NATIONAL RESEARCH UNIVERSITY
HIGHER SCHOOL OF ECONOMICS

Fedor Slepov, Sergey Kokovin

EQUILIBRIUM EXISTENCE AND UNIQUENESS IN ADDITIVE TRADE MODELS

BASIC RESEARCH PROGRAM
WORKING PAPERS

SERIES: ECONOMICS
WP BRP 262/EC/2023

Equilibrium existence and uniqueness in additive trade models*

November 14, 2023

Fedor Slepov[1] and Sergey Kokovin[2]

Abstract

This paper develops a modeling technique of "attainable profit" functions, applying it to two models of monopolistic competition. First, it revisits the Krugman's classical trade model in the most general form: several asymmetric countries and non-specified additive utility functions. We establish *the weakest conditions on utilities, sufficient for the existence* of equilibria. These conditions are also *necessary under symmetric preferences*. Equilibrium *uniqueness* is proved only for the case of two countries. Second, we study another, "indirectly additive" trade model (Bertoletti and Etro, 2015), and establish weak conditions on non-specified indirect utilities for the existence of equilibria in several asymmetric countries.

Keywords: international trade, monopolistic competition, variable elasticity of substitution, variable markups, existence of equilibria, attainable profits

JEL: F12, L13, D43

1 Introduction

This paper contributes to the foundations of New Trade theory and develops a technique of analysis. New Trade theory emerged when Krugman (1979) adapted ideas of monopolistic competition to international economics, thereby revealing trade gains from diversity. Subsequently, many papers studied various properties of Krugman's classical model with constant elasticity of substitution (CES) or variable elasticity of substitution (VES). Although after Melitz's (2003) paper, trade theory largely turned to heterogenous firms, Krugman's approach remains relevant for many research questions, especially in its general VES-version with non-specified utilities (lacking closed-form solutions). These questions include the Home-market effect, the microfoundations of gravity equations, the estimation of trade elasticity, pro-competitive effects (Mrazova and Neary, 2014), and welfare analysis (Arkolakis et al., 2019). In spite of the extensive use of the Krugman's general model, what remains unclear is the *weakest condition on utility functions sufficient for the existence of equilibria.*

We fill this lacuna with the help of our new concise reformulation of the model, performed through "attainable profit" functions, dependent only on market aggregators. Further, a similar technique allows us to also find sufficient conditions on *indirect* utility functions for the existence of equilibria in the Bertoletti and Etro trade model (Bertoletti and Etro, 2015). This interesting model has generated studies of income effects in trade (see Bertoletti et al., 2018).

Where can our theoretical findings be useful? First of all, our existence theorems delineate the possible/impossible choice of utility specifications in VES trade modeling, especially for empirical studies that calibrate VES trade models; see Arkolakis et al. (2019), Costinot and RodrÃguez-Clare (2014), and also Bertoletti et al. (2019). Indeed, one should know the range of utility/demand functions that can/cannot be exploited when searching for the best fit for the data. Second, the proof of the existence/uniqueness of equilibrium supports all theoretical papers devoted

*We are grateful for contributons by Evgeny Zhelobodko, Artem Razumovskii, Sergey Onenko, and Philipp Ushchev, as well as for comments on the draft by Pavel Molchanov and Konstantin Kucheryavyi. Ivan Serebrennikov provided research assistance on Bertoletti-Etro model. An earlier version of this paper (2020) did not include the uniqueness result and Bertoletti-Etro model. The study was financed by the HSE University Basic Research Program.

[1]HSE University, slepovfa@gmail.com.

[2]HSE University, skokov7@gmail.com, skokovin@hse.ru.

to comparative statics and the properties of Krugman's equilibria; otherwise these well-known results may be disputed. Third, our paper develops a convenient "indirect" method of modeling the demand side through "attainable profits", which is not common (a rare example is Asplund and Nocke, 2006). This approach allows us to reduce the equilibrium equations to a concise form involving only market aggregators. Prices and sales are subsequently derived from the equilibrium values of two groups of aggregators: marginal utility of income and wages in each country. This technique, as well as our non-trivial tricks for applying the fixed-point theorems (generalizable to various models of monopolistic competition), may be interesting for theorists.

Krugman's (1979) canonical VES setting is taken in its most general form: several countries with possibly asymmetric (linear) production/trade costs, possibly asymmetric unspecified additive utilities and asymmetric populations.

We provide the weakest known sufficient conditions on preferences for the existence of equilibria. These conditions are also necessary-and-sufficient for symmetric utilities (under the usual elementary utility function, i.e., neoclassical, thrice differentiable, and allowing for monopolistic competition). Essentially, it says that when consumption increases infinitely, the limit of the marginal utility is: (i) either non-positive; (ii) or positive but the second derivative of the utility multiplied by the squared consumption tends to negative infinity.[3]

When utilities are symmetric among countries, these sufficient conditions also appear to be necessary, but only when we need to ensure the existence of equilibria under *any parameters* of marginal, fixed, and trading costs, and any populations and any number of countries. At the same time, there can be *specific cost parameters* that generate equilibria even for some utilities violating our conditions (when profits remain finite). So, we almost *completely characterize the family of symmetric utility functions suitable for Krugman-type modeling of trade* with any linear costs. Still, under asymmetric utilities necessity remains an open question.

Our conditions for equilibria are the same as in a closed economy. This means that *trade per se does not add restrictions on the class of utilities suitable for monopolistic-competition modeling.*

Technically, we achieve our results through formulating equilibria in terms of "attainable profit functions", dependent on market aggregators, instead of consumption volumes as the main variables. This promising "indirect" formulation is shown to be equally applicable to indirectly-additive trade models, suggested by Betoletti and Etro (2016).

Betoletti and Etro's indirectly additive model is our second setting studied. Bertoletti and Etro (2015) suggested using an indirectly additive utility function as the main primitive of the model to achieve tractable income effects (elusive under Krugman's additive modelling). The indirect framework opens new horizons; being generalized to heterogenous firms, it allows for interesting empirical estimates connected to incomes (see Bertoletti et al., 2019). We apply our technique to the initial, homogenous version of the model and prove equilibria (which has not yet been done). The sufficient conditions on indirect utilities are found to be natural, the same as used by Bertoletti and Etro for studying income effects. Essentially, they require the elementary indirect utility to be decreasing in price, convex, and vanishing at the limit.

Literature. The growing VES literature on monopolistic competition and trade supplements the old CES tradition. VES studies establish more general theorems about the *robust* qualitative features of equilibria, independent of certain demand specifications, supported by VES empirics.[4] However, as we have noted, *the existence of trade equilibria* remains incompletely studied.

The seminal VES paper Krugman (1979) shows the existence of equilibria for two symmetric countries under increasingly elastic demand functions. Extending this proof to N symmetric countries is rather obvious. However, the asymmetric VES case is far from trivial, as the present paper shows. The reason is that the Brouwer fixed point theorem cannot be applied straightaway due to the lack of reliable constraints on the variables. These boundaries have to be skillfully constructed and implemented. This technical difficulty explains why typical VES-trade papers do not include propositions on the existence of equilibria, see Mrazova and Neary (2014), Arkolakis et al. (2019), Bykadorov et al. (2017). The only exception is Morgan et al. (2020). Adopting restrictive assumptions of two countries and choke price, they prove the existence of Krugman equilibria for two cases: two cost-asymmetric

[3]Essentially, condition (ii) ($\lim_{x\to\infty} x^2 u''(x) = -\infty$) in terms of utility amounts to unbounded attainable profit when marginal cost vanishes. Not every increasing concave elementary utility suits our conditions (i), (ii); we provide examples of functions violating both versions.

[4]As to heterogeneous firms Ã la Melitz (2003), see related VES model in Arkolakis et al. (2019). Other VES studies of trade implement multiple sectors of economy (Morgan et al. 2020), variable technology (Bykadorov et al. 2015, 2017), multi-product firms (Boehm et al. 2020), etc.

countries and two symmetric countries with K sectors. Our propositions generalize the first theorem, allowing multiple countries and the weakest possible restrictions on preferences.

Allen, Arkolakis and Lee (2015) formulate general methods of proving the existence and uniqueness of equilibria in trade models. Essentially, they advise formulating the demand mappings and apply fixed-point theorems when their conditions are satisfied. In our case, the crucial difficulty consists exactly in *finding conditions* on utilities and appropriate bounds on variables that make the demand mapping suitable for the application of general theorems.

As to studies of monopolistic competition without trade (a closed economy), Zhelobodko et al. (2012) find the necessary and sufficient conditions for the existence of equilibria, which is *cost-specific* (joint conditions on utilities and costs). By contrast, our paper finds a *cost-independent* assumption on utilities, which is necessary and sufficient both for a closed economy and Krugman's trade model. The existence of equilibria was not studied in the Bertoletti-Etro paper, as we have pointed out.

Sections 1-3 study Krugman's model (the existence, uniqueness and extensions of equilibria), Section 4 considers the Bertoletti-Etro model. Then the Conclusion and Appendix with auxiliary proofs follow.

2 Krugman's general model in terms of attainable profits

We study the classical Krugman (1979) model of trade in its most general non-CES form.

A single differentiated good is traded among k countries $k \in \{1, ..., K\}$, each having a specific population $l_k > 0$ and specific costs/utilities. The only production factor, labor, is immobile among countries. Free entry drives firms' profits to zero, thus determining the endogenous mass N_k of firms in each country. All consumers are identical. Each inelastically provides a unit of labor, in exchange for the endogenous country-specific wage w_k, determined by labor market clearing. Firms within every country are identical and each produces a unique variety of good.

2.1 Consumers, assumptions, demand

Now we introduce the standard Krugman trade model, and reformulate it through "elementary" revenue/consumption functions to present the reduced form of the model. We also explain why these functions should be continuous, monotone, etc.[5]

Consumers. Consumers maximize their utility which depends on the consumption of all varieties produced in the world, inelastically selling their labor endowment. Each identical consumer in country j solves the following maximization problem:

$$\max_{x_{\omega kj} \geq 0} U_j = \sum_{k=1}^{K} \int_0^{N_k} u_j(x_{\omega kj}) d\omega \quad \text{subject to} \quad \sum_{k=1}^{K} \int_0^{N_k} p_{\omega kj} x_{\omega kj} d\omega \leq w_j e_j,$$

where $p_{\omega kj}$ is the price of variety ω of country k in country j and $x_{\omega kj}$ is the consumption level of variety ω of country k which each consumer in country j chooses, and $u_j(x)$ is the sub-utility function of any consumer in country j. We impose several widely accepted assumptions on the sub-utility function.

Assumption 1. Sub-utility $u_j(x) : [0, X) \to \mathbb{R}$ is continuous on its domain and at least thrice differentiable on $(0, X)$ for some finite or infinite saturation point $X > 0$ of $u_j(x)$

$$u_j(0) = 0, \ u_j'(x) > 0, \ u_j''(x) < 0 \ \ \forall x \in (0, X), \tag{1}$$

$$u_j'''(x)x + 2u_j''(x) < 0 \ \ \forall x \in (0, X). \tag{2}$$

The traditional assumption (1) implies that utility is normalized at zero, increasing, and strictly concave. Assumption (2) ensures the strict concavity of profits, which provides similar (symmetric) choices of similar firms, as we shall see. These assumptions are essential for modeling monopolistic competition, without them the model becomes intractable.

[5]Such technique was sometimes used for non-Krugmanian models of monopolistic competition, e.g., Asplund and Nocke 2006, Mrazova and Neary, 2017.

Assumption 2. Finite or infinite point X is a saturation point in the sense that

$$\lim_{x \to X^-} u_j'(x) = 0. \tag{3}$$

Though this "saturation at the limit" assumption (about vanishing marginal utility) is not vital for monopolistic competition modeling per se, it is needed for our proof. Being not very restrictive, it is further attenuated in another section. Put together, Assumptions 1-2 admit *all* practically used utility functions, including CES, CARA, HARA, linear-quadratic, as well as many others.[6]

Under our assumptions, each consumer's utility maximization yields some demand function $\boldsymbol{x}_{\omega kj}(p_{\omega kj}, \lambda_j)$, which generates the following inverse demand function:

$$\boldsymbol{p}_{\omega kj}(x_{\omega kj}, \lambda_j) = \frac{u_j'(x_{\omega kj})}{\lambda_j},$$

where λ_j is the Lagrange multiplier of the budget constraint which could be interpreted as a price-aggregator in country j (the marginal utility of money). Plugging this price into the budget constraint yields

$$\lambda_j = \frac{1}{w_j e_j} \sum_{k=1}^{K} \int_0^{N_k} u_j'(x_{\omega kj}) x_{\omega kj} d\omega > 0.$$

Under our assumptions, the uniformity of demand functions across varieties ensures single-peaked producer profits. This, in view of identical firms in each country, guarantees the identical (symmetric) behavior of firms within each country. This means that all firms from any country k set the same price p_{kj} for consumers of country j. Consumers from country j, in turn, buy the same amount x_{kj} of all varieties from country k. So, we can drop index ω and simplify the inverse demand function as

$$\boldsymbol{p}_{kj}(x_{kj}, \lambda_j) = \frac{u_j'(x_{kj})}{\lambda_j}. \tag{4}$$

We now turn to producers.

2.2 Producers and attainable profits

As mentioned, firms are similar within every country j, each supplies one variety and has an affine production-cost function $C_j(q_j) = f_j + c_j q_j$, dependent upon the fixed cost $f_j > 0$, marginal cost c_j, and firm output q_j. Exporting also involves some destination-specific, possibly asymmetric, iceberg transportation cost coefficient τ_{jk} ($\tau_{jj} = 1$). This means that for supplying x_{jk} units to country k, a firm from country j must produce $\tau_{jk} x_{jk}$ units.

Facing the direct demand function $\boldsymbol{x}_{jk}(\cdot)$ and aggregator λ_j, any firm from country j can maximize its profit function in prices. Equivalently, the firm can use the inverse demand function $\boldsymbol{p}_{jk}(\cdot)$ from (4) and maximize its profit in output $q_j = \sum_{k=1}^{K} \tau_{jk} x_{jk} l_k$, i.e., in sales x:

$$\text{Profit}_j = \sum_{k=1}^{K} (\boldsymbol{p}_{jk}(x_{jk}, \lambda_j) x_{jk} l_k - w_j c_j \tau_{jk} x_{jk} l_k) - w_j f_j =$$

$$= \sum_{k=1}^{K} \frac{u_k'(x_{jk}) x_{jk} - \lambda_k w_j c_j \tau_{jk} x_{jk}}{\lambda_k} l_k - w_j f_j. \tag{5}$$

We need to derive the optimal choice of sales and the maximal attainable profit for all possible parameter values. The above derived representation implies that the choice of sales x_{jk} depends only on the value of one "composite cost" denoted as $\beta_{jk} := \lambda_k w_j c_j \tau_{jk}$, and on the properties of sub-utility function $u_k(x)$. This one-argument feature justifies the following "attainable functions" approach.

We now introduce and derive several functions related to any sub-utility function $u_k(x)$ (fixing, for brevity, index k). Let us ignore for a while multiplier $1/\lambda_k$, to define the "elementary revenue" function $R_k(x)$ and the "elementary profit" function $Pr_k(x, \beta)$ as follows:

$$R_k(x) := \begin{cases} u_k'(x) x, & x > 0 \\ 0, & x = 0 \end{cases}, \tag{6}$$

[6]CARA utlity is $u(x) = 1 - exp(-\rho x)$, HARA is $u(x) = (x + a)^\rho - a^\rho$, linear-quadratic $u(x) = ax - bx^2$.

$$Pr_k(x,\beta) := R_k(x) - \beta x,\ x \geq 0,\ \beta > 0.$$

These two functions are continuous by Assumption 1[7]. Restoring multiplier $1/\lambda_k$, we reformulate the producers' profits (5) in any country j through the elementary profit function:

$$\text{Profit}_j = \sum_{k=1}^{K} \frac{Pr_k(x_{jk}, \lambda_k w_j c_j \tau_{jk})}{\lambda_k} - w_j f_j. \tag{7}$$

To maximize $Pr_k(x,\beta)$ we use Assumption 1. Taking the derivative with respect to x on $(0, X)$ and equating it to zero, we get FOC: $u_k''(x)x + u_k'(x) - \beta = 0 \Longleftrightarrow \beta = u_k''(x)x + u_k'(x)$ and define a sort of "marginal revenue"

$$\beta_k(x) \equiv u_k''(x)x + u_k'(x).$$

By Assumption 1, this $\beta_k(\cdot)$ function is continuous and decreasing. It declines from $\overline{\beta}_k := \lim_{x\to 0^+} u_k''(x)x + u_k'(x) > 0$[8] to $\underline{\beta}_k := \lim_{x\to X^-} u_k''(x)x + u_k'(x) \leq 0$. This implies that $\beta_k(x)$ must have an inverse function:

$$\mathring{x}_k(\cdot) \equiv \beta_k^{-1}(\cdot) : \ (\underline{\beta}_k, \overline{\beta}_k) \to (0, X), , \tag{8}$$

where $\mathring{x}_k$ is continuous and decreasing from X to 0. Hereafter we exclude the lower (unneeded) part $(\underline{\beta}_k, 0]$ of the domain of $\mathring{x}_k(\beta)$ when $\underline{\beta}_k < 0$. The upper limit $\overline{\beta}_k$ can be either finite or infinite. If it is finite, we extend $\mathring{x}_k(\beta) \equiv 0$ for all $\beta \geq \overline{\beta}_k$, otherwise, $\mathring{x}_k(\beta)$ need not be extended. Note that the resulting $\mathring{x}_k(\beta)$ is continuous, non-increasing, and non-negative. A trivial observation is that $\mathring{x}_k(\beta) = \arg\max_{x\geq 0} Pr_k(x,\beta) \ \ \forall \beta > 0$. This allows us to treat $\mathring{x}_k(\beta)$ as the firm's *best-response function*, a sort of "optimal sales" (that motivates accent $\mathring{}$). It links each composite cost $\beta > 0$ with such sales x that a profit-maximizing producer would choose. So, we express the optimal producer's choice as

$$x_{jk} \equiv \mathring{x}_k(\lambda_k w_j c_j \tau_{jk}) \equiv \mathring{x}_k(\beta_{jk}). \tag{9}$$

We now plug the above best-response function – "optimal sales" – into our elementary revenue and profit functions to get "attainable elementary revenue" and "attainable elementary profit" functions:

$$\mathring{r}_k(\beta) := R_k(\mathring{x}_k(\beta)) \geq 0, \ \ \mathring{\pi}_k(\beta) := Pr_k(\mathring{x}_k(\beta), \beta) \geq 0. \tag{10}$$

These functions are obviously continuous. $\mathring{\pi}_k(\beta)$ is positive and decreasing on $(0, \overline{\beta}_k)$ and $\mathring{\pi}_k(\beta) \equiv 0$ for extension $\beta \geq \overline{\beta}_k$ if $\overline{\beta}_k$ is finite[9]. We now introduce one more useful "profit" function:

$$\mathring{\Pi}_k(\beta) := \frac{\mathring{\pi}_k(\beta)}{\beta}.$$

This $\mathring{\Pi}_k(\beta)$ is non-increasing on $(0, +\infty)$ and decreasing on $(0, \overline{\beta}_k)$, having the range $(0, +\infty)$. So, $\mathring{\Pi}_k(\beta)$ has an inverse function $\mathring{\Pi}_k^{-1} : (0, +\infty) \to (0, \overline{\beta}_k)$, which decreases on its domain. We call functions $\mathring{x}_k(\beta), \mathring{r}_k(\beta), \mathring{\pi}_k(\beta), \mathring{\Pi}_k(\beta)$ *the attainable functions*. In our expressions we shall often omit the arguments of the recently developed response-functions, using brief notations:

$$r_{jk} := \mathring{r}_k(\beta_{jk}), \ \ \pi_{jk} := \mathring{\pi}_k(\beta_{jk}), \ \ \Pi_{jk} := \mathring{\Pi}_k(\beta_{jk}).$$

Applying these notations to (7), the maximum profit achieved by any firm in country j can be expressed through our "attainable profit functions" as follows:

$$\max_{x_{jk}\geq 0} \text{Profit}_j = \max_{x_{jk}\geq 0} \sum_{k=1}^{K} \frac{Pr_k(x_{jk}, \lambda_k w_j c_j \tau_{jk})}{\lambda_k} - w_j f_j = \tag{11}$$

$$= \sum_{k=1}^{K} \frac{\mathring{\pi}_k(\lambda_k w_j c_j \tau_{jk})}{\lambda_k} l_k - w_j f_j = w_j f_j \left(\sum_{k=1}^{K} \mathring{\Pi}_k(\lambda_k w_j c_j \tau_{jk}) c_j \tau_{jk} \frac{l_k}{f_j} - 1 \right).$$

[7]For proof that $\lim_{x\to 0^+} u_k'(x)x = 0$ see Claim (5)
[8]See Claim (6)
[9]See Claim (7)

2.3 Equilibrium equations

This sub-section introduces the system of equilibrium equations.

