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PROPERTIES OF REGULARIZED EQUATIONS FOR BAROTROPIC GAS MIXTURES

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We consider regularizations of systems of equations for the multicomponent gas mixture dynamics in the barotropic multi-velocity and one-velocity cases and derive the energy balance equations. In the one-velocity case, we linearize the system on a constant solution and study the corresponding initial-boundary value problem with zero boundary data. We establish the existence, uniqueness, and L^2-dissipativity of weak solutions. Bibliography: 18 *titles.*

We consider the regularized quasigasdynamic and quasihydrodynamic systems of equations for the multi-component gas mixture dynamics in the barotropic case. The one-component barotropic case was considered in [1]–[3], and the case of barotropic binary two-velocity mixtures was studied in [4]. Various systems of equations for the barotropic mixture dynamics applied to computer simulations were discussed, in particular, in [5]–[8]. Quasigasdynamic and quasihydrodynamic systems of equations for homogeneous gas mixtures were studied in [4, 16] and [9]–[13]. Properties of weak solutions were studied in [14, 15].

The paper is organized as follows. In Section 1, we consider the quasigasdynamic and quasihydrodynamic systems of equations for the barotropic multi-velocity multi-component gas mixture dynamics and derive the equations of total energy balance in the presence of stationary potential mass forces. The case of the one-velocity mixture dynamics is considered In Sections 2–4. The total energy balance equations are derived in Section 2. In Section 3, the quasigasdynamic and quasihydrodynamic systems of equations are expanded with respect to the gradient of the sought functions. In Section 4, the linearization and symmetrization on a constant solution are performed. We establish the existence, uniqueness, and L^2-dissipativity of a weak solution to the initial-boundary value problem. To prove the results, we use the technique of [3, 16].

1 Barotropic Multi-Velocity Multi-Component Mixtures

Following [2]–[4], the quasigasdynamic ($\ell = 1$) and quasihydrodynamic ($\ell = 0$) systems of equations for a barotropic multi-velocity multi-component mixture can be represented as the mass and momentum balance equations for the mixture components

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Translated from *Problemy Matematicheskogo Analiza* **124**, 2023, pp. -.

$$\partial_t \rho_\alpha + \mathrm{div}\ j_{\ell\alpha} = 0, \quad \alpha = \overline{1, K}, \tag{1.1}$$

$$\partial_t(\rho_\alpha u_\alpha) + \mathrm{div}\,(j_{\ell\alpha} \otimes u_\alpha) + \nabla p_\alpha = \mathrm{div}\ \Pi_{\ell\alpha} + [\rho_\alpha - \ell\tau\,\mathrm{div}\,(\rho_\alpha u_\alpha)]F_\alpha + S_{u,\alpha}. \tag{1.2}$$

The main sought functions $\rho_\alpha > 0$ and $u_\alpha = (u_{\alpha 1}, \ldots, u_{\alpha n})$ ($\alpha = \overline{1, K}$) are interpreted as the density and velocity of component α and depend on $x = (x_1, \ldots, x_n) \in \Omega$ and $t \geqslant 0$; moreover, $K \geqslant 2$, $n = 1, 2, 3$, Ω is a bounded region in $\mathbb{R}^n$ with boundary $\partial\Omega$. Here, $p_\alpha = p_\alpha(\rho_\alpha)$ denotes the pressure of component α, $p'_\alpha(\rho_\alpha) > 0$. The operators div and $\nabla = (\partial_1, \ldots, \partial_n)$ are taken with respect to x. We use the notation $\partial_t = \partial/\partial t$ and $\partial_i = \partial/\partial x_i$. The symbols $\otimes$ and $\cdot$ mean the tensor and scalar vector products and the divergence of a tensor is taken with respect to its first index. We denote by $\langle\cdot\rangle$ the summation with respect to α, $\mathbb{I}$ is the unit tensor of order n. Further, $S_{u,\alpha}$, $\alpha = \overline{1, K}$, are the exchange terms depending on the sought functions and satisfying the relations

$$\langle S_{u,\alpha}\rangle = 0, \quad \langle S_{u,\alpha} u_\alpha\rangle \leqslant 0. \tag{1.3}$$

The mass flux vectors and viscous stress tensors have the form

$$j_{\ell\alpha} = \rho_\alpha(u_\alpha - w_{\ell\alpha}), \Pi_{\ell\alpha} = \Pi_\alpha^{NS} + \rho_\alpha u_\alpha \otimes \widehat{w}_\alpha + \ell\tau p'_\alpha(\rho_\alpha)\,\mathrm{div}\,(\rho_\alpha u_\alpha)\mathbb{I},$$

$$\Pi_\alpha^{NS} = 2\mu_\alpha \mathbb{D}(u_\alpha) + (\lambda_\alpha - \frac{2}{3}\mu_\alpha)(\mathrm{div}\ u_\alpha)\mathbb{I}, \quad \mathbb{D}(u_\alpha) = 0.5(\nabla u_\alpha + (\nabla u_\alpha)^T),$$

where Π_α^{NS} is the classical Navier–Stokes viscous stress tensor, $\mu_\alpha > 0$ and $\lambda_\alpha \geqslant 0$ are the viscosity coefficients of mixture components (possibly, depending on the sought functions), and $\nabla u = \{\partial_i u_j\}_{i,j=1}^n$.

The following regularizing velocities of the component α are used:

$$w_{\ell\alpha} = \ell\frac{\tau}{\rho_\alpha}\,\mathrm{div}\,(\rho_\alpha u_\alpha)u_\alpha + \widehat{w}_\alpha, \quad \widehat{w}_\alpha = \frac{\tau}{\rho_\alpha}[\rho_\alpha(u_\alpha \cdot \nabla)u_\alpha + \nabla p_\alpha - \rho_\alpha F_\alpha],$$

where $\tau = \tau(\rho_1, \ldots, \rho_K, u_1, \ldots, u_K) > 0$ is the regularization parameter. The mass force densities $F_\alpha = F_\alpha(x, t)$ are given functions.

