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On properties of aggregated regularized systems of equations for a homogeneous multicomponent gas mixture

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Two aggregated regularized systems of equations for a multicomponent homogeneous gas mixture are considered. An entropy balance equation with a non-negative entropy production is derived for them in the presence of diffusion fluxes. The existence, uniqueness and L^2-dissipativity of weak solutions to an initial-boundary value problem for the systems linearized on a constant solution are established. The Petrovskii parabolicity and the local in time classical unique solvability of the Cauchy problem are also proved for the aggregated systems themselves.

KEYWORDS
entropy balance equation, existence and uniqueness of solutions, homogeneous gas mixture, L^2-dissipativity, multicomponent gas dynamics equations, Petrovskii parabolicity

MSC CLASSIFICATION
35Q35; 35K45; 35K51

1 | INTRODUCTION

The classical Euler and Navier–Stokes equations for compressible one-component gas flows (e.g., see Landau and Lifschitz[1]) are widely used in practice and are actively studied mathematically. Regularized quasi-gasdynamic (QGD) and quasi-hydrodynamic (QHD) systems of equations for gas flows are presented in detail in monographs,[2–5] including their physical motivation and various applications which it is impossible to dwell on here. These equations are suitable for discretization and construction of simple and fairly effective explicit mesh methods and have been used for several decades in computer simulation of a broad variety of gas dynamics problems, with the QGD and QHD systems applied, respectively, for any and moderate Mach numbers. Some their important mathematical properties, similar to those considered in this paper, were proved in previous works.[3,4,6–8] Notice that alternative regularized systems of equations were studied, in particular, in papers[9–11]; but their practical applications have been rather limited until now, in contrast to the case of QGD and QHD systems.

Systems of equations describing non-stationary flows of compressible heat-conducting gas mixtures are also of great practical and significant mathematical interest, see, in particular, monographs.[1,12,13] Important mathematical results on the properties of their solutions were proved, in particular, in papers.[13–17] Various regularized QGD and QHD systems of equations for binary mixtures in the absence of chemical reactions, including both non-homogeneous mixtures (when not only the densities but also the velocities and temperatures of the components are different) and homogeneous ones

Abbreviations: QGD, QHD

(i.e., with the common velocity and temperature of the mixture components), have been constructed, discretized and successfully tested in numerical simulations including those in previous works.[3,18–23]

This paper studies the properties of aggregated QGD and QHD systems of equations for the homogeneous multicomponent gas mixture which are essential as a mathematical basis for the success of the mentioned discretizations. This is carried out in the unified manner for the both systems. Notice that the both systems are obtained by aggregating more complicated QGD and QHD systems of equations for inhomogeneous mixtures from Elizarova et al.[18] Moreover, the aggregated QHD system is new. An entropy balance equation with the non-negative entropy production is derived in the presence of diffusion fluxes between components, and the diffusion fluxes and additional heat flux are relatively simple and generalize those for binary mixtures considered in Landau and Lifschitz.[1] This indicates the physical and, to a certain extent, mathematical correctness of these systems. Similar result has recently been given in papers[20,21] in the case of the QGD system for binary mixtures and in the absence of diffusion fluxes only.

In the absence of diffusion fluxes, the existence, uniqueness and L^2-dissipativity of solutions to an initial-boundary value problem for systems linearized on a constant solution are established. The Petrovskii parabolicity (on compact sets of values of the sought functions) and the classical unique solvability of the Cauchy problem are also proved for the aggregated QGD and QHD systems themselves. These results substantiate the regularizing properties of these systems.

The paper is organized as follows. In Section 2, the aggregated QGD and QHD systems of equations are written down, their joint representation is given, and the form of diffusion fluxes and additional heat flux is discussed. The balance equations for the total mass, kinetic, and internal energies as well as the velocity and temperature are also given. In Section 3, the main results of the paper are stated. First, the entropy balance equation with the non-negative entropy production is presented. Next, in the absence of diffusion fluxes, an auxiliary expansion of the equivalent system of equations for the densities of the components, the common velocity and temperature is performed up to the squared modulus of the gradient of the sought functions leading to a reduced system of equations. This reduced system is applied in both subsequent main results. Second, a system of equations linearized on a constant solution is written down in the dimensionless symmetrized form. Then the existence, uniqueness and L^2-dissipativity of weak solutions to an initial-boundary value problem for this system are given. Third, the Petrovskii parabolicity for the aggregated QGD and QHD systems of equations for a homogeneous gas mixture and the existence and uniqueness theorem for a local in time classical solution to the Cauchy problem for them are stated. Here the results related to the linearized system are used essentially. The proofs are collected in Section 4 divided into seven Subsections. We emphasize that, in the multicomponent case, the reasoning is both essentially more cumbersome and significantly different from those given in the one-component case, and the results on the linearized system also strengthen the corresponding ones in papers.[6–8] The unified analysis of the QGD and QHD systems is also a new detail (previously, similar systems were studied separately).

2 | AGGREGATED QUASI-GASDYNAMIC AND QUASI-HYDRODYNAMIC SYSTEMS OF EQUATIONS FOR A HOMOGENEOUS GAS MIXTURE AND THEIR COROLLARIES

The aggregated quasi-gasdynamic (QGD) system of equations for a homogeneous multicomponent gas mixture consists of the following mass of components, total momentum and total energy balance equations

$$\partial_t\rho_\alpha + \operatorname{div}\left[\rho_\alpha(\mathbf{u}-\mathbf{w}_\alpha)+\mathbf{d}_\alpha\right] = 0,\ \alpha=\overline{1,K}, \tag{2.1}$$

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}\left[\rho(\mathbf{u}-\mathbf{w})\otimes\mathbf{u}\right] + \nabla p = \operatorname{div}\Pi + \left[\rho-\tau\operatorname{div}(\rho\mathbf{u})\right]\mathbf{f}, \tag{2.2}$$

$$\partial_t E + \operatorname{div}\left[\frac{1}{2}\rho|\mathbf{u}|^2(\mathbf{u}-\mathbf{w}) + \langle\rho_\alpha h_\alpha(\mathbf{u}-\mathbf{w}_\alpha)\rangle\right] = \operatorname{div}(-\mathbf{q}+\Pi\mathbf{u}) + \rho(\mathbf{u}-\mathbf{w})\cdot\mathbf{f} + Q. \tag{2.3}$$

Here the main sought functions $\rho_\alpha>0$ ($\alpha=\overline{1,K}$), $\mathbf{u}=(u_1,\dots,u_n)$ and $\theta>0$ are the densities of the mixture components, their common velocity and absolute temperature, respectively. These functions depend on $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ and $t\geq0$, where $K\geq2$ is the number of the components and $n=1,2,3$. Let $\rho := (\rho_1,\dots,\rho_K)$. The operators *div* and $\nabla=(\partial_1,\dots,\partial_n)$ are taken in x, $\partial_t=\partial/\partial t$ and $\partial_i=\partial/\partial x_i$. The symbols $\otimes$ and $\cdot$ denote the tensor and scalar products of vectors, and the tensor divergence is taken with respect to its first index.

The mixture components are assumed to be perfect polytropic gases with the equations of state

$$p_\alpha = (\gamma_\alpha - 1)\rho_\alpha\varepsilon_\alpha = R_\alpha\rho_\alpha\theta,\ \varepsilon_\alpha = c_{V\alpha}\theta,\ \alpha = \overline{1,K}, \tag{2.4}$$

where p_α and ε_α are the pressure and specific internal energy of the component α, with constant $\gamma_\alpha > 1$, $R_\alpha > 0$ and $c_{V\alpha} > 0$. Moreover, $h_\alpha = \varepsilon_\alpha + (p_\alpha/\rho_\alpha) = c_{p\alpha}\theta$ is the specific enthalpy of the component α, as well as $c_{V\alpha}$ and $c_{p\alpha} = c_{V\alpha} + R_\alpha$ are the specific heat capacities at constant volume and pressure, $\alpha = \overline{1,K}$.

The total density, pressure, specific internal energy, and total energy of the mixture are given by the formulas

$$\rho = \langle\rho_\alpha\rangle := \rho_1 + \ldots + \rho_K,\ p = \langle p_\alpha\rangle = R\rho\theta,\ \varepsilon = \left\langle \frac{\rho_\alpha}{\rho}\varepsilon_\alpha \right\rangle = c_V\theta,\ E = \frac{1}{2}\rho|\mathbf{u}|^2 + \rho\varepsilon, \tag{2.5}$$

the second of which is Dalton's law for mixtures, with

$$R := \left\langle \frac{\rho_\alpha}{\rho}R_\alpha \right\rangle,\ c_V := \left\langle \frac{\rho_\alpha}{\rho}c_{V\alpha} \right\rangle. \tag{2.6}$$

Here we have introduced the operation $\langle\cdot\rangle$ of summation over index $\alpha = \overline{1,K}$. We emphasize that, in contrast to the one-component case, R and c_V are *functions* rather than constants, and above $c_\alpha := \frac{\rho_\alpha}{\rho}$ are the mass concentrations of the mixture components.

In the equations, the regularizing velocities for the component α and total ones of the form

$$\mathbf{w}_\alpha = \frac{\tau}{\rho_\alpha}[\operatorname{div}(\rho_\alpha\mathbf{u}\otimes\mathbf{u}) + \nabla p_\alpha - \rho_\alpha\mathbf{f}],\ \hat{\mathbf{w}}_\alpha = \tau\left[(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho_\alpha}\nabla p_\alpha - \mathbf{f}\right], \tag{2.7}$$

$$\mathbf{w} := \left\langle \frac{\rho_\alpha}{\rho}\mathbf{w}_\alpha \right\rangle = \frac{\tau}{\rho}[\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla p - \rho\mathbf{f}],\ \hat{\mathbf{w}} = \left\langle \frac{\rho_\alpha}{\rho}\hat{\mathbf{w}}_\alpha \right\rangle = \tau\left[(\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p - \mathbf{f}\right], \tag{2.8}$$

are used, where $\tau = \tau(\rho, \mathbf{u}, \theta) > 0$ is a regularization parameter (which in general is a function).

We also consider a simpler aggregated quasi-hydrodynamic (QHD) system of equations for a homogeneous multicomponent gas mixture which consists of the following similar balance equations

$$\partial_t\rho_\alpha + \operatorname{div}\left[\rho_\alpha(\mathbf{u} - \hat{\mathbf{w}}_\alpha) + \mathbf{d}_\alpha\right] = 0,\ \alpha = \overline{1,K}, \tag{2.9}$$

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}\left[\rho(\mathbf{u} - \hat{\mathbf{w}})\otimes\mathbf{u}\right] + \nabla p = \operatorname{div}\Pi + \rho\mathbf{f}, \tag{2.10}$$

$$\partial_t E + \operatorname{div}\left[\frac{1}{2}\rho|\mathbf{u}|^2(\mathbf{u} - \hat{\mathbf{w}}) + \left\langle\rho_\alpha h_\alpha(\mathbf{u} - \hat{\mathbf{w}}_\alpha)\right\rangle\right] = \operatorname{div}(-\mathbf{q} + \Pi\mathbf{u}) + \rho(\mathbf{u} - \hat{\mathbf{w}})\cdot\mathbf{f} + Q. \tag{2.11}$$

In Section 1, it has already been noted that QGD systems and simpler QHD systems are used in practice and have proven themselves well, respectively, for any and moderate Mach numbers.

The viscosity tensor has the form $\Pi = \Pi^{NS} + \Pi^\tau_\ell$, and the heat flux is described by the formula $\mathbf{q} = \mathbf{q}^F + \mathbf{q}^d + \ell\mathbf{q}^\tau$. Hereafter, $\ell = 1$ for system (2.1)–(2.3) and $\ell = 0$ for system (2.9)–(2.11). The Navier–Stokes viscosity tensor and the Fourier heat flux are given by the standard formulas

$$\Pi^{NS} = \mu\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \frac{2}{3}(\operatorname{div}\mathbf{u})\mathbb{I}\right] + \lambda(\operatorname{div}\mathbf{u})\mathbb{I},\ -\mathbf{q}^F = \varkappa\nabla\theta,$$

where $\mu > 0$, $\lambda \geq 0$, and $\varkappa > 0$ are the total coefficients of dynamic and bulk viscosities and heat conductivity (which can depend on the sought functions), $\nabla\mathbf{u} = \{\partial_i u_j\}_{i,j=1}^n$ and $\mathbb{I}$ is the unit tensor of order n.

The regularizing viscosity tensor and heat flux are given by the formulas

$$\Pi^\tau_\ell = \rho\mathbf{u}\otimes\hat{\mathbf{w}} + \ell\tau\left[\mathbf{u}\cdot\nabla p + \langle\gamma_\alpha p_\alpha\rangle\operatorname{div}\mathbf{u} - \langle\gamma_\alpha Q_\alpha\rangle + Q\right]\mathbb{I}, \tag{2.12}$$

$$-\mathbf{q}^\tau = \tau\left\{\left[c_V\rho\nabla\theta - \theta\nabla(R\rho)\right]\cdot\mathbf{u} - Q\right]\right\}\mathbf{u}. \tag{2.13}$$

The density of mass force $\mathbf{f}$ and intensities of heat sources $Q_\alpha \geq 0$ are given functions, and $Q := \langle Q_\alpha \rangle \geq 0$. Let $\mathbf{Q} := (Q_1, \dots, Q_K)$.

Using the parameter $\ell = 0, 1$, we define the regularizing velocities

$$\mathbf{w}_{\ell\alpha} := \ell \frac{\tau}{\rho_\alpha} \mathrm{div}(\rho_\alpha \mathbf{u})\mathbf{u} + \hat{\mathbf{w}}_\alpha,\ \mathbf{w}_\ell := \left\langle \frac{\rho_\alpha}{\rho} \mathbf{w}_{l\alpha} \right\rangle$$

and rewrite the QGD system (2.1)–(2.3) and the QHD system (2.9)–(2.11) in the unified form

$$\partial_t \rho_\alpha + \mathrm{div}\,[\rho_\alpha(\mathbf{u} - \mathbf{w}_{\ell\alpha}) + \mathbf{d}_\alpha] = 0,\ \alpha = \overline{1, K}, \tag{2.14}$$

$$\partial_t(\rho\mathbf{u}) + \mathrm{div}\,[\rho(\mathbf{u} - \mathbf{w}_\ell) \otimes \mathbf{u}] + \nabla p = \mathrm{div}\Pi + [\rho - \ell\tau \mathrm{div}(\rho\mathbf{u})]\mathbf{f}, \tag{2.15}$$

$$\partial_t E + \mathrm{div}\left[\frac{1}{2}\rho|\mathbf{u}|^2(\mathbf{u} - \mathbf{w}_\ell) + \langle \rho_\alpha h_\alpha(\mathbf{u} - \mathbf{w}_{\ell\alpha})\rangle\right] = \mathrm{div}(-\mathbf{q} + \Pi\mathbf{u}) + \rho(\mathbf{u} - \mathbf{w}_\ell) \cdot \mathbf{f} + Q, \tag{2.16}$$

Below this will allow us to analyze both systems in a unified manner and not separately as usual. Note that the case $\ell = 0$ is simpler, and sometimes significantly simpler, than $\ell = 1$.

The presented model is a regularized Navier–Stokes system of equations for a mixture of viscous heat-conducting compressible gases and is transformed into it when $\tau = 0$. The case of regularized Euler equations, when the physical coefficients of viscosity and heat conductivity equal 0, is also covered: then the use of artificial coefficients μ, λ and $\varkappa$ proportional to τ is assumed, see previous works[2–4]; below their specific form is unimportant. In the case of a binary mixture ($K = 2$) with $\mathbf{d}_\alpha = 0$ and $\mathbf{q}^d = 0$, the discussed QGD equations have recently been derived in papers[20,21] by aggregating the QGD equations for inhomogeneous mixtures from Elizarova et al.[18] In the same case, the above QHD equations can be derived quite similarly and even simpler from the QHD equations for inhomogeneous mixtures in Elizarova et al,[18, formulas (19)–(21)] but we do not dwell on this in more detail here. The above QGD and QHD equations for multicomponent homogeneous mixtures are their natural generalizations.