These conditions for each country include: the zero-profit assumption (**ZP**), labor market clearing (**LM**), the budget constraint (**BC**) and the trade balance (**TB**). They are expressed first in terms of sales x_{jk}, then in terms of intensities of competition and wages λ_k, w_j as the main variables.

ZP states that firms enter the market until profits vanish:

$$[\mathbf{ZP}]: \quad \sum_{k=1}^{K}(p_{jk}x_{jk}l_k - w_jc_j\tau_{jk}x_{jk}l_k) - w_jf_j = \tag{12}$$

$$= w_jf_j\left(\sum_{k=1}^{K}\mathring{\Pi}_k(\lambda_k w_j c_j \tau_{jk})c_j\tau_{jk}\frac{l_k}{f_j} - 1\right) = 0\ \forall j.$$

Labor **M**arket clearing asserts that labor is fully employed:

$$[\mathbf{LM}]: \quad l_j = N_j \cdot (f_j + \sum_{k=1}^{K} c_j\tau_{jk}x_{jk}l_k)\ = \tag{13}$$

$$= N_j \cdot \left(f_j + \sum_{k=1}^{K} c_j\tau_{jk}\mathring{x}_k(\lambda_k w_j c_j\tau_{jk})l_k\right)\ \forall j.$$

Here the left-hand side l_j shows labor provided by all workers, while the right-hand side is the mass of firms multiplied by the total labor costs of each firm.

Budget **C**onstraint, as usual, requires that workers cannot spend more than they earn (naturally, at equilibria it becomes an equality):

$$[\mathbf{BC}]: \quad w_jl_j = \sum_{k=1}^{K} N_k p_{kj}x_{kj}l_j = \sum_{k=1}^{K} N_k\frac{\mathring{r}_j(\lambda_j w_k c_k \tau_{kj})}{\lambda_j}l_j\ \forall j. \tag{14}$$

In other words, all money earned in country j is spent on all goods purchased (identical workers spend their incomes identically).

Trade **B**alance implies that all imported goods are worth as much as all exported goods:

$$[\mathbf{TB}]: \quad N_j\sum_{k=1}^{K} p_{jk}x_{jk}l_k = \sum_{i=1}^{K} N_i p_{ij}x_{ij}l_j\ \forall j. \tag{15}$$

$$\Leftrightarrow\ N_j\sum_{k=1}^{K}\frac{\mathring{r}_k(\lambda_k w_j c_j\tau_{jk})}{\lambda_k}l_k = \sum_{i=1}^{K} N_i\frac{\mathring{r}_j(\lambda_j w_i c_i \tau_{ij})}{\lambda_j}l_j\ \forall j.$$

Definition. *Trade equilibrium* is a bundle

$$\left\{\lambda_j, w_j, N_j, (x_{jk})_{1\,\le\,k\,\le\,K}\,, (p_{jk})_{1\,\le\,k\,\le\,K}\right\}_{1\,\le\,j\,\le\,K} \in \mathbb{R}_+^{3K}\times\mathbb{R}_+^{2K^2}$$

that includes price-aggregates, wages, the numbers of firms, sales, prices, and satisfies: (i) utility-maximization (4); (ii) profit-maximization (9)-(10), (iii) Zero Profit condition (12); (iv) Labor Market clearing (13); (v) Budget Constraint (14); and (vi) Trade Balance (15).

Here, as usual, TB follows from the summation of BC (under labor balance and zero profit). This explains why our list of 6 (groups of) equations is not excessive for fitting only 5 (groups of) variables.[10]

[10] See Appendix.

2.4 Equilibrium existence in Krugman's model

In this subsection, to prove the existence of equilibrium, we reformulate our system of equations in the form suitable for applying the Brouwer fixed-point theorem.

First, using new convenient notation for variables $\mu_j \equiv \lambda_j w_j$, we rewrite ZP (12) as

$$w_j = w_j \sum_{k=1}^{K} \Pi_{jk} c_j \tau_{jk} \frac{l_k}{f_j} \ \forall j \ \Leftrightarrow$$

$$w_j = w_j \sum_{k=1}^{K} \mathring{\Pi}_k (\mu_k \frac{w_j}{w_k} c_j \tau_{jk}) c_j \tau_{jk} \frac{l_k}{f_j} \quad \forall j. \tag{16}$$

Second, we plug N_j from LM into BC to get

$$\lambda_j w_j = \sum_{k=1}^{K} \frac{l_k r_{kj}}{f_k + c_k \sum_{i=1}^{K} \tau_{ki} x_{ki} l_i} \ \Leftrightarrow$$

$$\mu_j = \sum_{k=1}^{K} \frac{\mathring{r}_j (\mu_j \frac{w_k}{w_j} c_k \tau_{kj}) l_k}{f_k + c_k \sum_{i=1}^{K} \tau_{ki} \mathring{x}_i (\mu_i \frac{w_k}{w_i} c_k \tau_{ki}) l_i} \quad \forall j. \tag{17}$$

Third, as we have said, when ZP, LM, and BC hold for some country, then TB is also satisfied in that country, so TB is superfluous in the further analysis. Moreover, these two $2K$ equilibrium equations of $\{\mu_j, w_j\}_{1 \le j \le K}$ are not independent. One of these symmetric equations (16), e.g., the first one, can be derived from others when other equations (16)-(17) hold.[11] Since wages come into equations as ratios, they allow for scaling, i.e., the free choice of wages/numerarie level. We can therefore reduce our setup to $2K - 1$ equations in $2K - 1$ unknowns $\left\{ (\mu_j)_{1 \le j \le K}, (w_j)_{2 \le j \le K} \right\}$ by normalizing the first wage as

$$w_1 \equiv 1. \tag{18}$$

This identity from now on supersedes the first one of equations (16). We will establish the existence of equilibrium using this system.

These considerations justify the three-stage *sequential* solution method:

Remark. Under Assumptions 1 and 2, any trade equilibrium in Krugman's model can be found in three stages: (1) finding aggregators/wages (μ, w) from the equation systems $\{(16), (17), (18)\}$, (2) finding sales x from equations (9) using (μ, w), (3) finding prices p and the number N of firms from equations (4), (13).

Thereby, the reduced form (16)-(18) of the equilibrium equations enable us to formulate the *sufficient conditions* for the existence of equilibria in a very general form, as follows (we postpone one more generalization − relaxing Assumption 2 − to the Extensions section).

Proposition 1. Under any positive populations/costs ($\{l_j, c_j, f_j\}_{1 \le j \le K} \gg 0$, $\tau_{jk} \ge 1$) and Assumptions 1,2 on sub-utility functions $u_j(\cdot)$, the reduced trade model $\{(16), (17), (18)\}$ has a positive equilibrium $\{\bar{\mu}_j, \bar{w}_j\}_{1 \le j \le K} \gg 0$ in terms of price-aggregators and wages, which determine prices, sales, and the number of firms as in Remark 1. Thereby, an equilibrium satisfying the initial equation systems $\{(12)–(15)\}$ exists.

Proof. The main idea of our proof is to use the Brouwer fixed point theorem. It states that *a continuous mapping of a non-empty convex compact into itself—always has a fixed point*. First, we define a convex compact set $\Omega \subset \mathbb{R}^{2K-1}_{++}$, sufficiently broad to include all possible solutions to equations $\{(16), (17)\}$. Second, we define "the Brouwer mapping" $F : \Omega \to \Omega$ whose fixed points (price-aggregators and wages) should be the equilibrium points. Third, we show that any fixed point of F is really an equilibrium point in our model.

The boundaries. To start with, we introduce useful notations for the extremal values of our parameters:

$$f_m := \min_j f_j, \ c_m := \min_j c_j, \ l_m := \min_j l_j, \ \tau_m := \min_j \min_k \tau_{jk} \le 1,$$

[11] See Lemma 1 in the Appendix.

$$f_M := \max_j f_j,\ c_M := \max_j c_j,\ l_M := \max_j l_j,\ \tau_M := \max_j \max_k \tau_{jk} \geq 1.$$

We now start constructing some constants $\mu_m, \mu_M, w^m, \frac{1}{w^m}$, used as boundaries for the convex compact set

$$\Omega := [\mu_m, \mu_M]^K \times [w^m, \frac{1}{w^m}]^{K-1} \subset \mathbb{R}^{2K-1}_{++}, \tag{19}$$

serving as a domain for our mapping F. The boundaries $\mu_m, \mu_M, w, \frac{1}{w}$ of our compact set will play a key role in our proof. The way they are constructed will allow us to show that any fixed point of mapping F is an equilibrium point. We start with μ_M using $R_j(\cdot)$ definition (6):

$$\mu_M = K \max_j \sup_{0 \leq x} \frac{R_j(x) l_M}{f_m + c_m \tau_m x l_m} > 0. \tag{20}$$

Here each of K supremums is finite because each fraction $\frac{R_j(x)}{a+bx}$ is continuous on $[0, X)$ and, under Assumption 2, vanishes at the saturation point X:[12]

$$\lim_{x \to X} \frac{u_j'(x) l_M}{\frac{f_m}{x} + c_m \tau_m l_m} = \frac{\lim_{x \to X} u_j'(x) l_M}{\lim_{x \to X} \left(\frac{f_m}{x} + c_m \tau_m l_m \right)} = 0.$$

Therefore, the upper bound μ_M is finite. Plugging constants into function $\mathring{\Pi}_j^{-1}$ (which is well-defined and positive, as previously established), we define the lower bound μ_m as

$$\mu_m = \frac{1}{2c_M} \min_j \mathring{\Pi}_j^{-1} \left(\frac{f_M}{c_m l_m} \right) > 0. \tag{21}$$

We now show that $\mu_m < \mu_M$. Denote by $\hat{i}$ an argminimum in the definition of μ_m. We can rewrite the definition of bound μ_m as:

$$\mu_m = \frac{1}{2c_M} \mathring{\Pi}_{\hat{i}}^{-1} \left(\frac{f_M}{c_m l_m} \right) \Rightarrow \frac{f_M}{c_m l_m} = \frac{\mathring{\pi}_{\hat{i}}(2\mu_m c_M)}{2\mu_m c_M} \Rightarrow$$

$$\Rightarrow \frac{2\mu_m c_M f_M}{c_m l_m} = \mathring{r}_{\hat{i}}(2\mu_m c_M) - 2\mu_m c_M \mathring{x}_{\hat{i}}(2\mu_m c_M) \Rightarrow \mu_m = \frac{1}{2} \frac{c_m}{c_M} \frac{\mathring{r}_{\hat{i}}(2\mu_m c_M) l_m}{f_M + c_m \mathring{x}_{\hat{i}}(2\mu_m c_M) l_m} =$$

$$= \frac{1}{2} \frac{c_m}{c_M} \frac{R_{\hat{i}}\big(\mathring{x}_{\hat{i}}(2\mu_m c_M)\big) l_m}{f_M + c_m \mathring{x}_{\hat{i}}(2\mu_m c_M) l_m} < K \max_j \sup_{0 \leq x} \frac{R_j(x) l_M}{f_m + c_m \tau_m x l_m} = \mu_M.$$

This strict inequality is guaranteed because the supremum in x at the right-hand side is taken, and multiplier $\frac{1}{2}$ is used. We now define bound w^m as:

$$w^m = \frac{1}{2\mu_M c_M \tau_M} \min_j \Pi_j^{-1} \left(\frac{f_M}{c_m \tau_m l_m} \right) > 0. \tag{22}$$

Since w^m and $\frac{1}{w^m}$ are our lower and upper boundaries for wages, we should make sure that $w^m < 1$. Denote by $\hat{i}$ the number where the minimum is achieved in the definition of w^m. With the same algebraic manipulations we get:

$$w^m = \frac{1}{2\mu_M c_M \tau_M} \mathring{\Pi}_{\hat{i}}^{-1} \left(\frac{f_M}{c_m \tau_m l_m} \right) \Rightarrow \frac{f_M}{c_m \tau_m l_m} = \frac{\mathring{\pi}_{\hat{i}}(2\mu_M w^m c_M \tau_M)}{2\mu_M w^m c_M \tau_M} \Rightarrow$$

$$\Rightarrow w^m = \frac{1}{\mu_M} \frac{1}{2} \frac{c_m \tau_m}{c_M \tau_M} \frac{R_{\hat{i}}\big(\mathring{x}_{\hat{i}}(2\mu_M w^m c_M \tau_M)\big) l_m}{f_M + c_m \tau_m \mathring{x}_{\hat{i}}(2\mu_M w^m c_M \tau_M) l_m} < \frac{1}{\mu_M} K \max_j \sup_{0 \leq x} \frac{R_j(x) l_M}{f_m + c_m \tau_m x l_m} = 1.$$

Thus, we have shown that $w^m < \frac{1}{w^m}$. So, Ω is correctly defined in (19), it is a non-empty subset of $\mathbb{R}^{2K-1}_{++}$. Since Ω is a Cartesian product of closed intervals, it is a *non-empty compact convex set*. Its elements will be denoted

$$\omega \equiv (\mu, w) \equiv (\mu_j, w_{j \neq 1})_{1 \leq j \leq K} \in \Omega.$$

[12] $X = +\infty$ for typical u. Vanishing is not guaranteed under Assumption 2* in our subsequent generalizations, yet, the limit remains finite.

The mapping. We are ready to define a mapping $F:\Omega\to\Omega$ for applying the Brouwer fixed point theorem as $F(\omega):=(\boldsymbol{\mu}(\omega),\boldsymbol{w}(\omega))$, namely

$$\boldsymbol{\mu}_j(\omega):=\max\left(\sum_{k=1}^{K}\frac{r_{kj}l_k}{f_k+c_k\sum_{i=1}^{K}\tau_{ki}x_{ki}l_i},\ \mu_m\right)\quad\forall j, \tag{23}$$

$$\boldsymbol{w}_j(\omega):=\min\left(\max\left(w_j\sum_{k=1}^{K}\Pi_{jk}c_j\tau_{jk}\frac{l_k}{f_j},\ w^m\right),\ \frac{1}{w^m}\right)\quad\forall j>1. \tag{24}$$

We have ensured that our functions $\mathring{r}_k(\omega)$, $\mathring{x}_k(\omega)$, $\mathring{\Pi}_k(\omega)$ are continuous, whereas *max*, *min* operators preserve continuity, thereby mapping F is continuous.

We should ensure that F maps its domain Ω into itself and not outside. Component $\boldsymbol{w}_j(\omega)$ of this mapping fits the boundaries $\left[w^m,\frac{1}{w^m}\right]$ by construction; operations *max*, *min* artificially restrict w. We should estimate another component, $\boldsymbol{\mu}_j(\omega)$, from its upper, unrestricted side. If not hitting the lower boundary μ_m, our variable $\boldsymbol{\mu}_j$ equals magnitude

$$\sum_{k=1}^{K}\frac{r_{kj}l_k}{f_k+c_k\sum_{i=1}^{K}\tau_{ki}x_{ki}l_i}\le\sum_{k=1}^{K}\frac{r_{kj}l_M}{f_m+c_m\tau_m\sum_{i=1}^{K}x_{ki}l_m}\le\sum_{k=1}^{K}\frac{r_{kj}l_M}{f_m+c_m\tau_m x_{kj}l_m}=$$

$$=\sum_{k=1}^{K}\frac{R_j(x_{kj})l_M}{f_m+c_m\tau_m x_{kj}l_m}\le K\max_j\sup_{0\le x}\frac{R_j(x)l_M}{f_m+c_m\tau_m x l_m}=\mu_M\ \ \forall\,\mu_j>0,\ \forall w_{j\neq1}>0\ j=1,...,K.$$

Thus, the upper bound is satisfied as $\boldsymbol{\mu}_j(\omega)\le\mu_M$ for all ω, and we have constructed a continuous mapping F from a convex compact set Ω into itself. So, by the Brouwer fixed point theorem, our mapping F must have at least one fixed point. It will be denoted $\hat{\omega}\equiv\{\hat{w}_j,\hat{\mu}_j\}_{1\le j\le K}$ (we include an additional component $\hat{w}_1:=1$). We shall further use the following intuitive notation with ˆ for all fixed-point components:

$$\hat{\beta}_{jk}:=\hat{\lambda}_k\hat{w}_jc_j\tau_{jk},\ \hat{x}_{jk}:=\mathring{x}_k(\hat{\beta}_{jk}),\ \hat{r}_{jk}:=\mathring{r}_k(\hat{\beta}_{jk}),\ \hat{\pi}_{jk}:=\mathring{\pi}_k(\hat{\beta}_{jk}).$$

Now we show why any fixed point of our mapping is a true equilibrium point.