The regularized systems of equations in the barotropic case are obtained by removing the full energy balance equations from the full system of regularized equations or by taking a formal generalization of the corresponding systems of equations for a binary mixture [4] and adding the equations of state, which express the dependence of the pressure of mixture components on the density.

We introduce the functions

$$P_{\alpha 0}(r) = \int_{r_{\alpha 0}}^{r} (r - s)\frac{p'_\alpha(s)}{s}\,ds, \quad r > 0,$$

where $r_{\alpha 0} > 0$, with the properties

$$P'_{\alpha 0}(r) = \int_{r_{\alpha 0}}^{r} \frac{p'_\alpha(s)}{s}\,ds, \quad P''_{\alpha 0}(r) = \frac{p'_\alpha(r)}{r}, \quad r > 0.$$

Since $p'_\alpha(r) > 0$, we have $P_{\alpha 0}(r) > 0$ for $r > 0$, $r \neq r_{\alpha 0}$, and $P_{\alpha 0}(r_{\alpha 0}) = 0$. In the isentropic case where $p_\alpha = p_{1\alpha} r^{\gamma_\alpha}$ with $\gamma_\alpha > 1$, we can take $r_{\alpha 0} = 0$. Then

$$P_{\alpha 0}(r) = \frac{1}{\gamma_\alpha - 1} p_{1\alpha} r^{\gamma_\alpha}, \quad P'_{\alpha 0}(r) = \frac{\gamma_\alpha - 1}{\gamma_\alpha} p_{1\alpha} r^{\gamma_\alpha - 1}, \quad \alpha = \overline{1, K}.$$

We assume that the mass force densities have the form $F_\alpha(x,t) = \nabla\Phi_\alpha(x) + f_\alpha(x,t)$, where $\nabla\Phi_\alpha$ are the densities of stationary mass forces and f_α are their perturbations. The functions Φ_α are defined up to a constant.

Theorem 1.1. *For the quasigasdynamic ($\ell = 1$) and quasihydrodynamic ($\ell =$)0 systems of equations for barotropic multi-velocity multi-component mixtures the energy balance equation holds*

$$\begin{aligned} &\partial_t \langle P_{\alpha 0}(\rho_\alpha) - \rho_\alpha \Phi_\alpha + 0.5\rho_\alpha |u_\alpha|^2 \rangle + B + \operatorname{div} A \\ &= \langle [\rho_\alpha - \ell\tau \operatorname{div}(\rho_\alpha u_\alpha)] f_\alpha \cdot u_\alpha + \tau f_\alpha \cdot [\rho_\alpha (u_\alpha \cdot \nabla) u_\alpha + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha \nabla\Phi_\alpha] \rangle, \end{aligned}$$

where

$$\begin{aligned} &A = \langle (P'_{\alpha 0}(\rho_\alpha) - \Phi_\alpha + 0.5|u_\alpha|^2) j_{\ell\alpha} - \Pi_{\ell\alpha} u_\alpha \rangle, \\ &B = \Big\langle 2\mu_\alpha |\mathbb{D}(u_\alpha)|^2 + (\lambda_\alpha - \frac{2}{3}\mu_\alpha)(\operatorname{div}\, u_\alpha)^2 + \frac{\tau}{\rho_\alpha} |\rho_\alpha (u_\alpha \cdot \nabla) u_\alpha + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha \nabla\Phi_\alpha|^2 \\ &+ \ell\tau \frac{p'_\alpha(\rho_\alpha)}{\rho_\alpha} [\operatorname{div}(\rho_\alpha u_\alpha)]^2 \Big\rangle - \langle S_{u,\alpha} u_\alpha \rangle \geqslant 0 \end{aligned}$$

and $|\mathbb{D}(u_\alpha)|^2 = \mathbb{D}(u_\alpha) : \mathbb{D}(u_\alpha)$, *where* $:$ *denotes the scalar product of tensors.*

Proof. Based on Statement 1 in [3], we obtain the required assertion by summarizing the corresponding energy balance equations for the mixture components (as in the case of a one-component gas) and considering the summands with the exchange terms. It is important that the energy dissipation B is nonnegative by the second property in (1.3). □

We suppose that the functions Φ_α are bounded in Ω. Then without loss of generality we can assume that $\Phi_\alpha \leqslant 0$, $\alpha = \overline{1, K}$.

Corollary 1.1. *Suppose that the boundary condition $A \cdot n|_{\partial\Omega} = 0$ is satisfied, where n is the outward normal to $\partial\Omega$ (which exists if $u|_{\partial\Omega} = 0$, $\rho_\alpha w_{\ell\alpha} \cdot n|_{\partial\Omega} = 0$, $\alpha = \overline{1, K}$). Then the integral energy balance equation for the multi-velocity mixture holds*

$$\begin{aligned} &\partial_t \int_\Omega \langle [P_{\alpha 0}(\rho_\alpha) - \rho_\alpha \Phi_\alpha + 0.5\rho_\alpha |u_\alpha|^2] \rangle dx + \int_\Omega \langle 2\mu_\alpha |\mathbb{D}(u_\alpha)|^2 + \Big(\lambda_\alpha - \frac{2}{3}\mu_\alpha\Big)(\operatorname{div}\, u_\alpha)^2 \rangle dx \\ &+ \int_\Omega \Big[\Big\langle \frac{\tau}{\rho_\alpha} |\rho_\alpha (u_\alpha \cdot \nabla) u_\alpha + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha \nabla\Phi_\alpha|^2 + \ell\tau \frac{p'_\alpha(\rho_\alpha)}{\rho_\alpha} [\operatorname{div}(\rho_\alpha u_\alpha)]^2 \Big\rangle - \langle S_{u,\alpha} u_\alpha \rangle \Big] dx \\ &= \int_\Omega \langle [\rho_\alpha - \ell\tau \operatorname{div}(\rho_\alpha u_\alpha)] f_\alpha \cdot u_\alpha + \tau f_\alpha \cdot [\rho_\alpha (u_\alpha \cdot \nabla) u_\alpha + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha \nabla\Phi_\alpha] \rangle dx. \end{aligned}$$