We introduce the diffusion fluxes and additional heat flux by the following formulas

$$-\mathbf{d}_\alpha := d_0 \left[\sum_{\beta:\,\beta\neq\alpha} \nabla(G_\alpha - G_\beta) + b_\alpha \nabla\theta\right] = d_0 \left[\nabla(KG_\alpha - G) + b_\alpha\nabla\theta\right] \text{ with } G := \langle G_\alpha \rangle, \tag{2.17}$$

$$\mathbf{q}^d = \left\langle \left(G_\alpha + K^{-1} b_\alpha \theta\right) \mathbf{d}_\alpha \right\rangle, \tag{2.18}$$

$$G_\alpha := \varepsilon_\alpha - s_\alpha\theta + \frac{p_\alpha}{\rho_\alpha} = (c_{p\alpha} - s_\alpha)\theta,\ s_\alpha = s_{\alpha 0} - R_\alpha \ln\frac{\rho_\alpha}{\rho_{\alpha 0}} + c_{V\alpha} \ln\frac{\theta}{\theta_0}, \tag{2.19}$$

where G_α and s_α are the Gibbs potential and specific entropy of the component $\alpha = \overline{1, K}$. The quantities $d_0 \geq 0$ and b_α are not specified here; essentially, they can depend on the sought functions, and it is assumed that $\langle b_\alpha \rangle = 0$. Also, $s_{\alpha 0}$, $\rho_{\alpha 0} > 0$ and $\theta_0 > 0$ are constants (reference values for s_α, ρ_α and θ).

The property $\langle \mathbf{d}_\alpha \rangle = 0$ is important; it immediately follows from (2.17) and $\langle b_\alpha \rangle = 0$.

Since

$$\nabla G_\alpha = (c_{p\alpha} - s_\alpha)\nabla\theta - \theta\nabla s_\alpha,\ \nabla s_\alpha = -R_\alpha \frac{1}{\rho_\alpha}\nabla\rho_\alpha + c_{V\alpha}\frac{1}{\theta}\nabla\theta,$$

the substitution $\tilde{b}_\alpha := b_\alpha - (Ks_\alpha - \langle s_\alpha \rangle)$ makes the above introduced expressions for fluxes explicitly independent of s_α:

$$\begin{aligned} -\mathbf{d}_\alpha &= d_0 \left\{\left[Kc_{p\alpha} - \langle c_{p\alpha}\rangle - (Ks_\alpha - \langle s_\alpha\rangle - b_\alpha)\right]\nabla\theta - \theta\nabla(Ks_\alpha - \langle s_\alpha \rangle)\right\} \\ &= d_0 \left[\theta\left(KR_\alpha \frac{1}{\rho_\alpha}\nabla\rho_\alpha - \left\langle R_\alpha \frac{1}{\rho_\alpha}\nabla\rho_\alpha \right\rangle\right) + \left(KR_\alpha - \langle R_\alpha\rangle + \tilde{b}_\alpha\right)\nabla\theta\right], \\ \mathbf{q}^d &= \left\langle \left(G_\alpha + s_\alpha\theta - K^{-1}\langle s_\alpha\rangle\theta + K^{-1}\tilde{b}_\alpha\theta\right)\mathbf{d}_\alpha \right\rangle = \left\langle \left(c_{p\alpha} + K^{-1}\tilde{b}_\alpha\right)\theta\mathbf{d}_\alpha \right\rangle \end{aligned} \tag{2.20}$$

taking into account that $\langle \mathbf{d}_\alpha \rangle = 0$. Here $\langle \tilde{b}_\alpha \rangle = 0$ too. Moreover, since $\rho_\alpha = \frac{pc_\alpha}{R\theta}$, then

$$\ln \rho_\alpha = \ln c_\alpha - \ln R + \ln p - \ln \theta,\ R = \langle R_\alpha c_\alpha \rangle,$$

and also the following formula holds

$$\begin{aligned}-\mathbf{d}_\alpha = d_0 \Bigg\{ \theta \left[KR_\alpha \frac{1}{c_\alpha} \nabla c_\alpha - \left\langle R_\alpha \frac{1}{c_\alpha} \nabla c_\alpha \right\rangle - (KR_\alpha - \langle R_\alpha \rangle) \frac{1}{R} \langle R_\alpha \nabla c_\alpha \rangle \right] \\ + (KR_\alpha - \langle R_\alpha \rangle) \frac{1}{R\rho} \nabla p + \tilde{b}_\alpha \nabla \theta \Bigg\}.\end{aligned} \tag{2.21}$$

In the case $K = 2$ (for binary mixtures), the formulas for fluxes (2.17) and (2.18) take the form equivalent to the well-known ones[1, Ch. VI]

$$-\mathbf{d}_1 = \mathbf{d}_2 = d_0 \left[\nabla (G_1 - G_2) + b_1 \nabla \theta \right],\ \mathbf{q}^d = (G_1 - G_2 + b_1 \theta) \mathbf{d}_1.$$

Besides, since $KR_1 - \langle R_\alpha \rangle = R_1 - R_2$ and

$$KR_1 \frac{1}{c_1} \nabla c_1 - \left\langle R_\alpha \frac{1}{c_\alpha} \nabla c_\alpha \right\rangle - (KR_1 - \langle R_\alpha \rangle) \frac{1}{R} \langle R_\alpha \nabla c_\alpha \rangle = \left(\frac{R_1}{c_1} + \frac{R_2}{c_2} \right) \nabla c_1 - \frac{(R_1 - R_2)^2}{R} \nabla c_1,$$

after simple algebraic transformations, formulas (2.21) and (2.20) lead to formulas of a more standard form for the binary mixtures

$$-\mathbf{d}_1 = \mathbf{d}_2 = d_0 \left[\frac{R_1 R_2 \theta}{R c_1 (1 - c_1)} \nabla c_1 - \frac{R_1 - R_2}{R\rho} \nabla p + \tilde{b}_1 \nabla \theta \right],\ \mathbf{q}^d = \left(c_{p1} - c_{p2} + \tilde{b}_1 \right) \theta \mathbf{d}_1.$$

In the particular case $\tilde{b}_1 = 0$ (i.e., in the absence of thermal diffusion), they are simplified.

The following collection of corollaries from Equations (2.14)–(2.16) is valid: *the total mass balance equation*

$$\partial_t \rho + \operatorname{div} [\rho (\mathbf{u} - \mathbf{w}_\ell)] = 0, \tag{2.22}$$

the total kinetic energy balance equation

$$\frac{1}{2} \partial_t (\rho |\mathbf{u}|^2) + \frac{1}{2} \operatorname{div} \left[\rho (\mathbf{u} - \mathbf{w}_\ell) |\mathbf{u}|^2 \right] + (\nabla p) \cdot \mathbf{u} = (\operatorname{div} \Pi) \cdot \mathbf{u} + [\rho - \ell \tau \operatorname{div} (\rho \mathbf{u})] \mathbf{f} \cdot \mathbf{u} \tag{2.23}$$

and *the total internal energy balance equation*

$$\partial_t (\rho \varepsilon) + \operatorname{div} \langle \rho_\alpha \varepsilon_\alpha (\mathbf{u} - \mathbf{w}_{\ell\alpha}) \rangle + p \operatorname{div} \mathbf{u} = \operatorname{div} (-\mathbf{q} + \langle p_\alpha \mathbf{w}_{\ell\alpha} \rangle) + \Pi : \nabla \mathbf{u} - \rho \hat{\mathbf{w}} \cdot \mathbf{f} + Q, \tag{2.24}$$

where the symbol : denotes the scalar product of tensors. For related systems, equations similar to these balance equations are well-known.

In addition, *the velocity balance equation*

$$\begin{aligned}\partial_t \mathbf{u} + [(\mathbf{u} - \mathbf{w}_\ell) \cdot \nabla] \mathbf{u} + \frac{1}{\rho} \nabla p = \frac{1}{\rho} \left\{ \operatorname{div} \Pi^{NS} + (\mathbf{u} \cdot \nabla)(\rho \hat{\mathbf{w}}) + (\operatorname{div} \mathbf{u}) \left(\rho \hat{\mathbf{w}} + \ell \nabla \langle \tau \gamma_\alpha p_\alpha \rangle \right) \right. \\ \left. + \ell \tau \langle \gamma_\alpha p_\alpha \rangle \nabla \operatorname{div} \mathbf{u} + \ell \nabla \left[\tau \mathbf{u} \cdot \nabla p - \tau (\langle \gamma_\alpha Q_\alpha \rangle - Q) \right] \right\} + \left[1 - \ell \frac{1}{\rho} \tau \operatorname{div} (\rho \mathbf{u}) \right] \mathbf{f}\end{aligned} \tag{2.25}$$

and *the temperature balance equation*

$$\partial_t \theta + \left(\mathbf{u} - \frac{\langle c_{V\alpha} \rho_\alpha \mathbf{w}_{\ell\alpha} \rangle}{c_V \rho} \right) \cdot \nabla \theta + \frac{R}{c_V} \theta \operatorname{div} \mathbf{u}$$

$$= \frac{1}{c_V\rho}\left[\langle c_{V\alpha}\mathrm{div}\mathbf{d}_\alpha\rangle\theta + \mathrm{div}(-\mathbf{q} + \langle p_\alpha \mathbf{w}_{\ell\alpha}\rangle) + \Pi : \nabla\mathbf{u} - \rho\hat{\mathbf{w}}\cdot\mathbf{f} + Q\right] \tag{2.26}$$

are valid. In what follows, it is essential that the systems of equations (2.14), (2.15), (2.16) and (2.14), (2.25), (2.26) (taking into account formula (2.12)) are equivalent for smooth solutions. The derivation of all Equations (2.22)–(2.26) will be given at the beginning of Section 4.

3 | THE MAIN RESULTS

In this section, we state our main results. Their proofs will be given in the next Section 4.

3.1 | The entropy balance equation in the presence of diffusion fluxes

The first main result of the paper concerns the total specific entropy $S := \langle c_\alpha s_\alpha\rangle$ and the total entropy balance equation for ρS.

Theorem 3.1. *Let $d_0 > 0$. The following regularized entropy balance equation for a homogeneous multicomponent mixture in the presence of diffusion fluxes holds*

$$\partial_t(\rho S) + \mathrm{div}\left[\langle \rho_\alpha s_\alpha(\mathbf{u} - \mathbf{w}_{\ell\alpha})\rangle + \frac{1}{K}\langle b_\alpha \mathbf{d}_\alpha\rangle + \frac{1}{\theta}(\mathbf{q}^F + \ell\mathbf{q}^\tau)\right] = \mathcal{P}^{NS} + \langle \mathcal{P}^\tau_{\ell\alpha}\rangle \tag{3.1}$$

with the entropy production $\mathcal{P}^{NS} + \langle \mathcal{P}^\tau_{\ell\alpha}\rangle$, where

$$\mathcal{P}^{NS} = \frac{1}{\theta}\left[\frac{\mu}{2}|\nabla\mathbf{u} + \nabla\mathbf{u}^T|^2 + \left(\lambda - \frac{2}{3}\mu\right)(\mathrm{div}\mathbf{u})^2\right] + \frac{1}{\theta^2}\varkappa|\nabla\theta|^2 + \frac{1}{Kd_0\theta}\left\langle |\mathbf{d}_\alpha|^2\right\rangle \geq 0,$$

$$\mathcal{P}^\tau_{\ell\alpha} = \frac{\rho_\alpha}{\tau\theta}|\hat{\mathbf{w}}_\alpha|^2 + \ell\tau\frac{R_\alpha}{\rho_\alpha}[\mathrm{div}(\rho_\alpha\mathbf{u})]^2 + \ell\tau c_{V\alpha}\rho_\alpha\left[\mathbf{u}\cdot\nabla\ln\theta + (\gamma_\alpha - 1)\mathrm{div}\mathbf{u} - \frac{(\gamma_\alpha - 1)Q_\alpha}{2p_\alpha}\right]^2$$
$$+\frac{Q_\alpha}{\theta}\left(1 - \ell\frac{\tau(\gamma_\alpha - 1)Q_\alpha}{4p_\alpha}\right);$$

moreover, $\mathcal{P}^\tau_{\ell\alpha} \geq 0$ for $\ell = 0$ as well as for $\ell = 1$ provided that $\tau(\gamma_\alpha - 1)Q_\alpha \leq 4p_\alpha$, $\alpha = \overline{1,K}$.

Clearly, here $\mathcal{P}^{NS}$ and $\langle \mathcal{P}^\tau_{\ell\alpha}\rangle$ are the Navier–Stokes and regularizing contributions to the entropy production. Notice that $\langle \mathcal{P}^\tau_{\ell\alpha}\rangle \geq 0$ for $\ell = 1$ under the more general unique condition $\tau\left\langle \frac{(\gamma_\alpha-1)Q_\alpha^2}{4p_\alpha}\right\rangle \leq Q$.

The entropy balance Equation (3.1) remains true for $\mathbf{d}_\alpha = 0$, $\alpha = \overline{1,K}$ and/or $\tau = 0$. In these substantially simpler cases, the terms with respectively $\mathbf{d}_\alpha$ and $\mathbf{w}_{\ell\alpha}$, $\mathbf{q}^\tau$, $\mathcal{P}^\tau_{\ell\alpha}$ should be omitted on its left and right sides. The first case for $K = 2$ and $\ell = 1$ has recently been considered in papers.[20,21]

3.2 | The linearized QGD and QHD systems and their properties

The properties studied below are related to the regularizing properties of the τ-terms in the QGD and QHD systems. Below we restrict ourselves to the case $\mathbf{d}_\alpha = 0$, $\alpha = \overline{1,K}$. Let below also $\mathbf{f} = 0$ and $\mathbf{Q} = 0$ (except for Theorem 3.3).

We introduce the vector of the sought functions $\mathbf{z} = (\rho, \mathbf{u}, \theta)$ and present an auxiliary reduction of Equations (2.14) and (2.25)–(2.26) with the accuracy $O(|\nabla\mathbf{z}|^2)$. They are as follows: the reduced equations for the component densities

$$\partial_t\rho_\alpha + \nabla\rho_\alpha\cdot\mathbf{u} + \rho_\alpha\mathrm{div}\mathbf{u} = \tau\left[R_\alpha\theta\Delta\rho_\alpha + \ell[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\nabla]\rho_\alpha\right.$$
$$\left.+(\ell+1)\rho_\alpha(\mathbf{u}\cdot\nabla)\mathrm{div}\mathbf{u} + R_\alpha\rho_\alpha\Delta\theta\right] + O(|\nabla\mathbf{z}|^2),\ \alpha = \overline{1,K}, \tag{3.2}$$

the reduced equation for the velocity

$$\partial_t\mathbf{u} + \frac{\theta}{\rho}\langle R_\alpha\nabla\rho_\alpha\rangle + (\mathbf{u}\cdot\nabla)\mathbf{u} + R\nabla\theta$$

$$= (\ell+1)\tau\frac{\theta}{\rho}(\mathbf{u}\cdot\nabla)\langle R_\alpha\nabla\rho_\alpha\rangle + \frac{\mu}{\rho}\Delta\mathbf{u} + \frac{\chi}{\rho}\nabla\mathrm{div}\mathbf{u} + \ell\tau\frac{\langle\gamma_\alpha p_\alpha\rangle}{\rho}\nabla\mathrm{div}\mathbf{u}$$

$$+\tau[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\nabla]\mathbf{u} + (\ell+1)\tau R(\mathbf{u}\cdot\nabla)\nabla\theta + O(|\nabla\mathbf{z}|^2) \tag{3.3}$$

and the reduced equation for the temperature

$$\partial_t\theta + \frac{R}{c_V}\theta\mathrm{div}\mathbf{u} + \mathbf{u}\cdot\nabla\theta = \tau\frac{\theta^2}{c_V\rho}\langle R_\alpha^2\Delta\rho_\alpha\rangle$$

$$+(\ell+1)\tau\frac{R\theta}{c_V}(\mathbf{u}\cdot\nabla)\mathrm{div}\mathbf{u} + \ell\tau[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\nabla]\theta + \left(\frac{\varkappa}{c_V\rho} + \tau\frac{\langle R_\alpha p_\alpha\rangle}{c_V\rho}\right)\Delta\theta + O(|\nabla\mathbf{z}|^2). \tag{3.4}$$

Hereafter, $\Delta = \mathrm{div}\nabla$ is the Laplace operator and $\chi := \frac{1}{3}\mu + \lambda$.

Below the reduced system of Equations (3.2)–(3.4) is a very convenient basis for both the linearization of the original QGD and QHD systems and their parabolicity analysis. Note that the left-hand sides of these equations are independent of τ and ℓ.