Boundaries do not bind. Our construction of mapping F artificially restricts the set of values that F can take. This creates the possibility that a fixed point lying on any boundary of Ω will not be a true equilibrium, i.e., not a solution to the initial equations (16)–(17). To exclude this case, we will show that none of the artificial constraints is binding at any fixed point. We start by showing (by contradiction) that hitting the boundary $\hat{w}_j=w^m$ is impossible for any $j>1$. Suppose that $\hat{w}_i=w^m$ for some $i>1$. Then

$$\hat{w}_i=w^m=\min\left(\max\left(\hat{w}_i\sum_{k=1}^{K}\hat{\Pi}_{ik}c_i\tau_{ik}\frac{l_k}{f_i},w^m\right),\frac{1}{w^m}\right)\ge$$

$$\ge\hat{w}_i\sum_{k=1}^{K}\mathring{\Pi}_k\left(\hat{\mu}_k\frac{\hat{w}_i}{\hat{w}_k}c_i\tau_{ik}\right)c_i\tau_{ik}\frac{l_k}{f_i}=$$

$$=w^m\sum_{k=1}^{K}\mathring{\Pi}_k\left(\hat{\mu}_k\frac{w^m}{\hat{w}_k}c_i\tau_{ik}\right)c_i\tau_{ik}\frac{l_k}{f_i}\ge$$

$$\ge w^m\mathring{\Pi}_1(\hat{\mu}_1w^mc_i\tau_{i1})c_i\tau_{i1}\frac{l_1}{f_i}\ge w^m\mathring{\Pi}_1(\mu_Mw^mc_M\tau_M)c_m\tau_m\frac{l_m}{f_M}\Rightarrow$$

$$\Rightarrow\frac{f_M}{c_m\tau_ml_m}\ge\mathring{\Pi}_1(\mu_Mw^mc_M\tau_M)\Rightarrow$$

$$\Rightarrow w^m\ge\frac{1}{\mu_Mc_M\tau_M}\mathring{\Pi}_1^{-1}\left(\frac{f_M}{c_m\tau_ml_m}\right)\ge\frac{1}{\mu_Mc_M\tau_M}\min_j\mathring{\Pi}_j^{-1}\left(\frac{f_M}{c_m\tau_ml_m}\right)=2w^m.$$

Since $w^m>0$, we came to a contradiction $w^m\ge2w^m$. We conclude that at the fixed point the lower bound is not reached: $\hat{w}_j>w^m\ \forall j>1$. We now prove that reaching the upper bound $\hat{w}_j=\frac{1}{w^m}$ for any j is also impossible. We apply Lemma (1) from Appendix to our fixed point. The Lemma considers some vector $y=(w,x,p)$ and restores, as an inequality, the omitted ZP condition for the country $j = 1$, using K inequalties like BC (23) and $K-1$ inequalties like the ZP condition (24). We use the fixed-point values for vector y:

$$y \equiv \left\{ w_j, (x_{jk})_{1 \le k \le K}, (p_{jk})_{1 \le k \le K} \right\}_{1 \le j \le K} =$$

$$= \left\{ \hat{w}_j, (\hat{x}_{jk})_{1 \le k \le K}, \left(\hat{p}_{jk} := \frac{u'_k\left(\hat{x}_{jk}\right)\hat{w}_k}{\hat{\mu}_k} \right)_{1 \le k \le K} \right\}_{1 \le j \le K}.$$

For applying Lemma 1, we use the fact that the lower bound w^m is not reached by wages:

$$\hat{w}_j = \min\left(\hat{w}_j \sum_{k=1}^{K} \hat{\Pi}_{jk} c_j \tau_{jk} \frac{l_k}{f_j}, \frac{1}{w^m} \right) \le \hat{w}_j \sum_{k=1}^{K} \mathring{\Pi}_k \left(\hat{\mu}_k \frac{\hat{w}_j}{\hat{w}_k} c_j \tau_{jk} \right) c_j \tau_{jk} \frac{l_k}{f_j} \Longleftrightarrow$$

$$\Longleftrightarrow \sum_{k=1}^{K} (\hat{p}_{jk}\hat{x}_{jk} l_k - \hat{w}_j c_j \tau_{jk} \hat{x}_{jk} l_k) - \hat{w}_j f_j \ge 0 \quad \forall j > 1,$$

which shows that the first set of inequalities required for Lemma 1 is provided. To guarantee the second set, observe

$$\hat{\mu}_j = \max\left(\sum_{k=1}^{K} \frac{\hat{r}_{kj} l_k}{f_k + c_k \sum_{i=1}^{K} \tau_{ki}\hat{x}_{ki} l_i}, \mu_m \right) \ge \sum_{k=1}^{K} \frac{\hat{r}_{kj} l_k}{f_k + c_k \sum_{i=1}^{K} \tau_{ki}\hat{x}_{ki} l_i} \Longleftrightarrow$$

$$\Longleftrightarrow \hat{w}_j l_j \ge \sum_{k=1}^{K} \frac{l_k}{f_k + \sum_{i=1}^{K} c_k \tau_{ki}\hat{x}_{ki} l_i} \hat{p}_{kj}\hat{x}_{kj} l_j \quad \forall j.$$

Thus, the Lemma is applicable. It yields the following inequality, satisfied for values of our fixed point (we use identity $\hat{w}_1 = 1$):

$$\sum_{k=1}^{K} (\hat{p}_{1k}\hat{x}_{1k} l_k - \hat{w}_1 c_1 \tau_{1k} \hat{x}_{1k} l_k) - \hat{w}_1 f_1 \le 0 \Longleftrightarrow$$

$$1 \ge \sum_{k=1}^{K} \Pi_k \left(\frac{\hat{\mu}_k}{\hat{w}_k} c_1 \tau_{1k} \right) c_1 \tau_{1k} \frac{l_k}{f_1}. \tag{25}$$

Suppose the upper bound is achieved as $\hat{w}_i = \frac{1}{w^m}$ for some i. Then we have:

$$1 \ge \sum_{k=1}^{K} \Pi_k \left(\frac{\hat{\mu}_k}{\hat{w}_k} c_1 \tau_{1k} \right) c_1 \tau_{1k} \frac{l_k}{f_1} \ge \Pi_i \left(\frac{\hat{\mu}_i}{\hat{w}_i} c_1 \tau_{1i} \right) c_1 \tau_{1i} \frac{l_i}{f_1} = \Pi_i \left(\hat{\mu}_i w^m c_1 \tau_{1i} \right) c_1 \tau_{1i} \frac{l_i}{f_1} \Rightarrow$$

$$\Rightarrow w^m \ge \frac{1}{\mu_M c_M \tau_M} \Pi_i^{-1} \left(\frac{f_M}{c_m \tau_m l_m} \right) \ge \frac{1}{\mu_M c_M \tau_M} \min_j \Pi_j^{-1} \left(\frac{f_M}{c_m \tau_m l_m} \right) = 2w^m.$$

Since we arrived at a contradiction, we conclude that $\hat{w}_j < \frac{1}{w^m}$ holds for $\forall j > 1$. At this point, we have established that our mapping F generates real, not artificial values for wages:

$$\hat{w}_j = \hat{w}_j \sum_{k=1}^{K} \Pi_k \left(\hat{\mu}_k \frac{\hat{w}_j}{\hat{w}_k} c_j \tau_{jk} \right) c_j \tau_{jk} \frac{l_k}{f_j} \quad \forall j > 1.$$

Together with ((25)), it implies that

$$\hat{w}_j \ge \hat{w}_j \sum_{k=1}^{K} \Pi_k \left(\hat{\mu}_k \frac{\hat{w}_j}{\hat{w}_k} c_j \tau_{jk} \right) c_j \tau_{jk} \frac{l_k}{f_j} \quad \forall j.$$

Now we want to show that for the price aggregators μ the lower boundary $\hat{\mu}_j = \mu_m$ is also not attained. Suppose that $\hat{\mu}_i = \mu_m$ for some i. Then by applying the familiar trick, we come to a contradiction, as shown below:

$$\hat{w}_i \ge \hat{w}_i \sum_{k=1}^{K} \Pi_k \left(\frac{\hat{\mu}_k}{\hat{w}_k} \hat{w}_i c_i \tau_{ik} \right) c_i \tau_{ik} \frac{l_k}{f_i} \ge$$

$$\geq \hat{w}_i \Pi_i(\hat{\mu}_i c_i) c_i \frac{l_i}{f_i} = \hat{w}_i \Pi_i(\mu_m c_i) c_i \frac{l_i}{f_i} \geq$$

$$\geq \hat{w}_i \Pi_i(\mu_m c_M) c_m \frac{l_m}{f_M} \Rightarrow \mu_m \geq \frac{1}{c_M} \Pi_i^{-1}\left(\frac{f_M}{c_m l_m}\right) \geq \frac{1}{c_M} \min_j \Pi_j^{-1}\left(\frac{f_M}{c_M l_m}\right) = 2\mu_m.$$

Since $\mu_m > 0$, it is a contradiction. So, we have proved that none of the artificial boundaries used to restrict our mapping F is binding at any fixed point. Without these artificial restrictions, our mapping exactly expresses the needed equilibrium equations (16)–(17). Thus, any fixed point $\hat{\omega}$ (which exists, as we have shown) is an equilibrium point. Using $\hat{\omega}$, other equilibrium variables – prices, sales and masses of firms – are easily found with Remark 1. This completes the proof. **Q.E.D.**

3 Necessary condition for Krugman's equilibrium, equilibrium uniqueness

3.1 Generalized condition on marginal utility

For equilibria existence we imposed Assumption 2 on u_j' (demand saturation at the limit). However, our proof can be extended to a broader class of utility functions u by using the following "weakest sufficient" condition.

Assumption 2*: Elementary utility u_j in any country j satisfy

$$\lim_{x \to X} u_j'(x) \leq 0 \quad or \quad \begin{cases} X = +\infty \\ \lim_{x\to+\infty} u_j'(x) > 0 \\ \lim_{x\to+\infty} -u_j''(x)x^2 = +\infty \end{cases}.$$

This assumption means that, whenever our marginal utility (inverse demand u') is not vanishing at the limit, its (absolute value of) derivative $\left(-u_j''\right)$ must not decrease too fast. This property guarantees that the attainable profit increases infinitely when marginal cost vanishes (one can see this from combining the FOC $u'(x) + xu''(x) = \beta$ with the "elementary profit" function $xu'(x) - \beta x$), which is used as follows.

Proposition 2. *To prove the existence of equilibria as in Proposition 1, we may replace Assumption 2 (asymptotic saturation) by Assumption 2*, the conclusion remains true.*

Proof. We mention only amendments to the previous proof. We used Assumption 2 *(asymptotic saturation)* to construct in (9) our sales function $\mathring{x}_j$ (the inverse function to the marginal revenue) in such a way that its domain be $(0, \infty)$. For those countries where Assumption 2 is satisfied for u_j, no amendment is needed. We should check domains and ranges of $\mathring{x}_j$ for other countries.

Define T as the set of countries with $\lim_{x\to+\infty} u_t'(x) > 0$. As in definition (9), for any such country $t \in T$ we construct the best-response function $\mathring{x}_t(\beta)$, but now it has domain $(\underline{\beta}_t, +\infty)$ instead of the usual $(0, +\infty)$ (where the lower limit is $\underline{\beta}_t \equiv \lim_{x\to+\infty} u_t''(x)x + u_t'(x) = \lim_{x\to+\infty} u_t'(x) > 0$).[13] We define the upper limit as $\overline{\beta}_t \equiv \lim_{x\to 0} u_t''(x)x + u_t'(x) > 0$ under our assumptions. Our new best-response function $\mathring{x}_t(\beta)$ spans $(0, +\infty)$ on $(\underline{\beta}_t, \overline{\beta}_t)$. We set $\mathring{x}_t(\cdot) \equiv 0$ on the upper interval $[\overline{\beta}_t, +\infty)$ in case $\overline{\beta}_t$ is finite.

As in definitions (10), we construct "attainable" functions $\mathring{r}_t(\beta)$ and $\mathring{\pi}_t(\beta)$ from $\mathring{x}_t(\beta)$. As before, function $\mathring{\pi}_t(\cdot)$ *decreases* on $(\underline{\beta}_t, \overline{\beta}_t)$, whereas it equals zero for larger $\beta > \overline{\beta}_t$, in case $\overline{\beta}_t$ is finite.

Using identity $(u_t''(\mathring{x}_t(\beta))\mathring{x}_t(\beta) + u_t'(\mathring{x}_t(\beta))) \equiv \beta$ we can express the maximal attainable profit as

$$\lim_{x\to+\infty} -u_t''(x)x^2 = \lim_{x\to+\infty} \left(u_t'(x)x - \left(u_t''(x)x + u_t'(x)\right)x\right) =$$

$$= \lim_{\beta \to \underline{\beta}_t} \left(u_t'(\mathring{x}_t(\beta))\mathring{x}_t(\beta) - \beta \mathring{x}_t(\beta)\right) = \lim_{\beta \to \underline{\beta}_t} \pi_t(\beta) = +\infty,$$

because $\lim_{x\to+\infty} -u_t''(x)x^2 = +\infty$, by Assumption 2*.

[13] For proof that $\lim_{x\to+\infty} u_t''(x)x = 0$ see Claim (8) in Appendix.

Thus, our decreasing function $\mathring{\Pi}_t(\beta) \equiv \frac{\mathring{\pi}_t(\beta)}{\beta}$ has range $(0, +\infty)$ on $(\underline{\beta}_t, \overline{\beta}_t)$ (taking value 0 for $\beta > \overline{\beta}_t$ if finite) and therefore $\mathring{\Pi}_t$ has the inverse function $\mathring{\Pi}_t^{-1} : (0, +\infty) \leftrightarrow (\underline{\beta}_t, \overline{\beta}_t)$.

We are going to make our previous proof of the existence of equilibrium applicable to the case when set T is non-empty. For all $t \in T$, we should extend the definition of our "attainable" functions $\mathring{x}_t, \mathring{r}_t, \mathring{\pi}_t, \mathring{\Pi}_t$ — from $(\underline{\beta}_t, +\infty)$ to the broader domain $(0, +\infty)$. The latter should contain *all* positive arguments needed in our proof, even the small ones.

To construct such an extension we need to choose some value $\widehat{x}_t$ of function $\mathring{x}_t(\cdot)$ that can be attributed to all small arguments $(0, \underline{\beta}_t]$. However, since $\mathring{x}_t(\underline{\beta}_t) = \infty$ it cannot play the role of $\widehat{x}_t$. So, we need an argument $\widehat{\beta}_t$ somewhat larger than $\underline{\beta}_t$ to define $\widehat{x}_t := \mathring{x}_t(\widehat{\beta}_t)$, e.g.,

$$\widehat{\beta}_t := \frac{1}{2}\left(\mathring{\Pi}_t^{-1}\left(\frac{f_M}{c_m \tau_m l_m}\right) + \underline{\beta}_t\right) > \underline{\beta}_t.$$

Using this new constant, we can define artificial functions $\overline{x}_t, \overline{r}_t, \overline{\pi}_t, \overline{\Pi}_t$ *extending* their values at $\widehat{\beta}_t$ to all arguments below $\widehat{\beta}_t$, in the following way:

$$\overline{x}_t(\beta) := \mathring{x}_t(\max\{\beta,\, \widehat{\beta}_t\}) \;\; \overline{r}_t(\beta) := R_t(\overline{x}_t(\beta)),$$

$$\overline{\pi}_t(\beta) := Pr_t(\overline{x}_t(\beta), \max\{\beta,\, \widehat{\beta}_t\}), \quad \overline{\Pi}_t(\beta) := \frac{\overline{\pi}_t(\beta)}{\beta}.$$

From now on, in our proof we replace the original "attainable" functions by these new "extended" ones, for the countries from set T (where Assumption 2* holds). We keep the old functions intact for other countries (where Assumption 2 holds). All the elements of the argumentation from the proof of Proposition 1 remain the same with these new functions, exactly as we proceeded with the old functions $\mathring{x}_t, \mathring{r}_t, \mathring{\pi}_t, \mathring{\Pi}_t$. We repeat the same construction of our (continuous) mapping F and the borders of its (compact, rectangular) domain Ω. Then, applying the Brouwer fixed point theorem again, we get the fixed point and ensure that it is an equilibrium point $\{\widetilde{w}_j, \widetilde{\mu}_j\}_{1 \le j \le K}$.

It remains to show that at this point our "extended" functions $\overline{x}_t, \overline{r}_t, \overline{\pi}_t, \overline{\Pi}_t$ coincide with the initial functions $\mathring{x}_t, \mathring{r}_t, \mathring{\pi}_t, \mathring{\Pi}_t$, which are used on their normal domain, so that their artificial values do not come to play.