We derive the mixture total mass and total momentum balance equations taking the sum over the index α of the equations in the system (1.1)-(1.2):

$$\partial_t\rho + \mathrm{div}\,\langle j_{\ell\alpha}\rangle = 0, \quad \alpha = \overline{1,K},$$
$$\partial_t\langle\rho_\alpha u_\alpha\rangle + \mathrm{div}\,\langle j_{\ell\alpha}\otimes u_\alpha\rangle + \nabla p = \mathrm{div}\,\Pi_\ell + \langle[\rho_\alpha - \ell\tau\,\mathrm{div}\,(\rho_\alpha u_\alpha)]F_\alpha\rangle, \tag{1.4}$$

where the total density, pressure, and viscosity tensor are as follows:

$$\rho = \langle\rho_\alpha\rangle, \quad p = \langle p_\alpha\rangle, \quad \Pi_\ell = \langle\Pi_{\ell\alpha}\rangle$$

and the first property of (1.3) is applied.

2 Barotropic One-Velocity Multi-Component Mixtures

Within the scope of the one-velocity model, it is assumed that $u_\alpha = u$, $\alpha = \overline{1,K}$. As a result, the mass balance equations (1.1) and the mixture momentum balance equation (1.4) are simplified and take the form

$$\partial_t\rho_\alpha + \mathrm{div}\; j_{\ell\alpha} = 0, \quad \alpha = \overline{1,K}, \tag{2.1}$$
$$\partial_t(\rho u) + \mathrm{div}\,(j_\ell\otimes u) + \nabla p = \mathrm{div}\,\Pi_\ell + \langle[\rho_\alpha - \ell\tau\,\mathrm{div}\,(\rho_\alpha u)]F_\alpha\rangle,, \tag{2.2}$$

where

$$j_{\ell\alpha} = \rho_\alpha(u - w_{\ell\alpha}), \quad j_\ell = \langle j_{\ell\alpha}\rangle = \rho(u - w_\ell),$$
$$w_{\ell\alpha} = \ell\frac{\tau}{\rho_\alpha}\,\mathrm{div}\,(\rho_\alpha u)u + \widehat{w}_\alpha, \quad \widehat{w}_\alpha = \frac{\tau}{\rho_\alpha}[\rho_\alpha(u\cdot\nabla)u + \nabla p_\alpha - \rho_\alpha F_\alpha],$$
$$w_\ell = \Big\langle\frac{\rho_\alpha}{\rho}w_{\ell\alpha}\Big\rangle = \ell\frac{\tau}{\rho}\,\mathrm{div}\,(\rho u)u + \widehat{w}, \quad \widehat{w} = \frac{\tau}{\rho}[\rho(u\cdot\nabla)u + \nabla p - \langle\rho_\alpha F_\alpha\rangle],$$
$$\Pi_\ell = \Pi^{NS} + \rho u\otimes\widehat{w} + \ell\tau\langle p'_\alpha(\rho_\alpha)\,\mathrm{div}\,(\rho_\alpha u)\rangle\mathbb{I},$$
$$\Pi^{NS} = \mu(\nabla u + (\nabla u)^T) + \Big(\lambda - \frac{2}{3}\mu\Big)(\mathrm{div}\; u)\mathbb{I}, \quad \mu = \langle\mu_\alpha\rangle, \quad \lambda = \langle\lambda_\alpha\rangle.$$

The total mass balance equation holds

$$\partial_t\rho + \mathrm{div}\; j_\ell = 0. \tag{2.3}$$

We prove counterparts of Theorem 1.1 and Corollary 1.1.

Theorem 2.1. *For the quasigasdynamic ($\ell = 1$) and quasihydrodynamic ($\ell = 0$) systems of equations for barotropic one-velocity multi-component mixtures the energy balance equation holds*

$$\partial_t[\langle P_{\alpha 0}(\rho_\alpha) - \rho_\alpha\Phi_\alpha\rangle + 0.5\rho|u|^2] + B + \mathrm{div}\; A$$
$$= \langle[\rho_\alpha - \ell\tau\,\mathrm{div}\,(\rho_\alpha u)]f_\alpha\cdot u + \tau f_\alpha\cdot[\rho_\alpha(u\cdot\nabla)u + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha]\rangle$$

where

$$A = \langle (P'_{\alpha 0}(\rho_\alpha) - \Phi_\alpha + 0.5|u|^2) j_{\ell\alpha}\rangle - \Pi_\ell u, \tag{2.4}$$

$$B = 2\mu|\mathbb{D}(u)|^2 + (\lambda - \frac{2}{3}\mu)(\text{div } u)^2 + \Big\langle \frac{\tau}{\rho_\alpha}|\rho_\alpha(u\cdot\nabla)u + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha|^2 + \ell\tau\frac{p'_\alpha(\rho_\alpha)}{\rho_\alpha}[\text{div }(\rho_\alpha u)]^2\Big\rangle. \tag{2.5}$$

Proof. We follow the scheme of the proof of Statement 1 in [3]. We multiply Equations (2.1) by $P'_{0\alpha}(\rho_\alpha) - \Phi_\alpha$ and, taking into account the formula

$$(\text{div } j_{\ell\alpha})(P'_{0\alpha}(\rho_\alpha) - \Phi_\alpha) = \text{div }[(P'_{0\alpha}(\rho_\alpha) - \Phi_\alpha) j_{\ell\alpha}] - \Big(\frac{1}{\rho_\alpha}\nabla p_\alpha(\rho_\alpha) - \nabla\Phi_\alpha\Big) j_{\ell\alpha},$$

get

$$\partial_t(P_{0\alpha}(\rho_\alpha) - \rho_\alpha\Phi_\alpha) + \text{div }[(P'_{0\alpha}(\rho_\alpha) - \Phi_\alpha) j_{\ell\alpha}] - (\nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha)(u - w_{\ell\alpha}) = 0. \tag{2.6}$$