In the case under consideration, when $\mathbf{f} = 0$ and $\mathbf{Q} = 0$, the QGD and QHD system of equations (2.14)–(2.16) has constant solutions

$$(\rho,\mathbf{u},\theta)(x,t) \equiv \mathbf{z}_0 = (\rho_{10},\ \dots\ ,\rho_{K0},\mathbf{u}_0,\theta_0),\ \rho_{10} > 0,\ \dots\ ,\rho_{K0} > 0,\ \theta_0 > 0.$$

We linearize the solution to this system on a background solution $\mathbf{z}_0$ writing it in the form

$$\rho_\alpha = \rho_{\alpha 0} + \rho_{\alpha *}\tilde{\rho}_\alpha\ (\alpha = \overline{1,K}),\ \mathbf{u} = \mathbf{u}_0 + u_*\widetilde{\mathbf{u}},\ \theta = \theta_0 + \theta_*\tilde{\theta}, \tag{3.5}$$

where $\rho_{\alpha *} > 0$, $u_* > 0$ and $\theta_* > 0$ are normalizing parameters selected below. We introduce the vector of dimensionless perturbations $\widetilde{\mathbf{z}} := (\tilde{\rho},\widetilde{\mathbf{u}},\tilde{\theta})$ with $\tilde{\rho} := (\tilde{\rho}_1,\ \dots\ ,\tilde{\rho}_K)$. Below it is essential that the choice $\rho_{\alpha *} \neq \rho_{\alpha 0}$ is possible, in contrast to the one used in previous works.[6–8]

We introduce the additional notation for the background normalized solution and values of $\rho,\ c_\alpha,\ R$ and c_V as well as introduce the background mean values of $R_\alpha\gamma_\alpha$ and R_α^2

$$\hat{\rho}_{\alpha 0} := \frac{\rho_{\alpha 0}}{\rho_{\alpha *}},\ \hat{\mathbf{u}}_0 = (\hat{u}_{10},\ \dots\ ,\hat{u}_{n0}) := \frac{\mathbf{u}_0}{u_*},\ \hat{\theta}_0 := \frac{\theta_0}{\theta_*},\ \rho_0 := \langle\rho_{\alpha 0}\rangle,\ c_{\alpha_0} := \frac{\rho_{\alpha 0}}{\rho_0},$$

$$R_0 := \left\langle\frac{\rho_{\alpha 0}}{\rho_0}R_\alpha\right\rangle,\ c_{V0} := \left\langle\frac{\rho_{\alpha 0}}{\rho_0}c_{V\alpha}\right\rangle,\ (R\gamma)_0 = \left\langle\frac{\rho_{\alpha 0}}{\rho_0}R_\alpha\gamma_\alpha\right\rangle,\ (R^2)_0 = \left\langle\frac{\rho_{\alpha 0}}{\rho_0}R_\alpha^2\right\rangle.$$

Let also $\tau_0,\ \mu_0,\ \chi_0$ and $\varkappa_0$ be the background values of $\tau,\ \mu,\ \chi$ and $\varkappa$, that is, their values on the background solution.

We substitute the solution in the form (3.5) directly into the reduced system (3.2)–(3.4) rather than into the original QGD and QHD systems or the system of Equations (2.14), (2.25), and (2.26), in contrast to the approach used in papers.[6–8] Since

$$\nabla\mathbf{z} = (\rho_{1*}\nabla\tilde{\rho}_1,\ \dots\ ,\rho_{K*}\nabla\tilde{\rho}_K,u_*\nabla\widetilde{\mathbf{u}},\theta_*\nabla\tilde{\theta}),\ O(|\nabla\mathbf{z}|^2) = O(|\nabla\widetilde{\mathbf{z}}|^2), \tag{3.6}$$

after discarding the terms of the second order of smallness with respect to the vector function $\widetilde{\mathbf{z}}$ and its first- and second-order derivatives and dividing the equations by $\rho_{\alpha *},\ u_*$ and θ_*, respectively, we can easily obtain the linearized system of equations.

Furthermore, to simplify the analysis of the resulting system of linearized equations, the possibility of symmetrization for both the convective terms (i.e., containing the first derivatives with respect to x) and dissipative terms (i.e., containing the second derivatives) is crucial. This is achieved under the following choice of normalizing parameters containing a free parameter

$$\rho_{\alpha *} = b\sqrt{\frac{\rho_{\alpha 0}c_{V0}\rho_0}{R_\alpha}},\ \alpha = \overline{1,K},\ u_* = b\sqrt{c_{V0}\theta_0},\ \theta_* = b\theta_0\ \forall b > 0. \tag{3.7}$$

This choice is assumed to be valid below allowing us to simplify significantly the form of the linearized system of equations as follows

$$\partial_t\tilde{\rho}_\alpha + u_*\left(\hat{\mathbf{u}}_0\cdot\nabla\tilde{\rho}_\alpha + \hat{\rho}_{\alpha 0}\mathrm{div}\widetilde{\mathbf{u}}\right) = \tau_0 u_*^2\left[a_\alpha\hat{\theta}_0\Delta\tilde{\rho}_\alpha + \ell(\hat{\mathbf{u}}_0\cdot\nabla)^2\tilde{\rho}_\alpha + (\ell+1)\hat{\rho}_{\alpha 0}(\hat{\mathbf{u}}_0\cdot\nabla)\mathrm{div}\widetilde{\mathbf{u}} + a_\alpha\hat{\rho}_{\alpha 0}\Delta\tilde{\theta}\right],\ \alpha=\overline{1,K}, \tag{3.8}$$

$$\partial_t\widetilde{\mathbf{u}} + u_*\left(\langle\hat{\rho}_{\alpha 0}\nabla\tilde{\rho}_\alpha\rangle + (\hat{\mathbf{u}}_0\cdot\nabla)\widetilde{\mathbf{u}} + a_0\nabla\tilde{\theta}\right) = u_*^2\left[(\ell+1)\tau_0(\hat{\mathbf{u}}_0\cdot\nabla)\langle\hat{\rho}_{\alpha 0}\nabla\tilde{\rho}_\alpha\rangle + \bar{\mu}_0\Delta\widetilde{\mathbf{u}} + \left(\bar{\chi}_0 + \ell\tau_0\hat{\theta}_0(a\gamma)_0\right)\nabla\mathrm{div}\widetilde{\mathbf{u}} + \tau_0(\hat{\mathbf{u}}_0\cdot\nabla)^2\widetilde{\mathbf{u}} + (\ell+1)\tau_0 a_0(\hat{\mathbf{u}}_0\cdot\nabla)\nabla\tilde{\theta}\right], \tag{3.9}$$

$$\partial_t\tilde{\theta} + u_*\left(a_0\mathrm{div}\widetilde{\mathbf{u}} + \hat{\mathbf{u}}_0\cdot\nabla\tilde{\theta}\right) = u_*^2\left[\tau_0\langle a_\alpha\hat{\rho}_{\alpha 0}\Delta\tilde{\rho}_\alpha\rangle + (\ell+1)\tau_0 a_0(\hat{\mathbf{u}}_0\cdot\nabla)\mathrm{div}\widetilde{\mathbf{u}} + \ell\tau_0(\hat{\mathbf{u}}_0\cdot\nabla)^2\tilde{\theta} + (\tau_0(a^2)_0 + \bar{\varkappa}_0)\Delta\tilde{\theta}\right], \tag{3.10}$$

where, for convenience, we have introduced the constants

$$a_\alpha := \frac{R_\alpha\theta_*}{u_*^2},\ a_0 := \left\langle\frac{\rho_{\alpha 0}}{\rho_0}a_\alpha\right\rangle = \frac{R_0\theta_*}{u_*^2},\ (a\gamma)_0 = \left\langle\frac{\rho_{\alpha 0}}{\rho_0}a_\alpha\gamma_\alpha\right\rangle,\ (a^2)_0 := \left\langle\frac{\rho_{\alpha 0}}{\rho_0}a_\alpha^2\right\rangle,$$

$$\bar{\mu}_0 := \frac{\mu_0}{\rho_0 u_*^2},\ \bar{\chi}_0 := \frac{\chi_0}{\rho_0 u_*^2},\ \bar{\varkappa}_0 := \frac{\varkappa_0}{c_{V0}\rho_0 u_*^2}.$$

Also, the common factors u_* and u_*^2 are carried out of the convective and dissipative terms, respectively, and the following operator is used

$$(\hat{\mathbf{u}}_0\cdot\nabla)^2 = (\hat{\mathbf{u}}_0\cdot\nabla)(\hat{\mathbf{u}}_0\cdot\nabla) = \sum_{i,j=1}^n \hat{u}_{0i}\hat{u}_{0j}\partial_i\partial_j.$$

Let Ω be a domain in $\mathbb{R}^n$. We introduce inner products and norms $(\cdot,\cdot)_\Omega = (\cdot,\cdot)_{L^2(\Omega)}$ and $\|\cdot\|_\Omega = \|\cdot\|_{L^2(\Omega)}$ as well as $(\cdot,\cdot)_\Omega = (\cdot,\cdot)_{\mathbf{L}^2(\Omega)}$ and $\|\cdot\|_\Omega = \|\cdot\|_{\mathbf{L}^2(\Omega)}$ in the Lebesgue spaces of, respectively, functions and vector-functions defined on Ω. Let $\mathbf{H}^1(\Omega) = \mathbf{W}_2^1(\Omega)$ be a Sobolev space of vector functions, and $\mathbf{H}_0^1(\Omega)$ be the closure in the $\mathbf{H}^1(\Omega)$-norm of the space $\mathbf{D}(\Omega)$ of smooth vector-functions with a compact support in Ω. Recall that $\mathbf{H}_0^1(\mathbb{R}^n) = \mathbf{H}^1(\mathbb{R}^n)$.

We consider the system of Equations (3.8)–(3.10) in the cylinder $\Omega\times(0,\infty)$ supplemented with the boundary and initial conditions

$$\widetilde{\mathbf{z}}|_{\partial\Omega\times(0,\infty)} = 0,\ \widetilde{\mathbf{z}}|_{t=0} = \widetilde{\mathbf{z}}^{(0)}(x). \tag{3.11}$$

Let us analyze such an initial-boundary value problem. For $\partial_t\widetilde{\mathbf{z}}(\cdot,t), \nabla\widetilde{\mathbf{z}}(\cdot,t)\in\mathbf{L}^2(\Omega)$, equations (3.8)–(3.10) correspond to the integral identity

$$\left(\partial_t\widetilde{\mathbf{z}}(\cdot,t),\mathbf{z}\right)_\Omega + u_*\mathcal{B}_\Omega(\widetilde{\mathbf{z}}(\cdot,t),\mathbf{z}) + u_*^2\mathcal{A}_\Omega(\widetilde{\mathbf{z}}(\cdot,t),\mathbf{z}) = 0,\ \forall t>0,\ \forall\mathbf{z}\in\mathbf{H}_0^1(\Omega), \tag{3.12}$$

with $\mathbf{z} = (\rho_1,\ldots,\rho_K,\mathbf{u},\theta)(x)$ (it should not be confused with the solution to the QGD and QHD systems which was denoted above in the same way) and the following bilinear forms

$$\mathcal{B}_\Omega(\widetilde{\mathbf{z}},\mathbf{z}) := \left\langle\left(\hat{\mathbf{u}}_0\cdot\nabla\tilde{\rho}_\alpha + \hat{\rho}_{\alpha 0}\mathrm{div}\widetilde{\mathbf{u}},\rho_\alpha\right)_\Omega\right\rangle + \left(\langle\hat{\rho}_{\alpha 0}\nabla\tilde{\rho}_\alpha\rangle + (\hat{\mathbf{u}}_0\cdot\nabla)\widetilde{\mathbf{u}} + a_0\nabla\tilde{\theta},\mathbf{u}\right)_\Omega + \left(a_0\mathrm{div}\widetilde{\mathbf{u}} + \hat{\mathbf{u}}_0\cdot\nabla\tilde{\theta},\theta\right)_\Omega$$

and

$$\mathcal{A}_\Omega(\widetilde{\mathbf{z}},\mathbf{z}) := \bar{\mu}_0\left(\nabla\widetilde{\mathbf{u}},\nabla\mathbf{u}\right)_\Omega + \bar{\chi}_0\left(\mathrm{div}\widetilde{\mathbf{u}},\mathrm{div}\mathbf{u}\right)_\Omega + \bar{\varkappa}_0\left(\nabla\tilde{\theta},\nabla\theta\right)_\Omega + \tau_0\left[\left\langle\left(a_\alpha\hat{\theta}_0\nabla\tilde{\rho}_\alpha,\nabla\rho_\alpha\right)_\Omega\right\rangle + \ell\left\langle\left(\hat{\mathbf{u}}_0\cdot\nabla\tilde{\rho}_\alpha,(\hat{\mathbf{u}}_0\cdot\nabla)\rho_\alpha\right)_\Omega\right\rangle + (\ell+1)\left((\hat{\mathbf{u}}_0\cdot\nabla)\widetilde{\mathbf{u}},\langle\hat{\rho}_{\alpha 0}\nabla\rho_\alpha\rangle\right)_\Omega\right.$$

$$+\big(\nabla\tilde{\theta},\langle a_\alpha\hat{\rho}_{\alpha 0}\nabla\rho_\alpha\rangle\big)_\Omega+(\ell+1)\big(\langle\hat{\rho}_{\alpha 0}\nabla\tilde{\rho}_\alpha\rangle,(\hat{\mathbf{u}}_0\cdot\nabla)\mathbf{u}\big)_\Omega+\ell\big(\hat{\theta}_0(a\gamma)_0\mathrm{div}\tilde{\mathbf{u}},\mathrm{div}\mathbf{u}\big)_\Omega$$

$$+\big((\hat{\mathbf{u}}_0\cdot\nabla)\tilde{\mathbf{u}},(\hat{\mathbf{u}}_0\cdot\nabla)\mathbf{u}\big)_\Omega+(\ell+1)\big(a_0\nabla\tilde{\theta},(\hat{\mathbf{u}}_0\cdot\nabla)\mathbf{u}\big)_\Omega$$

$$+(\langle a_\alpha\hat{\rho}_{\alpha 0}\nabla\tilde{\rho}_\alpha\rangle,\nabla\theta)_\Omega+(\ell+1)\big((\hat{\mathbf{u}}_0\cdot\nabla)\tilde{\mathbf{u}},a_0\nabla\theta\big)_\Omega+\ell\big(\hat{\mathbf{u}}_0\cdot\nabla\tilde{\theta},\hat{\mathbf{u}}_0\cdot\nabla\theta\big)_\Omega+\big((a^2)_0\nabla\tilde{\theta},\nabla\theta\big)_\Omega\Big].$$

Here the tensors $\nabla\tilde{\mathbf{u}}$ and $\nabla\mathbf{u}$ are treated as vectors of length n^2. Notice that we restrict ourselves to the real case and will not introduce matrix notation of forms (unlike previous works[6–8]).

The next lemma on the properties of the defined bilinear forms plays the key role below.

