Техника Келдыша

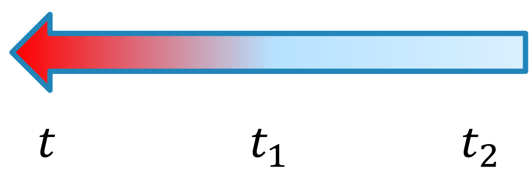
Основные понятия

Хронологическое упорядочение

$$U(t,t_0)=1-i\int_{t_0}^{t}\hat{V}(t_1)dt_1+i^2\int_{t_0}^{t}\hat{V}(t_1)dt_1\int_{t_0}^{t_1}\hat{V}(t_2)dt_2+\ldots.$$

$$\boxed{U(t,t_0)=T\,exp\left[-i\int_{t_0}^{t}\hat{V}(t')dt'\right]}$$

$$\int_{t_0}^{t}\hat{V}(t_1)dt_1\int_{t_0}^{t_1}\hat{V}(t_2)dt_2=\tfrac{1}{2}T\left[\int_{t_0}^{t}\hat{V}(t_1)dt_1\int_{t_0}^{t}\hat{V}(t_2)dt_2\right]$$



$$\mathrm{T}(\widehat{A}(t_1)\widehat{B}(t_2))=\theta(t_1-t_2)\widehat{A}(t_1)\widehat{B}(t_2)+\theta(t_2-t_1)\widehat{B}(t_2)\widehat{A}(t_1)$$

Корреляционные функции

Коррелятор t > t’

$$< \hat{A}_H(t)\hat{B}_H(t') >= Sp[\rho_0 S^+(t,t_0)\hat{A}S(t,t')\hat{B}S(t',t_0)]$$

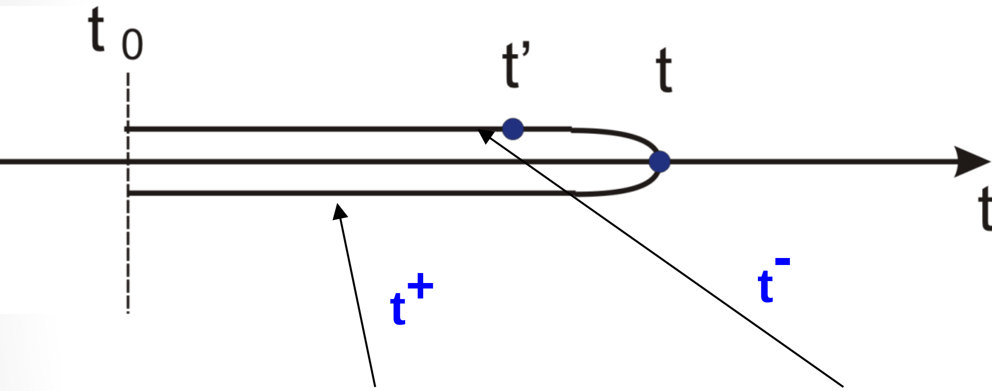
$$= Sp[\rho_0 U^+(t,t_0)\hat{A}(t)U(t,t')\hat{B}(t')U(t',t_0)]$$

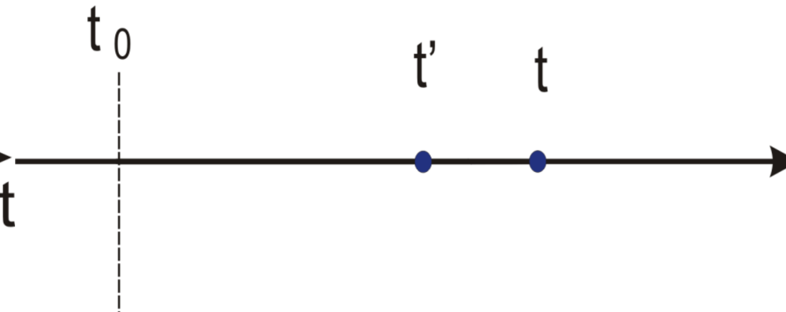
То же для T - упорядочения $< T\hat{A}_H(t)\hat{B}_H(t') >$

$$T\hat{A}(t)\hat{B}(t') = \theta(t-t')\hat{A}(t)\hat{B}(t') \mp \theta(t'-t)\hat{B}(t')\hat{A}(t)$$

$$< T\hat{A}_H(t)\hat{B}_H(t') >= Sp[\rho_0 U^+(t,t_0)T\{U(t,t_0)\hat{A}(t)\hat{B}(t')\}]$$