We take the inner product of Equation (2.2) and u and use the formulas

$$\partial_t(\rho u)\cdot u = 0.5\partial_t(\rho|u|^2) + 0.5(\partial_t\rho)|u|^2,$$
$$[\text{div }(j_\ell\otimes u)]\cdot u = 0.5\,\text{div }(|u|^2 j_\ell) + 0.5(\text{div } j_\ell)|u|^2,$$
$$(\text{div } \Pi_\ell)\cdot u = \text{div }(\Pi_\ell u) - \Pi_\ell : \nabla u$$

and Equation (2.3). Once we have regrouped the terms, we find

$$0.5\partial_t(\rho|u|^2) + 0.5\,\text{div }(|u|^2 j_\ell) + (\nabla p - \langle\rho_\alpha\nabla\Phi_\alpha\rangle)\cdot u + \Pi_\ell : \nabla u$$
$$+ \ell\tau\langle\text{div }(\rho_\alpha u)\nabla\Phi_\alpha\rangle\cdot u - \text{div }(\Pi_\ell u) = \langle[\rho_\alpha - \ell\tau\,\text{div }(\rho_\alpha u)]f_\alpha\rangle\cdot u.$$

Adding this equality to (2.6) for all $\alpha = \overline{1,K}$, we derive

$$\partial_t\left[\langle P_{0\alpha}(\rho_\alpha) - \rho_\alpha\Phi_\alpha\rangle + 0.5\rho|u|^2\right] + \langle(\nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha)\cdot w_{\ell\alpha}\rangle + \Pi_\ell : \nabla u$$
$$+ \ell\tau\langle\text{div }(\rho_\alpha u)\nabla\Phi_\alpha\rangle\cdot u + \text{div } A = \langle[\rho_\alpha - \ell\tau\,\text{div }(\rho_\alpha u)]f_\alpha\rangle\cdot u,$$

where the vector-valued function A has the form (2.4). We extract the terms with f_α from the expressions for $w_{\ell\alpha}$ and $\Pi_{\ell\alpha}$ and write the last equation as

$$\partial_t[\langle P_{0\alpha}(\rho_\alpha) - \rho_\alpha\Phi_\alpha\rangle + 0.5\rho|u|^2] + \Pi^{NS} : \nabla u + \langle\Psi_\alpha(\rho_\alpha, u)\rangle + \text{div } A$$
$$= \langle(\rho_\alpha - \ell\tau\,\text{div }(\rho_\alpha u)f_\alpha\rangle\cdot u + \tau\langle f_\alpha\cdot[\nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha + \rho_\alpha(u\nabla)u]\rangle, \tag{2.7}$$

where

$$\Psi_\alpha(\rho_\alpha, u) = (\nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha)(w_{\ell\alpha} + \tau f_\alpha)$$
$$+ [\rho_\alpha u\otimes(\widehat{w}_\alpha + \tau f_\alpha) + \ell\tau p'_\alpha(\rho_\alpha)\,\text{div }(\rho_\alpha u)\mathbb{I}] : \nabla u + \ell\tau\langle\text{div }(\rho_\alpha u)\nabla\Phi_\alpha\rangle\cdot u$$
$$= (\nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha)(w_{\ell\alpha} + \tau f_\alpha) + \rho_\alpha(u\cdot\nabla)u\cdot(\widehat{w}_\alpha + \tau f_\alpha)$$
$$+ \ell\tau\nabla\Phi_\alpha\cdot\text{div }(\rho_\alpha u)u + \ell\tau p'_\alpha(\rho_\alpha)[\text{div }(\rho_\alpha u)]\,\text{div } u.$$

Then we use the formulas

$$\Pi^{NS} : \nabla u = 2\mu|\mathbb{D}(u)|^2 + (\lambda - \frac{2}{3}\mu)(\mathrm{div}\ u)^2,$$

$$w_{\ell\alpha} + \tau f_\alpha = \widehat{w}_\alpha + \tau f_\alpha + \ell\frac{\tau}{\rho_\alpha}\,\mathrm{div}\,(\rho_\alpha u)u,$$

$$\widehat{w}_\alpha + \tau f_\alpha = \frac{\tau}{\rho_\alpha}[\rho_\alpha(u\nabla)u + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha]$$

and transform $\Psi_\alpha(\rho_\alpha, u)$ to the form

$$\Psi_\alpha(\rho_\alpha, u) = [\rho_\alpha(u\nabla)u + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha]\cdot(\widehat{w}_\alpha + \tau f_\alpha)$$

$$+\,\ell\frac{\tau}{\rho_\alpha}[\nabla p_\alpha(\rho_\alpha)]\,\mathrm{div}\,(\rho_\alpha u)u + \ell\tau p'_\alpha(\rho_\alpha)[\mathrm{div}\,(\rho_\alpha u)]\,\mathrm{div}\ u$$

$$= \frac{\tau}{\rho_\alpha}|\rho_\alpha(u\cdot\nabla)u + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha|^2 + \ell\frac{p'_\alpha(\rho_\alpha)}{\rho_\alpha}[\mathrm{div}\,(\rho_\alpha u)]^2.$$

The theorem is proved. □

Corollary 2.1. *Suppose that the boundary condition* $A\cdot n|_{\partial\Omega} = 0$ *is satisfied (which certainly takes place if* $u|_{\partial\Omega} = 0,\ \rho_\alpha w_{\ell\alpha}\cdot n|_{\partial\Omega} = 0,\ \alpha = \overline{1,K}$*). Then the integral energy balance equation for the one-velocity mixture holds*

$$\partial_t\int\limits_\Omega[\langle P_{\alpha 0}(\rho_\alpha) - \rho_\alpha\Phi_\alpha\rangle + 0.5\rho|u|^2]dx + \int\limits_\Omega\Big[2\mu|\mathbb{D}(u)|^2 + (\lambda - \frac{2}{3}\mu)(\mathrm{div}\ u)^2\Big]dx$$

$$+\int\limits_\Omega\Big\langle\frac{\tau}{\rho_\alpha}|\rho_\alpha(u\nabla)u + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha|^2 + \ell\frac{p'_\alpha(\rho_\alpha)}{\rho_\alpha}[\mathrm{div}\,(\rho_\alpha u)]^2\Big\rangle dx$$

$$= \int\limits_\Omega\langle[\rho_\alpha - \ell\tau\,\mathrm{div}\,(\rho_\alpha u)]f_\alpha\cdot u + \tau f_\alpha\cdot[\rho_\alpha(u\cdot\nabla)u + \nabla p_\alpha(\rho_\alpha) - \rho_\alpha\nabla\Phi_\alpha]\rangle dx.$$

3 Expansion of Quasigasdynamic and Quasihydrodynamic Systems

The properties studied below are related to the regularizing properties of the terms with multiplier τ in the quasigasdynamic and quasihydrodynamic systems. Let $p_\alpha \in C^2(0, +\infty)$, $F_\alpha = 0,\ \alpha = \overline{1,K}$.