Lemma 3.1. *The following properties hold*

$$\mathcal{B}_\Omega(\mathbf{z},\mathbf{z})=0\ \forall\mathbf{z}\in\mathbf{H}_0^1(\Omega), \tag{3.13}$$

$$\mathcal{A}_\Omega(\tilde{\mathbf{z}},\mathbf{z})=\mathcal{A}_\Omega(\mathbf{z},\tilde{\mathbf{z}})\ \forall\tilde{\mathbf{z}},\mathbf{z}\in\mathbf{H}^1(\Omega). \tag{3.14}$$

Moreover, let $\mathbf{z}\in\mathbf{H}^1(\Omega)$ *and* $\mathbf{u}\in\mathbf{H}_0^1(\Omega)$. *The following formula holds*

$$\mathcal{A}_\Omega(\mathbf{z},\mathbf{z})=\bar{\mu}_0\|\nabla\mathbf{u}\|_\Omega^2+\bar{\chi}_0\|\mathrm{div}\mathbf{u}\|_\Omega^2+\bar{\varkappa}_0\|\nabla\theta\|_\Omega^2$$
$$+\tau_0\Big[\ell\left\langle\|\hat{\mathbf{u}}_0\cdot\nabla\rho_\alpha+\hat{\rho}_{\alpha 0}\mathrm{div}\mathbf{u}\|_\Omega^2\right\rangle+\left\langle\|(\hat{\theta}_0a_\alpha)^{1/2}\nabla\rho_\alpha+\sqrt{c_{\alpha 0}}(\hat{\mathbf{u}}_0\cdot\nabla)\mathbf{u}+\sqrt{c_{\alpha 0}}a_\alpha\nabla\theta\|_\Omega^2\right\rangle$$
$$+\ell\left\|a_0\mathrm{div}\mathbf{u}+\hat{\mathbf{u}}_0\cdot\nabla\theta\right\|_\Omega^2+\ell g_0\|\mathrm{div}\mathbf{u}\|_\Omega^2\Big]\geq 0, \tag{3.15}$$

where

$$g_0:=\frac{\theta_0}{u_*^2}\left\langle\frac{\rho_{\alpha 0}}{\rho_0}c_{V\alpha}(\gamma_\alpha-\bar{\gamma}_0)^2\right\rangle\geq 0,\ \bar{\gamma}_0:=\frac{R_0}{c_{V0}}-1. \tag{3.16}$$

Consequently, the following lower bounds hold

$$\mathcal{A}_\Omega(\mathbf{z},\mathbf{z})\geq\delta_1\tau_0\left\langle\|\nabla\rho_\alpha\|_\Omega^2\right\rangle, \tag{3.17}$$

$$\mathcal{A}_\Omega(\mathbf{z},\mathbf{z})\geq\bar{\mu}_0\|\nabla\mathbf{u}\|_\Omega^2+\bar{\chi}_0\|\mathrm{div}\mathbf{u}\|_\Omega^2+\bar{\varkappa}_0\|\nabla\theta\|_\Omega^2, \tag{3.18}$$

and, moreover,

$$\mathcal{A}_\Omega(\mathbf{z},\mathbf{z})\geq\delta_2\left(\left\langle\|\nabla\rho_\alpha\|_\Omega^2\right\rangle+\|\nabla\mathbf{u}\|_\Omega^2+\|\nabla\theta\|_\Omega^2\right)$$

with $\delta_2:=\frac{1}{2}\min\{\delta_1\tau_0,\bar{\mu}_0,\bar{\varkappa}_0\}>0$ *and*

$$\delta_1:=\frac{1}{2}(1+\max\{2\delta_0-1,0\})^{-1}\hat{\theta}_0\min_{\alpha=\overline{1,K}}a_\alpha,\ \delta_0:=\tau_0\max\left\{\frac{|\hat{\mathbf{u}}_0|^2}{\bar{\mu}_0},\frac{(a^2)_0}{\bar{\varkappa}_0}\right\}.$$

Let $\mathbf{V}(Q_T)$ be the space of vector functions $\tilde{\mathbf{z}}\in L^2((0,T);\mathbf{H}_0^1(\Omega))$ having a distributional derivative $\partial_t\tilde{\mathbf{z}}\in L^2((0,T);\mathbf{H}^{-1}(\Omega))$, where $Q_T=\Omega\times(0,T)$ is a cylinder, the domain Ω is bounded and $\mathbf{H}^{-1}(\Omega)=(\mathbf{H}_0^1(\Omega))^*$, for example, see these notions in Gaewski et al.[24]

For the initial-boundary value problem for the system of equations (3.8)–(3.10) in $Q:=\Omega\times(0,\infty)$ supplemented with conditions (3.11), we introduce *the weak solution* $\tilde{\mathbf{z}}\in\mathbf{V}(Q_T)$, for any $T>0$, satisfying the integral identity

$$\int_0^T\left\langle\partial_t\tilde{\mathbf{z}}(\cdot,t),\mathbf{z}(\cdot,t)\right\rangle_\Omega dt+u_*\mathcal{B}_{Q_T}(\tilde{\mathbf{z}},\mathbf{z})+u_*^2\mathcal{A}_{Q_T}(\tilde{\mathbf{z}},\mathbf{z})=0\ \forall\mathbf{z}\in\mathbf{L}^2((0,T);\mathbf{H}_0^1(\Omega)), \tag{3.19}$$

for any $T>0$, and the initial condition $\widetilde{\mathbf{z}}|_{t=0}=\widetilde{\mathbf{z}}^{(0)}\in\mathbf{L}^2(\Omega)$. Here $\langle\cdot,\cdot\rangle_\Omega$ is the duality relation on $\mathbf{H}^{-1}(\Omega)\times\mathbf{H}_0^1(\Omega)$, and the inner products in the bilinear forms $\mathcal{B}_{Q_T}$ and $\mathcal{A}_{Q_T}$ are taken over Q_T instead of Ω as above. The well-known embedding $\mathbf{V}(Q_T)\subset C([0,T];\mathbf{L}^2(\Omega))$[24] allows one to understand the fulfillment of the initial condition by continuity in $\mathbf{L}^2(\Omega)$. Note that formally identity (3.19) is derived from (3.12) for $\mathbf{z}=\mathbf{z}(\cdot,t)$ by integration over $(0,T)$.

Now we state the second main result.

Theorem 3.2. *The introduced weak solution $\widetilde{\mathbf{z}}\in\mathbf{V}(Q_T)$, for any $T>0$, to the initial-boundary value problem (3.8)–(3.11) for the linearized system of equations exists, is unique and satisfies the following energy equality*

$$\frac{1}{2}\|\widetilde{\mathbf{z}}(\cdot,T)\|^2_{\mathbf{L}^2(\Omega)}+u_*^2\mathcal{A}_{Q_T}(\widetilde{\mathbf{z}},\widetilde{\mathbf{z}})=\frac{1}{2}\|\widetilde{\mathbf{z}}^{(0)}\|^2_{\mathbf{L}^2(\Omega)}\ \forall T>0. \tag{3.20}$$

Consequently, the function $\|\widetilde{\mathbf{z}}(\cdot,t)\|_{\mathbf{L}^2(\Omega)}$ does not increase for $t\geq 0$, that is, the initial-boundary value problem is the $\boldsymbol{L}^2(\Omega)$-dissipative, and the following energy bound holds

$$\max\left\{\max_{t\geq 0}\|\widetilde{\mathbf{z}}(\cdot,t)\|_{\mathbf{L}^2(\Omega)},\ \sqrt{2\delta_1\tau_0}u_*\left\langle\|\nabla\tilde{\rho}_\alpha\|^2_{\mathbf{L}^2(Q)}\right\rangle^{1/2},\right.$$
$$\left.\sqrt{2}u_*\left(\bar{\mu}_0\|\nabla\mathbf{u}\|^2_{\mathbf{L}^2(Q)}+\bar{\chi}_0\|\mathrm{div}\mathbf{u}\|^2_{L^2(Q)}+\bar{\varkappa}_0\|\nabla\theta\|^2_{\mathbf{L}^2(Q)}\right)^{1/2}\right\}\leq\|\widetilde{\mathbf{z}}^{(0)}\|_{\mathbf{L}^2(\Omega)}. \tag{3.21}$$

Corollary 3.1. *There exists the derivative $\partial_t\left(\|\widetilde{\mathbf{z}}(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)}\right)\in L^1(0,\infty)$, and another form of the energy equality holds*

$$\frac{1}{2}\partial_t\left(\|\widetilde{\mathbf{z}}(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)}\right)+u_*^2\mathcal{A}_\Omega(\widetilde{\mathbf{z}}(\cdot,t),\widetilde{\mathbf{z}}(\cdot,t))=0 \tag{3.22}$$

for almost all $t>0$ and, as a consequence, the following sharpened form of the $\mathbf{L}^2(\Omega)$-dissipativity property holds: $\partial_t\left(\|\widetilde{\mathbf{z}}(\cdot,t)\|^2_{\mathbf{L}^2(\Omega)}\right)\leq 0$ for almost all $t>0$.

The energy equality (3.22) and Lemma 3.1 imply the exponential decay $\|\widetilde{\mathbf{z}}(\cdot,t)\|_{\mathbf{L}^2(\Omega)}\leq e^{-\delta_2u_*^2t}\|\widetilde{\mathbf{z}}^{(0)}\|_{\mathbf{L}^2(\Omega)}$ for $t\geq 0$.

Methods of theory of linear parabolic equations (see, in particular, previous works[24–26]) can be applied to establish various regularity theorems for $\widetilde{\mathbf{z}}$ which we will not dwell on.

3.3 | The Petrovskii parabolicity and the classical local in time existence and uniqueness of solutions to the Cauchy problem for the QGD and QHD systems of equations

For the QGD and QHD systems of equations in the one-component case, the Petrovskii parabolicity was analyzed in papers.[6–8] We perform a similar analysis for the QGD and QHD system (2.14)–(2.16) based on the above results and first turn to the equivalent system of Equations (2.14) and (2.25)–(2.26). This quasilinear system can be written in the canonical vector form

$$\partial_t\mathbf{z}=\sum_{i,j=1}^n A_{ij}(\mathbf{z})\partial_i\partial_j\mathbf{z}+\mathbf{R}(x,t,\mathbf{z},\nabla\mathbf{z}) \tag{3.23}$$

(cf. the derivation of the reduced system (3.2)–(3.4) in Section 4 below), where A_{ij} and $\mathbf{R}$ are matrices and a vector of order $K+n+1$ which we do not write out explicitly. For $\tau,\mu,\lambda,\varkappa\in C^2(\mathcal{D}_+)$ with $\mathcal{D}_+:=(0,\infty)^K\times\mathbb{R}^n\times(0,\infty)$, we have $A_{ij}\in C^2(\mathcal{D}_+)$ and $\mathbf{R}$ is C^1-smooth with respect to its arguments $\mathbf{z}$ and $\nabla\mathbf{z}$. The dependence of $\mathbf{R}$ on the variables (x,t) is related only to the dependence of the functions $\mathbf{f}$, $\nabla\mathbf{f}$, $\mathbf{Q}$ and $\nabla\mathbf{Q}$ on them which are included in the expression for $\mathbf{R}$ (in this subsection, we once again consider general $\mathbf{f}$ and $\mathbf{Q}$).

To analyze the parabolicity of this system, it is sufficient to discard the convective terms on the left and remainders $O(|\nabla\mathbf{z}|^2)$ on the right in the equations of the reduced system (3.2)–(3.4). Then, in the resulting simplified system of equations containing only the derivatives ∂_t and $\partial_i\partial_j$, we "freeze" the coefficients depending on the solution $\mathbf{z}$ in front of $\partial_i\partial_j$ and accomplish the Fourier transform with respect to x

$$\mathcal{F}\mathbf{z}(\zeta,t)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\mathbf{z}(x,t)e^{-\mathrm{i}x\cdot\zeta}\,dx,\ \zeta\in\mathbb{R}^n,$$

where i is the imaginary unit. The coefficients are "freezed" at some (any) point $\mathbf{z}_0 = (\rho_{10}, \dots, \rho_{K0}, \mathbf{u}_0, \theta_0) \in \mathcal{D}_+$ already taken above as a background solution. As a result, we obtain a system of ordinary differential equations of the form

$$\partial_t \mathcal{F}\mathbf{z}(\zeta, t) + |\zeta|^2 u_*^2 A(\mathbf{z}_0, \xi)\mathcal{F}\mathbf{z}(\zeta, t) = 0,\ t > 0, \tag{3.24}$$

where

$$u_*^2 A(\mathbf{z}_0, \xi) = \sum_{i,j=1}^{n} A_{ij}(\mathbf{z}_0)\xi_i\xi_j,\ \xi = \frac{\zeta}{|\zeta|},$$

with the parameter $\zeta \in \mathbb{R}^n \setminus \{0\}$, the normalization factor $u_* = u_*(\mathbf{z}_0) > 0$ (we assume that in (3.7) the parameter b does not depend on $\mathbf{z}_0$) and a real matrix $A(\mathbf{z}_0, \xi)$ of order $K + n + 1$ (its explicit form is not required as we explain soon).

Let $\lambda[A(\mathbf{z}_0, \xi)]$ be the eigenvalues of the introduced matrix, and $\Re\,\lambda$ be the real part of λ. According to monograhps[26, Ch. 3, Section 4,27] the property of *the uniform Petrovskii parabolicity* in some bounded subdomain $\mathcal{D} \subset \mathcal{D}_+$ means that

$$\inf_{\mathbf{z}_0 \in \mathcal{D}} \left(u_*^2 \inf_{|\xi|=1} \Re\,\lambda[A(\mathbf{z}_0, \xi)] \right) > 0. \tag{3.25}$$

Let $-u_*^2 \hat{A}(\mathbf{z}_0, \xi)$ be the matrix arising directly by application of $\mathcal{F}$ to the right-hand sides of the symmetrized linearized system (3.8)–(3.10) (instead of the simplified reduced system (3.2)–(3.4) as above). Then the following properties hold

$$\hat{A}(\mathbf{z}_0, \xi) = [\hat{A}(\mathbf{z}_0, \xi)]^T,\ \lambda[\hat{A}(\mathbf{z}_0, \xi)] = \lambda[A(\mathbf{z}_0, \xi)] \in \mathbb{R}. \tag{3.26}$$

The next lemma allows to study the uniform Petrovskii parabolicity. Let $s = s(\xi) := \hat{\mathbf{u}}_{0\cdot\xi}$.

Lemma 3.2. *For any block vector* $\mathbf{b} := (\mathbf{r}^T, \mathbf{v}, q)^T$ *with* $\mathbf{r}^T = (r_1, \dots, r_K) \in \mathbb{R}^K$, $\mathbf{v} \in \mathbb{R}^n$, $q \in \mathbb{R}$, *the following formula holds*

$$\hat{A}(\mathbf{z}_0, \xi)\mathbf{b}\cdot\mathbf{b} = \bar{\mu}_0|\mathbf{v}|^2 + \bar{\chi}_0(\xi\cdot\mathbf{v})^2 + \bar{\varkappa}_0 q^2 + \tau_0 \left\{ \ell|s\mathbf{r} + (\xi\cdot\mathbf{v})\hat{\rho}_0|^2 \right.$$
$$\left. + \left\langle \left| (\hat{\theta}_0 a_\alpha)^{1/2} r_\alpha \xi + \sqrt{c_{\alpha 0}}s\mathbf{v} + \sqrt{c_{\alpha 0}}a_\alpha q\xi \right|^2 \right\rangle + \ell[a_0(\xi\cdot\mathbf{v}) + sq]^2 + \ell g_0(\xi\cdot\mathbf{v})^2 \right\} \geq 0 \tag{3.27}$$

and, as a consequence, the following lower bounds hold

$$\hat{A}(\mathbf{z}_0, \xi)\mathbf{b}\cdot\mathbf{b} \geq \delta_1\tau_0|\mathbf{r}|^2,\ \hat{A}(\mathbf{z}_0, \xi)\mathbf{b}\cdot\mathbf{b} \geq \bar{\mu}_0|\mathbf{v}|^2 + \bar{\chi}_0(\xi\cdot\mathbf{v})^2 + \bar{\varkappa}_0 q^2 \tag{3.28}$$

and even more $\hat{A}(\mathbf{z}_0, \xi)\mathbf{b}\cdot\mathbf{b} \geq \delta_2|\mathbf{b}|^2$, *with* $\delta_1 > 0$ *and* $\delta_2 > 0$ *introduced in Lemma 3.1.*

Corollary 3.2. *For* $\tau, \mu, \lambda, \varkappa \in C(\bar{\mathcal{D}})$, *condition (3.25) is satisfied. Therefore, the equivalent quasilinear QGD and QHD system of Equations (2.14) and (2.25)–(2.26) satisfies in* $\mathcal{D}$ *(in fact, in* $\bar{\mathcal{D}}$*) the uniform Petrovskii parabolicity condition.*

The last result plays an important role including the possibility to obtain a theorem on the local in time unique classical solvability of the Cauchy problem for the QGD and QHD system of Equations (2.14)–(2.16) considered in the layer $\Pi_T := \mathbb{R}^n \times (0, T)$, supplemented with the initial conditions

$$\rho|_{t=0} = \rho^{(0)}(x),\ \mathbf{u}|_{t=0} = \mathbf{u}^{(0)}(x),\ \theta|_{t=0} = \theta^{(0)}(x),\ x \in \mathbb{R}^n.$$

To formulate it, we need to recall the required function spaces, for example, see monographs.[25,26] Let $C_b(\mathbb{R}^n)$ and $C_b(\bar{\Pi}_T)$ be the spaces of functions continuous and bounded in $\mathbb{R}^n$ and $\bar{\Pi}_T$, respectively, and $0 < \beta < 1$ and $0 \leq \beta_t < 1$ be parameters. Let $C^{2+\beta}(\mathbb{R}^n)$ be the Hölder space of functions $w \in C_b(\mathbb{R}^n)$ with $\partial_i w, \partial_i\partial_j w \in C_b(\mathbb{R}^n)$ and $\partial_i\partial_j w$ satisfying the Hölder condition of order β uniformly in $\mathbb{R}^n$ for all $i, j = \overline{1, n}$. We also introduce the space of functions $C^{(m,\beta,\beta_t)}(\bar{\Pi}_T)$, $m = 0, 1, 2$, having derivatives of order $k = 0, \dots, m$ with respect to x from $C_b(\bar{\Pi}_T)$ satisfying the Hölder condition of order β with respect to x and β_t with respect to t (for $0 < \beta_t < 1$) uniformly in $\bar{\Pi}_T$. As usual, we assume that a vector-function belongs to some space provided that each of its components belongs to this space.