Контур Келдыша

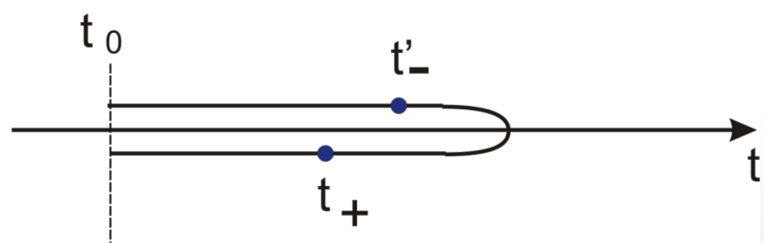




$$Sp[\rho_0 \tilde{T} exp\left[+i\int_{t_0}^{t}\hat{V}(t_2')dt_2'\right] T\left\{exp\left[-i\int_{t_0}^{t}\hat{V}(t_1')dt_1'\right]\hat{A}(t)\hat{B}(t')\right\}]$$

$$G^{+-}(rt,r't') = -i\langle\hat{\psi}(rt)\hat{\psi}^{+}(r't')\rangle$$

$$G^{-+}(rt,r't') = i\langle\hat{\psi}^{+}(r't')\hat{\psi}(rt)\rangle$$



Контур Келдыша

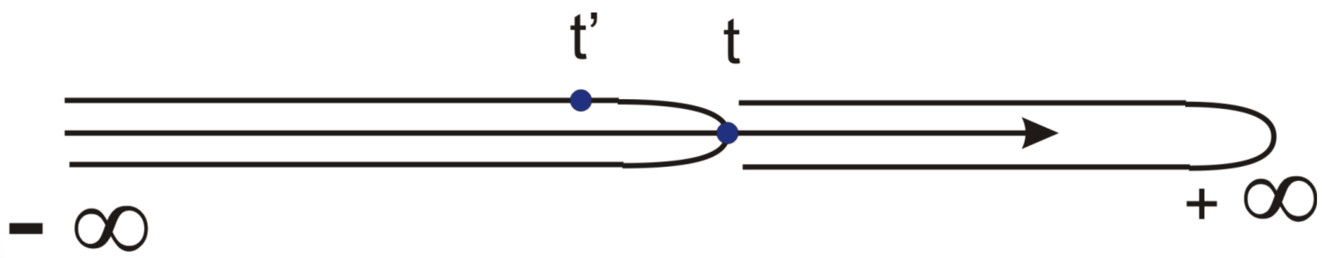
Группы операторов стоят в определенном порядке

$$Sp[\rho_0 T_C \left\{ exp\left[-i\int_C \hat{V}(t_c)dt_c \right] \hat{A}(t^-)\hat{B}(t'^-) \right\}]$$

Если $t_0 \to -\infty$

$$< \hat{A}_H(t)\hat{B}_H(t') >= Sp[\rho_0 S^+(t,-\infty)S^+(+\infty,t)S(+\infty,t)\hat{A}S(t,t')\hat{B}S(t',-\infty)] =$$

$$= Sp[\rho_0 U^+(+\infty,-\infty)T\{U(+\infty,-\infty)\hat{A}(t)\hat{B}(t')\}]$$



Представление Ларкина и Овчинникова

$$\hat{G}^R = \hat{G}^{--} - \hat{G}^{-+} = \hat{G}^{+-} - \hat{G}^{++}$$

$$\hat{G}^A = \hat{G}^{--} - \hat{G}^{+-} = \hat{G}^{-+} - \hat{G}^{++}$$

$$\hat{G}^K = \hat{G}^{-+} + \hat{G}^{+-} = \hat{G}^{<} + \hat{G}^{>}$$

$$\hat{G}^{\alpha,\beta} = \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix} \qquad \hat{G} \Rightarrow L\hat{G}L^{-1} = \begin{pmatrix} 0 & G^A \\ G^R & G^K \end{pmatrix}$$

$$\hat{G} \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{G} \qquad \hat{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}$$

