Manipulability of Aggregation Procedures for the Case of Large Numbers of Voters

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1 Introduction

Manipulability occurs when during a voting an agent misrepresents his/her preferences and gets a better outcome of the procedure.

It was proven in Gibbard [10] and Satterthwaite [19] that for the case of single-valued social choice every non-dictatorial aggregation procedure is manipulable. Then, Duggan and Schwartz [8] showed the same result for the case of multi-valued choice when ties between alternatives are possible. Thus, a question arises: which aggregation procedure is the least manipulable one?

Since then, various papers studying manipulability have been published, a non-exhaustive list includes [2, 6, 7, 9, 16, 18, 20, 21].

Two approaches have been used in most papers. The first one is to look for an analytical solution, i.e., to find a formula for a certain manipulability index for a certain aggregation procedure [9, 16]. The main difficulty is that analytical solutions are usually found only for most popular positional rules, for example, Plurality rule, Inverse Plurality rule, Borda's rule. However, it is known that such rules are more manipulable [2], and the less manipulable aggregation procedures, for example, Hare's procedure, Nanson's procedure [3] are more complicated to find an analytical formula for them.

That is why there is the second approach: computer modeling. One of the first papers in this area [2] used computer modeling to generate all possible profiles for small cases: $3 \ldots 5$ alternatives and $3 \ldots 10$ agents. Later, researchers started generating a large random number of profiles (one million) to get approximate

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estimations of manipulability indices. Such an approach allowed to estimate the degree of manipulability of many aggregation procedures without deriving any analytical formulae for the cases of $3 \ldots 100$ agents ([1, 3]; Aleskerov et al. [4]).

The results from computer modeling showed that the least manipulable aggregation procedures in most cases are Inverse Borda's rule, Nanson's procedure and Hare's procedure [3].

However, some questions have arisen from the results for the case of $3 \ldots 100$ agents. For example, if there is no least manipulable aggregation procedure for the case of small number of agents, maybe there is one for the case of large number of agents? It was shown in Aleskerov and Kurbanov [2] and Aleskerov et al. [3] that the manipulability indices for even and odd numbers of agents may vary 3–5 times. Does that result hold for larger number of agents?

In this paper, we address these and other questions by comparing the previously known results in literature from computer modeling for the case of $3 \ldots 100$ agents with our new results from computer modeling for the cases of $3 \ldots 10{,}000$ agents for the case of Impartial Culture and 3 alternatives.

2 Main Notions

2.1 Aggregation Procedures

We use similar notation as in Aleskerov and Kurbanov [2] and Aleskerov et al. [3]. We denote the number of agents as n, and the number of alternatives as m. Each agent has a preference (linear order) over the set of alternatives. For the case of n agents there are $m!$ possible preferences. The set of agents with their preferences over the set of alternatives comprise a profile, P. An aggregation procedure $C(P)$ determines the winner taking a profile P as an input.

We consider nine scoring aggregation procedures and use the same definitions as in Aleskerov and Kurbanov [2] and Aleskerov et al. [3]:

1. Plurality rule

   The alternative which is the first best for the largest number of agents is chosen.
2. q-Approval rule with q = 2

   The alternative which is the first best or the second best for the largest number of agents is chosen.
3. Borda's rule

   For each alternative Borda's count is calculated: for each agent for whom it is first best, the alternative gets $m - 1$ points, for each agent for whom it is second best, the alternative gets $m - 2$ points ... for each agent for whom it is m-th best, the alternative gets 0 points. The result of the aggregation procedure is the alternative with the highest Borda's count, i.e., the alternative(s) with the highest sum of points.

4. Black's procedure
   The procedure chooses the unique Condorcet winner if it exists, and uses Borda's rule otherwise.
5. Threshold rule
   The alternative which is the worst for the smallest number of agents is chosen. If there are more than one such alternatives, the number of agents for whom they are second-worst alternatives are compared, etc., until the winner is found.
6. Hare's procedure
   If there is an alternative which is the first best for the simple majority of agents, it is the winner. Otherwise, the alternative with the least number of first-best votes is eliminated, and the procedure repeats.
7. Inverse Borda's rule
   Borda's count is calculated for all alternatives. The alternative with the lowest Borda's count is eliminated, and the procedure repeats until the winner is found.
8. Nanson's procedure
   Borda's count is calculated for each alternative. Then, the average Borda's count among alternatives is calculated. The alternatives with Borda's count less than the average are eliminated, and the procedure repeats.
9. Coombs' procedure
   The alternative which is the worst for the largest number of agents is eliminated. Procedure repeats until the winner is found.

In Aleskerov et al. [3] 10 aggregation procedures are studied. In addition to 9 aggregation procedures described above, Inverse Plurality rule is included. However, for the case of 3 alternatives which we consider in this paper, Inverse Plurality rule has the same definition as q-Approval rule with q = 2.

2.2 Manipulation and Manipulability Index (NK-Index)

If P is a profile where all agents cast their sincere preferences, and $P^{'}$ is a profile, where an agent misrepresents her preferences, then manipulation happens if $C(P^{'}) \succ C(P)$ for the manipulating agent.