We introduce a parallelepiped $\mathcal{D} := (\underline{\rho}_1, \bar{\rho}_1) \times \dots \times (\underline{\rho}_K, \bar{\rho}_K) \times (-\bar{u}_1, \bar{u}_1) \times \dots \times (-\bar{u}_n, \bar{u}_n) \times (\underline{\theta}, \bar{\theta})$ such that $\bar{\mathcal{D}} \subset \mathcal{D}_+$. The next theorem is the third main result of the paper.

Theorem 3.3. *Let $\tau, \mu, \lambda, \varkappa \in C^2(\mathcal{D})$. Let the initial data $\rho^{(0)}, \mathbf{u}^{(0)}, \theta^{(0)} \in C^{2+\beta}(\mathbb{R}^n)$, and their values $(\rho^{(0)}, \mathbf{u}^{(0)}, \theta^{(0)})(x)$ belong to a compact set in $\mathcal{D}$, and $\mathbf{f}, \mathbf{Q} \in C^{(1,\beta,0)}(\bar{\Pi}_T)$.*

Then, for a sufficiently small $T > 0$, the Cauchy problem for the QGD and QHD system of equations (2.14)–(2.16) in the layer Π_T has a unique classical solution $\rho, \mathbf{u}, \theta \in C^{(2,\beta,\beta/2)}(\bar{\Pi}_T)$ with $\partial_t\rho, \partial_t\mathbf{u}, \partial_t\theta \in C^{(0,\beta,0)}(\bar{\Pi}_T)$, and its values $(\rho, \mathbf{u}, \theta)(x,t)$ belong to $\mathcal{D}$.

4 | PROOFS OF THE RESULTS

It will be convenient to divide all the proofs into seven Subsections.

4.1 | The derivation of corollaries from the main equations

We first give the derivation of corollaries from Equations (2.14)–(2.16), which will be essential below. Application of the operation $\langle\cdot\rangle$ to Equation (2.14) (i.e., its summation over $\alpha = \overline{1,K}$) together with the property $\langle \mathbf{d}_\alpha \rangle = 0$ lead to the important *total mass balance equation* (2.22).

We take the inner product of the momentum balance Equation (2.15) and $\mathbf{u}$, use the formulas

$$[\partial_t(\rho\mathbf{u})] \cdot \mathbf{u} = \frac{1}{2}\partial_t(\rho|\mathbf{u}|^2) + \frac{1}{2}(\partial_t\rho)|\mathbf{u}|^2,$$

$$\operatorname{div}[\rho(\mathbf{u} - \mathbf{w}_\ell) \otimes \mathbf{u}] \cdot \mathbf{u} = \frac{1}{2}\operatorname{div}\left[\rho(\mathbf{u} - \mathbf{w}_\ell)|\mathbf{u}|^2\right] + \frac{1}{2}\{\operatorname{div}[\rho(\mathbf{u} - \mathbf{w}_\ell)]\}|\mathbf{u}|^2$$

and the total mass balance Equation (2.22) and obtain *the total kinetic energy balance equation* (2.23). Then, we subtract it from the total energy balance Equation (2.16), apply the formulas

$$\langle \rho_\alpha h_\alpha(\mathbf{u} - \mathbf{w}_{\ell\alpha}) \rangle = \langle \rho_\alpha \varepsilon_\alpha(\mathbf{u} - \mathbf{w}_{\ell\alpha}) \rangle + p\mathbf{u} - \langle p_\alpha \mathbf{w}_{\ell\alpha} \rangle, \ \operatorname{div}(p\mathbf{u}) = (\nabla p) \cdot \mathbf{u} + p\operatorname{div}\mathbf{u},$$

$$\operatorname{div}(\Pi\mathbf{u}) = (\operatorname{div}\Pi) \cdot \mathbf{u} + \Pi : \nabla\mathbf{u}, \ \rho\mathbf{w} = \tau\operatorname{div}(\rho\mathbf{u})\mathbf{u} + \rho\hat{\mathbf{w}} \tag{4.1}$$

and get *the total internal energy balance equation* (2.24).

In addition, similarly to the second formula (4.1), we have

$$\rho_\alpha\mathbf{w}_\alpha = \tau\operatorname{div}(\rho_\alpha\mathbf{u})\mathbf{u} + \rho_\alpha\hat{\mathbf{w}}_\alpha = \tau(\nabla\rho_\alpha \cdot \mathbf{u} + \rho_\alpha\operatorname{div}\mathbf{u})\mathbf{u} + \rho_\alpha\hat{\mathbf{w}}_\alpha. \tag{4.2}$$

We multiply these equalities by $R_\alpha\theta$ and sum up over α. Since also $\nabla(R\rho) = \langle R_\alpha \nabla\rho_\alpha \rangle$, we get

$$-\mathbf{q}^\tau + \langle p_\alpha \mathbf{w}_\alpha \rangle = \tau\left(c_V\rho\nabla\theta \cdot \mathbf{u} + p\operatorname{div}\mathbf{u} - Q\right)\mathbf{u} + \langle p_\alpha \hat{\mathbf{w}}_\alpha \rangle. \tag{4.3}$$

Differentiating the terms on the left in the momentum balance Equation (2.15), we obtain

$$(\partial_t\rho)\mathbf{u} + \rho\partial_t\mathbf{u} + \operatorname{div}[\rho(\mathbf{u} - \mathbf{w}_\ell)]\mathbf{u} + \rho[(\mathbf{u} - \mathbf{w}_\ell) \cdot \nabla]\mathbf{u} + \nabla p = \operatorname{div}(\Pi^{NS} + \Pi^\tau_\ell) + [\rho - \ell\tau\operatorname{div}(\rho\mathbf{u})]\mathbf{f}.$$

In virtue of the total mass balance Equation (2.22) and after calculating $\operatorname{div}\Pi^\tau_\ell$ and dividing the result by ρ, we derive *the velocity balance equation* (2.25).

Next, consider Equation (2.24). Since $\rho\varepsilon = \langle c_{V\alpha}\rho_\alpha \rangle\theta$ and $\rho_\alpha\varepsilon_\alpha = c_{V\alpha}\rho_\alpha\theta$, we have

$$\partial_t(\rho\varepsilon) = \langle c_{V\alpha}\partial_t\rho_\alpha \rangle\theta + \langle c_{V\alpha}\rho_\alpha \rangle\partial_t\theta,$$

$$\operatorname{div}\langle \rho_\alpha\varepsilon_\alpha(\mathbf{u} - \mathbf{w}_{\ell\alpha}) \rangle = \langle c_{V\alpha}\operatorname{div}[\rho_\alpha(\mathbf{u} - \mathbf{w}_{\ell\alpha})] \rangle\theta + \langle c_{V\alpha}\rho_\alpha(\mathbf{u} - \mathbf{w}_{\ell\alpha}) \rangle \cdot \nabla\theta.$$

Using the mass balance equations for the components (2.14) and dividing the result by $\langle c_{V\alpha}\rho_\alpha \rangle = c_V\rho$, we derive *the temperature balance equation* (2.26).

4.2 Proof of Theorem 3.1

We are ready to prove Theorem 3.1. Let us perform sequentially the following transformations in virtue of the formula $\rho S = \langle \rho_\alpha s_\alpha \rangle$, the mass balance equations for the mixture components (2.14) and the definition of s_α, see (2.19):

$$\begin{aligned}
\partial_t(\rho S)&+\mathrm{div}\,\langle \rho_\alpha s_\alpha(\mathbf{u}-\mathbf{w}_{\ell\alpha})\rangle = \langle \partial_t(\rho_\alpha s_\alpha) + \mathrm{div}\,[\rho_\alpha s_\alpha(\mathbf{u}-\mathbf{w}_{\ell\alpha})]\rangle \\
&= \langle \{\partial_t\rho_\alpha + \mathrm{div}\,[\rho_\alpha(\mathbf{u}-\mathbf{w}_{\ell\alpha})]\}\, s_\alpha + \rho_\alpha\,[\partial_t s_\alpha + \nabla s_\alpha\cdot(\mathbf{u}-\mathbf{w}_{\ell\alpha})]\rangle \\
&= \left\langle -s_\alpha \mathrm{div}\mathbf{d}_\alpha - R_\alpha\,[\partial_t\rho_\alpha + \nabla\rho_\alpha\cdot(\mathbf{u}-\mathbf{w}_{\ell\alpha})] + \frac{1}{\theta}\,[\rho_\alpha\partial_t\varepsilon_\alpha + \rho_\alpha\nabla\varepsilon_\alpha\cdot(\mathbf{u}-\mathbf{w}_{\ell\alpha})]\right\rangle \\
&= \langle -s_\alpha \mathrm{div}\mathbf{d}_\alpha - R_\alpha\,\{\partial_t\rho_\alpha + \mathrm{div}\,[\rho_\alpha(\mathbf{u}-\mathbf{w}_{\ell\alpha})] - \rho_\alpha\mathrm{div}(\mathbf{u}-\mathbf{w}_{\ell\alpha})\} \\
&\quad + \frac{1}{\theta}\,\{\partial_t(\rho_\alpha\varepsilon_\alpha) + \mathrm{div}\,[\rho_\alpha\varepsilon_\alpha(\mathbf{u}-\mathbf{w}_{\ell\alpha})] - \varepsilon_\alpha[\partial_t\rho_\alpha + \mathrm{div}\,[\rho_\alpha(\mathbf{u}-\mathbf{w}_{\ell\alpha})]]\}\Big\rangle .
\end{aligned}$$

We also use the formula $\langle \rho_\alpha\varepsilon_\alpha\rangle = \rho\varepsilon$, the balance equations for the mass of components (2.14) and the total internal energy (2.24) as well as the formula $\frac{1}{\theta}\varepsilon_\alpha = c_{V\alpha}$ and get

$$\begin{aligned}
\partial_t(\rho S)&+\mathrm{div}\,\langle \rho_\alpha s_\alpha(\mathbf{u}-\mathbf{w}_{\ell\alpha})\rangle = \langle (-s_\alpha + R_\alpha + c_{V\alpha})\mathrm{div}\mathbf{d}_\alpha + R_\alpha\rho_\alpha\mathrm{div}(\mathbf{u}-\mathbf{w}_{\ell\alpha})\rangle \\
&+ \frac{1}{\theta}\left[-p\mathrm{div}\mathbf{u} + \mathrm{div}(-\mathbf{q} + \langle p_\alpha\mathbf{w}_{\ell\alpha}\rangle) + \Pi : \nabla\mathbf{u} - \rho\hat{\mathbf{w}}\cdot\mathbf{f} + Q\right].
\end{aligned}$$

Applying the definition of G_α, see (2.19), and the formulas $\langle R_\alpha\rho_\alpha\rangle = \frac{p}{\theta}$ and $R_\alpha\rho_\alpha = \frac{p_\alpha}{\theta}$, further we have

$$\begin{aligned}
\partial_t(\rho S)&+\mathrm{div}\,\langle \rho_\alpha s_\alpha(\mathbf{u}-\mathbf{w}_{\ell\alpha})\rangle = \left\langle \frac{G_\alpha}{\theta}\mathrm{div}\mathbf{d}_\alpha\right\rangle - \frac{1}{\theta}\langle p_\alpha\mathrm{div}\mathbf{w}_{\ell\alpha}\rangle \\
&+ \frac{1}{\theta}\left[\mathrm{div}(-\mathbf{q} + \langle p_\alpha\mathbf{w}_{\ell\alpha}\rangle) + \Pi : \nabla\mathbf{u} - \rho\hat{\mathbf{w}}\cdot\mathbf{f} + Q\right].
\end{aligned}$$

Next we perform the following transformations

$$\frac{G_\alpha}{\theta}\mathrm{div}\mathbf{d}_\alpha = \mathrm{div}\left(\frac{G_\alpha}{\theta}\mathbf{d}_\alpha\right) + \frac{1}{\theta}\nabla\theta\cdot\frac{G_\alpha}{\theta}\mathbf{d}_\alpha - \frac{1}{\theta}\nabla G_\alpha\cdot\mathbf{d}_\alpha,$$

$$\frac{1}{\theta}\mathrm{div}(-\mathbf{q}) = \mathrm{div}\left[-\frac{1}{\theta}(\mathbf{q}^F + \ell\mathbf{q}^\tau) - \frac{\mathbf{q}^d}{\theta}\right] + \frac{1}{\theta^2}\nabla\theta\cdot(-\mathbf{q}^F - \ell\mathbf{q}^\tau) + \frac{1}{\theta}\nabla\theta\cdot\left(-\frac{\mathbf{q}^d}{\theta}\right),$$

$$\left\langle\frac{G_\alpha}{\theta}\mathbf{d}_\alpha\right\rangle - \frac{\mathbf{q}^d}{\theta} = -\frac{1}{K}\langle b_\alpha\mathbf{d}_\alpha\rangle,$$

$$\mathrm{div}(p_\alpha\mathbf{w}_{\ell\alpha}) = p_\alpha\mathrm{div}\mathbf{w}_{\ell\alpha} + \nabla p_\alpha\cdot\mathbf{w}_{\ell\alpha}.$$

We apply the definitions of $-\mathbf{q}^F$ and Π and get

$$\begin{aligned}
&\partial_t(\rho S) + \mathrm{div}\left[\left\langle \rho_\alpha s_\alpha(\mathbf{u}-\mathbf{w}_{\ell\alpha}) + \frac{1}{K}b_\alpha\mathbf{d}_\alpha\right\rangle + \frac{1}{\theta}(\mathbf{q}^F + \ell\mathbf{q}^\tau)\right] \\
&= -\frac{1}{K\theta}\nabla\theta\cdot\langle b_\alpha\mathbf{d}_\alpha\rangle - \frac{1}{\theta}\langle\nabla G_\alpha\cdot\mathbf{d}_\alpha\rangle + \frac{1}{\theta^2}\varkappa|\nabla\theta|^2 + \frac{1}{\theta}\Pi^{NS} : \nabla\mathbf{u} + \frac{1}{\theta}B^\tau,
\end{aligned} \tag{4.4}$$

where

$$B^\tau := \langle\nabla p_\alpha\cdot\mathbf{w}_{\ell\alpha}\rangle - \ell\mathbf{q}^\tau\cdot\frac{1}{\theta}\nabla\theta + \Pi_\ell^\tau : \nabla\mathbf{u} - \rho\hat{\mathbf{w}}\cdot\mathbf{f} + Q.$$

It remains to show that the right-hand sides of equalities (4.4) and (3.1) coincide. First, using the property $\langle\nabla G\cdot\mathbf{d}_\alpha\rangle = \nabla G\cdot\langle\mathbf{d}_\alpha\rangle = 0$, we can write

$$-\frac{1}{K\theta}\nabla\theta\cdot\langle b_\alpha\mathbf{d}_\alpha\rangle - \frac{1}{\theta}\langle\nabla G_\alpha\cdot\mathbf{d}_\alpha\rangle = -\frac{1}{K\theta}\left\langle\left[\nabla(KG_\alpha - G) + b_\alpha\nabla\theta\right]\cdot\mathbf{d}_\alpha\right\rangle = \frac{1}{Kd_0\theta}\langle|\mathbf{d}_\alpha|^2\rangle. \tag{4.5}$$

Second, the following well-known formula holds

$$\Pi^{NS} : \nabla \mathbf{u} = \frac{\mu}{2}|\nabla \mathbf{u} + \nabla \mathbf{u}^T|^2 + \left(\lambda - \frac{2}{3}\mu\right)(\mathrm{div}\mathbf{u})^2. \tag{4.6}$$