Suppose it is not so: the artificial value is used at the fixed point. Then for some country i and some country $t \in T$ we have the inequality: $\widetilde{\mu}_t \frac{\widetilde{w}_i}{\widetilde{w}_t} c_i \tau_{it} \le \widehat{\beta}_t$. Consider the zero-profit condition for country i:

$$\widetilde{w}_i = \widetilde{w}_i \sum_{k=1}^{K} \Pi_k\left(\widetilde{\mu}_k \frac{\widetilde{w}_i}{\widetilde{w}_k} c_i \tau_{ik}\right) c_i \tau_{ik} \frac{l_k}{f_i} \ge$$

$$\widetilde{w}_i \overline{\Pi}_t\left(\widetilde{\mu}_t \frac{\widetilde{w}_i}{\widetilde{w}_t} c_i \tau_{it}\right) c_i \tau_{it} \frac{l_t}{f_i} \ge \widetilde{w}_i \overline{\Pi}_t\left(\widetilde{\mu}_t \frac{\widetilde{w}_i}{\widetilde{w}_t} c_i \tau_{it}\right) c_m \tau_m \frac{l_m}{f_M} \Rightarrow$$

$$\Rightarrow \frac{f_M}{c_m \tau_m l_m} \ge \overline{\Pi}_t\left(\widetilde{\mu}_t \frac{\widetilde{w}_i}{\widetilde{w}_t} c_i \tau_{it}\right) = \frac{\overline{\pi}_t\left(\widetilde{\mu}_t \frac{\widetilde{w}_i}{\widetilde{w}_t} c_i \tau_{it}\right)}{\widetilde{\mu}_t \frac{\widetilde{w}_i}{\widetilde{w}_t} c_i \tau_{it}}$$

$$= \frac{\pi_t(\widehat{\beta}_t)}{\widetilde{\mu}_t \frac{\widetilde{w}_i}{\widetilde{w}_t} c_i \tau_{it}} \ge \frac{\pi_t(\widehat{\beta}_t)}{\widehat{\beta}_t} = \Pi_t(\widehat{\beta}_t) \Rightarrow$$

$$\Rightarrow \widehat{\beta}_t \ge \Pi_t^{-1}\left(\frac{f_M}{c_m \tau_m l_m}\right) > \frac{1}{2}\left(\Pi_t^{-1}\left(\frac{f_M}{c_m \tau_m l_m}\right) + \underline{c}_t\right) = \widehat{\beta}_t.$$

This is a contradiction. So, at any equilibrium point obtained with our "extended" functions *none* of the artificial values are engaged. We conclude that it is an equilibrium point with the original functions as well. **Q.E.D.**

3.2 The necessity of the generalized condition on marginal utility

Our generalized condition (Assumption 2*) is probably necessary and sufficient in all cases. However, we are able to prove its necessity for the existence of equilibria only in the simple case, when countries are symmetric in utilities (not in costs or populations).

Proposition 3. *If all sub-utilities, satisfying Assumption 1 are symmetric among countries, i.e. $u_j(x) = u(x)$, then Assumption 2* is sufficient for equilibrium existence under* ***all*** *positive parameters $(c, f, \tau, l) \in R^{4K}_{++}$, and also necessary for this conclusion (whereas under specific cost parameters Assumption 2* may be superfluous).*

Proof. The sufficiency of Assumption 2* is stated in Proposition 2 (which includes asymmetric countries also). Consider necessity. We must prove that for *any* $u(\cdot)$ that satisfies Assumption 1, but violate Assumption 2*, one can find such positive parameters c, f, τ, l that equilibrium is absent.

Violation of Assumption 2* by some $u'(\cdot)$ means that its limit is positive and attained too quickly in the sense

$$\begin{cases} \lim_{x\to+\infty} u'(x) > 0 \\ \lim_{x\to+\infty} -u''(x)x^2 \neq +\infty \end{cases}.$$

Denote $S \equiv \lim_{x\to+\infty} -u''(x)x^2$ where $S > 0$ is some positive number (the existence of this limit follows from the monotonicity of $-u''(x)x^2$, guaranteed by Assumption 1). Therefore, $\lim_{c\to \underline{c}^+} \pi(c) = S$ and $\Pi(c) < \frac{S}{\underline{c}}$.[14]

Now consider the zero-profit condition, it states that

$$w_j = \sum_{k=1}^{K} \frac{\pi_{jk}}{\frac{s_k}{w_k}} \frac{l_k}{f_j} \Rightarrow f_j = \sum_{k=1}^{K} \Pi_j(s_k \frac{w_j}{w_k} c_j \tau_{jk}) l_k =$$

$$= \sum_{k=1}^{K} \Pi(s_k \frac{w_j}{w_k} c_j \tau_{jk}) c_j \tau_{jk} l_k < K \frac{S}{\underline{c}} c_M t_M l_M$$

This inequality becomes impossible when f_j is chosen big enough, relative to other parameters. Thus, under any utility u that violates Assumption 2* we can always find such cost/population parameters (c, f, τ, l), such that an equilibrium becomes impossible.

To prove that Assumption 2* can be "superfluous" under *some* specific parameters, we need a counterexample to its necessity. Indeed, take two symmetric countries ($K = 2$, $u_i = u_j = u$) and utility $u(x) = 0.5\ln(x + 1) + \underline{m}x$ that violates Assumption 2*. One can check that equilibria are absent under parameters ($\underline{m} = 1$, $c_1 = c_2 = 1$, $\tau = 1$, $l_1 = l_2 = 1$, $f_1 = f_2 = 5$), but exist under other parameters, e.g., ($\underline{m} = 1$, $c_1 = c_2 = 1$, $\tau = 1$, $l_1 = l_2 = 1$, $f_1 = f_2 = 0.5$). This completes the proof. **Q.E.D.**

3.3 The uniqueness of a trade equilibrium with two countries

In this subsection we consider only two countries, being unable to prove uniqueness in the more general case. Here we reformulate our equilibrium in terms of market aggregators $\lambda_j = \mu_j/w_j$ instead of variables μ_j used previously. The related equilibrium $(w_1, w_2, \lambda_1, \lambda_2)$ in Krugman's model with two countries, defined in (16)–(17), can be reformulated as the following system:

$$\frac{l_1 \frac{\mathring{r}_2(\lambda_2 w_1 c_1 \tau_{12}) l_2}{\lambda_2}}{f_1 + c_1 \mathring{x}_1(\lambda_1 w_1 c_1) l_1 + c_1 \tau_{12} \mathring{x}_2(\lambda_2 w_1 c_1 \tau_{12}) l_2} =$$

$$= \frac{l_2 \frac{\mathring{r}_1(\lambda_1 w_2 c_2 \tau_{21}) l_1}{\lambda_1}}{f_2 + c_2 \mathring{x}_2(\lambda_2 w_2 c_2) l_2 + c_2 \tau_{21} \mathring{x}_1(\lambda_1 w_2 c_2 \tau_{21}) l_1}, \tag{26}$$

$$f_1 = \frac{\mathring{\pi}_1(\lambda_1 w_1 c_1) l_1}{\lambda_1 w_1} + \frac{\mathring{\pi}_2(\lambda_2 w_1 c_1 \tau_{12}) l_2}{\lambda_2 w_1}, \tag{27}$$

$$f_2 = \frac{\mathring{\pi}_2(\lambda_2 w_2 c_2) l_2}{\lambda_2 w_2} + \frac{\mathring{\pi}_1(\lambda_1 w_2 c_2 \tau_{21}) l_1}{\lambda_1 w_2}. \tag{28}$$

Two ZP conditions are reformulated in variables λ, whereas BC is transformed into TB, with the help of ZP conditions. What we need is to show that the solution to this system is unique in the following sense.

[14] See Appendix.

Proposition 4. *Under two trading countries ($K = 2$) and Assumption 1 there is not more than one bundle of sales $(x_{11}, x_{12}, x_{21}, x_{22})$ consistent with the equilibrium system. The aggregators (λ, w) generating these sales—are determined uniquely up to scale when countries do trade ($x_{12} > 0$, $x_{21} > 0$).*

Proof. Suppose the opposite: we have two different equilibria $\left(\hat{w}_1, \hat{w}_2, \hat{\lambda}_1, \hat{\lambda}_2\right)$ and $\left(\widehat{w}_1, \widehat{w}_2, \widehat{\lambda}_1, \widehat{\lambda}_2\right)$.

We normalize the variables of the first equilibrium dividing $\hat{\lambda}_j$ by $\hat{\lambda}_2$ and multiplying both wages $\hat{w}_j$ by $\hat{\lambda}_2$, to get $\left(\tilde{w}_1 = \hat{\lambda}_2 \hat{w}_1,\, \tilde{w}_2 = \hat{\lambda}_2 \hat{w}_2,\, \tilde{\lambda}_1 = \frac{\hat{\lambda}_1}{\hat{\lambda}_2}, \tilde{\lambda}_2 = 1\right)$. This normalized bundle still satisfies the same equilibrium system (26)–(28), as one can see from the equations. Similarly we normalize the second equilibrium as $\left(\widetilde{w}_1 = \widehat{\lambda}_2 \widehat{w}_1, \widetilde{w}_2 = \widehat{\lambda}_2 \widehat{w}_2, \widetilde{\lambda}_1 = \frac{\widehat{\lambda}_1}{\widehat{\lambda}_2}, \widetilde{\lambda}_2 = 1\right)$.

Let $\widetilde{\lambda}_1 \geq \tilde{\lambda}_1$ without loss of generality. We connect these two equilibria by some continuous trajectory and check some sort of monotonicity. We want to introduce such functions ($w_1(t) > 0$, $w_2(t) > 0$, $\lambda_1(t) > 0$, $\lambda_2(t) > 0$) so that aggregators λ_j are connected linearly as $\lambda_1(t) = (1-t)\tilde{\lambda}_1 + t\widetilde{\lambda}_1$, $\lambda_2(t) \equiv 1$, whereas wages are calculated from two equations 27 and 28 for every point $0 \leq t \leq 1$. The beginning of the trajectory is $w_1(0) = \tilde{w}_1$, $w_2(0) = \tilde{w}_2$, and the end is $w_1(1) = \widetilde{w}_1$, $w_2(1) = \widetilde{w}_2$.

We should establish existence and uniqueness of these functions w_j, and their properties. First, we substitute $\lambda_1(t)$ and $\lambda_2(t)$ into 27. We get

$$f_1 = \frac{\pi_1(\lambda_1(t) w_1 c_1) l_1}{\lambda_1(t) w_1} + \frac{\pi_2(w_1 c_1 \tau_{12}) l_2}{w_1} \Rightarrow w_1 f_1 = \frac{\pi_1(\lambda_1(t) w_1 c_1) l_1}{\lambda_1(t)} + \pi_2(w_1 c_1 \tau_{12}) l_2.$$

Here the LHS grows with w_1 while the RHS does not. So, we ascertain that there is at most one solution w_1 for each $\lambda_1(t)$ (and thereby for each t). We know that at $t = 0$ this equation could be resolved with respect to w_1 as $w_1 = \tilde{w}_1$. If $w_1(t)$ were increasing, it would contradict the above equation, because the LHS would increase but the RHS could not (keeping in mind that λ_1 is non-decreasing). Thus, if $w_1(t)$ exists, it must not grow.

Looking again at the same equation in the form

$$f_1 = \frac{\pi_1(\lambda_1(t) w_1(t) c_1) l_1}{\lambda_1(t) w_1(t)} + \frac{\pi_2(w_1(t) c_1 \tau_{12}) l_2}{w_1(t)} =$$
$$= \Pi_1\left(\lambda_1(t) w_1(t) c_1\right) c_1 l_1 + \Pi_2\left(w_1(t) c_1 \tau_{12}\right) c_1 \tau_{12} l_2,$$

we want to establish that $\lambda_1(t) w_1(t)$ *must not decrease*. Remember that $\Pi(c) = \frac{\pi(c)}{c}$ decreases unless it has reached zero. Here we cannot have both summands equal to zero at the same time. If the first summand is positive, then decreasing $\lambda_1(t) w_1(t)$ would increase it, and we would end up with a constant LHS and an increasing RHS (contradiction). In the case when first summand is zero, function $w_1(t)$ must be constant to maintain LHS=RHS. In this case, $\lambda_1(t) w_1(t)$ cannot decrease either, because neither $\lambda_1(t)$ nor $w_1(t)$ decreases. To understand that $w_1(t)$ must exist, observe that it exists for $t = 0$. When $\lambda_1(t)$ grows and term $\Pi_1\left(\lambda_1(t) w_1(t) c_1\right)$ is positive, there exist a decrease in $w_1(t)$ that compensates this shift, the second term not hindering finding such $w_1(t)$. When term $\Pi_1\left(\lambda_1(t) w_1(t) c_1\right) = 0$, growing λ_1 just does not matter, and solution $w_1(t)$ exists anyway. It must be continuous by continuity of all functions involved.

We know at this point that there is unique function $w_1(t)$ that maintains equation 27, and this $w_1(t)$ happens to be continuous and non-increasing, for $0 \leq t \leq 1$.

Consider partial derivatives in t and w_1 of function $F(t, w_1) = \Pi_1\left(\lambda_1(t) w_1 c_1\right) c_1 l_1 + \Pi_2\left(w_1 c_1 \tau_{12}\right) c_1 \tau_{12} l_2 - f_1$. They should exist and be equal to (except for maybe finite number of points)

$$\frac{\partial F(t, w_1)}{\partial t} = -\frac{r_1(\lambda_1(t) w_1 c_1)}{\left(\lambda_1(t) w_1 c_1\right)^2 c_1} \left(\widetilde{\lambda}_1 - \tilde{\lambda}_1\right) l_1,$$

$$\frac{\partial F(t, w_1)}{\partial w_1} = -\frac{r_1(\lambda_1(t) w_1 c_1)}{\left(\lambda_1(t) w_1 c_1\right)^2 c_1} l_1 - \frac{r_2(w_1 c_1 \tau_{12})}{w_1^2 c_1 \tau_{12}} l_2 \neq 0.$$

These functions are continuous around any $(t, w_1(t))$. The term $\frac{\partial F(t, w_1)}{\partial w_1}$ cannot be zero at any point $(t, w_1(t))$ as it is zero iff $\mathring{x}_1\left(\lambda_1(t) w_1(t) c_1\right) = 0$ and $\mathring{x}_2\left(w_1(t) c_1 \tau_{12}\right) = 0$ together, which cannot hold. Thus, the implicit function theorem states that $w_1(t)$ is *differentiable* on $[0, 1]$ except for maybe a finite number of points.

We proceed to the equation 28:

$$f_2 = \frac{\pi_2(w_2 c_2) l_2}{w_2} + \frac{\pi_1(\lambda_1(t) w_2 c_2 \tau_{21}) l_1}{\lambda_1(t) w_2} \Rightarrow w_2 f_2 = \pi_2(w_2 c_2) l_2 + \frac{\pi_1(\lambda_1(t) w_2 c_2 \tau_{21}) l_1}{\lambda_1(t)}.$$

We see that $w_2(t)$ must not increase as $\lambda_1(t)$ increases, because the LHS in this case would increase and the RHS would not. We also note that there could not be two solutions $w_2(t)$ for any t. Looking at

$$f_2 = \frac{\pi_2(w_2(t)c_2)l_2}{w_2(t)} + \frac{\pi_1(\lambda_1(t)w_2(t)c_2\tau_{21})l_1}{\lambda_1(t)w_2(t)}$$

we conclude (similarly to the w_1 case) that $\lambda_1(t)w_2(t)$ is non-decreasing and that $w_2(t)$ exists for all $0 < t < 1$. We again establish that $w_2(t)$ is differentiable for all t except for maybe a finite number of points.

Having ascertained the properties of w_j, we note that $x_{12}(t) := \mathring{x}_2\left(w_1(t)c_1\tau_{12}\right)$ and $x_{21}(t) := \mathring{x}_1\left(\lambda_1(t)w_2(t)c_2\tau_{21}\right)$ must be differentiable everywhere except for maybe a finite number of points.

We introduce notations

$$x_{11}(t) := \mathring{x}_1\left(\lambda_1(t)w_1(t)c_1\right),$$

$$x_{12}(t) := \mathring{x}_2\left(\lambda_2(t)w_1(t)c_1\tau_{12}\right) = \mathring{x}_2\left(w_1(t)c_1\tau_{12}\right),$$

$$x_{22}(t) := \mathring{x}_2\left(\lambda_2(t)w_2(t)c_2\right) = \mathring{x}_2\left(w_2(t)c_2\right),$$

$$x_{21}(t) := \mathring{x}_1\left(\lambda_1(t)w_2(t)c_2\tau_{21}\right),$$
$$r_{12}(t) := \mathring{r}_2\left(w_1(t)c_1\tau_{12}\right),$$
$$r_{21}(t) := \mathring{r}_1\left(\lambda_1(t)w_2(t)c_2\tau_{21}\right).$$

Now we are going to check how the remaining, unused equation (26) changes on our trajectory. Subtract the LHS of 26 at $t = 0$ from a similar LHS at point $t = 1$:

$$\frac{l_1\frac{\mathring{r}_2(\lambda_2(1)w_1(1)c_1\tau_{12})l_2}{\lambda_2(1)}}{f_1 + c_1\mathring{x}_1\left(\lambda_1(1)w_1(1)c_1\right)l_1 + c_1\tau_{12}\mathring{x}_2\left(\lambda_2(1)w_1(1)c_1\tau_{12}\right)l_2} -$$

$$-\frac{l_1\frac{\mathring{r}_2(\lambda_2(0)w_1(0)c_1\tau_{12})l_2}{\lambda_2(0)}}{f_1 + c_1\mathring{x}_1\left(\lambda_1(0)w_1(0)c_1\right)l_1 + c_1\tau_{12}\mathring{x}_2\left(\lambda_2(0)w_1(0)c_1\tau_{12}\right)l_2} =$$

$$= \frac{l_1\mathring{r}_2\left(w_1(1)c_1\tau_{12}\right)l_2}{f_1 + c_1\mathring{x}_1\left(\lambda_1(1)w_1(1)c_1\right)l_1 + c_1\tau_{12}\mathring{x}_2\left(w_1(1)c_1\tau_{12}\right)l_2} -$$

$$-\frac{l_1\mathring{r}_2\left(w_1(0)c_1\tau_{12}\right)l_2}{f_1 + c_1\mathring{x}_1\left(\lambda_1(0)w_1(0)c_1\right)l_1 + c_1\tau_{12}\mathring{x}_2\left(w_1(0)c_1\tau_{12}\right)l_2} \geq$$

$$\geq \frac{l_1\mathring{r}_2\left(w_1(1)c_1\tau_{12}\right)l_2}{f_1 + c_1\mathring{x}_1\left(\lambda_1(0)w_1(0)c_1\right)l_1 + c_1\tau_{12}\mathring{x}_2\left(w_1(1)c_1\tau_{12}\right)l_2} -$$

$$-\frac{l_1\mathring{r}_2\left(w_1(0)c_1\tau_{12}\right)l_2}{f_1 + c_1\mathring{x}_1\left(\lambda_1(0)w_1(0)c_1\right)l_1 + c_1\tau_{12}\mathring{x}_2\left(w_1(0)c_1\tau_{12}\right)l_2} =$$

$$= \int_0^1 \left(\frac{l_1u_2'(x_{12}(t))x_{12}(t)l_2}{f_1 + c_1\tilde{x}_{11}l_1 + c_1\tau_{12}x_{12}(t)l_2}\right)_t' dt =$$

$$= l_1l_2\int_0^1 \frac{\left(u_2''(x_{12}(t))x_{12}(t) + u_2'(x_{12}(t))\right)\left(f_1 + c_1\tilde{x}_{11}l_1 + c_1\tau_{12}x_{12}(t)l_2\right)}{\left(f_1 + c_1\tilde{x}_{11}l_1 + c_1\tau_{12}x_{12}(t)l_2\right)^2} \cdot \frac{dx_{12}(t)}{dt}dt -$$

$$-l_1l_2\int_0^1 \frac{c_1\tau_{12}l_2\left(u_2'(x_{12}(t))x_{12}(t)\right)}{\left(f_1 + c_1\tilde{x}_{11}l_1 + c_1\tau_{12}x_{12}(t)l_2\right)^2} \cdot \frac{dx_{12}(t)}{dt}dt =$$

$$= l_1l_2\int_0^1 \frac{c_1\tau_{12}\left(w_1(t)\left(f_1 + c_1\tilde{x}_{11}l_1 + c_1\tau_{12}x_{12}(t)l_2\right) - r_{12}(t)l_2\right)}{\left(f_1 + c_1\tilde{x}_{11}l_1 + c_1\tau_{12}x_{12}(t)l_2\right)^2}\frac{dx_{12}(t)}{dt}dt.$$

Since $w_1(t)\left(f_1 + c_1\tilde{x}_1l_1 + c_1\tau_{12}x_{12}(t)l_2\right) - r_{12}(t)l_2 \geq 0$ and $\frac{dx_{12}(t)}{dt} \geq 0$ the final and initial expressions are both non-negative.