We introduce a vector of functions $z = (\boldsymbol{\rho}, u)$, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_K)$ and perform an auxiliary reduction of Equations (2.1)-(2.2) with accuracy $O(|\nabla z|^2)$. We write the mass balance equations for the mixture component (2.1) as

$$\partial_t\rho_\alpha + \mathrm{div}\,(\rho_\alpha u) = \mathrm{div}\,(\rho_\alpha w_{\ell\alpha}),\quad \alpha = \overline{1,K}.$$

We represent the function on the right-hand side in the form

$$\rho_\alpha w_{\ell\alpha} = \ell\tau\,\mathrm{div}\,(\rho_\alpha u)u + \rho_\alpha\widehat{w}_\alpha = \ell\tau(\nabla\rho_\alpha\cdot u + \rho_\alpha\,\mathrm{div}\ u)u + \rho_\alpha\widehat{w}_\alpha.$$

We perform the following expansions:

$$
\begin{aligned}
&\operatorname{div}[\tau(\nabla\rho_\alpha\cdot u)u] = \tau[\nabla(\nabla\rho_\alpha\cdot u)]u + O(|\nabla z|^2) = \tau[u\cdot(u\cdot\nabla)\nabla]\rho_\alpha + O(|\nabla z|^2),\\
&\operatorname{div}[\tau\rho_\alpha(\operatorname{div} u)u] = \tau\rho_\alpha(u\cdot\nabla)\operatorname{div} u + O(|\nabla z|^2),\\
&\partial_k(\rho_\alpha\widehat{w}_\alpha) = \tau\partial_k\{\rho_\alpha(u\cdot\nabla)u + \nabla p_\alpha\} + O(|\nabla z|^2)\\
&\qquad = \tau\big\{\rho_\alpha(u\cdot\nabla)\partial_k u + p'_\alpha(\rho_\alpha)\partial_k\nabla\rho_\alpha]\big\} + O(|\nabla z|^2), \quad k=\overline{1,n}.
\end{aligned} \tag{3.1}
$$

Summing up over $k=\overline{1,n}$ in the last formula yields

$$
\operatorname{div}(\rho_\alpha\widehat{w}_\alpha) = \tau[\rho_\alpha(u\cdot\nabla)\operatorname{div} u + p'_\alpha\Delta\rho_\alpha] + O(|\nabla z|^2),
$$

where $\Delta = \operatorname{div}\nabla$ denotes the Laplace operator. Therefore, the reduced equations for the component densities take the form

$$
\begin{aligned}
\partial_t\rho_\alpha + \nabla\rho_\alpha\cdot u + \rho_\alpha\operatorname{div} u = \tau[p'_\alpha\Delta\rho_\alpha + \ell[u\cdot(u\cdot\nabla)\nabla]\rho_\alpha&\\
+ (\ell+1)\rho_\alpha(u\cdot\nabla)\operatorname{div} u] + O(|\nabla z|^2), \quad \alpha = \overline{1,K}.&
\end{aligned} \tag{3.2}
$$

Similarly to Equation (2.25) in [16], we transform the total momentum balance equation (2.2) to the form

$$
\begin{aligned}
&\partial_t u + [(u-w_\ell)\cdot\nabla]\,u + \frac{1}{\rho}\nabla p\\
&= \frac{1}{\rho}\operatorname{div}\{\operatorname{div}\Pi^{NS} + (u\cdot\nabla)(\rho\widehat{w}) + (\operatorname{div} u)(\rho\widehat{w}) + \ell\tau\nabla(p'_\alpha\operatorname{div}(\rho_\alpha u))\}.
\end{aligned} \tag{3.3}
$$

Using (3.1), we expand the functions on the right-hand side of (3.3) as follows:

$$
\begin{aligned}
&\operatorname{div}\Pi^{NS} = \mu\Delta u + \chi\nabla\operatorname{div} u + O(|\nabla z|^2),\\
&(u\cdot\nabla)(\rho\widehat{w}) = (u\cdot\nabla)\langle\rho_\alpha\widehat{w}_\alpha\rangle = \tau\langle\rho_\alpha[u\cdot(u\cdot\nabla)\nabla]u + p'_\alpha(u\cdot\nabla)\nabla\rho_\alpha\rangle + O(|\nabla z|^2),\\
&\nabla(p'_\alpha\operatorname{div}(\rho_\alpha u)) = p'_\alpha\nabla(\rho_\alpha\operatorname{div}(u) + u\cdot\nabla\rho_\alpha) + O(|\nabla z|^2)\\
&\qquad = p'_\alpha\rho_\alpha\nabla\operatorname{div} u + p'_\alpha(u\cdot\nabla)\nabla\rho_\alpha + O(|\nabla z|^2),
\end{aligned}
$$

where $\chi := \frac{1}{3}\mu + \lambda$. Decomposing the left-hand side of (3.3) and using the estimate

$$
|w_\ell\cdot\nabla u| + |(\operatorname{div} u)(\rho\widehat{w})| = O(|\nabla z|^2),
$$

we derive the reduced equation for the velocity

$$
\begin{aligned}
\partial_t u + \frac{1}{\rho}\langle p'_\alpha\nabla\rho_\alpha\rangle + (u\cdot\nabla)u = \tau(\ell+1)\frac{1}{\rho}\langle p'_\alpha(u\cdot\nabla)\nabla\rho_\alpha\rangle + \frac{\mu}{\rho}\Delta u + \frac{\chi}{\rho}\nabla\operatorname{div} u&\\
+ \tau\frac{\ell}{\rho}\langle p'_\alpha\rho_\alpha\rangle\nabla\operatorname{div} u + \tau[u\cdot(u\cdot\nabla)\nabla]u + O(|\nabla z|^2).&
\end{aligned} \tag{3.4}
$$