Third, in virtue of definitions (2.5), (2.6), (2.8), (2.12), (2.13) and $Q = \langle Q_\alpha \rangle$, we have $B^\tau = \langle B^\tau_\alpha \rangle$, where

$$\begin{aligned}
B^\tau_\alpha &:= \nabla p_\alpha \cdot \mathbf{w}_{\ell\alpha} - \ell \mathbf{q}^\tau_\alpha \cdot \frac{1}{\theta}\nabla\theta + \Pi^\tau_{\ell\alpha} : \nabla\mathbf{u} - \rho\hat{\mathbf{w}}_\alpha \cdot \mathbf{f} + Q_\alpha,\\
-\mathbf{q}^\tau_\alpha &:= \tau\left[(c_{V\alpha}\rho_\alpha\nabla\theta - R_\alpha\theta\nabla\rho_\alpha)\cdot\mathbf{u} - Q_\alpha\right]\mathbf{u},\\
\Pi^\tau_{\ell\alpha} : \nabla\mathbf{u} &:= (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\rho\hat{\mathbf{w}}_\alpha + \ell\tau\left[\mathbf{u}\cdot\nabla p_\alpha + \gamma_\alpha p_\alpha \mathrm{div}\mathbf{u} - (\gamma_\alpha - 1)Q_\alpha\right]\mathrm{div}\mathbf{u},
\end{aligned}$$

since $\mathbf{q}^\tau = \langle \mathbf{q}^\tau_\alpha \rangle$ and

$$\Pi^\tau_\ell : \nabla\mathbf{u} = (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\rho\hat{\mathbf{w}} + \ell\tau\left[\mathbf{u}\cdot\nabla p + \langle\gamma_\alpha p_\alpha\rangle\mathrm{div}\mathbf{u} - \langle\gamma_\alpha Q_\alpha\rangle + Q\right]\mathrm{div}\mathbf{u} = \langle \Pi^\tau_{\ell\alpha} : \nabla\mathbf{u}\rangle.$$

It is known that the expression for $\frac{1}{\theta}B^\tau_\alpha$, referring to a component α only, can be transformed to the form

$$\begin{aligned}
\frac{1}{\theta}B^\tau_\alpha = &\frac{\rho_\alpha}{\tau\theta}|\hat{\mathbf{w}}_\alpha|^2 + \ell\tau\frac{R_\alpha}{\rho_\alpha}[\mathrm{div}(\rho_\alpha\mathbf{u})]^2\\
&+\ell\tau c_{V\alpha}\rho_\alpha\left[\mathbf{u}\cdot\nabla\ln\theta + (\gamma_\alpha - 1)\mathrm{div}\mathbf{u} - \frac{(\gamma_\alpha - 1)Q_\alpha}{2p_\alpha}\right]^2 + \frac{Q_\alpha}{\theta}\left(1 - \ell\frac{\tau(\gamma_\alpha - 1)Q_\alpha}{4p_\alpha}\right).
\end{aligned} \tag{4.7}$$

For $\ell = 0$, this is simple. For $\ell = 1$ and $Q_\alpha = 0$, see the equivalent formula in;[3,4] its most concise derivation is given in Chetverushkin and Zlotnik.[28] The case $Q_\alpha \neq 0$ is covered in the same way as in Zlotnik and Gavrilin.[29]

Substitution of formulas (4.5)–(4.7) into the right-hand side of equality (4.4) taking into account the formula $B^\tau = \langle B^\tau_\alpha \rangle$ completes the proof of Theorem 3.1.

4.3 | The derivation of the reduced system of equations

Now we derive the reduced system of Equations (3.2)–(3.4). We rewrite the mass balance equations for the components (2.14) as

$$\partial_t\rho_\alpha + \mathrm{div}(\rho_\alpha\mathbf{u}) = \mathrm{div}(\rho_\alpha\mathbf{w}_{\ell\alpha}),\ \alpha = \overline{1,K}, \tag{4.8}$$

and expand the terms of their right-hand sides, see (4.2):

$$\begin{aligned}
\mathrm{div}[\tau(\nabla\rho_\alpha\cdot\mathbf{u})\mathbf{u}] =&\tau[\nabla(\nabla\rho_\alpha\cdot\mathbf{u})]\cdot\mathbf{u} + O(|\nabla\mathbf{z}|^2) = \tau[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\nabla]\rho_\alpha + O(|\nabla\mathbf{z}|^2),\\
&\mathrm{div}[\tau\rho_\alpha(\mathrm{div}\mathbf{u})\mathbf{u}] = \tau\rho_\alpha(\mathbf{u}\cdot\nabla)\mathrm{div}\mathbf{u} + O(|\nabla\mathbf{z}|^2),\\
&\partial_k(\rho_\alpha\hat{\mathbf{w}}_\alpha) = \tau\partial_k\{\rho_\alpha(\mathbf{u}\cdot\nabla)\mathbf{u} + R_\alpha[(\nabla\rho_\alpha)\theta + \rho_\alpha\nabla\theta]\} + O(|\nabla\mathbf{z}|^2)\\
&=\tau\{\rho_\alpha(\mathbf{u}\cdot\nabla)\partial_k\mathbf{u} + R_\alpha[(\partial_k\nabla\rho_\alpha)\theta + \rho_\alpha\partial_k\nabla\theta]\} + O(|\nabla\mathbf{z}|^2),\ k = \overline{1,n}.
\end{aligned} \tag{4.9}$$

Summing up over $k = \overline{1,n}$ the kth components in the last formula, we get

$$\mathrm{div}(\rho_\alpha\hat{\mathbf{w}}_\alpha) = \tau\left[\rho_\alpha(\mathbf{u}\cdot\nabla)\mathrm{div}\mathbf{u} + R_\alpha\theta\Delta\rho_\alpha + R_\alpha\rho_\alpha\Delta\theta\right] + O(|\nabla\mathbf{z}|^2),$$

where $\Delta = \mathrm{div}\nabla$ is the Laplace operator. Thus, we obtain the reduced equations for the component densities (3.2).

We expand the terms on the right-hand side of Equation (2.25) using formula (4.9):

$$\begin{aligned}
\operatorname{div}\Pi^{NS} =&\mu\operatorname{div}\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right)+\left(\lambda-\frac{2}{3}\mu\right)\nabla\operatorname{div}\mathbf{u}+O(|\nabla\mathbf{z}|^2)=\mu\Delta\mathbf{u}+\chi\nabla\operatorname{div}\mathbf{u}+O(|\nabla\mathbf{z}|^2),\\
&(\mathbf{u}\cdot\nabla)(\rho\widehat{\mathbf{w}})=(\mathbf{u}\cdot\nabla)\langle\rho_\alpha\widehat{\mathbf{w}}_\alpha\rangle\\
&=\tau\left\langle\rho_\alpha[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\nabla]\mathbf{u}+R_\alpha\theta(\mathbf{u}\cdot\nabla)\nabla\rho_\alpha+R_\alpha\rho_\alpha(\mathbf{u}\cdot\nabla)\nabla\theta\right\rangle+O(|\nabla\mathbf{z}|^2)\\
&=\tau\left\{\rho[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\nabla]\mathbf{u}+\theta(\mathbf{u}\cdot\nabla)\langle R_\alpha\nabla\rho_\alpha\rangle+\langle R_\alpha\rho_\alpha\rangle(\mathbf{u}\cdot\nabla)\nabla\theta\right\}+O(|\nabla\mathbf{z}|^2),\\
&\nabla(\tau\mathbf{u}\cdot\nabla p)=\tau(\mathbf{u}\cdot\nabla)\nabla p+O(|\nabla\mathbf{z}|^2)=\tau\langle R_\alpha\theta(\mathbf{u}\cdot\nabla)\nabla\rho_\alpha+R_\alpha\rho_\alpha(\mathbf{u}\cdot\nabla)\nabla\theta\rangle+O(|\nabla\mathbf{z}|^2)\\
&=\tau\left[\theta(\mathbf{u}\cdot\nabla)\langle R_\alpha\nabla\rho_\alpha\rangle+\langle R_\alpha\rho_\alpha\rangle(\mathbf{u}\cdot\nabla)\nabla\theta\right]+O(|\nabla\mathbf{z}|^2),
\end{aligned}$$

where $\chi := \frac{1}{3}\mu+\lambda$. Expanding the left-hand side of Equation (2.25) and using the estimate

$$|\mathbf{w}\cdot\nabla\mathbf{u}|+|(\operatorname{div}\mathbf{u})\left(\rho\widehat{\mathbf{w}}+\nabla\langle\tau\gamma_\alpha p_\alpha\rangle\right)|=O(|\nabla\mathbf{z}|^2),$$

we derive the reduced equation for the velocity (3.3).

We expand the terms on the right-hand side of Equation (2.26) taking into account formula (4.3) and the previous expansion (4.9):

$$\begin{aligned}
\operatorname{div}(-\mathbf{q}^F) =&\varkappa\Delta\theta+O(|\nabla\mathbf{z}|^2),\\
&\operatorname{div}\left[\tau\left(c_V\rho\nabla\theta\cdot\mathbf{u}+p\operatorname{div}\mathbf{u}\right)\mathbf{u}\right]=\tau\left\{c_V\rho[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\nabla]\theta+p(\mathbf{u}\cdot\nabla)\operatorname{div}\mathbf{u}\right\}+O(|\nabla\mathbf{z}|^2),\\
&\operatorname{div}\langle p_\alpha\widehat{\mathbf{w}}_\alpha\rangle=\left\langle\frac{p_\alpha}{\rho_\alpha}\operatorname{div}(\rho_\alpha\widehat{\mathbf{w}}_\alpha)\right\rangle+O(|\nabla\mathbf{z}|^2)\\
&=\tau\left[p(\mathbf{u}\cdot\nabla)\operatorname{div}\mathbf{u}+\langle R_\alpha(R_\alpha\theta^2\Delta\rho_\alpha+p_\alpha\Delta\theta)\rangle\right]+O(|\nabla\mathbf{z}|^2)\\
&=\tau\left[p(\mathbf{u}\cdot\nabla)\operatorname{div}\mathbf{u}+\theta^2\langle R_\alpha^2\Delta\rho_\alpha\rangle+\langle R_\alpha p_\alpha\rangle\Delta\theta\right]+O(|\nabla\mathbf{z}|^2).
\end{aligned}$$

Expanding also the left-hand side of Equation (2.26) and using the estimate

$$|\langle c_{V\alpha}\rho_\alpha\mathbf{w}_\alpha\rangle\cdot\nabla\theta|+|\Pi:\nabla\mathbf{u}|=O(|\nabla\mathbf{z}|^2),$$

we derive the reduced equation for the temperature (3.4).

4.4 | The justification of the linearized system of equations

Now we justify the linearized system of Equations (3.8)–(3.10). Substituting the solution in the form (3.5) into the reduced system (3.2)–(3.4), using formulas (3.6), discarding the terms of the second order of smallness with respect to the vector-function $\widetilde{\mathbf{z}}$ and its first and second order derivatives and dividing the equations by $\rho_{\alpha*}$, u_* and θ_*, respectively, we easily obtain the original linearized system of equations

$$\begin{aligned}
&\partial_t\tilde{\rho}_\alpha+u_*\left(\widehat{\mathbf{u}}_0\cdot\nabla\tilde{\rho}_\alpha+\hat{\rho}_{\alpha0}\operatorname{div}\widetilde{\mathbf{u}}\right)\\
&=\tau_0u_*^2\left[\frac{R_\alpha\theta_0}{u_*^2}\Delta\tilde{\rho}_\alpha+\ell(\widehat{\mathbf{u}}_0\cdot\nabla)^2\tilde{\rho}_\alpha+(\ell+1)\hat{\rho}_{\alpha0}(\widehat{\mathbf{u}}_0\cdot\nabla)\operatorname{div}\widetilde{\mathbf{u}}+\frac{R_\alpha\hat{\rho}_{\alpha0}\theta_*}{u_*^2}\Delta\tilde{\theta}\right],\ \alpha=\overline{1,K},\\
&\partial_t\widetilde{\mathbf{u}}+u_*\left(\frac{\theta_0}{\rho_0u_*^2}\langle R_\alpha\rho_{\alpha*}\nabla\tilde{\rho}_\alpha\rangle+(\widehat{\mathbf{u}}_0\cdot\nabla)\widetilde{\mathbf{u}}+\frac{R_0\theta_*}{u_*^2}\nabla\tilde{\theta}\right)=u_*^2\left[(\ell+1)\tau_0\frac{\theta_0}{\rho_0u_*^2}(\widehat{\mathbf{u}}_0\cdot\nabla)\langle R_\alpha\rho_{\alpha*}\nabla\tilde{\rho}_\alpha\rangle\right.\\
&\left.+\frac{\mu_0}{\rho_0u_*^2}\Delta\widetilde{\mathbf{u}}+\left(\frac{\chi_0}{\rho_0u_*^2}+\ell\tau_0\frac{(R\gamma)_0\theta_0}{u_*^2}\right)\nabla\operatorname{div}\widetilde{\mathbf{u}}+\tau_0(\widehat{\mathbf{u}}_0\cdot\nabla)^2\widetilde{\mathbf{u}}+(\ell+1)\tau_0\frac{R_0\theta_*}{u_*^2}(\widehat{\mathbf{u}}_0\cdot\nabla)\nabla\tilde{\theta}\right],\\
&\partial_t\tilde{\theta}+u_*\left(\frac{R_0\hat{\theta}_0}{c_{V0}}\operatorname{div}\widetilde{\mathbf{u}}+\widehat{\mathbf{u}}_0\cdot\nabla\tilde{\theta}\right)=u_*^2\left[\tau_0\frac{\hat{\theta}_0^2\theta_*}{c_{V0}\rho_0u_*^2}\langle R_\alpha^2\rho_{\alpha*}\Delta\tilde{\rho}_\alpha\rangle\right.\\
&\left.+(\ell+1)\tau_0\frac{R_0\hat{\theta}_0}{c_{V0}}(\widehat{\mathbf{u}}_0\cdot\nabla)\operatorname{div}\widetilde{\mathbf{u}}+\ell\tau_0(\widehat{\mathbf{u}}_0\cdot\nabla)^2\tilde{\theta}+\left(\frac{\varkappa_0}{c_{V0}\rho_0u_*^2}+\tau_0\frac{(R^2)_0\theta_0}{c_{V0}u_*^2}\right)\Delta\tilde{\theta}\right].
\end{aligned}$$

The common factors u_* and u_*^2 are carried out of the convective and dissipative terms, respectively.

Furthermore, to simplify the analysis of the resulting system of linearized equations, the possibility of symmetrization for both the convective and dissipative terms is essential. It is achieved under the following conditions

$$\hat{\rho}_{\alpha 0} = \frac{\theta_0}{\rho_0 u_*^2} R_\alpha \rho_{\alpha *} \iff \frac{u_*^2}{\rho_{\alpha *}^2} = \frac{R_\alpha \theta_0}{\rho_{\alpha 0}\rho_0}, \ \alpha = \overline{1,K}, \ \frac{R_0\theta_*}{u_*^2} = \frac{R_0\hat{\theta}_0}{c_{V0}} \iff \frac{\theta_*^2}{u_*^2} = \frac{\theta_0}{c_{V0}} \tag{4.10}$$

for the convective terms and under the conditions

$$\hat{\rho}_{\alpha 0} = \frac{\theta_0}{\rho_0 u_*^2} R_\alpha \rho_{\alpha *}, \ \frac{R_\alpha \hat{\rho}_{\alpha 0}\theta_*}{u_*^2} = \frac{\hat{\theta}_0^2\theta_*}{c_{V0}\rho_0 u_*^2} R_\alpha^2 \rho_{\alpha *}, \ \alpha = \overline{1,K}, \ \frac{R_0\theta_*}{u_*^2} = \frac{R_0\theta_0}{c_{V0}\theta_*}. \tag{4.11}$$

for the dissipative terms. The first and third conditions (4.11) coincide with (4.10), and the second one is a consequence of conditions (4.10); this ensures the simultaneous symmetrization of both the convective and dissipative terms. The above given choice (3.7) containing a free parameter satisfies conditions (4.10), and below conditions (4.10) are assumed to be valid. This allows us to simplify significantly the form of the original linearized system of equations down to (3.2)–(3.4).

Next we consider the system of Equations (3.8)–(3.10) in the cylinder $\Omega\times(0,\infty)$ supplemented with the boundary and initial conditions (3.11) and analyze such an initial-boundary value problem. For $\partial_t\widetilde{\mathbf{z}}(\cdot,t), \nabla\widetilde{\mathbf{z}}(\cdot,t) \in \mathbf{L}^2(\Omega)$, equations (3.8)–(3.10) correspond to the integral identity (3.12). Formally, it is derived in a standard way. Namely, we multiply these equations by the corresponding components of an "arbitrary" vector function $\mathbf{z} = (\rho_1, \ldots, \rho_K, \mathbf{u}, \theta)(x)$ such that $\mathbf{z}|_{\partial\Omega} = 0$ (it should not be confused with the solutions to the QGD and QHD systems which was denoted above in the same way). Then we integrate the results over Ω, sum them up, integrate by parts in terms with the second derivatives and thus obtain the integral identity, with the above given form $\mathcal{B}_\Omega(\widetilde{\mathbf{z}},\mathbf{z})$ and, after placing the terms without the factor τ_0 first and rearranging the other factors, form $\mathcal{A}_\Omega(\widetilde{\mathbf{z}},\mathbf{z})$.