Оператор движения свободных частиц

$$\begin{aligned} a_\nu^\dagger(t) &\equiv e^{iHt}\, a_\nu^\dagger\, e^{-iHt}, \\ a_\nu(t) &\equiv e^{iHt}\, a_\nu\, e^{-iHt}. \end{aligned}$$

$$\{c_{\nu_j}^\dagger, c_{\nu_k}^\dagger\} = 0, \qquad \{c_{\nu_j}, c_{\nu_k}\} = 0,$$
$$\{c_{\nu_j}, c_{\nu_k}^\dagger\} = \delta_{\nu_j,\nu_k}.$$

$$H = \sum_\nu \varepsilon_\nu a_\nu^\dagger a_\nu.$$

$$\begin{aligned} \dot{a}_\nu(t) &= i[H, a_\nu(t)] = ie^{iHt}[H, a_\nu]e^{-iHt} \\ &= ie^{iHt}\sum_{\nu'}\varepsilon_{\nu'}\left[a_{\nu'}^\dagger a_{\nu'}, a_\nu\right]e^{-iHt} = ie^{iHt}\sum_{\nu'}\varepsilon_{\nu'}\left(-\delta_{\nu,\nu'}\right)a_{\nu'}e^{-iHt} \\ &= -i\varepsilon_\nu e^{iHt}a_\nu e^{-iHt} = -i\varepsilon_\nu a_\nu(t). \end{aligned}$$

$$\boxed{a_\nu(t) = e^{-i\varepsilon_\nu t}a_\nu,}$$

$$\boxed{a_\nu^\dagger(t) = e^{+i\varepsilon_\nu t}a_\nu^\dagger.}$$

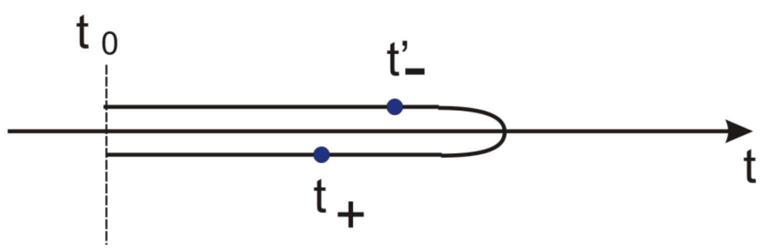
Функции Грина свободных частиц

$$\hat{G}(r,t_\alpha;r't_\beta{}') = -i < T_c \Psi(\vec{r},t_\alpha)\Psi^+(\vec{r}',t'_\beta) >= \hat{G}^{\alpha,\beta}(r,t;r't')$$

$$\Psi(r,t) = \sum_p \hat{a}_p(t)e^{ipr} \qquad \hat{a}_p^+(t) = e^{i\varepsilon_p t}\hat{a}_p^+ \qquad \hat{a}_p(t) = e^{-i\varepsilon_p t}\hat{a}_p$$

$$G^{+-}(rt,r't') = -i\langle\hat{\psi}\,(rt)\hat{\psi}^+(r't')\rangle$$

$$G^{-+}(rt,r't') = i\langle\hat{\psi}^+(r't')\hat{\psi}\,(rt)\rangle$$



Эти функции не упорядочены с точки зрения настоящего времени
Но упорядочены с точки зрения пути вдоль контура

Функции Грина свободных частиц

$$\hat{G}_p^{--}(t,t') = -\mathrm{i}\langle T_C \hat{a}_p(t^-)\hat{a}_p^+(t'^-)\rangle = -\mathrm{i}\langle T\hat{a}_p(t)\hat{a}_p^+(t')\rangle =$$

$$= -\mathrm{i}\left[\theta(t-t') \pm n_p\right] \exp\left[-\mathrm{i}\varepsilon_p(t-t')\right],$$

$$\hat{G}_p^{++}(t,t') = -\mathrm{i}\langle \tilde{T}\hat{a}_p(t)\hat{a}_p^+(t')\rangle =$$

$$= -\mathrm{i}\left[\theta(t'-t) \pm n_p\right] \exp\left[-\mathrm{i}\varepsilon_p(t-t')\right],$$

$$\hat{G}_p^{+-}(t,t') = -\mathrm{i}\langle \hat{a}_p(t)\hat{a}_p^+(t')\rangle = -\mathrm{i}(1 \pm n_p) \exp\left[-\mathrm{i}\varepsilon_p(t-t')\right],$$

$$\hat{G}_p^{-+}(t,t') = \mp\mathrm{i}\langle \hat{a}_p^+(t')\hat{a}_p(t)\rangle = \mp\mathrm{i}n_p \exp\left[-\mathrm{i}\varepsilon_p(t-t')\right].$$