4 Linearized Quasigasdynamic and Quasihydrodynamic Systems

In the case $F_\alpha = 0$, $\alpha=\overline{1,K}$, the quasigasdynamic and quasihydrodynamic systems of equations (2.1)–(2.2) have constant solutions

$$
(\boldsymbol{\rho}, u)(x,t) \equiv z_0 = (\rho_{10},\dots,\rho_{K0},u_0), \quad \rho_{10}>0,\dots,\rho_{K0}>0.
$$

To linearize the system on the background solution z_0, we write the solution as

$$\rho_\alpha = \rho_{\alpha 0} + \rho_{\alpha *}\widetilde{\rho}_\alpha \quad (\alpha = \overline{1,K}), \quad u = u_0 + u_*\widetilde{u}, \tag{4.1}$$

where $\rho_{\alpha *} > 0$ and $u_* > 0$ are the dimensionless parameters which will be chosen below. We introduce the dimensionless perturbation vector $\widetilde{z} := (\widetilde{\boldsymbol{\rho}}, \widetilde{u})$ with $\widetilde{\boldsymbol{\rho}} := (\widetilde{\rho}_1, \ldots, \widetilde{\rho}_K)$, the background normalized solution

$$\widehat{\rho}_{\alpha 0} := \frac{\rho_{\alpha 0}}{\rho_{\alpha *}}, \quad \widehat{u}_0 = (\widehat{u}_{10}, \ldots, \widehat{u}_{n0}) := \frac{u_0}{u_*},$$

and set $\rho_0 := \langle \rho_{\alpha 0} \rangle$. We substitute the solution in form (4.1) into the reduced system (3.2), (3.4), as in [16]. Since

$$\nabla z = (\rho_{1*}\nabla\widetilde{\rho}_1, \ldots, \rho_{K*}\nabla\widetilde{\rho}_K, u_*\nabla\widetilde{u}), \quad O(|\nabla z|^2) = O(|\nabla \widetilde{z}|^2),$$

after discarding the second order of smallness terms with respect to the vector-valued function $\widetilde{z}$ and its first and second order derivatives and dividing the equations by $\rho_{\alpha *}$ and u_* respectively, we easily obtain the linearized system of equations

$$\begin{aligned}
&\partial_t \widetilde{\rho}_\alpha + u_*(\widehat{u}_0 \cdot \nabla\widetilde{\rho}_\alpha + \widehat{\rho}_{\alpha 0} \operatorname{div} \widetilde{u}) \\
&= \tau_0 u_*^2 \Big[\frac{p'_{\alpha 0}}{u_*^2}\Delta\widetilde{\rho}_\alpha + \ell(\widehat{u}_0 \cdot \nabla)^2\widetilde{\rho}_\alpha + (\ell+1)\widehat{\rho}_{\alpha 0}(\widehat{u}_0 \cdot \nabla) \operatorname{div} \widetilde{u}\Big], \quad \alpha = \overline{1,K}, \\
&\partial_t \widetilde{u} + u_*\Big(\frac{1}{\rho_0 u_*^2}\langle p'_{\alpha 0}\rho_{\alpha *}\nabla\widetilde{\rho}_\alpha\rangle + (\widehat{u}_0 \cdot \nabla)\widetilde{u}\Big) \\
&= u_*^2\Big[(\ell+1)\frac{\tau_0}{\rho_0 u_*^2}(\widehat{u}_0 \cdot \nabla)\langle p'_{\alpha 0}\rho_{\alpha *}\nabla\widetilde{\rho}_\alpha\rangle + \frac{\mu_0}{\rho_0 u_*^2}\Delta\widetilde{u} \\
&+ \Big(\frac{\chi_0}{\rho_0 u_*^2} + \ell\frac{\tau_0}{\rho_0 u_*^2}\langle p'_{\alpha 0}\rho_{\alpha 0}\rangle\Big)\nabla \operatorname{div} \widetilde{u} + \tau_0(\widehat{u}_0 \cdot \nabla)^2\widetilde{u}\Big].
\end{aligned}$$

In these equations, τ_0, μ_0, χ_0, and $p'_{\alpha 0}$ are the values of τ, μ, χ, and p'_α at the background solution. Also, from the convective terms (i.e., containing the first order derivatives with respect to x) and the dissipative terms (i.e., containing the second order derivatives), the common factor u_* and the factor u_*^2 are carried out respectively. To simplify the analysis of the resulting system of linearized equations, the possibility of symmetrizing both convective and dissipative terms is essential. It is simultaneously fulfilled under the conditions

$$\widehat{\rho}_{\alpha 0} = \frac{1}{\rho_0 u_*^2}\rho_{\alpha *}p'_{\alpha 0}, \quad \alpha = \overline{1,K}. \tag{4.2}$$