Identity (3.12) holds for both real and complex-valued functions $\widetilde{\mathbf{z}}$ and $\mathbf{z}$; in the first case, both of the forms are bilinear whereas, in the second case, they are sesquilinear. These forms can be considered for any $\widetilde{\mathbf{z}},\mathbf{z} \in \mathbf{H}^1(\Omega)$ and even $\widetilde{\mathbf{z}},\mathbf{z} \in \mathbf{L}^{1,loc}(\Omega)$ with $\nabla\widetilde{\mathbf{z}}, \nabla\mathbf{z} \in \mathbf{L}^2(\Omega)$. Here we restrict ourselves to the real case (unlike papers[6–8]).

4.5 | Proof of Lemma 3.1

We turn to proving Lemma 3.1. Rearranging the terms and integrating by parts, we obtain

$$\begin{aligned}\mathcal{B}_\Omega(\widetilde{\mathbf{z}},\mathbf{z}) &= \left(\hat{\mathbf{u}}_0, \langle(\nabla\tilde{\rho}_\alpha)\rho_\alpha\rangle + (\nabla\widetilde{\mathbf{u}})\mathbf{u} + (\nabla\tilde{\theta})\theta\right)_\Omega \\ &-\left(\widetilde{\mathbf{u}}, \langle\hat{\rho}_{\alpha 0}\nabla\rho_\alpha\rangle\right)_\Omega + (\langle\hat{\rho}_{\alpha 0}\nabla\tilde{\rho}_\alpha\rangle, \mathbf{u})_\Omega - a_0\left(\tilde{\theta}, \operatorname{div}\mathbf{u}\right)_\Omega + a_0(\operatorname{div}\widetilde{\mathbf{u}}, \theta)_\Omega \ \forall \widetilde{\mathbf{z}},\mathbf{z} \in \mathbf{H}_0^1(\Omega).\end{aligned}$$

Hence, since $\hat{\mathbf{u}}_0 = \text{const}$, we have

$$\begin{aligned}\mathcal{B}_\Omega(\mathbf{z},\mathbf{z}) &= \left(\hat{\mathbf{u}}_0, \langle(\nabla\rho_\alpha)\rho_\alpha\rangle + (\nabla\mathbf{u})\mathbf{u} + (\nabla\theta)\theta\right)_\Omega \\ &= \left(\hat{\mathbf{u}}_0, \frac{1}{2}\nabla\left(\langle\rho_\alpha^2\rangle + |\mathbf{u}|^2 + \theta^2\right)\right)_\Omega = 0 \ \forall \mathbf{z} \in \mathbf{H}_0^1(\Omega),\end{aligned}$$

that is, property (3.13) is proved. The symmetry property (3.14) of $\mathcal{A}_\Omega(\widetilde{\mathbf{z}},\mathbf{z})$ is obvious.

Furthermore, for any $\mathbf{z}\in\mathbf{H}^1(\Omega)$, the corresponding quadratic form is as follows

$$\begin{aligned}\mathcal{A}_\Omega(\mathbf{z},\mathbf{z}) &= \bar{\mu}_0\|\nabla\mathbf{u}\|_\Omega^2 + \bar{\chi}_0\|\operatorname{div}\mathbf{u}\|_\Omega^2 + \bar{\varkappa}_0\|\nabla\theta\|_\Omega^2 \\ &+ \tau_0\left[\hat{\theta}_0\left\langle a_\alpha\|\nabla\rho_\alpha\|_\Omega^2\right\rangle + \ell\langle\|\hat{\mathbf{u}}_0\cdot\nabla\rho_\alpha\|_\Omega^2\rangle + \ell\hat{\theta}_0(a\gamma)_0\|\operatorname{div}\mathbf{u}\|_\Omega^2 + \|(\hat{\mathbf{u}}_0\cdot\nabla)\mathbf{u}\|_\Omega^2\right. \\ &+ \ell\|\hat{\mathbf{u}}_0\cdot\nabla\theta\|_\Omega^2 + (a^2)_0\|\nabla\theta\|_\Omega^2 \\ &\left.+2(\ell+1)\left(\langle\hat{\rho}_{\alpha 0}\nabla\rho_\alpha\rangle + a_0\nabla\theta, (\hat{\mathbf{u}}_0\cdot\nabla)\mathbf{u}\right)_\Omega + 2(\langle a_\alpha\hat{\rho}_{\alpha 0}\nabla\rho_\alpha\rangle, \nabla\theta)_\Omega\right].\end{aligned} \tag{4.12}$$

The following lemma plays a key role in justifying the positive definiteness of $\mathcal{A}_\Omega(\mathbf{z},\mathbf{z})$.

Lemma 4.1. *The following pointwise formula is valid*

$$\begin{aligned}
&\hat{\theta}_0 \left\langle a_\alpha|\nabla\rho_\alpha|^2\right\rangle + \ell\langle(\hat{\mathbf{u}}_0\cdot\nabla\rho_\alpha)^2\rangle + \ell\hat{\theta}_0(a\gamma)_0(\mathrm{div}\mathbf{u})^2 + |(\hat{\mathbf{u}}_0\cdot\nabla)\mathbf{u}|^2 + \ell(\hat{\mathbf{u}}_0\cdot\nabla\theta)^2 + (a^2)_0|\nabla\theta|^2\\
&\quad + 2\ell\left[\hat{\mathbf{u}}_0\cdot(\langle\hat{\rho}_{\alpha 0}\nabla\rho_\alpha\rangle + a_0\nabla\theta)\right]\mathrm{div}\mathbf{u}\\
&\quad + 2\left(\langle\hat{\rho}_{\alpha 0}\nabla\rho_\alpha\rangle + a_0\nabla\theta, (\hat{\mathbf{u}}_0\cdot\nabla)\mathbf{u}\right) + 2\left(\langle a_\alpha\hat{\rho}_{\alpha 0}\nabla\rho_\alpha\rangle, \nabla\theta\right)\\
&\quad = \ell\left\langle\left(\hat{\mathbf{u}}_0\cdot\nabla\rho_\alpha + \hat{\rho}_{\alpha 0}\mathrm{div}\mathbf{u}\right)^2 + \left|(\hat{\theta}_0 a_\alpha)^{1/2}\nabla\rho_\alpha + \sqrt{c_{\alpha 0}}(\hat{\mathbf{u}}_0\cdot\nabla)\mathbf{u} + \sqrt{c_{\alpha 0}}a_\alpha\nabla\theta\right|^2\right\rangle\\
&\quad + \ell\left(a_0\mathrm{div}\mathbf{u} + \hat{\mathbf{u}}_0\cdot\nabla\theta\right)^2 + \ell g_0(\mathrm{div}\mathbf{u})^2,
\end{aligned} \tag{4.13}$$

with g_0 *defined in (3.16).*

Proof. We uncover the squared terms on the right-hand side of the equality. First, we write down the formula

$$\begin{aligned}
&\left\langle\left(\hat{\mathbf{u}}_0\cdot\nabla\rho_\alpha + \hat{\rho}_{\alpha 0}\mathrm{div}\mathbf{u}\right)^2\right\rangle + \left(a_0\mathrm{div}\mathbf{u} + \hat{\mathbf{u}}_0\cdot\nabla\theta\right)^2 + g_0(\mathrm{div}u)^2 = \left\langle\left(\hat{\mathbf{u}}_0\cdot\nabla\rho_\alpha\right)^2\right\rangle\\
&+ \left(\left\langle\hat{\rho}_{\alpha 0}^2\right\rangle + a_0^2 + g_0\right)(\mathrm{div}\mathbf{u})^2 + \left(\hat{\mathbf{u}}_0\cdot\nabla\theta\right)^2 + 2\left(\hat{\mathbf{u}}_0\cdot\langle\hat{\rho}_{\alpha 0}\nabla\rho_\alpha\rangle\right)\mathrm{div}\mathbf{u} + 2a_0(\hat{\mathbf{u}}_0\cdot\nabla\theta)\mathrm{div}\mathbf{u}.
\end{aligned} \tag{4.14}$$

In virtue of the second and fourth of equalities (4.10) as well as the definitions of $\hat{\theta}_0$ and γ_α, the following formulas holds

$$\left\langle\hat{\rho}_{\alpha 0}^2\right\rangle + a_0^2 = \frac{R_0\theta_0}{u_*^2} + \left(\frac{R_0\theta_*}{u_*^2}\right)^2 = \frac{R_0\theta_0}{u_*^2} + \frac{R_0^2}{c_{V0}}\frac{\theta_0}{u_*^2},$$

$$\hat{\theta}_0(a\gamma)_0 = \hat{\theta}_0\left\langle\frac{\rho_{\alpha 0}}{\rho_0}\frac{R_\alpha\theta_*}{u_*^2}\left(1 + \frac{R_\alpha}{c_{V\alpha}}\right)\right\rangle = \frac{R_0\theta_0}{u_*^2} + \left\langle\frac{\rho_{\alpha 0}}{\rho_0}\frac{R_\alpha^2}{c_{V\alpha}}\right\rangle\frac{\theta_0}{u_*^2}.$$

Therefore, we have

$$\left\langle\hat{\rho}_{\alpha 0}^2\right\rangle + a_0^2 + \left[\frac{1}{c_{V0}}\left\langle\frac{\rho_{\alpha 0}}{\rho_0}\frac{R_\alpha^2}{c_{V\alpha}}\right\rangle - \left(\frac{R_0}{c_{V0}}\right)^2\right]\frac{c_{V0}\theta_0}{u_*^2} = \hat{\theta}_0(a\gamma)_0.$$

In virtue of the equality $\frac{1}{c_{V0}}\left\langle\frac{\rho_{\alpha 0}}{\rho_0}c_{V\alpha}\right\rangle = 1$, it is easy to check the identity

$$\frac{1}{c_{V0}}\left\langle\frac{\rho_{\alpha 0}}{\rho_0}c_{V\alpha}e_\alpha^2\right\rangle - \tilde{e}^2 = \frac{1}{c_{V0}}\left\langle\frac{\rho_{\alpha 0}}{\rho_0}c_{V\alpha}(e_\alpha - \tilde{e})^2\right\rangle \text{ with } \tilde{e} := \frac{1}{c_{V0}}\left\langle\frac{\rho_{\alpha 0}}{\rho_0}c_{V\alpha}e_\alpha\right\rangle,$$

for any numbers $e_1, \ldots, e_K$. According to it, for $e_\alpha := \frac{R_\alpha}{c_{V\alpha}} = \gamma_\alpha - 1$, together with definitions (3.16), the following formula holds

$$\left\langle\hat{\rho}_{\alpha 0}^2\right\rangle + a_0^2 + g_0 = \hat{\theta}_0(a\gamma)_0. \tag{4.15}$$

Second, according to the second of equalities (4.10), we have

$$\frac{\hat{\rho}_{\alpha 0}^2}{c_{\alpha 0}} = \frac{\rho_{\alpha 0}\rho_0}{\rho_{\alpha *}^2} = \frac{R_\alpha\theta_0}{u_*^2} = \hat{\theta}_0 a_\alpha. \tag{4.16}$$

Due to the equality $\langle c_{\alpha 0}\rangle = 1$ and the definitions of $(a^2)_0$ and a_0, we can write

$$\left\langle\left|\frac{\hat{\rho}_{\alpha 0}}{\sqrt{c_{\alpha 0}}}\nabla\rho_\alpha + \sqrt{c_{\alpha 0}}(\hat{\mathbf{u}}_0\cdot\nabla)\mathbf{u} + \sqrt{c_{\alpha 0}}a_\alpha\nabla\theta\right|^2\right\rangle$$

$$= \left\langle \frac{\hat{\rho}_{\alpha 0}^2}{c_{\alpha 0}} |\nabla \rho_\alpha|^2 \right\rangle + |(\hat{\mathbf{u}}_0 \cdot \nabla)\mathbf{u}|^2 + (a^2)_0 |\nabla \theta|^2$$

$$+2\left(\langle \hat{\rho}_{\alpha 0} \nabla \rho_\alpha \rangle, (\hat{\mathbf{u}}_0 \cdot \nabla)\mathbf{u}\right) + 2\left(\langle a_\alpha \hat{\rho}_{\alpha 0} \nabla \rho_\alpha \rangle, \nabla \theta\right) + 2\left((\hat{\mathbf{u}}_0 \cdot \nabla)\mathbf{u}, a_0 \nabla \theta\right). \tag{4.17}$$

Finally, we multiply equality (4.14) by ℓ and add to (4.17), take into account formulas (4.15) and (4.16) and obtain equality (4.13). □

Now we complete the proof of Lemma 3.1 by reducing $\mathcal{A}_\Omega(\mathbf{z}, \mathbf{z})$ to a sum of squared terms and deriving its positive definiteness on $\mathbf{H}_0^1(\Omega)$. First, notice that the following formula hold

$$(\hat{\mathbf{u}}_0 \cdot \nabla \varphi, \mathrm{div}\mathbf{u})_\Omega = (\nabla \varphi, (\hat{\mathbf{u}}_0 \cdot \nabla)\mathbf{u})_\Omega \ \forall \varphi \in H^1(\Omega), \mathbf{u} \in \mathbf{H}_0^1(\Omega). \tag{4.18}$$

Indeed, the following integration by parts performed two times

$$(\hat{\mathbf{u}}_0 \cdot \nabla \varphi, \mathrm{div}\mathbf{u})_\Omega = (\hat{u}_{0i} \partial_i \varphi, \partial_j u_j)_\Omega = -(\hat{u}_{0i} \partial_i \partial_j \varphi, u_j)_\Omega = (\partial_j \varphi, \hat{u}_{0i} \partial_i u_j)_\Omega = (\nabla \varphi, (\hat{\mathbf{u}}_0 \cdot \nabla)\mathbf{u})_\Omega$$

proves it for $\varphi \in H^2(\Omega)$ and $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$. Here the summation over the repeated indices i and j is assumed. For $\mathbf{u} \in \mathbf{D}(\Omega)$, the formula is also true for $\varphi \in H^1(\Omega)$, since for any bounded subdomain Ω_1 such that $\bar{\Omega}_1 \subset \Omega$, there exist functions $\varphi_k \in C^2(\Omega)$, $k \geq 1$, satisfying $\|\varphi - \varphi_k\|_{H^1(\Omega_1)} \to 0$ for $k \to \infty$. Then, by the definition of $\mathbf{H}_0^1(\Omega)$, the formula is obtained for $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ as well.

To transform all the nine terms with the factor τ_0 in expression (4.12) for $\mathcal{A}_\Omega(\mathbf{z}, \mathbf{z})$, we integrate equality (4.13) over Ω. Then, we transform the seventh term on the left using the formula

$$2\ell\left(\hat{\mathbf{u}}_0 \cdot (\langle \hat{\rho}_{\alpha 0} \nabla \rho_\alpha \rangle + a_0 \nabla \theta), \mathrm{div}\mathbf{u}\right)_\Omega = 2\ell\left[\left(\langle \hat{\rho}_{\alpha 0} \nabla \rho_\alpha \rangle + a_0 \nabla \theta, (\hat{\mathbf{u}}_0 \cdot \nabla)\mathbf{u}\right)_\Omega\right],$$

see (4.18); after that, this term coincides with the eighth term up to the multiplier ℓ that leads to formula (3.15).