Линейная зависимость функций Грина

$$\hat{G}_p^{-+} + \hat{G}_p^{+-} = \hat{G}_p^{--} + \hat{G}_p^{++}$$

$$\hat{G}_p^{R} = \hat{G}_p^{--} - \hat{G}_p^{-+} \qquad \hat{G}_p^{A} = \hat{G}_p^{--} - \hat{G}_p^{+-}$$

$$\hat{G}_p^{R}(t,t') = -i < \hat{a}_p(t)\hat{a}_p^{+}(t') + \hat{a}_p^{+}(t')\hat{a}_p(t) > \theta(t-t') = -ie^{-i\varepsilon_p(t-t')}\theta(t-t')$$

$$G^{R}(p,\omega) = \frac{1}{\omega - \varepsilon(p) + i\delta}$$

$$G^{-+}(p,\omega) = 2\pi i n(\omega)\delta(\omega - \varepsilon(p)) = -2in(\omega)\operatorname{Im} G^{R}(p,\omega)$$

Усреднение операторов

$$Sp\left\{\rho_0\tilde{T}\left[\frac{i^n}{n!}\iiint\hat{V}(t_1)\hat{V}(t_2).....\hat{V}(t_n)dt_1dt_2...dt_n\right]T\left[\hat{A}(t)\hat{B}(t')\frac{(-i)^m}{m!}\iiint\hat{V}(t_1)\hat{V}(t_2).....\hat{V}(t_m)dt_1dt_2...dt_m\right]\right\}$$

Внешнее поле

$$\hat{V}=\int\Psi^+(r)U(r,t)\Psi(r)=\sum_p a^+_{p+q}a_pU_q(t)$$

$$\hat{A}=a_p\ \hat{B}=a^+_{p'}$$

$$\hat{G}^{--}_p(t,t')=-i<T\,\hat{a}_p(t)\hat{a}^+_p(t')>$$

Первый порядок

$$-i<Ta_p(t)a^+_{p'}(t')(-i)\int\sum_{p_1}a^+_{p_1+q}(t_1)U_q(t)a_{p_1}(t_1)dt_1>$$

$$-i<(+i)\tilde{T}\int\sum_{p_1}a^+_{p_1+q}(t_1)U_q(t)a_{p_1}(t_1)dt_1T[a_p(t)a^+_{p'}(t')]>$$

Теорема Вика

$$\langle T[\hat{A}\hat{B}\hat{C}\hat{D}\dots]\rangle_o = \hat{A}\hat{B}\hat{C}\hat{D}\cdots + \hat{A}\hat{B}\hat{C}\hat{D}\cdots + \hat{A}\hat{B}\hat{C}\hat{D}\cdots + \cdots .$$

$$A = \langle T\,\hat{c}_{\mathbf{k}\sigma}(\tau)\hat{c}^{\dagger}_{\mathbf{k}\sigma}(0)\hat{c}^{\dagger}_{\mathbf{k}'\sigma'}(\tau_1)\hat{c}_{\mathbf{k}'\sigma'}(\tau_1)\rangle_0$$

$$A = \langle T\,\hat{c}_{\mathbf{k}\sigma}(\tau)\hat{c}^{\dagger}_{\mathbf{k}\sigma}(0)\rangle_0\langle T\,\hat{c}^{\dagger}_{\mathbf{k}'\sigma'}(\tau_1)\hat{c}_{\mathbf{k}'\sigma'}(\tau_1)\rangle_0$$

$$- \langle T\,\hat{c}_{\mathbf{k}\sigma}(\tau)\hat{c}^{\dagger}_{\mathbf{k}'\sigma'}(\tau_1)\rangle_0\langle T\,\hat{c}^{\dagger}_{\mathbf{k}\sigma}(0)\hat{c}_{\mathbf{k}'\sigma'}(\tau_1)\rangle_0.$$

$$\hat{c}_{\mathbf{k}\sigma}(\tau_1)\,\hat{c}_{\mathbf{k}'\sigma'}(\tau_2) = \hat{c}^{\dagger}_{\mathbf{k}\sigma}(\tau_1)\,\hat{c}^{\dagger}_{\mathbf{k}'\sigma'}(\tau_2) = 0.$$

Если нет сверхпроводимости!