Substract now the RHS of 26 at $t=0$ from a similar RHS at $t=1$:

$$\frac{l_2 \frac{\mathring{r}_1(\lambda_1(1) w_2(1) c_2 \tau_{21}) l_1}{\lambda_1(1)}}{f_2 + c_2 \mathring{x}_2 \left(\lambda_2(1) w_2(1) c_2\right) l_2 + c_2 \tau_{21} \mathring{x}_1 \left(\lambda_1(1) w_2(1) c_2 \tau_{21}\right) l_1} -$$

$$-\frac{l_2 \frac{\mathring{r}_1(\lambda_1(0) w_2(0) c_2 \tau_{21}) l_1}{\lambda_1(0)}}{f_2 + c_2 \mathring{x}_2 \left(\lambda_2(0) w_2(0) c_2\right) l_2 + c_2 \tau_{21} \mathring{x}_1 \left(\lambda_1(0) w_2(0) c_2 \tau_{21}\right) l_1} =$$

$$= \frac{l_2 \frac{\mathring{r}_1(\lambda_1(1) w_2(1) c_2 \tau_{21}) l_1}{\lambda_1(1)}}{f_2 + c_2 \mathring{x}_2 \left(w_2(1) c_2\right) l_2 + c_2 \tau_{21} \mathring{x}_1 \left(\lambda_1(1) w_2(1) c_2 \tau_{21}\right) l_1} -$$

$$-\frac{l_2 \frac{\mathring{r}_1(\lambda_1(0) w_2(0) c_2 \tau_{21}) l_1}{\lambda_1(0)}}{f_2 + c_2 \mathring{x}_2 \left(w_2(0) c_2\right) l_2 + c_2 \tau_{21} \mathring{x}_1 \left(\lambda_1(0) w_2(0) c_2 \tau_{21}\right) l_1} \leq$$

$$\leq \frac{l_2 \frac{\mathring{r}_1(\lambda_1(1) w_2(1) c_2 \tau_{21}) l_1}{\lambda_1(1)}}{f_2 + c_2 \mathring{x}_2 \left(w_2(0) c_2\right) l_2 + c_2 \tau_{21} \mathring{x}_1 \left(\lambda_1(1) w_2(1) c_2 \tau_{21}\right) l_1} -$$

$$-\frac{l_2 \frac{\mathring{r}_1(\lambda_1(0) w_2(0) c_2 \tau_{21}) l_1}{\lambda_1(0)}}{f_2 + c_2 \mathring{x}_2 \left(w_2(0) c_2\right) l_2 + c_2 \tau_{21} \mathring{x}_1 \left(\lambda_1(0) w_2(0) c_2 \tau_{21}\right) l_1} =$$

$$= \int_0^1 \left(\frac{1}{\lambda_1(t)} \frac{l_2 u_1'(x_{21}(t)) x_{21}(t) l_1}{f_2 + c_2 \tilde{x}_{22} l_2 + c_2 \tau_{21} x_{21}(t) l_1} \right)_t' dt =$$

$$= l_1 l_2 \int_0^1 -\frac{1}{\left(\lambda_1(t)\right)^2} \left(\widetilde{\lambda}_1 - \tilde{\lambda}_1 \right) \frac{u_1'(x_{21}(t)) x_{21}(t)}{f_2 + c_2 \tilde{x}_{22} l_2 + c_2 \tau_{21} x_{21}(t) l_1} dt +$$

$$+ l_1 l_2 \int_0^1 \frac{1}{\lambda_1(t)} \frac{\left(u_1''(x_{21}(t)) x_{21}(t) + u_1'(x_{21}(t))\right) \left(f_2 + c_2 \tilde{x}_{22} l_2 + c_2 \tau_{21} x_{21}(t) l_1\right)}{\left(f_2 + c_2 \tilde{x}_{22} l_2 + c_2 \tau_{21} x_{21}(t) l_1\right)^2} \cdot \frac{dx_{21}(t)}{dt} dt -$$

$$- l_1 l_2 \int_0^1 \frac{1}{\lambda_1(t)} \frac{c_2 \tau_{21} l_1 u_1'(x_{21}(t)) x_{21}(t)}{\left(f_2 + c_2 \tilde{x}_{22} l_2 + c_2 \tau_{21} x_{21}(t) l_1\right)^2} \cdot \frac{dx_{21}(t)}{dt} dt =$$

$$= l_1 l_2 \int_0^1 -\frac{1}{\left(\lambda_1(t)\right)^2} \left(\widetilde{\lambda}_1 - \tilde{\lambda}_1 \right) \frac{u_1'(x_{21}(t)) x_{21}(t)}{f_2 + c_2 \tilde{x}_{22} l_2 + c_2 \tau_{21} x_{21}(t) l_1} dt +$$

$$+ l_1 l_2 \int_0^1 \frac{1}{\lambda_1(t)} \frac{c_2 \tau_{21} \left(\lambda_1(t) w_2(t) \left(f_2 + c_2 \tilde{x}_{22} l_2 + c_2 \tau_{21} x_{21}(t) l_1\right) - r_{21}(t) l_1\right)}{\left(f_2 + c_2 \tilde{x}_{22} l_2 + c_2 \tau_{21} x_{21}(t) l_1\right)^2} \frac{dx_{21}(t)}{dt} dt$$

As $\lambda_1(t) w_2(t) \left(f_2 + c_2 \tilde{x}_{22} l_2 + c_2 \tau_{21} x_{21}(t) l_1\right) - r_{21}(t) l_1 \geq 0$ and $\frac{dx_{21}(t)}{dt} \leq 0$ the second summand obtained is non-positive. First summand is also non-positive, moreover, it is negative unless $\widetilde{\lambda}_1 - \widetilde{\lambda}_1 = 0$ or $\frac{u_1'(x_{21}(t)) x_{21}(t)}{f_2 + c_2 \tilde{x}_{22} l_2 + c_2 \tau_{21} x_{21}(t) l_1} = 0$ almost everywhere. This second equality is equivalent to $x_{21}(t) = 0$ almost everywhere. As $x_{21}(t)$ is continuous, this is only possible if $x_{21}(t) \equiv 0$.

Thus, our suggestion about two equilibria leads to the conclusion that the ends of our trajectory coincide in the sense $\tilde{\lambda}_1 = \tilde{\lambda}_1$ or $x_{21}(t) \equiv 0$. The first option trivially leads to coinciding equilibria $\tilde{\cdot}$ and $\widetilde{\cdot}$, because wages $\tilde{w}_j = \widetilde{w}_j$ are uniquely determined from ZP conditions. The second option (no trade) also trivially leads to two identical equalibria: $\tilde{x}_{21} \equiv x_{21}(0) = x_{21}(1) = 0$ implies $x_{12}(0) = x_{12}(1) = 0$. Then, the remaining variables x_{11} and x_{22} are determined *uniquely* by the ZP condition, which exploits the same product $(\lambda_j w_j)$ at the beginning and at the end of our trajectory. Thus, the assumption of two equilibria leads to the conclusion that they are essentially coinciding. **Q.E.D**.

4 Bertoletti-Etro trade model: the existence of equilibria

Indirectly additive utilities were suggested in [?] for modeling *income effects* in trade. This method makes the consumer's expenditure function, instead of the utility function, the main primitive of the model. This interesting and promising way of modeling is extended to heterogeneous firms in (Bertoletti et al., 2019), where the model is calibrated on data. However, the *existence* of equilibria remains unknown for both versions of the model: homogeneous and heterogeneous. This section fills this lacuna for the homogeneous case. Namely, we apply our methodology of "attainable profit functions" to the initial trade model proposed in [?] (we generalize the [?] setup by including more than two trading countries).

4.1 Varieties, consumers, and demand

As in Krugman's model, the world consists of K countries, each having its specific costs and specific population l_k of consumers (identical within the country). In each country $k \leq K$, the endogenous mass N_k of firms is homogeneous within the country. Each firm produces its specific variety, facing fixed and marginal costs of production, plus iceberg transportation costs. Consumers enjoy consumption of all varieties produced in the world. Each consumer has a labor endowment $e_k > 0$ and maximizes some implicit (hidden from us) utility function U_k.

The only difference from Krugman's model is that in the Bertoletti-Etro setup, the utility maximization is presented *indirectly*; we observe only the result of this optimization. Namely, the consumer's preferences in country j are represented by a symmetric and additively separable *indirect utility function* of the following form:

$$V_j \equiv \sum_{k=1}^{K} \int_0^{N_k} v_j\left(\frac{p_{\omega kj}}{w_j e_j}\right) d\omega.$$

Here $p_{\omega kj}$ is the price of variety ω of country k sold in country j. Function $v_j(\cdot)$ is the indirect sub-utility function of each consumer in country j. This function is the main primitive of this model. The argument of $v_j(\cdot)$ is the "real" price, i.e., the nominal price divided by income:

$$s_{\omega kj} := \frac{p_{\omega kj}}{w_j e_j}. \tag{29}$$

As in [?], we impose the following widely accepted (indirect) assumptions on preferences.

Assumption 3. In any country j, its indirect sub-utility function satisfies five conditions:
(i) $v_j(\cdot) : (0, +\infty) \to \mathbb{R}$ is *thrice differentiable* on $(0, \bar{s}_j)$, where $\bar{s}_j > 0$ is either finite or infinite real *choke price*;
(ii) v_j *vanishes* at the choke price, together with its derivative:

$$\lim_{s \to \bar{s}_j^-} v_j(s) = 0, \quad \lim_{s \to \bar{s}_j^-} v_j'(s) = 0. \tag{30}$$

(iii) v_j *becomes zero* above the choke price (when $\bar{s}_j$ is finite)[15]:

$$v_j(s) \equiv 0 \ \forall s \geq \bar{s}_j; \tag{31}$$

(iv) v_j is *strictly convex* below the choke price:

$$v''_j(s) > 0 \ \ \forall s \in (0, \bar{s}_j); \tag{32}$$

(v) v_j has a moderately convex derivative (below the choke price), in the sense:

$$2(v''_j(s))^2 - v_j'(s) v_j'''(s) > 0 \quad \forall s \in (0, \bar{s}_j). \tag{33}$$

Among these conditions, (iv) (convexity) means that an initial increase in price has more impact on expenditures than a subsequent increase, this assumption provides increasing marginal revenue for any firm. Assumptions (ii), (iii) indicate that $\bar{s}_j$ is indeed a choke price, i.e. such a value that a consumer ceases consuming a variety as soon as its price reaches $\bar{s}_j$. Moreover, any price fluctuations become almost immaterial near the choke price. Assumptions (ii) and (iv) together ensure that v_j is *decreasing* below the choke price[16]:

$$v_j'(s) < 0 \ \ \forall s \in (0, \bar{s}_j). \tag{34}$$

Finally, (v) is a technical assumption, which ensures that producers' profit functions derived from v become concave or at least single-peaked; otherwise modeling competition becomes quite tedious.

We are now in a position to derive the demand functions. Using the Roy identity, we get the following individual demand function for variety ω of country k that any consumer from country j demonstrates:

$$x_{\omega kj}\left(s_{\omega kj}\right) = \frac{-v_j'\left(s_{\omega kj}\right)}{\mu_j}, \ \text{ where} \tag{35}$$

[15] $\lim_{s \to 0^+} \frac{v_j(\bar{s}_j + s) - v_j(\bar{s}_j)}{s} = \lim_{s \to 0^+} \frac{v_j(\bar{s}_j + s)}{s} = \lim_{s \to 0^+} -\frac{v_j'(\bar{s}_j + s)}{1} = 0$ due to L'HÃŽpital's rule, so $v_j'(s) \equiv 0$ for $s \geq \bar{s}_j$

[16] It follows then that $v_j(s) > 0$ for $s : 0 < s < \bar{s}_j$

$$\mu_j := \sum_{k=1}^{K} \int_0^{N_k} s_{\omega kj} \cdot \big(-v_j'\big(s_{\omega kj}\big)\big)\, d\omega. \tag{36}$$

Here $\mu_j > 0$ is a price aggregator in country j. Intending to rely on symmetry and drop index ω, we note that demand functions are *the same* in regard to varieties and producers' profit functions are single-peaked. Other conditions, including costs, that producers from one country face, are identical. So, producers within any country should behave *identically.* Thus, all firms from any country k would set *the same* prices p_{kj} for consumers from country j. Accordingly, all consumers from country j buy *the same* amount x_{kj} of all varieties produced in country k. Therefore, we drop from now on index ω of a specific firm, and formulate the symmetric demand function as:

$$\boldsymbol{x}_{kj}(s_{kj}, \mu_j) \equiv x_{kj} = \frac{-v_j'(s_{kj})}{\mu_j}. \tag{37}$$

4.2 Producers and attainable profits

We use the above demand function, as in Krugman's setup, for any firm in any country j and formulate the following profit function:

$$\text{Profit}_j := \sum_{k=1}^{K} (p_{jk}\boldsymbol{x}_{jk} l_k - w_j c_j \tau_{jk} \boldsymbol{x}_{jk} l_k) - w_j f_j,$$

where p_{jk} is the price for country k, and function $\boldsymbol{x}_{jk} = \boldsymbol{x}_{jk}(s_{jk}, \mu_j)$ is the individual demand in country k for any variety from country j. Further, we plug the demand (37) into this profit function:

$$\text{Profit}_j = \sum_{k=1}^{K} \left(\frac{-(p_{jk} - w_j c_j \tau_{jk}) v_k'\big(\frac{p_{jk}}{w_k e_k}\big) l_k}{\mu_k} \right) - w_j f_j =$$

$$= \sum_{k=1}^{K} \left(\frac{-\Big(s_{jk} - \frac{w_j}{e_k w_k} c_j \tau_{jk}\Big) v_k'(s_{jk}) l_k e_k w_k}{\mu_k} \right) - w_j f_j \;\rightarrow\; \max_{\{s_{j\cdot}\} \geq 0}. \tag{38}$$

As usual, producers choose prices p_{jk} (or, equivalently, real prices s_{jk}) to maximize their profits. We start by solving this maximization problem to derive any firm's best-response function. From this optimal pricing rule we shall derive the "attainable profit" function.

We proceed as in Section 2.2, fixing here index k (related to *any* destination country of producer j). We again define composite (real) marginal cost $\beta_{jk} := \frac{w_j}{e_k w_k} c_j \tau_{jk}$. Next, we introduce the auxiliary "elementary revenue" and "elementary profit" functions (this auxiliary profit from sales in country k will turn into real profit after multiplication by $1/\mu_k$):

$$R_k(s) := -v_k'(s)s, \;\; s > 0,$$
$$Pr_k(s, \beta) := R_k(s) - (-v_k'(s) \cdot \beta), \;\; s > 0, \; \beta > 0.$$

Obviously, both of these functions are continuous on their domains ($s > 0$, $\beta > 0$).

Now, plugging all *elementary* profit functions operating in countries $k = 1, ..., K$ into expression (38) and adding multiplier $1/\mu_k$, we reformulate the *composite* Profit_j of any producer from any country j:

$$\text{Profit}_j = \sum_{k=1}^{K} \frac{Pr_k\Big(s_{jk}, \frac{w_j}{e_k w_k} c_j \tau_{jk}\Big) l_k e_k w_k}{\mu_k} - w_j f_j \;\rightarrow\; \max_{\{s_{j\cdot}\} \geq 0}.$$

We observe that this additive function can be maximized for each variable s_{jk} separately. Taking one elementary profit function $Pr_k(s, \beta)$, we maximize it in the standard way using assumption (v). We equate the derivative with respect to s on $(0, \bar{s}_k)$ to zero: $v_k''(s)s + v_k'(s) - v_k''(s) \cdot \beta = 0 \iff$

$$\beta_k(s) := \beta = s + \frac{v_k'(s)}{v_k''(s)}.$$

The obtained function $\beta_k(s)$ means "the cost that generates the real price s". This function is continuous and increasing on $(0, \bar{s}_k)$ due to assumption (v) (33) (i.e., higher costs correspond to higher optimal real prices). This function $\beta_k(\cdot)$ grows from the lower limit

$$\underline{\beta}_k = \lim_{s \to 0_+} s + \frac{v_k'(s)}{v_k''(s)} \leq 0$$

to the upper limit $\overline{\beta}_k = \lim_{s\to\bar{s}_-} s + \frac{v_k'(s)}{v_k''(s)} > 0$[17]. By monotonicity, $\beta_k(\cdot)$ should have an inverse function

$$\mathring{s}_k(\beta) : (\underline{\beta}_k, \overline{\beta}_k) \to (0, \bar{s}_k),$$

which is also continuous and increasing (the optimal real price is higher when its generating cost is higher). In case of the negative lower bound $\underline{\beta}_k < 0$, we exclude interval $(\underline{\beta}_k, 0]$ (unneeded for our analysis) from the domain of $\mathring{s}_k(\cdot)$. For the upper bound of $\mathring{s}_k(\cdot)$ domain, two cases are possible: bound $\overline{\beta}_k$ can be either finite or infinite. If finite, we extend $\mathring{s}_k(\cdot)$ by setting the maximal real price as $\mathring{s}_k(\beta) \equiv \bar{s}_k$ for higher arguments $\beta \geq \overline{\beta}_k$ (if infinite such an extension is not needed).