We further assume that these conditions are satisfied. This allows us to simplify considerably the above form of the linearized system of equations; namely,

$$\begin{aligned}
&\partial_t \widetilde{\rho}_\alpha + u_*(\widehat{u}_0 \cdot \nabla\widetilde{\rho}_\alpha + \widehat{\rho}_{\alpha 0} \operatorname{div} \widetilde{u}) \\
&= \tau_0 u_*^2 [m_{\alpha 0}^2 \Delta\widetilde{\rho}_\alpha + \ell(\widehat{u}_0 \cdot \nabla)^2\widetilde{\rho}_\alpha + (\ell+1)\widehat{\rho}_{\alpha 0}(\widehat{u}_0 \cdot \nabla) \operatorname{div} \widetilde{u}], \quad \alpha = \overline{1,K}, \qquad (4.3) \\
&\partial_t \widetilde{u} + u_*(\langle \widehat{\rho}_{\alpha 0}\nabla\widetilde{\rho}_\alpha\rangle + (\widehat{u}_0 \cdot \nabla)\widetilde{u}) \\
&= u_*^2[(\ell+1)\tau_0(\widehat{u}_0 \cdot \nabla)\langle \widehat{\rho}_{\alpha 0}\nabla\widetilde{\rho}_\alpha\rangle + \overline{\mu}_0\Delta\widetilde{u} + (\overline{\chi}_0 + \ell\tau_0\langle \widehat{\rho}_{\alpha 0}^2\rangle)\nabla \operatorname{div} \widetilde{u} + \tau_0(\widehat{u}_0 \cdot \nabla)^2\widetilde{u}], \qquad (4.4)
\end{aligned}$$

where the following constants were introduced for the sake of convenience:

$$m_{\alpha 0} := \frac{\sqrt{p'_{\alpha 0}}}{u_*}, \quad \overline{\mu}_0 := \frac{\mu_0}{\rho_0 u_*^2}, \quad \overline{\chi}_0 := \frac{\chi_0}{\rho_0 u_*^2}.$$

The resulting system of equations is a simplification of the corresponding more general system of equations in [16] in the case of the zero perturbation of the total mixture temperature. The coefficients $m_{\alpha 0}^2$ have different forms there, but this is not substantial. This suggests that the properties of symmetrized linearized system (4.3)-(4.4) below hold.

We introduce the inner products and norms $(\cdot,\cdot)_\Omega = (\cdot,\cdot)_{L^2(\Omega)}$, $\|\cdot\|_\Omega = \|\cdot\|_{L^2(\Omega)}$, $(\cdot,\cdot)_{\mathbf{\Omega}} = (\cdot,\cdot)_{L^2(\Omega)}$, and $|||\cdot\|_{\mathbf{\Omega}} = |||\cdot||_{L^2(\Omega)}$ in the Lebesgue spaces of functions and vector-valued functions given on Ω. Let $H^1(\Omega) = W_2^1(\Omega)$ be the Sobolev space of vector-valued functions, and let $H_0^1(\Omega)$ be the closure in the $H^1(\Omega)$-norm of the space of smooth vector-valued functions having finite support in Ω.

We consider the system of equations (4.3)–(4.4) in the cylinder $\Omega\times(0,\infty)$ under the boundary and initial conditions

$$\widetilde{z}|_{\partial\Omega\times(0,\infty)} = 0, \quad \widetilde{z}|_{t=0} = \widetilde{z}^{(0)}(x). \tag{4.5}$$

Let us analyze this initial-boundary value problem. If $\partial_t\widetilde{z}(\cdot,t), \nabla\widetilde{z}(\cdot,t) \in L^2(\Omega)$, then the integral identity

$$(\partial_t\widetilde{z}(\cdot,t), z)_{\mathbf{\Omega}} + u_*\mathscr{B}_\Omega(\widetilde{z}(\cdot,t), z) + u_*^2\mathscr{A}_\Omega(\widetilde{z}(\cdot,t), z) = 0 \quad \forall t > 0 \ \ \forall z \in H_0^1(\Omega) \tag{4.6}$$

corresponds to Equations (4.3)–(4.4). It contains the bilinear forms

$$\mathscr{B}_\Omega(\widetilde{z}, z) := \langle(\widehat{u}_0\cdot\nabla\widetilde{\rho}_\alpha + \widehat{\rho}_{\alpha 0}\,\mathrm{div}\,\widetilde{u}, \rho_\alpha)_\Omega\rangle + (\langle\widehat{\rho}_{\alpha 0}\nabla\widetilde{\rho}_\alpha\rangle + (\widehat{u}_0\cdot\nabla)\widetilde{u}, u)_{\mathbf{\Omega}},$$

$$\begin{aligned}\mathscr{A}_\Omega(\widetilde{z}, z) :=\ &\overline{\mu}_0(\nabla\widetilde{u}, \nabla u)_{\mathbf{\Omega}} + \overline{\chi}_0(\mathrm{div}\,\widetilde{u}, \mathrm{div}\,u)_\Omega + \tau_0[\langle(m_{\alpha 0}^2\nabla\widetilde{\rho}_\alpha, \nabla\rho_\alpha)_{\mathbf{\Omega}}\rangle\\ &+ \ell\langle(\widehat{u}_0\cdot\nabla\widetilde{\rho}_\alpha, (\widehat{u}_0\cdot\nabla)\rho_\alpha)_\Omega\rangle + (\ell+1)((\widehat{u}_0\cdot\nabla)\widetilde{u}, \langle\widehat{\rho}_{\alpha 0}\nabla\rho_\alpha\rangle)_{\mathbf{\Omega}}\\ &+ (\ell+1)(\langle\widehat{\rho}_{\alpha 0}\nabla\widetilde{\rho}_\alpha\rangle, (\widehat{u}_0\cdot\nabla)u)_{\mathbf{\Omega}} + \ell(\langle\widehat{\rho}_{\alpha 0}^2\rangle\,\mathrm{div}\,\widetilde{u}, \mathrm{div}\,u)_\Omega + ((\widehat{u}_0\cdot\nabla)\widetilde{u}, (\widehat{u}_0\cdot\nabla)u)_{\mathbf{\Omega}}].\end{aligned}$$

Here, we consider the tensors $\nabla\widetilde{u}$ and ∇u as vectors of length n^2.

We indicate some properties of these bilinear forms which follow from Lemma 3.1 of [16].