The following inequalities hold

$$\|\mathbf{y}_\rho\|_\Omega^2 \leq 2\left[\|\mathbf{y}_\rho + \mathbf{y}_u + \mathbf{y}_\theta\|_\Omega^2 + 2(\|\mathbf{y}_u\|_\Omega^2 + \|\mathbf{y}_\theta\|_\Omega^2)\right] \ \forall \mathbf{y}_\rho, \mathbf{y}_u, \mathbf{y}_\theta \in \mathbf{L}^2(\Omega),$$

$$\left\langle \|\sqrt{c_{\alpha 0}}(\hat{\mathbf{u}}_0 \cdot \nabla)\mathbf{u}\|_\Omega^2 + \|\sqrt{c_{\alpha 0}} a_\alpha \nabla \theta\|_\Omega^2 \right\rangle = \|(\hat{\mathbf{u}}_0 \cdot \nabla)\mathbf{u}\|_\Omega^2 + (a^2)_0 \|\nabla \theta\|_\Omega^2$$

$$\leq |\hat{\mathbf{u}}_0|^2 \|\nabla \mathbf{u}\|_\Omega^2 + (a^2)_0 \|\nabla \theta\|_\Omega^2 \leq \frac{\delta_0}{\tau_0}\left(\bar{\mu}_0 \|\nabla \mathbf{u}\|_\Omega^2 + \bar{\varkappa}_0 \|\nabla \theta\|_\Omega^2\right)$$

with δ_0 introduced in the statement of Lemma 3.1. Setting

$$\mathbf{y}_{\rho\alpha} := (\hat{\theta}_0 a_\alpha)^{1/2} \nabla \rho_\alpha, \ \mathbf{y}_{u\alpha} := \sqrt{c_{\alpha 0}}(\hat{\mathbf{u}}_0 \cdot \nabla)\mathbf{u}, \ \mathbf{y}_{\theta\alpha} := \sqrt{c_{\alpha 0}} a_\alpha \nabla \theta$$

and applying the both last inequalities and property (3.15), we obtain

$$\tau_0 \left\langle \|(\hat{\theta}_0 a_\alpha)^{1/2} \nabla \rho_\alpha\|_\Omega^2 \right\rangle = \tau_0 \left\langle \|\mathbf{y}_{\rho\alpha}\|_\Omega^2 \right\rangle \leq 2\left[\tau_0 \left\langle \|\mathbf{y}_{\rho\alpha} + \mathbf{y}_{u\alpha} + \mathbf{y}_{\theta\alpha}\|_\Omega^2 \right\rangle + 2\tau_0 \left\langle \|\mathbf{y}_{u\alpha}\|_\Omega^2 + \|\mathbf{y}_{\theta\alpha}\|_\Omega^2 \right\rangle\right]$$
$$\leq 2\left[\mathcal{A}_\Omega(\mathbf{z}, \mathbf{z}) + \max\{2\delta_0 - 1, 0\} \mathcal{A}_\Omega(\mathbf{z}, \mathbf{z})\right],$$

whence inequality (3.17) follows. Inequality (3.18) is obvious that completes the proof of Lemma 3.1.

4.6 | Proofs of Theorem 3.2 and Corollary 3.1

Let us prove Theorem 3.2 and Corollary 3.1. The existence and uniqueness of the introduced weak solution to the linearized problem follows from Gaewski et al.[24, Ch. VI, Theorem 1.1 and Section 1.3] based on Lemma 3.1.

Taking $\mathbf{z} = \widetilde{\mathbf{z}}(\cdot, t)$ in identity (3.19) and applying properties (3.13) and (3.15), we obtain the energy equality (3.20). Moreover, the property $\widetilde{\mathbf{z}} \in C([0,T]; \mathbf{L}^2(\Omega))$ and the formula

$$\int_0^T \left\langle \partial_t \widetilde{\mathbf{z}}(\cdot, t), \mathbf{z} \right\rangle_\Omega dt = \frac{1}{2}\|\widetilde{\mathbf{z}}(\cdot, T)\|^2_{\mathbf{L}^2(\Omega)} - \frac{1}{2}\|\widetilde{\mathbf{z}}^{(0)}\|^2_{\mathbf{L}^2(\Omega)} \; \forall T > 0$$

follows from Gaewski et al.[24, Ch. IV, Theorem 1.17 and Remark 1.22] The function $u_*^2 \mathcal{A}_{Q_T}(\widetilde{\mathbf{z}}, \widetilde{\mathbf{z}})$ does not decrease for $T \geq 0$ and, summed up with $0.5\|\widetilde{\mathbf{z}}(\cdot, T)\|^2_{\mathbf{L}^2(\Omega)}$, is constant in T in equality (3.20), thus $\|\widetilde{\mathbf{z}}(\cdot, T)\|_{\mathbf{L}^2(\Omega)}$ does not increase for $T \geq 0$.

Due to the $\mathbf{L}^2(\Omega)$-dissipativity property and energy equality, we have

$$\max_{t \geq 0} \|\widetilde{\mathbf{z}}(\cdot, t)\|_{\mathbf{L}^2(\Omega)} = \|\widetilde{\mathbf{z}}^{(0)}\|_{\mathbf{L}^2(\Omega)}, \; \sqrt{2} u_* \mathcal{A}_Q^{1/2}(\widetilde{\mathbf{z}}, \widetilde{\mathbf{z}}) \leq \|\widetilde{\mathbf{z}}^{(0)}\|_{\mathbf{L}^2(\Omega)}. \tag{4.19}$$

Therefore, inequalities (3.17)–(3.18) entail the energy estimate (3.21). Notice that the uniqueness of the solution follows from (3.20) or (3.21).

Concerning Corollary 3.1, indeed, in virtue of the energy equality, we have

$$\frac{1}{2}\|\widetilde{\mathbf{z}}(\cdot, T)\|^2_{\mathbf{L}^2(\Omega)} = \frac{1}{2}\|\widetilde{\mathbf{z}}^{(0)}\|^2_{\mathbf{L}^2(\Omega)} - u_*^2 \int_0^T \mathcal{A}_\Omega(\widetilde{\mathbf{z}}(\cdot, t), \widetilde{\mathbf{z}}(\cdot, t))\, dt \; \forall T \geq 0,$$

where $\mathcal{A}_\Omega(\widetilde{\mathbf{z}}(\cdot, t), \widetilde{\mathbf{z}}(\cdot, t)) \in L^1(0, \infty)$ due to the second bound (4.19). Therefore, there exists the derivative $\partial_t \left(\|\widetilde{\mathbf{z}}(\cdot, t)\|^2_{\mathbf{L}^2(\Omega)} \right) \in L^1(0, \infty)$, and differentiation of equality (3.20) leads to (3.22). Note that formally equality (3.22) is obtained simply from identity (3.12) for $\mathbf{z} = \widetilde{\mathbf{z}}(\cdot, t)$.

4.7 | Proofs of Lemma 3.2 and Theorem 3.3

Finally, we prove Lemma 3.2 and Theorem 3.3.

We introduce the diagonal matrix $D \equiv D(\mathbf{z}_0) := \mathrm{diag}\{\rho_{1*}, \dots, \rho_{K*}, u_*, \dots, u_*, \theta_*\}$ of order $K + n + 1$ with the diagonal elements satisfying conditions (4.10). We accomplish the change of unknowns $\mathbf{z} = D\widetilde{\mathbf{z}}$, then $\mathcal{F}\mathbf{z} = D\mathcal{F}\widetilde{\mathbf{z}}$ and, after multiplying system (3.24) on the left by D^{-1}, obtain the following equivalent system

$$\partial_t \mathcal{F}\widetilde{\mathbf{z}}(\zeta, t) + |\zeta|^2 u_*^2 \hat{A}(\mathbf{z}_0, \xi)\mathcal{F}\widetilde{\mathbf{z}}(\zeta, t) = 0, \; t > 0,$$

with the matrix $\hat{A}(\mathbf{z}_0, \xi) = D^{-1}A(\mathbf{z}_0, \xi)D$ similar to the matrix $A(\mathbf{z}_0, \xi)$.

In virtue of the above derivation of the symmetrized linearized system (3.8)–(3.10), the matrix $-u_*^2\hat{A}(\mathbf{z}_0, \xi)$ arises directly by application of $\mathcal{F}$ to the right-hand sides of the system (instead of the simplified reduced system (3.2)–(3.4) as above). Therefore, properties (3.26) hold.

We define the diagonal matrix $\Lambda = \mathrm{diag}\{a_1, \dots, a_K\}$. Let I_k be the identity matrix of order k, $\hat{\rho}_0 := (\hat{\rho}_{10}, \dots, \hat{\rho}_{K0})^T$, $s = s(\xi) = \hat{\mathbf{u}}_0 \cdot \xi$ and the above defined ξ be considered as a column vector. According to the mentioned direct appearance of $\hat{A}(\mathbf{z}_0, \xi)$, its explicit 3×3-block form is written as follows

$$\hat{A}(\mathbf{z}_0, \xi) = \begin{pmatrix} \tau_0(\hat{\theta}_0\Lambda + \ell s^2 I_K) & (\ell+1)\tau_0 s\hat{\rho}_0 \otimes \xi & \tau_0\Lambda\hat{\rho}_0 \\ (\ell+1)\tau_0 s\xi \otimes \hat{\rho}_0 & (\bar{\mu}_0 + \tau_0 s^2)I_n + \left[\bar{\chi}_0 + \ell\tau_0\hat{\theta}_0(a\gamma)_0\right]\xi\xi^T & (\ell+1)\tau_0 a_0 s\xi \\ \tau_0\hat{\rho}_0^T\Lambda & (\ell+1)\tau_0 a_0 s\xi^T & \bar{\varkappa}_0 + \tau_0\left[\ell s^2 + (a^2)_0\right] \end{pmatrix}.$$

The quadratic form with the matrix $\hat{A}(\mathbf{z}_0, \xi)$ has the form

$$\begin{aligned} \hat{A}(\mathbf{z}_0, \xi)\mathbf{b} \cdot \mathbf{b} = {} & \bar{\mu}_0|\mathbf{v}|^2 + \bar{\chi}_0(\xi \cdot \mathbf{v})^2 + \bar{\varkappa}_0 q^2 + \tau_0 \left\{ \hat{\theta}_0 \langle a_\alpha r_\alpha^2 \rangle + \ell s^2|\mathbf{r}|^2 + 2(\ell+1)s(\hat{\rho}_0 \cdot \mathbf{r})\xi \cdot \mathbf{v} \right. \\ & \left. + 2\langle a_\alpha \hat{\rho}_{\alpha 0} r_\alpha \rangle q + s^2|\mathbf{v}|^2 + \ell\hat{\theta}_0(a\gamma)_0(\xi \cdot \mathbf{v})^2 + 2(\ell+1)a_0 s(\xi \cdot \mathbf{v})q + \left[\ell s^2 + (a^2)_0\right] q^2 \right\}, \end{aligned} \tag{4.20}$$

where the terms without the factor τ_0 are put forward, for any block vector $\mathbf{b} := (\mathbf{r}^T, \mathbf{v}, q)^T$ with $\mathbf{r}^T = (r_1, \ldots, r_K) \in \mathbb{R}^K$, $\mathbf{v} \in \mathbb{R}^n$, $q \in \mathbb{R}$, taking into account the formulas

$$\Lambda \mathbf{r} \cdot \mathbf{r} = \langle a_\alpha r_\alpha^2 \rangle,\ \Lambda \hat{\rho}_0 \cdot \mathbf{r} = \langle a_\alpha \hat{\rho}_{\alpha 0} r_\alpha \rangle.$$

Next, we transform this quadratic form to a sum of squared terms and prove its positive definiteness, that is, justify Lemma 3.2. We calculate the squared sums on the right in formula (3.27) (recall that $\hat{\theta}_0 a_\alpha = \frac{\hat{\rho}_{\alpha 0}^2}{c_{\alpha 0}}$ due to formula (4.16)) and get three equalities

$$|s\mathbf{r} + (\xi \cdot \mathbf{v})\hat{\rho}_0|^2 = s^2 |\mathbf{r}|^2 + 2s(\hat{\rho}_0 \cdot \mathbf{r})\xi \cdot \mathbf{v} + |\hat{\rho}_0|^2 (\xi \cdot \mathbf{v})^2,$$

$$[a_0(\xi \cdot \mathbf{v}) + sq]^2 = a_0^2 (\xi \cdot \mathbf{v})^2 + 2sa_0(\xi \cdot \mathbf{v})q + s^2 q^2,$$

$$\left\langle \left| \frac{\hat{\rho}_{\alpha 0}}{\sqrt{c_{\alpha 0}}} r_\alpha \xi + \sqrt{c_{\alpha 0}} s\mathbf{v} + \sqrt{c_{\alpha 0}} a_\alpha q \xi \right|^2 \right\rangle = \left\langle \frac{\hat{\rho}_{\alpha 0}^2}{c_{\alpha 0}} r_\alpha^2 \right\rangle + \langle c_{\alpha 0} \rangle s^2 |\mathbf{v}|^2 + \langle c_{\alpha 0} a_\alpha^2 \rangle q^2$$
$$+2s\langle \hat{\rho}_{\alpha 0} r_\alpha \rangle (\xi \cdot \mathbf{v}) + 2\langle a_\alpha \hat{\rho}_{\alpha 0} r_\alpha \rangle q + 2s\langle a_\alpha \rangle (\xi \cdot \mathbf{v}) q$$
$$= \left\langle \frac{\hat{\rho}_{\alpha 0}^2}{c_{\alpha 0}} r_\alpha^2 \right\rangle + s^2 |\mathbf{v}|^2 + (a^2)_0 q^2 + 2s(\hat{\rho}_0 \cdot \mathbf{r})(\xi \cdot \mathbf{v}) + 2\langle a_\alpha \hat{\rho}_{\alpha 0} r_\alpha \rangle q + 2sa_0(\xi \cdot \mathbf{v})q.$$

We multiply the first and second equalities by ℓ, sum up them with the third equality and add $\ell g_0(\xi \cdot \mathbf{v})^2$ to the both sides of the result. Applying formulas (4.15) and (4.16), we pass from formula (4.20) to (3.27). These transformations are similar to those performed in the proof of Lemma 4.1.

The first inequality (3.28) is derived similarly to (3.17) and follows from the chain of relations

$$\hat{\theta}_0 a_{\min} |\mathbf{r}|^2 \le \hat{\theta}_0 \left\langle a_\alpha r_\alpha^2 \right\rangle = \left\langle \left| (\hat{\theta}_0 a_\alpha)^{1/2} r_\alpha \xi \right|^2 \right\rangle$$
$$\le 2 \left\langle \left| (\hat{\theta}_0 a_\alpha)^{1/2} r_\alpha \xi + \sqrt{c_{\alpha 0}} s\mathbf{v} + \sqrt{c_{\alpha 0}} a_\alpha q \xi \right|^2 + 2\left(c_{\alpha 0} s^2 |\mathbf{v}|^2 + c_{\alpha 0} a_\alpha^2 q^2 \right) \right\rangle$$
$$\le 2(1 + \max\{2\delta_0 - 1, 0\}) \hat{A}(\mathbf{z}_0, \xi)\mathbf{b} \cdot \mathbf{b},$$

with the same δ_0 as in Lemma 3.1, since also

$$\left\langle c_{\alpha 0} s^2 |\mathbf{v}|^2 + c_{\alpha 0} a_\alpha^2 q^2 \right\rangle \le |\hat{\mathbf{u}}_0|^2 |\mathbf{v}|^2 + (a^2)_0 q^2 \le \delta_0 (\bar{\mu}_0 |\mathbf{v}|^2 + \bar{\varkappa}_0 q^2).$$

Thus, Lemma 3.2 is proved.

Using the Fourier transform, it can be shown that Lemma 3.1 for $\Omega = \mathbb{R}^n$ and Lemma 3.2 are equivalent, and thus the analogy between the proofs of Lemmas 3.1 and 4.1 versus Lemma 3.2 is natural. Note that the system of equations with the matrix $\hat{A}(\mathbf{z}_0, \xi)$ corresponds to a strongly parabolic system of equations in the sense of Vishik, see, for example, Eidelman[26, Ch. 1, Section 3.3)] but this does not concern system (3.23).

For $\tau, \mu, \lambda, \varkappa \in C(\bar{D})$, according to Lemma 3.2 and the classical Rayleigh formula for the minimal eigenvalue of a symmetric matrix, we have $\inf_{|\xi|=1} \lambda[\hat{A}(\mathbf{z}_0, \xi)] \ge \min_{\mathbf{z}_0 \in \bar{D}} \delta_2 > 0$ and thus, due to the second property (3.26), condition (3.25) is satisfied. Therefore, the equivalent quasilinear QGD and QHD systems of Equations (2.14) and (2.25)–(2.26) satisfies in D (in fact, in $\bar{D}$) the uniform Petrovskii parabolicity condition, that is, Corollary 3.2 is valid.

Theorem 3.3 follows directly from the general result on the local in time unique solvability of the Cauchy problem for quasilinear Petrovskii parabolic systems, see Theorem 6.3 and Remark 1 to it in Eidelman.[26, Ch. 3, Section 4] Reducing D to the cube as in Eidelman[26] can be performed by an affine transformation of each of the sought functions.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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