Пример

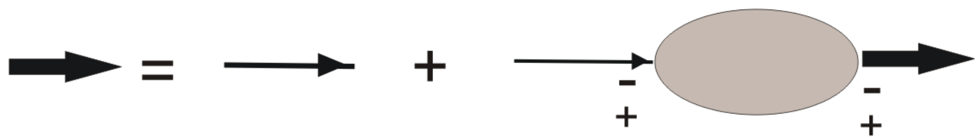
Ферми

$$0+$$
$$-n\,\theta(t-t_1)+n\,\theta(t_2-t_1)+n\,\theta(t-t')-n\,\theta(t_2-t')-$$
$$-\theta(t-t')\theta(t_2-t_1)+\theta(t-t_1)\theta(t_2-t')$$

$$\Downarrow$$

$$-[\theta(t-t')-n][\theta(t_2-t_1)-n]+[\theta(t-t_1)-n][\theta(t_2-t')-n]$$

Уравнение Дайсона



$$\hat{G}^{\alpha,\beta} = \hat{G}_0^{\alpha,\beta} + \hat{G}_0^{\alpha,\gamma}\Sigma^{\gamma,\delta}\hat{G}^{\delta,\beta}$$

$$\Sigma^{--} + \Sigma^{++} + \Sigma^{-+} + \Sigma^{+-} = 0$$

$$\hat{G}^{\alpha,\beta} = \begin{pmatrix} G^{--} & G^{-+} \\ G^{+-} & G^{++} \end{pmatrix}$$

$$\hat{G}^{R} = \hat{G}^{--} - \hat{G}^{-+} = \hat{G}^{+-} - \hat{G}^{++}$$

$$\hat{G}^{A} = \hat{G}^{--} - \hat{G}^{+-} = \hat{G}^{-+} - \hat{G}^{++}$$

$$\hat{G}^{K} = \hat{G}^{-+} + \hat{G}^{+-} = \hat{G}^{<} + \hat{G}^{>}$$

$$\hat{G} \Rightarrow L\hat{G}L^{-1} = \begin{pmatrix} 0 & G^{A} \\ G^{R} & G^{K} \end{pmatrix}$$

Удобней: $\hat{G} \Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{G}$ $\qquad \hat{G} = \begin{pmatrix} G^{R} & G^{K} \\ 0 & G^{A} \end{pmatrix}$

Еще удобней

$$\hat{G} = \begin{pmatrix} G^R & G^< \\ 0 & G^A \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \Sigma^R & \Sigma^< \\ 0 & \Sigma^A \end{pmatrix} \qquad \Sigma^< = -\Sigma^{-+}$$

$$G^< \equiv G^{-+} \qquad \Sigma^R = \Sigma^{--} + \Sigma^{-+} \qquad \Sigma^A = \Sigma^{--} + \Sigma^{+-}$$

$$\hat{G}^{\alpha,\beta} = \hat{G}_0^{\alpha,\beta} + \hat{G}_0^{\alpha,\gamma}\Sigma^{\gamma,\delta}\hat{G}^{\delta,\beta}$$

Элемент «11» матрицы

$$\hat{G}^R = \hat{G}_0^R + \hat{G}_0^R \Sigma^R \hat{G}^R$$

Элемент «12» матрицы

$$\hat{G}^< = \hat{G}_0^< + [\hat{G}_0 \Sigma \hat{G}]^<$$

$$= \hat{G}_0^< + \hat{G}_0^< \Sigma^A \hat{G}^A + \hat{G}_0^R \Sigma^< \hat{G}^A + \hat{G}_0^R \Sigma^R \hat{G}^<$$

Об обратных операторах

$$G_R^{-1} G_0{}^R = 1$$

$$\hat{G}_p^R(t,t') = -i\theta(t-t')e^{-i\varepsilon_p(t-t')} \qquad (\hat{G}_0^R)^{-1} = [i\frac{\partial}{\partial t} - \varepsilon_p]$$

$$(\hat{G}_0^R)^{-1}\hat{G}_p^{0R} = [i\frac{\partial}{\partial t} - \varepsilon_p]\hat{G}_p^{0R}(t,t') = \delta(t-t')$$

$$G_p{}^R(\omega) = \frac{1}{\omega - \varepsilon(p) + i\gamma} \Longrightarrow (\omega - \varepsilon(p))G_p{}^R(\omega) = 1$$