Note that the resulting function $\mathring{s}_k(\cdot) : (0, +\infty) \to R_+$ is continuous, non-decreasing, and positive on $(0, +\infty)$. Since it is built from maximization, a trivial observation is that $\mathring{s}_k(\beta) \in \underset{0<s}{\arg\max}\, Pr_k(s, \beta) \;\; \forall \beta > 0$. It allows us to treat $\mathring{s}_k(\cdot)$ as the best-response function, which yields a profit maximizing relative price for each positive composite cost $\beta > 0$.[18] Slightly abusing notation (dropping arguments of function $\mathring{s}_k$ for brevity), we denote the values s_{jk} of our optimal pricing functions $\mathring{s}_k$ as

$$s_{jk} := \mathring{s}_k\Big(\frac{w_j}{e_k w_k} c_j \tau_{jk}\Big) = \mathring{s}_k(\beta_{jk}). \tag{39}$$

Following the reasoning in 2.2, we insert maximizers $\mathring{s}_k(\beta)$ into $R_k(s)$ and $Pr_k(s, \beta)$, to define the "attainable elementary revenue" and "attainable elementary profit" functions, dependent on costs:

$$\mathring{r}_k(\beta) := R_k(\mathring{s}_k(\beta)) \geq 0, \;\; \mathring{\pi}_k(\beta) := Pr_k(\mathring{s}_k(\beta), \beta).$$

Here function $\mathring{\pi}_k(\beta)$ is positive and decreasing on $(0, \overline{\beta}_k)$, whereas $\mathring{\pi}_k(\beta) = 0$ for high costs $\beta \geq \overline{\beta}_k$ (when $\overline{\beta}_k$ is finite).[19] We also define one more, "normalized" attainable profit function $\mathring{\Pi}_k(\beta)$:

$$\mathring{\Pi}_k(\beta) := \frac{\mathring{\pi}_k(\beta)}{\beta}, \;\; \mathring{\Pi}_k(\cdot) : (0, +\infty) \to (0, +\infty).$$

This new function is continuous and decreasing on $(0, \overline{\beta}_k)$. Therefore, there is a continuous decreasing inverse function $\Pi_k^{-1} : (0, +\infty) \leftrightarrow (0, \overline{\beta}_k)$. We call functions $\mathring{s}_k(\beta)$, $\mathring{r}_k(\beta)$, $\mathring{\pi}_k(\beta)$, $\mathring{\Pi}_k(\beta)$ *the attainable functions*. For brevity of future narration, we introduce the following simplified notation (without explicit arguments) for our optimal functions of outputs, revenues, and profits:

$$x_{jk} := \frac{-v_k'(s_{jk})}{\mu_k} \geq 0, \; r_{jk} := \mathring{r}_k(\beta_{jk}), \; \pi_{jk} := \mathring{\pi}_k(\beta_{jk}), \; \Pi_{jk} := \mathring{\Pi}_k(\beta_{jk}).$$

Now we use the elementary *attainable* functions to formulate the maximal composite profit that producers from each country j can obtain:

$$\max_{s_{j\cdot} \geq 0} \text{Profit}_j = \max_{s_{j\cdot} \geq 0} \sum_{k=1}^{K} \frac{Pr_k\Big(s_{jk}, \frac{w_j}{e_k w_k} c_j \tau_{jk}\Big) l_k e_k w_k}{\mu_k} - w_j f_j = \tag{40}$$

$$= \sum_{k=1}^{K} \frac{\mathring{\pi}_k\Big(\frac{w_j}{e_k w_k} c_j \tau_{jk}\Big) l_k e_k w_k}{\mu_k} - w_j f_j = w_j f_j \Big(\sum_{k=1}^{K} \frac{\Pi_{jk}}{\mu_k} c_j \tau_{jk} \frac{l_k}{f_j} - 1\Big).$$

Now we can formulate the equilibrium equations in the Bertoletti-Etro model in terms of *attainable* functions.

---

[17]See Claim (10)

[18]Actually, if $\overline{\beta}_k$ is finite, there are multiple maximizers for $\beta \geq \overline{\beta}_k$: $\underset{0<s}{\arg\max}\, Pr_k(s, \beta) = \{\tilde{s} : \tilde{s} \geq \bar{s}_k\}$. Out of them we pick the magnitude that secures continuity of $\mathring{s}_k(\beta)$, this property being important for our proof. This choice does not diminish generality of our future reformulation of equilibrium equations as all these maximizers ensure $v_k'(\tilde{s}) = 0$ while s-variables enter these equations only in the form of $v_j'(s)s$ or $v_j'(s)$.

[19]See Claim (11)

4.3 Equilibrium equations

We now specify the system of equations that determine equilibria under indirectly additive preferences, incorporating the optimal pricing rule through the use of *attainable* functions. First, we formulate the ZP condition $\sum_{k=1}^{K}(p_{jk}x_{jk}l_k - w_j c_j \tau_{jk} x_{jk} l_k) - w_j f_j = 0$.

Second, we reformulate the BC: $w_j e_j l_j = \sum_{k=1}^{K} \frac{e_k l_k}{f+\sum_{i=1}^{K} c_k \tau_{ki} x_{ki} l_i} p_{kj} x_{kj} l_j$.

Combining these equations with our "attainable" functions, we come to the following "reduced system" of equilibrium equations, to be used further:

$$[\mathbf{ZP}]: \quad w_j = w_j \sum_{k=1}^{K} \frac{\mathring{\Pi}_k\left(\frac{w_j}{e_k w_k} c_j \tau_{jk}\right)}{\mu_k} c_j \tau_{jk} \frac{l_k}{f_j} \quad \forall j > 1, \; w_1 = 1, \tag{41}$$

$$[\mathbf{BC}]: \quad \mu_j = \sum_{k=1}^{K} \frac{\mathring{r}_k\left(\frac{w_k}{e_j w_j} c_k \tau_{kj}\right) e_k l_k}{f_k + c_k \sum_{i=1}^{K} \tau_{ki} \frac{-v_k'\left(\mathring{s}_k\left(\frac{w_k}{e_i w_i} c_k \tau_{ki}\right)\right)}{\mu_k} l_i} \quad \forall j. \tag{42}$$

Our reformulation enables us to seek for equilibria sequentially. First one can find a bundle of auxiliary variables $\{\mu_j, w_j\}_{1 \le j \le K}$ that satisfies equations (41) and (42); then this bundle can be translated into some equilibrium prices and consumptions. Namely, the main variables of any equilibrium could be found through (39), (29), (35). Consumption should satisfy two more conditions, as in Krugman's model, LM and TB. **L**abour **M**arket clearing asserts that labour is fully employed:

$$[\mathbf{LM}]: \quad l_j = N_j \cdot (f_j + \sum_{k=1}^{K} c_j \tau_{jk} x_{jk} l_k) \; \forall j. \tag{43}$$

This condition gives us number N_j when we know sales x.

Trade **B**alance implies that all imported goods worth as much as all exported goods:

$$[\mathbf{TB}]: \quad N_j \sum_{k=1}^{K} p_{jk} x_{jk} l_k = \sum_{i=1}^{K} N_i p_{ij} x_{ij} l_j \; \forall j. \tag{44}$$

As in Krugman's model, TB follows from BC when other equilibrium conditions hold.

Definition. *Trade equilibrium* in the Bertoletti-Etro model is a bundle

$$\left\{\mu_j, w_j, N_j, (x_{jk})_{1 \le k \le K}, (p_{jk})_{1 \le k \le K}\right\}_{1 \le j \le K} \in \mathbb{R}_+^{3K} \times \mathbb{R}_+^{2K^2} \tag{45}$$

that includes price-aggregates, wages, number of firms, sales, and prices, and satisfies: (i) utility-maximization (35); (ii) profit-maximization (39),(29), (iii) the ZP condition (41); (iv) LM clearing (43); v) BC (42); and (vi) TB (44).

We are now in a position to impose one more condition (*vi*) on sub-utilities and parameters of this model, namely,

$$\text{(vi)} \;\; \bar{s}_j > \frac{c_j}{e_j} \, \forall j. \tag{46}$$

This inequality makes economies "productive", saying that the benefits outweigh the costs in each country taken separately.

As in Krugman's model, we have normalized $w_1 \equiv 1$, otherwise we could get an inconvenient system of $2K-1$ independent equations in $2K$ unknowns $\{\mu_{j \le K}, w_{1 \le j \le K}\}$ with an indefinite level of w_j.

4.4 The existence of equilibrium in the Bertoletti-Etro model

We are now going to apply the Brouwer fixed-point theorem to equations defined above to prove the existence of equilibria.

Proposition 5. *In the Bertoletti-Etro model there is an equilibrium, defined by* (45)*, for all sets of indirect utility functions* $v_j(\cdot)$ *and all positive parameter values* $\left\{e_j,\ f_j,\ c_j,\ l_j,\ (\tau_{jk})_{1\leq k\leq K}\right\}_{1\leq j\leq K} \gg 0$*, satisfying assumptions (i)-(vi) (*(30)*,* (31)*,* (32)*, (46)).*

Proof. As will be evident later, the peculiarities of this model bring about some new difficulties not encountered in Krugman's model. Nonetheless, we adhere to the scheme of his proof. Namely, we again define a convex compact set Ω of the Euclidean space R^{2K-1}, suitable for a mapping F which operates from Ω into Ω, we find a fixed point, and then show that any fixed point of this mapping is a true equilibrium of the Bertoletti-Etro model.

Let us construct the **Brouwer mapping** and its domain. To begin with, we introduce the following constants:

$$f_m := \min_j f_j,\ c_m := \min_j c_j,\ l_m := \min_j l_j,\ e_m := \min_j e_j,\ \tau_m := \min_j \min_k \tau_{jk} \leq 1, \tag{47}$$

$$f_M := \max_j f_j,\ c_M := \max_j c_j,\ l_{M:} = \max_j l_j,\ e_M := \max_j e_j,\ \tau_M := \max_j \max_k \tau_{jk} \geq 1.$$

As before, we introduce a convex compact set

$$\Omega := [\mu_m, \mu_M]^K \times [\underline{w}, \frac{1}{\underline{w}}]^{K-1}$$

which will serve as the domain of our mapping. Here μ_m, μ_M, $\underline{w}$, $\frac{1}{\underline{w}}$ are constants (not defined so far) that serve as the boundaries of our rectangular compact set. These boundaries play a key role in our proof, since their specific construction will allow us to show that any fixed point of the mapping F is an equilibrium point. Namely, we define the upper bound μ_M as a magnitude that satisfies two conditions:

$$\mu_M = K \max_j \sup_{s>0} \frac{R_j(s) e_M l_M}{f_m - c_m \tau_m l_m \frac{v_j'(s)}{\mu_M}} > 0,$$

$$K \max_j \sup_{s>0} \frac{R_j(s) e_M l_M}{f_m - c_m \tau_m l_m \frac{v_j'(s)}{\mu}} < \mu_M \quad \forall \mu :\ 0 < \mu < \mu_M.$$

Using assumptions (i)–(v), a separate Lemma (2) in Appendix shows that there always exists such a number μ_M.

Further, we define the lower bound μ_m as[20]

$$\mu_m = \frac{c_m l_m}{2 f_M} \min_j \Pi_j \left(\frac{c_j}{e_j}\right) > 0.$$

It is important to establish that $\mu_m < \mu_M$. Suppose that the minimum taken in the definition of μ_m is achieved at some $j = i$. Then we get the following result:

$$\mu_m = \frac{c_m l_m}{2 f_M} \Pi_i \left(\frac{c_i}{e_i}\right) = \frac{e_i l_m}{2 f_M} \frac{c_m}{c_i} \pi_i \left(\frac{c_i}{e_i}\right)$$

$$= \frac{e_i l_m}{2 f_M} \frac{c_m}{c_i} \left(r_i\left(\frac{c_i}{e_i}\right) + \frac{c_i}{e_i} v_i'\left(s_i\left(\frac{c_i}{e_i}\right)\right)\right) \Rightarrow$$

$$\Rightarrow 1 = \frac{c_m}{c_i} \frac{r_i\left(\frac{c_i}{e_i}\right) e_i l_m}{2\mu_m f_M - c_m v_i'\left(s_i\left(\frac{c_i}{e_i}\right)\right) l_m} = \frac{c_m}{c_i} \frac{R_i\left(s_i\left(\frac{c_i}{e_i}\right)\right) e_i l_m}{2\mu_m f_M - c_m v_i'\left(s_i\left(\frac{c_i}{e_i}\right)\right) l_m}.$$

Remember that

$$\mu_M = K \max_j \sup_{s>0} \frac{R_j(s) e_M l_M}{f_m - c_m \tau_m \frac{v_j'(s)}{\mu_M} l_m} \Rightarrow 1 = K \max_j \sup_{s>0} \frac{R_j(s) e_M l_M}{\mu_M f_m - c_m \tau_m v_j'(s) l_m}.$$

Suppose $\mu_m \geq \mu_M$. This would imply:

[20] The positivity of the following minimum is guaranteed by assumption (vi).

$$1 = \frac{c_m \tau_m}{c_i} \frac{R_i\left(s_i\left(\frac{c_i}{e_i}\right)\right) e_i l_m}{2\mu_m f_m - c_m v_i'\left(s_i\left(\frac{c_i}{e_i}\right)\right) l_m} \leq \frac{c_m \tau_m}{c_i} \frac{R_i\left(s_i\left(\frac{c_i}{e_i}\right)\right) e_i l_m}{2\mu_M f_m - c_m v_i'\left(s_i\left(\frac{c_i}{e_i}\right)\right) l_m} <$$

$$< K \max_j \sup_{s>0} \frac{R_j(s) e_M l_M}{\mu_M f_m - c_m \tau_m v_j'(s) l_m} = 1.$$

Obviously, this is a contradiction, and therefore $\mu_m < \mu_M$.

Now we define the bounds on wages w. We set the lower boundary for wages in the following way:

$$\underline{w} := \frac{e_m}{2 c_M \tau_M} \min_j \Pi_j^{-1}\left(\frac{\mu_M f_M}{c_m \tau_m l_m}\right) > 0.$$

Now we must show that $\underline{w} < 1$. We suppose that the minimum above is achieved in some country $j = i$ and reformulate the $\underline{w}$ definition:

$$\underline{w} = \frac{e_m}{2 c_M \tau_M} \Pi_i^{-1}\left(\frac{\mu_M f_M}{c_m \tau_m l_m}\right) \Rightarrow \frac{\mu_M f_M}{c_m \tau_m l_m} = \frac{\pi_i(2\underline{w}\frac{c_M \tau_M}{e_m})}{2\underline{w}\frac{c_M \tau_M}{e_m}} \Rightarrow$$

$$\Rightarrow \underline{w} = \frac{1}{2} \frac{c_m \tau_m}{c_M \tau_M} \frac{R_i\left(s_i\left(2\underline{w}\frac{c_M \tau_M}{e_m}\right)\right) e_m l_m}{\mu_M f_M - c_m \tau_m v_i'\left(s_i\left(2\underline{w}\frac{c_M \tau_M}{e_m}\right)\right) l_m} =$$

$$= \frac{1}{\mu_M} \frac{1}{2} \frac{c_m \tau_m}{c_M \tau_M} \frac{R_i\left(s_i\left(2\underline{w}\frac{c_M \tau_M}{e_m}\right)\right) e_m l_m}{f_M - \frac{c_m \tau_m}{\mu_M} v_i'\left(s_i\left(2w\frac{c_M \tau_M}{e_m}\right)\right) l_m} <$$

$$< \frac{1}{\mu_M} K \max_j \sup_{s>0} \frac{R_j(s) e_M l_M}{f_m - c_m \tau_m \frac{v_j'(s)}{\mu_M} l_m} = 1.$$

Thus, we can conclude that $\underline{w} < 1$ and $\frac{1}{\underline{w}} > \underline{w}$, so, our interval for wages w and for μ is non-empty.

Using our domain Ω, we now can define the required Brouwer mapping $F\left(\{w_{j\neq 1}, \mu_j\}_{1 \leq j \leq K}\right) :=$

$$\boldsymbol{\mu}_j = \max\left(\sum_{k=1}^{K} \frac{r_{kj} e_k l_k}{f_k + c_k \sum_{i=1}^{K} \tau_{ki} x_{ki} l_i}, \mu_m\right) \forall j,$$

$$\boldsymbol{w}_j = \min\left(\max\left(w_j \sum_{k=1}^{K} \frac{\Pi_{jk}}{\mu_k} c_j \tau_{jk} \frac{l_k}{f_j}, \underline{w}\right), \frac{1}{\underline{w}}\right) \forall j > 1.$$

This mapping is continuous by construction and restricted to fit the borders $\mu_m, \underline{w}, \frac{1}{\underline{w}}$. However, we must make sure that it maps Ω into itself, to use the Brouwer fixed point theorem. Obviously, we only need to check for the non-violation of the upper border for $\boldsymbol{\mu}_j\left(\{w_{j\neq 1}, \mu_j\}_{1 \leq j \leq K}\right) \leq \mu_M$:

$$\boldsymbol{\mu}_j = \sum_{k=1}^{K} \frac{r_{kj} e_k l_k}{f_k + c_k \sum_{i=1}^{K} \tau_{ki} x_{ki} l_i} \leq \sum_{k=1}^{K} \frac{r_{kj} e_M l_M}{f_m + c_m \sum_{i=1}^{K} \tau_m x_{ki} l_m} \leq$$

$$\leq \sum_{k=1}^{K} \frac{r_{kj} e_M l_M}{f_m + c_m \tau_m x_{kj} l_m} = \sum_{k=1}^{K} \frac{R_j(s_{kj}) e_M l_M}{f_m + \tau_m c_m \frac{-v_j'(s_{kj})}{\mu_j} l_m} \leq$$

$$\leq K \max_j \sup_{s>0} \frac{R_j(s) e_M l_M}{f_m + c_m \tau_m \frac{-v_j'(s)}{\mu_j} l_m} \leq \mu_M \;\forall 0 < \mu_j \leq \mu_M,\; \forall w_{j\neq 1} > 0 \; j = 1, ..., K.$$

So, we have constructed a mapping F suitable for the application of the Brouwer fixed point theorem. Thus, this mapping must have at least one fixed point. We again (as in Krugman's model) denote any such fixed point by variables with a hat: $\{\hat{w}_j, \hat{\mu}_j\}_{1 \le j \le K}$, for convenience additionally defining $\hat{w}_1 := 1$. We also shall use the notation

$$\hat{\beta}_{jk} := \frac{\hat{w}_j}{e_k \hat{w}_k} c_j \tau_{jk},\ \hat{s}_{jk} := s_k(\hat{\beta}_{jk}),\ \hat{x}_{jk} := \frac{-v_k'(\hat{s}_{jk})}{\hat{\mu}_k},\ \hat{r}_{jk} := r_k(\hat{\beta}_{jk}),\ \hat{\pi}_{jk} := \pi_k(\hat{\beta}_{jk}),\ \hat{\Pi}_{jk} := \Pi_k(\hat{\beta}_{jk}).$$

We now need to demonstrate that any fixed point is an equilibrium point.

