding the primordial dark energy driving inflation in the early Universe:
  a number of inflationary models having only one free parameter can explain all existing observational data.

- The typical inflationary predictions that $|n_s - 1|$ is small and of the order of N_H^{-1}, and that r does not exceed $\sim 8(1 - n_s)$ are confirmed. Typical consequences following without assuming additional small parameters: $H_{55} \sim 10^{14}\,\mathrm{GeV},\ m_{infl} \sim 10^{13}\,\mathrm{GeV}$.
- Though the Einstein gravity plus a minimally coupled inflaton remains sufficient for description of inflation with existing observational data, modified (in particular, scalar-tensor or $f(R)$) gravity can do it as well.
- From the scalar power spectrum $P_\xi(k)$, it is possible to reconstruct an inflationary model both in the Einstein and $f(R)$ gravity up to one arbitrary physical constant of integration.

- In the Einstein gravity, the simplest inflationary models permitted by observational data are two-parametric, no preferred quantitative prediction for r, apart from its parametric dependence on $n_s - 1$, namely, $\sim (n_s - 1)^2$ or larger.
- The simplest one-parametric inflationary models (the $R + R^2$, Higgs and combined Higgs-R^2 ones) use modified (scalar-tensor) gravity and have the preferred target value for r: $r = \frac{12}{N^2} = 3(n_s - 1)^2 \approx 0.004.$