Lemma 4.1. *For any* $z, \widetilde{z} \in H_0^1(\Omega)$

$$\begin{aligned}&\mathscr{B}_\Omega(z,z) = 0, \quad \mathscr{A}_\Omega(\widetilde{z}, z) = \mathscr{A}_\Omega(z, \widetilde{z}),\\ &\mathscr{A}_\Omega(z,z) = \overline{\mu}_0\|\nabla u\|_{\mathbf{\Omega}}^2 + \overline{\chi}_0\|\,\mathrm{div}\,u\|_\Omega^2 + \tau_0[\langle\|(m_{\alpha 0}\nabla\rho_\alpha\\ &\qquad + \sqrt{c_{\alpha 0}}(\widehat{u}_0\cdot\nabla)u\|_{\mathbf{\Omega}}^2 + \ell\langle\|\widehat{u}_0\cdot\nabla\rho_\alpha + \widehat{\rho}_{\alpha 0}\,\mathrm{div}\,u\|_\Omega^2\rangle] \geqslant 0,\end{aligned}$$

where $c_{\alpha 0} = \dfrac{\rho_{\alpha 0}}{\rho_0}$. *As a consequence, the lower estimate holds*

$$\mathscr{A}_\Omega(z,z) \geqslant \max\{\delta_1\tau_0\langle\|\nabla\rho_\alpha\|_{\mathbf{\Omega}}^2\rangle, \overline{\mu}_0\|\nabla u\|_{\mathbf{\Omega}}^2 + \overline{\chi}_0\|\,\mathrm{div}\,u\|_\Omega^2\};$$

moreover,

$$\mathscr{A}_\Omega(z,z) \geqslant \delta_2(\langle\|\nabla\rho_\alpha\|_{\mathbf{\Omega}}^2\rangle + \|\nabla u\|_{\mathbf{\Omega}}^2)$$

with $\delta_2 := \frac{1}{2}\min\{\delta_1\tau_0, \overline{\mu}_0\} > 0$, $\delta_1 := \frac{1}{2}(1 + \max\{2\delta_0 - 1, 0\})^{-1}\min\limits_{\alpha=\overline{1,K}} m_{\alpha 0}^2$, *and* $\delta_0 := \tau_0\frac{|\widehat{u}_0|^2}{\overline{\mu}_0}$.

We denote by $V(Q_T)$ the space of vector-functions $\widetilde{z} \in L^2((0,T); H_0^1(\Omega))$ with distributional derivatives $\partial_t \widetilde{z} \in L^2((0,T); H^{-1}(\Omega))$, where $Q_T = \Omega \times (0,T)$ and $H^{-1}(\Omega) = (H_0^1(\Omega))^*$ (see, for example, [17]). For the initial-boundary value problem for the system of equations (4.3)–(4.4) in $Q := \Omega \times (0,\infty)$ with the conditions (4.5), we introduce a weak solution $\widetilde{z} \in V(Q_T)$ for all $T > 0$ satisfying the integral identity

$$\int_0^T \langle \partial_t \widetilde{z}(\cdot,t), z(\cdot,t)\rangle_\Omega \, dt + u_* \mathscr{B}_{Q_T}(\widetilde{z}, z) + u_*^2 \mathscr{A}_{Q_T}(\widetilde{z}, z) = 0 \quad \forall z \in L^2((0,T); H_0^1(\Omega)) \tag{4.7}$$

for any $T > 0$, with the initial function $\widetilde{z}^{(0)} \in L^2(\Omega)$. In this identity, $\langle \cdot,\cdot\rangle_\Omega$ is the duality relation on $H^{-1}(\Omega) \times H_0^1(\Omega)$ and, in the bilinear forms $\mathscr{B}_{Q_T}$ and $\mathscr{A}_{Q_T}$, we take the inner products over Q_T instead of Ω as above. The property $V(Q_T) \subset C([0,T]; L^2(\Omega))$ [17] allows one to understand the validity of the initial condition of continuity in $L^2(\Omega)$, and the identity (4.7) is formally obtained from (4.6) for $z = z(\cdot,t)$ by integration over $(0,T)$.

Theorem 4.1. *The weak solution $\widetilde{z} \in V(Q_T)$, $T > 0$, to the initial-boundary value problem for the linearized system of equations* (4.3)*–*(4.5) *exists and is unique. The energy equality holds*

$$\tfrac{1}{2}\|\widetilde{z}(\cdot,T)\|^2_{L^2(\Omega)} + u_*^2 \mathscr{A}_{Q_T}(\widetilde{z}, \widetilde{z}) = \tfrac{1}{2}\|\widetilde{z}^{(0)}\|^2_{L^2(\Omega)} \quad \forall T > 0.$$

The function $\|\widetilde{z}(\cdot,t)\|_{L^2(\Omega)}$ does not decrease for $t \geqslant 0$, and the energy estimate holds

$$\max\{\max_{t\geqslant 0} \|\widetilde{z}(\cdot,t)\|_{L^2(\Omega)},\ \sqrt{2\delta_1\tau_0}\, u_* \langle \|\nabla\widetilde{\rho}_\alpha\|^2_{L^2(Q)}\rangle^{1/2},$$
$$\sqrt{2}u_*(\overline{\mu}_0 \|\nabla u\|^2_{L^2(Q)} + \overline{\chi}_0 \|\operatorname{div}\, u\|^2_{L^2(Q)})^{1/2}\} \leqslant \|\widetilde{z}^{(0)}\|_{L^2(\Omega)}.$$

Corollary 4.1. *There derivative $\partial_t(\|\widetilde{z}(\cdot,t)\|^2_{L^2(\Omega)}) \in L^1(0,\infty)$ exists, and the energy equality can be written in the alternative form*

$$\tfrac{1}{2}\partial_t(\|\widetilde{z}(\cdot,t)\|^2_{L^2(\Omega)}) + u_*^2 \mathscr{A}_\Omega(\widetilde{z}(\cdot,t), \widetilde{z}(\cdot,t)) = 0,$$

valid for almost all $t > 0$. The sharpened form of the $L^2(\Omega)$-dissipativity holds:

$$\partial_t(\|\widetilde{z}(\cdot,t)\|^2_{L^2(\Omega)}) \leqslant 0$$

for almost all $t > 0$.

These results are derived in the same manner as the Theorem 3.2 and Corollary 3.1 in [16].

In conclusion, we emphasize that for the system of equations (2.1)-(2.2) the results about the Petrovskii parabolicity and the classical solvability of the Cauchy problem, local in time, are rather similar to those proved in [16]. We recall that the latter is based on general theorems from [18].

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