To show that any fixed point of the mapping is an equilibrium of the model, we recall that mapping F is artificially restricted. Its fixed point situated on a boundary of our set Ω could be not a true equilibrium. Therefore we show that a fixed point never lies on the boundaries of Ω, and the existence of an equilibrium existence is proven. Adhering to this strategy, we start by proving that the lower boundary $\underline{w}$ is not reached for any $j > 1$ by stationary wages $\hat{w}_j$. Suppose the opposite: there is such wage $\hat{w}_i = \underline{w}$ for some country $i > 1$:

$$\underline{w} = \hat{w}_i = \max\left(\hat{w}_i \sum_{k=1}^{K} \frac{\hat{\Pi}_{ik}}{\hat{\mu}_k} c_i \tau_{ik} \frac{l_k}{f_i}, \underline{w}\right)\Bigg) \ge \hat{w}_i \sum_{k=1}^{K} \frac{\mathring{\Pi}_k\left(\frac{\hat{w}_i}{e_k \hat{w}_k} c_i \tau_{ik}\right)}{\hat{\mu}_k} c_i \tau_{ik} \frac{l_k}{f_i} \ge$$

$$\ge \hat{w}_i \frac{\mathring{\Pi}_1\left(\frac{\hat{w}_i}{e_1} c_i \tau_{i1}\right)}{\hat{\mu}_i} c_i \tau_{i1} \frac{l_1}{f_i} = \underline{w} \frac{\mathring{\Pi}_1\left(\frac{\underline{w}}{e_1} c_i \tau_{i1}\right)}{\hat{\mu}_i} c_i \tau_{i1} \frac{l_1}{f_i} \ge \underline{w} \frac{\mathring{\Pi}_1\left(\underline{w} \frac{c_M \tau_M}{e_m}\right)}{\mu_M} c_m \tau_m \frac{l_m}{f_M} \Rightarrow$$

$$\Rightarrow \underline{w} \ge \frac{e_m}{c_M \tau_M} \mathring{\Pi}_1^{-1}\left(\frac{\mu_M f_M}{c_m \tau_m l_m}\right) \ge \frac{e_m}{c_M \tau_M} \min_j \mathring{\Pi}_j^{-1}\left(\frac{\mu_M f_M}{c_m \tau_m l_m}\right) = 2\underline{w}.$$

This is a contradiction due to the positivity of $\underline{w}$; therefore, reaching the lower bound $\hat{w}_j = \underline{w}$ for wages is impossible for any country $j > 1$. As for the upper bound, we show that $\hat{w}_j = \frac{1}{\underline{w}}$ is also impossible for any $j > 1$. To this end, we apply Lemma ((1)) from Appendix to our fixed point (this Lemma uses $K-1$ ZP conditions and all BC, to derive the remaining ZP condition, as an inequality). Consider our fixed point $(\hat{w}, \hat{\mu})$ and the related sales/prices in the role of the point $y = \{w, x, p\}$ studied in Lemma 1:

$$y = \left\{\hat{w}_j, (\hat{x}_{jk})_{1 \le k \le K}, (\hat{s}_{jk} \hat{w}_j e_j)_{1 \le k \le K}\right\}_{1 \le j \le K}.$$

The assumptions (inequalities) of this Lemma are satisfied because

$$\hat{w}_j = \min\left(\hat{w}_j \sum_{k=1}^{K} \frac{\hat{\Pi}_{jk}}{\hat{\mu}_k} c_j \tau_{jk} \frac{l_k}{f_j}, \frac{1}{w}\right) \le \sum_{k=1}^{K} \frac{\Pi_k\left(\frac{\hat{w}_j}{e_k \hat{w}_k} c_j \tau_{jk}\right)}{\hat{\mu}_k} c_j \tau_{jk} \frac{l_k}{f_j} \iff$$

$$\iff \sum_{k=1}^{K} (\hat{p}_{jk} \hat{x}_{jk} l_k - \hat{w}_j c_j \tau_{jk} \hat{x}_{jk} l_k) - \hat{w}_j f_j \ge 0 \quad \forall j > 1,$$

$$\hat{\mu}_j = \max\left(\sum_{k=1}^{K} \frac{\hat{r}_{kj} e_k l_k}{f_k + c_k \sum_{i=1}^{K} \tau_{ki} \hat{x}_{ki} l_i}, \mu_m\right) \ge \sum_{k=1}^{K} \frac{\hat{r}_{kj} e_k l_k}{f_k + c_k \sum_{i=1}^{K} \tau_{ki} \hat{x}_{ki} l_i} \iff$$

$$\iff \hat{w}_j e_j l_j \ge \sum_{k=1}^{K} \frac{l_k e_k}{f_k + c_k \sum_{i=1}^{K} \tau_{ki} \hat{x}_{ki} l_i} \hat{p}_{kj} \hat{x}_{kj} l_j \quad \forall j.$$

Thus, Lemma 1 provides the following result:

$$\sum_{k=1}^{K} (\hat{p}_{1k} \hat{x}_{1k} l_k - \hat{w}_1 c_1 \tau_{1k} \hat{x}_{1k} l_k) - \hat{w}_1 f_1 \le 0 \iff 1 \ge \sum_{k=1}^{K} \frac{\Pi_k\left(\frac{1}{e_k \hat{w}_k} c_1 \tau_{1k}\right)}{\hat{\mu}_k} c_1 \tau_{1k} \frac{l_k}{f_1}. \tag{48}$$

Suppose that the upper bound $\hat{w}_i = \frac{1}{\underline{w}}$ is achieved for some $i > 1$:

$$1 \ge \sum_{k=1}^{K} \frac{\Pi_k\left(\frac{1}{e_k \hat{w}_k} c_1 \tau_{1k}\right)}{\hat{\mu}_k} c_1 \tau_{1k} \frac{l_k}{f_1} \ge \frac{\Pi_i\left(\frac{1}{e_i \hat{w}_i} c_1 \tau_{1i}\right)}{\hat{\mu}_i} c_1 \tau_{1i} \frac{l_i}{f_1} = \frac{\Pi_i\left(w \frac{c_1 \tau_{1i}}{e_i}\right)}{\hat{\mu}_i} c_1 \tau_{1i} \frac{l_i}{f_1} \Rightarrow$$

$$\Rightarrow w \geq \frac{e_m}{c_M \tau_M} \Pi_1^{-1}\left(\frac{\mu_M f_M}{c_m \tau_m l_m}\right) \geq \frac{e_m}{c_M \tau_M} \min_j \Pi_j^{-1}\left(\frac{\mu_M f_M}{c_m \tau_m l_m}\right) = 2w.$$

Observing this contradiction, we conclude that $\hat{w}_j = \frac{1}{\underline{w}}$ is impossible for any country $j > 1$. We have established that at the fixed point

$$\hat{w}_j = \hat{w}_j \sum_{k=1}^{K} \frac{\Pi_k\left(\frac{\hat{w}_j}{e_k \hat{w}_k} c_j \tau_{jk}\right)}{\hat{\mu}_k} c_j \tau_{jk} \frac{l_k}{f_j} \quad \forall j > 1.$$

So, together with (48), this implies that

$$\hat{w}_j \geq \hat{w}_j \sum_{k=1}^{K} \frac{\Pi_k\left(\frac{\hat{w}_j}{e_k \hat{w}_k} c_j \tau_{jk}\right)}{\hat{\mu}_k} c_j \tau_{jk} \frac{l_k}{f_j} \quad \forall j.$$

Now we turn to proving that the lower bound $\hat{\mu}_j = \mu_m$ is not attained for any j. Suppose there is some i : $\hat{\mu}_i = \mu_m$. Then

$$\hat{w}_i \geq \hat{w}_i \sum_{k=1}^{K} \frac{\hat{\Pi}_{ik}}{\hat{\mu}_k} c_i \tau_{ik} \frac{l_k}{f_i} = \hat{w}_i \sum_{k=1}^{K} \frac{\Pi_k\left(\frac{\hat{w}_i}{e_k \hat{w}_k} c_i \tau_{ik}\right)}{\hat{\mu}_k} c_i \tau_{ik} \frac{l_k}{f_i} \geq$$

$$\geq \hat{w}_i \frac{\Pi_i\left(\frac{c_i}{e_i}\right)}{\hat{\mu}_i} c_i \frac{l_i}{f_i} \geq \hat{w}_i \frac{\Pi_i\left(\frac{c_i}{e_i}\right)}{\mu_m} c_m \frac{l_m}{f_M} \Rightarrow$$

$$\Rightarrow \mu_m \geq \Pi_i\left(\frac{c_i}{e_i}\right) c_m \frac{l_m}{f_M} \geq \frac{c_m}{f_M} \min_j \Pi_j\left(\frac{c_j}{e_j}\right) l_m = 2\mu_m,$$

which is a contradiction because $\mu_m > 0$ by definition. Thus, equality $\hat{\mu}_j = \mu_m$ is impossible for any country j. Thus, we have shown that none of the fixed points of mapping F lie on any of the boundaries (i.e., that the artificial restriction of the values of mapping F have not played a role in determining its fixed point).

Based on this, we conclude that under the conditions listed, this model of international trade always has at least one equilibrium point in terms of variables (μ, w). The remaining variables (x, p, N) are found directly from by the equations, mentioned in the definition of equilibria. **Q.E.D**.

5 Conclusion

This paper establishes the existence of an equilibrium under very weak assumptions on elementary utility functions for two tractable broad classes of models: directly additive (Krugman's model) or indirectly additive preferences (the Bertoletti-Etro model). Though in general the *sufficient* conditions established are not *necessary*, they appear to be the "weakest" conditions. Moreover, under directly additive preferences we formulate the exact *necessary* conditions for the existence of an equilibrium in the case of symmetric (identical) preferences across countries. In addition, we find sufficient conditions for the *uniqueness* of the equilibrium for two countries. Thereby, we *comprehensively characterize the classes of preferences suitable for modeling international trade* under monopolistic competition.

Our characterization of "almost all" preferences classes suitable for such modeling—can be useful for empirical research. Indeed, for model calibration/identification, an empiricist should freely choose the functional forms of demand that better fit the data. From the purely theoretical side, one achievement of this article is a method for analyzing international trade models through the "attainable revenue" and "attainable profit" functions. We hope that this method will be useful in other models of international trade.

As to extensions, we are successfully working on the application of our method to a Melitz-like setup with heterogeneous firms.

References

6 Appendix

In the proof of Proposition 1 we used the following lemma, assuming $e_k \equiv 1\,\forall k$. In other claims we use this lemma with other constants $e_k > 0$. Essentially, using $K-1$ zero-profit conditions and all budget constraints, this lemma derives the remaining zero-profit condition (as inequalities).

Lemma 1. *Consider some vector of wages, sales, prices*

$$y \equiv \left\{ w_j \geq 0, (x_{jk} \geq 0)_{1 \leq k \leq K}, (p_{jk} \geq 0)_{1 \leq k \leq K} \right\}_{1 \leq j \leq K}.$$

If at y the following $(2K-1)$ inequalities hold:

$$\sum_{k=1}^{K} (p_{jk} x_{jk} l_k - w_j c_j \tau_{jk} x_{jk} l_k) - w_j f_j \geq 0 \;\; \forall j \neq 1,$$

$$w_j e_j l_j \geq \sum_{k=1}^{K} \frac{e_k l_k}{f_k + \sum_{i=1}^{K} c_k \tau_{ki} x_{ki} l_i} p_{kj} x_{kj} l_j \;\; \forall j,$$

then the following inequality must hold at point y:[21]

$$\sum_{k=1}^{K} (p_{1k} x_{1k} l_k - w_1 c_1 \tau_{1k} x_{1k} l_k) - w_1 f_1 \leq 0. \tag{49}$$

If instead of initial inequalities at point y similar equalities hold, then the resulting inequality (49) turns into equality as well.

Proof. Consider the first set of inequalities:

$$\sum_{k=1}^{K} (p_{jk} x_{jk} l_k - w_j c_j \tau_{jk} x_{jk} l_k) - w_j f_j \geq 0 \Rightarrow$$

[21] Note the opposite sign here.

$$\sum_{k=1}^{K} p_{jk}x_{jk}l_k \geq w_j\left(f_j + c_j\sum_{k=1}^{K}\tau_{jk}x_{jk}l_k\right) \Rightarrow N_j\sum_{k=1}^{K} p_{jk}x_{jk}l_k \geq w_jl_je_j.$$

Here we have used the following notation:

$$N_i = \frac{e_il_i}{\left(f_i + \sum_{k=1}^{K} c_i\tau_{ik}x_{ik}l_k\right)}.$$

Putting all these inequalities together, we get inequality

$$\sum_{j=1,\, j\neq i}^{K} N_j\sum_{k=1}^{K} p_{jk}x_{jk}l_k \geq \sum_{j=1,\, j\neq i}^{K} w_jl_je_j. \tag{50}$$

Consider now the second set of inequalities:

$$w_jl_je_j \geq \sum_{k=1}^{K}\frac{e_kl_k}{f_k + \sum_{i=1}^{K} c_k\tau_{ki}x_{ki}l_i}p_{kj}x_{kj}l_j \Rightarrow w_jl_je_j \geq \sum_{k=1}^{K} N_kp_{kj}x_{kj}l_j.$$

Summing up over j, we get inequality

$$\sum_{j=1}^{K} w_jl_je_j \geq \sum_{j=1}^{K}\sum_{k=1}^{K} N_kp_{kj}x_{kj}l_j. \tag{51}$$

Subtracting (50) from (51), we arrive at

$$w_il_ie_i \geq N_i\sum_{k=1}^{K} p_{ik}x_{ik}l_k \Rightarrow w_i(f_i + c_i\sum_{k=1}^{K}\tau_{ik}x_{ik}l_k) \geq \sum_{k=1}^{K} p_{ik}x_{ik}l_k \Rightarrow$$

$$\sum_{k=1}^{K}(p_{ik}x_{ik}l_k - w_ic_i\tau_{ik}x_{ik}l_k) - w_if_i \leq 0,$$

as needed. Finally, it is straightfoward to see that if "$\geq$" and "$\leq$" signs are replaced by "=" sign everywhere, same algerbraic manipulations prove the second statement of the Lemma. **Q.E.D.**

The following lemma is used in the proof for Bertoletti-Etro model.

Lemma 2. *There is always such μ_M that*

$$\mu_M = K\max_j\sup_{s>0}\frac{R_j(s)e_Ml_M}{f_m - c_m\tau_ml_m\frac{v_j'(s)}{\mu_M}},$$

$$K\max_j\sup_{s>0}\frac{R_j(s)e_Ml_M}{f_m - c_m\tau_ml_m\frac{v_j'(s)}{\mu}} < \mu_M \quad \mu : 0 < \mu < \mu_M$$

Proof. Step 1. We set up our tools.
Consider

$$\phi(\mu) := \max_j\sup_{s>0}\frac{-v_j'(s)s}{-v_j'(s) + B\mu},\ B := \frac{f_m}{c_m\tau_ml_m} > 0,\ \mu > 0.$$

Define $g_j(s,\mu) := \frac{-v_j'(s)s}{-v_j'(s)+B\mu}$. Since $\lim_{s\to 0^+} -v_j'(s) > 0$ [22], we know that[23]

$$\lim_{s\to 0^+} g_j(s,\mu) = \lim_{s\to 0^+}\frac{s}{1+\frac{B\mu}{-v_j'(s)}} = 0,$$

[22] Immediately follows from $-v_j'(s) > 0$ and $-v_j''(s) < 0$ in the vicinity of zero

[23] See Claim (9)

$$\lim_{s\to+\infty} g_j(s,\mu) = \frac{\lim_{s\to+\infty} -v_j'(s)s}{\lim_{s\to+\infty} -v_j'(s) + B\mu} = 0.$$

As $g_j(s,\mu)$ are defined for $\mu > 0$, these observations trivially prove that $\arg\max_{s>0} g_j(s,\mu) \subset \mathbb{R}$ exits for any $\mu > 0$ and j. Define then $s_j(\mu) := \max(\arg\max_{s>0} g_j(s,\mu))$. We prove that $g_j(s_j(\mu),\mu)$ are continuous on $\mu > 0$. In order to do that, we introduce functions $\hat{g}_j(s,\mu) : \overline{\mathbb{R}}_+ \times \mathbb{R}_{++} \to \mathbb{R}$ such that $\hat{g}_j(s,\mu) = g_j(s,\mu)$ for $0 < s \in \mathbb{R}$ and $\hat{g}_j(+\infty,\mu) = 0$, $\hat{g}_j(0,\mu) = 0$ for all $\mu > 0$. It is easy to see that $g_j(s_j(\mu),\mu)$ and $\sup_{s\in[0,+\infty]} \hat{g}_j(s,\mu)$ coincide on $\mu > 0$. At the same time, the latter function is continuous by maximum theorem due to continuity of $\hat{g}_j(s,\mu)$ on its domain. Thus, $g_j(s_j(\mu),\mu)$ are continuous.

We now introduce some other useful functions:

$$j(\mu) = \max(\arg\max_j g_j(s_j(\mu),\mu)),\ \ s(\mu) = s_{j(\mu)}(\mu).$$

Obviously,

$$\phi(\mu) = \max_j \sup_{s>0} g_j(s,\mu) = \frac{-v_{j(\mu)}'(s(\mu))s(\mu)}{-v_{j(\mu)}'(s(\mu)) + B\mu}.$$

And we are in a position to conclude that $\phi(\mu)$ is continuous on its domain $\mu > 0$ as a maximum of a few continuous functions.

Step 2. We prove that $\phi(\mu)$ is decreasing on $(0,+\infty)$.

Consider $0 < \mu_1 < \mu_2$. Then we have the following due to the definitions of $j(\mu)$ and $s(\mu)$ and the fact that $-v_{j(\mu)}'(s(\mu))s(\mu) > 0$ for all $\mu > 0$[24]:

$$\phi(\mu_1) = \frac{-v_{j(\mu_1)}'(s(\mu_1))s(\mu_1)}{-v_{j(\mu_1)}'(s(\mu_1)) + B\mu_1} \le \frac{-v_{j(\mu_2)}'(s(\mu_2))s(\mu_2)}{-v_{j(\mu_2)}'(s(\mu_2)) + B\mu_1} < \frac{-v_{j(\mu_2)}'(s(\mu_2))s(\mu_2)}{-v_{j(\mu_2)}'(s(\mu_2)) + B\mu_2} = \phi(\mu_2).$$

Step 3. We prove that $\lim_{\mu\to+\infty} \phi(\mu) = 0$.

This limit exists because of monotonicity of $\phi(\mu)$. Moreover, it is greater or equal to zero as $\phi(\mu) > 0$. If this limit is greater than zero, then there is some $t > 0$ such that $\phi(\mu) > t$ in the vicinity of $+\infty$. Thus, we have

$$\frac{-v_{j(\mu)}'(s(\mu))s(\mu)}{-v_{j(\mu)}'(s(\mu)) + B\mu} > t \Rightarrow -v_{j(\mu)}'(s(\mu))\big(s(\mu) - t\big) > Bt\mu \Rightarrow$$

$$\Rightarrow Bt\mu < -v_{j(\mu)}'(s(\mu))\big(s(\mu) - t\big) \le \max_j \sup_{s>0} -v_j'(s)\big(s - t\big) = \max_j \pi_j(t).$$

This is a contradiction as the RHS is a number while the LHS can be made arbitrarily big. Therefore, $\lim_{\mu\to+\infty} \phi(\mu) = 0$.

Step 4. We prove that $\lim_{\mu\to 0^+} \phi(\mu) = \bar{s}_M$. This limit exists due to monotonicity of $\phi(\mu)$.

Case 1: There is such a country i that $\bar{s}_i = +\infty$.

Then $-v_i'(s) > 0$ for $s > 0$. Consider $g_i(s,\mu) = \frac{-v_i'(s)s}{-v_i'(s)+B\mu}$. We show that one can choose $s_1 > 0,\ \mu_1 > 0$ in such a way that $g_i(s_1,\mu_1) \ge M$, where M is an arbitrarily big number. Take, for example, $s_1 = M + \epsilon$, $\mu_1 = \frac{-v_i'(s_1)\epsilon}{BM}$. Then

$$g_i(s_1,\mu_1) = \frac{-v_i'(s_1)s_1}{-v_i'(s_1) + B\mu_1} = \frac{s_1}{1 + \frac{B\mu_1}{-v_i'(s_1)}} = \frac{M+\epsilon}{1 + \frac{\epsilon}{M}} = M \ge M.$$

Since $\phi(\mu) \ge g_j(s,\mu)$ for all $j,\ s,\ \mu > 0$ and $\phi(\mu)$ is a decreasing function, we must have that $\lim_{\mu\to 0^+} \phi(\mu) = +\infty$.

Case 2: All $\bar{s}_j$ are finite.

We fix j and cover all countries. $-v_j'(s) > 0$ for $s : 0 < s < \bar{s}_j$ and $-v_j'(s) = 0$ for $s \ge \bar{s}_j$. Consider again $g_j(s,\mu) = \frac{-v_j'(s)s}{-v_j'(s)+B\mu}$. We show that it is possible to choose s_1 and μ_1 in such a way that $\bar{s}_j - g_j(s_1,\mu_1) \le \epsilon$. Take $s_1 = \bar{s}_j - \epsilon/2$, $\mu_1 = \frac{-v_j'(s_1)\epsilon}{2(\bar{s}_j-\epsilon)B}$. Then

$$\bar{s}_j - g_j(s_1,\mu_1) = \bar{s}_j - \frac{-v_j'(s_1)s_1}{-v_j'(s_1) + B\mu_1} = \bar{s}_j - \frac{s_1}{1 + \frac{B\mu_1}{-v_j'(s_1)}} = \bar{s}_j - \frac{\bar{s}_j - \epsilon/2}{1 + \frac{\epsilon}{2(\bar{s}_j-\epsilon)}} = \epsilon \le \epsilon.$$

[24] $\phi(\mu) > 0 \Rightarrow -v_{j(\mu)}'(s(\mu))s(\mu) > 0$ and $\phi(\mu) > 0\ \forall\mu > 0$ is straightforward

As previously, since $\phi(\mu) \geq g_j(s,\mu)$ for all $j,\ s,\ \mu > 0$ and $\phi(\mu)$ is a decreasing function, we have that $\lim_{\mu\to 0^+}\phi(\mu) \geq \bar{s}_M$. Remember now that $\phi(\mu) > 0 \Rightarrow s(\mu) < \bar{s}_M$ and so

$$\phi(\mu) = \frac{-v'_{j(\mu)}(s(\mu))s(\mu)}{-v'_{j(\mu)}(s(\mu)) + B\mu} = \frac{s(\mu)}{1 + B\mu\frac{B\mu}{-v'_{j(\mu)}(s(\mu))}} \leq s(\mu) < \bar{s}_M \Rightarrow \lim_{\mu\to 0^+}\phi(\mu) \leq \bar{s}_M,$$

that finishes the proof of this step's claim.

Further, consider the following function:

$$\mu\phi(\mu) = \frac{-v'_{j(\mu)}(s(\mu))s(\mu)}{\frac{-v'_{j(\mu)}(s(\mu))}{\mu} + B}.$$

Step 5. We prove that $\mu\phi(\mu)$ is increasing.

As done earlier, take $0 < \mu_1 < \mu_2$. Then

$$\mu_2\phi(\mu_2) = \mu_2\frac{-v'_{j(\mu_2)}(s(\mu_2))s(\mu_2)}{-v'_{j(\mu_2)}(s(\mu_2)) + B\mu_2} \geq \mu_2\frac{-v'_{j(\mu_1)}(s(\mu_1))s(\mu_1)}{-v'_{j(\mu_1)}(s(\mu_1)) + B\mu_2} =$$

$$= \frac{-v'_{j(\mu_1)}(s(\mu_1))s(\mu_1)}{\frac{-v'_{j(\mu_1)}(s(\mu_1))}{\mu_2} + B} > \frac{-v'_{j(\mu_1)}(s(\mu_1))s(\mu_1)}{\frac{-v'_{j(\mu_1)}(s(\mu_1))}{\mu_1} + B} = \mu_1\phi(\mu_1),$$

due to the definitions of $j(\mu)$ and $s(\mu)$ and the fact that $-v'_{j(\mu)}(s(\mu))s(\mu) > 0\ \forall\mu > 0$.

Step 6. We prove the Lemma.

Define $A := K\frac{e_M l_M}{c_m\tau_m l_m}$. Remember that we want to show that there is always such a μ_M that

$$\mu_M = K\max_j\sup_{s>0}\frac{R_j(s)e_M l_M}{f_m - c_m\tau_m l_m\frac{v'_j(s)}{\mu_M}} = \frac{e_M l_M}{c_m\tau_m l_m}K\max_j\sup_{s>0}\frac{-v'_j(s)s}{\frac{f_m}{c_m\tau_m l_m} - \frac{v'_j(s)}{\mu_M}} = A\mu_M\phi(\mu_M),$$

$$A\mu\phi(\mu) < \mu_M \quad \mu : 0 < \mu < \mu_M.$$

Firstly, as $\lim_{\mu\to 0^+}\phi(\mu) = \bar{s}_M > \frac{c_m\tau_m}{e_M} \geq \frac{1}{K}\frac{c_m\tau_m l_m}{e_M l_M} = \frac{1}{A}$[25], there is always such a neighborhood of zero that $A\phi(\mu) > 1 \Longleftrightarrow A\mu\phi(\mu) > \mu$.

Secondly, as $\lim_{\mu\to+\infty}\phi(\mu) = 0$, there is always such a neighborhood of infinity that $A\phi(\mu) < 1 \Longleftrightarrow A\mu\phi(\mu) < \mu$.

These two observations put together with the fact that $\mu\phi(\mu)$ is increasing and continuous on $(0, +\infty)$ prove the Lemma. **Q.E.D.**

The next lemma is used in the Claims that follow it.

Lemma 3. *If some differentiable function $f(x)$ in some neighbourhood Θ of $+\infty$ has the following properties: 1) $f(x) > 0$, 2) $f'(x) < 0$, 3) $\lim_{x\to+\infty} f'(x)x$ exists, then $\lim_{x\to+\infty} f'(x)x = 0$.*

Proof. Note that $f(x)$ is bounded from below by zero and decreasing in Θ, so $\lim_{x\to+\infty} f(x) = L \geq 0$ exists and is finite. It is straightforward to see that $\lim_{x\to+\infty} f'(x)x \leq 0$. Suppose that $\lim_{x\to+\infty} f'(x)x < 0$. This means that in some neighbourhood Δ of $+\infty$ $f'(x)x < t < 0$ holds. Let $[a,b] \in \Theta\bigcap\Delta$. Then

$$f(b) - f(a) = \int_a^b f'(x)dx = \int_a^b \frac{f'(x)x}{x}dx < \int_a^b \frac{t}{x}dx = t(\ln(b) - \ln(a)).$$

Note that $\lim_{b\to+\infty} f(b) - f(a) = L - f(a)$ is finite while $\lim_{b\to+\infty} t(\ln(b) - \ln(a)) = -\infty$. Thus, this inequality cannot hold if we set b sufficiently big. Therefore, $\lim_{x\to+\infty} f'(x)x = 0$. **Q.E.D.**

Claim 4. If ZP, LM and BC hold in some country i, BP is also satisfied there.

[25]The first inequality is a trivial corollary of the assumption (vi).

Proof. From zero-profit condition we get

$$\sum_{k=1}^{K}(p_{ik}x_{ik}l_k - w_ic_i\tau_{ik}x_{ik}l_k) - w_if_i = 0 \Rightarrow \sum_{k=1}^{K}p_{ik}x_{ik}l_k = w_i\left(f_i + \sum_{k=1}^{K}c_i\tau_{ik}x_{ik}l_k\right) \Rightarrow$$

$$\Rightarrow \frac{1}{\left(f_i + \sum_{k=1}^{K}c_i\tau_{ik}x_{ik}l_k\right)}\sum_{k=1}^{K}p_{ik}x_{ik}l_k = w_i \Rightarrow N_i\sum_{k=1}^{K}p_{ik}x_{ik}l_k = w_ie_il_i.$$

At the last step we multiplied both sides by e_il_i and plugged in N_i from the labour market balance. Now if we plug the RHS into the budget constraint, we will get the balance of payments. **Q.E.D.**

Claim 5. *For u, satisfying Assumption 1,* $\lim_{x\to 0^+} u_j'(x)x = 0$.

Proof. Before anything else, note that $\lim_{x\to 0^+} u_j'(x)x = \lim_{x\to+\infty} u_j'(\frac{1}{x})\frac{1}{x}$. Let then $f(x) = u_j(\frac{1}{x})$. So, $f(x) > 0$ and $f'(x) = -u_j'(\frac{1}{x})\frac{1}{x^2} < 0$ on $(0,+\infty)$. Consider $(f'(x)x)' = f''(x)x + f'(x) = (u_j''(\frac{1}{x})\frac{1}{x} + u_j'(\frac{1}{x}))\frac{1}{x^2}$. This function cannot change its sign more than once as $u_j''(\frac{1}{x})\frac{1}{x} + u_j'(\frac{1}{x})$ is increasing. So, its antiderivative $f'(x)x$ is monotonic in the vicinity of $+\infty$ and $\lim_{x\to+\infty} f'(x)x$ exists. Thus, $f(x)$ satisfies all conditions of Lemma 3 and $\lim_{x\to 0^+} u_j'(x)x = \lim_{x\to+\infty} u_j'(\frac{1}{x})\frac{1}{x} = -\lim_{x\to+\infty} f'(x)x = 0$. **Q.E.D.**

Claim 6. $\lim_{x\to 0^+} u_k''(x)x + u_k'(x) > 0$

Proof. Since $u_k''(x)x+u_k'(x)$ is decreasing on $(0, X)$, it is either positive or negative in some neighborhood Θ_k of zero. If it is negative, we cannot have $\lim_{x\to 0^+} u_k'(x)x = 0$[26] and $u_k'(x)x > 0$ in the vicinity of zero simultaneously. It is due to the fact that $u_k'(x)x$ is an antiderivative of $u_k''(x)x+u_k'(x)$. Thus, $\lim_{x\to 0^+} u_k''(x)x+u_k'(x)$ is positive in Θ_k. We showed that $u_k''(x)x+u_k'(x)$ is postitive somewhere, so, as it is decreasing, $\lim_{x\to 0^+} u_k''(x)x+u_k'(x) > 0$. **Q.E.D.**

Claim 7. $\pi_k(\beta)$ *is positive and decreasing on* $(0, \overline{\beta}_k)$ *and* $\pi_k(\beta) = 0$ *for* $\beta \geq \overline{\beta}$ *if* $\overline{\beta}$ *is finite.*

Proof. Remember that $\{x_k(\beta)\} = \arg\max_{x\geq 0} Pr_k(x,\beta)$ for all $\beta > 0$. As $x_k(\beta) > 0$ for $\beta : \overline{\beta}_k > \beta$, it must be that $Pr_k(x_k(\beta),\beta) > Pr_k(0,\beta) = 0$. Thus, $\pi_k(\beta) > 0$ when $\overline{\beta}_k > \beta$. Let now $\overline{\beta} > \beta_1 > \beta_2$, then

$$\pi(\beta_2) = Pr_k(x_k(\beta_2),\beta_2) \geq Pr_k(x_k(\beta_1),\beta_2) > Pr_k(x_k(\beta_1),\beta_1) = \pi_k(\beta_1)$$

as $Pr_k(x_k(\beta_1),\beta) = u_k'(x_k(\beta_1))x_k(\beta_1) - \beta x_k(\beta_1)$ decreases in β. In case $\overline{\beta}_k$ is finite,

$$\pi_k(\beta) = Pr_k(x_k(\beta),\beta) = Pr_k(0,\beta) = 0\,\forall\beta \geq \overline{\beta}_k.$$

Q.E.D.

Claim 8. $\lim_{x\to+\infty} u_t''(x)x = 0$.

Proof. Let $f(x) = u_t'(x)$. Then $f(x) > 0$ and $f'(x) = u_t''(x) < 0$ on $(0,+\infty)$. Since $\lim_{x\to+\infty} u_t''(x)x + u_t'(x)$ exists due to monotonicity of $u_t''(x)x+u_t'(x)$ and $\lim_{x\to+\infty} u_t'(x)$ is finite, $\lim_{x\to+\infty} f'(x)x = \lim_{x\to+\infty} u_t''(x)x$ must exist. Thus, $f(x)$ satisfies all conditions of Lemma 3 and $\lim_{x\to+\infty} u_t''(x)x = \lim_{x\to+\infty} f'(x)x = 0$. **Q.E.D.**

Claim 9. $\lim_{s\to\bar{s}_j} v_j'(s)s = \lim_{s\to+\infty} v_j'(s)s = 0$.

Proof. If $\bar{s}_j \neq +\infty$, validity of this claim is obvious. Consider the case when $\bar{s}_j = +\infty$. Let $f(x) = v_j(x)$. Then $f(x) > 0$ and $f'(x) = v_j'(x) < 0$ on $(0,+\infty)$. As $s + \frac{v_k'(s)}{v_k''(s)}$ is increasing and $v_j''(s) > 0$, $(v_j'(x)x)' = v_j''(s)s + v_j'(s)$ cannot change its sign more than once. So, its antiderivative $v_j'(x)x$ is monotonic in the vicinity of $+\infty$ and $\lim_{x\to+\infty} f'(x)x = \lim_{x\to+\infty} v_j'(x)x$ exists. Thus, $f(x)$ satisfies all conditions of Lemma 3 and $\lim_{x\to+\infty} v_j'(x)x = \lim_{x\to+\infty} f'(x)x = 0$. **Q.E.D.**

Claim 10. $\lim_{s\to\bar{s}_k^-} s + \frac{v_k'(s)}{v_k''(s)} > 0$.

[26] See Claim (5)

Proof. Consider $\beta_k(s) = s + \frac{v_k'(s)}{v_k''(s)}$. This function is increasing on $(0, \bar{s}_k)$, so $\beta_k(s)$ can change its sign there at most once. As $v_k''(s) > 0$ in the left half-neighborhood of $\bar{s}_k$, $v_k''(s)s + v_k'(s)$ is either positive or negative in some left half-neighborhood Θ_k of $\bar{s}$. If it is negative, then we cannot have $\lim_{s \to \bar{s}_j^-} v_k'(s)s = 0$[27] and $v_k'(s)s < 0$ on $(0, \bar{s})$ at the same time. It is so because $v_k'(s)s$ is an antiderivative of $v_k''(s)s + v_k'(s)$. Thus, $v_k''(s)s + v_k'(s) > 0$ in Θ_k. As $v_k''(s)s + v_k'(s)$ is positive somewhere and $s + \frac{v_k'(s)}{v_k''(s)}$ is increasing, $\lim_{s \to \bar{s}_k^-} s + \frac{v_k'(s)}{v_k''(s)} > 0$. **Q.E.D.**

Claim 11. $\pi_k(\beta)$ is positive and decreasing on $(0, \overline{\beta})$ and $\pi_k(\beta) = 0$ for $\beta \geq \overline{\beta}_k$ if $\overline{\beta}_k$ is finite.

Proof. Remember that $\{s_k(\beta)\} = \arg\max_{s>0} Pr_k(s, \beta)$ for all $\overline{\beta}_k > \beta$. As $s_k(\beta) < \bar{s}_k$ for $\beta : \overline{\beta}_k > \beta$, it must be that $Pr_k(s_k(\beta), \beta) > Pr_k(\bar{s}_k, \beta) = 0$. Thus, $\pi_k(\beta) > 0$ when $\overline{\beta}_k > \beta$. Let now $\overline{\beta}_k > \beta_1 > \beta_2$, then

$$\pi(\beta_2) = Pr_k(s_k(\beta_2), \beta_2) \geq Pr_k(s_k(\beta_1), \beta_2) > Pr_k(s_k(\beta_1), \beta_1) = \pi_k(\beta_1)$$

due to the fact that $Pr_k(s_k(\beta_1), \beta) = -v_k'(s_k(\beta_1))(s_k(\beta_1) - \beta)$ decreases in β. In case $\overline{\beta}_k$ is finite,

$$\pi_k(\beta) = Pr_k(s_k(\beta), \beta) = Pr_k(\bar{s}_k, \beta) = 0 \; \forall \beta \geq \overline{\beta}_k.$$

Q.E.D.

[27] See Claim (9)