Every profile is marked either as manipulable or non-manipulable. A profile is called manipulable, if there exists at least one way of successful manipulation by at least one agent. If there are no ways for any agent to misrepresent her preferences and to get a better outcome, then the profile is non-manipulable.

The most widely used index to estimate the degree of manipulability of an aggregation procedure is Nitzan-Kelly index (NK-index) which was introduced in Nitzan [17], Kelly [14, 15]. NK-index stands for the share of manipulable profiles in the total number of profiles:

$$NK = \frac{number\ of\ manipulable\ profiles}{total\ number\ of\ profiles}$$

2.3 Extended Preferences

What happens in the case of a tie between two or more alternatives in the aggregation procedure? One approach is to use an additional tie-breaking rule, for example, alphabetical tie-breaking rule. This approach, however, is not neutral to the set of alternatives. For this reason, we use another approach known in the literature, namely, the so-called extended preferences. Such an approach allows ties between alternatives: if there is a tie, then all such alternatives constitute the social choice. For example, if both alternatives $\{a\}$ and $\{c\}$ under Plurality rule have equal number of votes, then the result of the procedure is $\{a, c\}$. It means that the result of an aggregation procedure $C(P)$ is a non-empty subset of the set of alternatives (multi-valued choice). Then, extended preferences are used to allow an agent to compare two multi-valued choices (the results of the aggregation procedure before and after a manipulation attempt). Detailed research regarding preferences extension axioms and extended preferences can be found in Barberà et al. [7].

We consider four ways of constructing extended preferences (EP) for the case of three alternatives: Leximin, Leximax, Risk-lover, Risk-averse. Their definitions are the same as in Aleskerov et al. [3].

Suppose that an agent has the sincere preferences $a \succ b \succ c$. Then, we consider four EPs:

1. Leximin: multi-valued choices are compared alphabetically, and the choice where the worst alternative is better is more preferred. EP under Leximin (three multi-valued choices where the ordering is different among four EPs are underlined):

$$\{a\} \succ \{a, b\} \succ \underline{\{b\} \succ \{a, c\} \succ \{a, b, c\}} \succ \{b, c\} \succ \{c\}$$

2. Leximax: multi-valued choices are compared alphabetically, and the choice where the best alternative is better is more preferred. EP under Leximax:

$$\{a\} \succ \{a, b\} \succ \underline{\{a, b, c\} \succ \{a, c\} \succ \{b\}} \succ \{b, c\} \succ \{c\}$$

3. Risk-averse ("PWorst" in Aleskerov et al. [3]: multi-valued choices are compared by the probability of the worst alternative, and the choice where the probability of the worst alternative alphabetically, and the choice, where probability of the worst alternative is lower, is better. EP under Risk-averse:

$$\{a\} \succ \{a, b\} \succ \underline{\{b\} \succ \{a, b, c\} \succ \{a, c\}} \succ \{b, c\} \succ \{c\}$$

4. Risk-lover ("PBest" in Aleskerov et al. [3]: multi-valued choices are compared by the probability of the best alternative, and the choice where the probability of the best alternative, and the choice, where probability of the best alternative is higher, is better. EP under Risk-lover:

$$\{a\} \succ \{a,b\} \succ \underline{\{a,c\} \succ \{a,b,c\} \succ \{b\}} \succ \{b,c\} \succ \{c\}$$

For example, if in some profile an agent has sincere prefenrece $a \succ b \succ c$, the result with sincere preferences is $\{a, c\}$ and the result with insincere preferences is $\{b\}$, then it is manipulation for Leximax EP and Risk-lover EP, and there is no manipulation for Leximin EP and Risk-averse EP.

2.4 Impartial Culture and Computer Modeling Scheme

What is the probability of a certain profile? We use Impartial Culture (IC) probabilistic model which was first studied in Guilbaud [11]. Under IC all preferences are equally likely. If each of n agents has one of $m!$ preferences, the total number of profiles under IC is equal to $(m!)^n$.

It can be noticed, that the number of profiles is growing very fast. Even a modern supercomputer cannot generate all possible profiles for cases of $n > 30$. That is why computer modelling papers use random profile generation. Instead of generating all possible profiles which is impossible for large n, 1,000,000 profiles are generated randomly. As it was shown in Karabekyan [13], such an approach allows to estimate the degree of manipulability of aggregation procedures with the precision of 0.001 in manipulability indices. We use this approach for estimating manipulability indices.

As a result, the computer modelling scheme to estimate the degree of manipulability of a given aggregation procedure for a given number of agents and for a given number of alternatives usually consists of the following steps:

1. Random 1,000,000 profiles are generated
2. For each profile, the sincere choice is calculated
3. For each profile each possible manipulation attempt by each agent is generated, i.e. for each of n agents $m\,! - 1$ ways of misrepresenting preferences are considered
4. If there is at least one successful manipulation attempt for at least one agent, the profile is marked as manipulable. If all possible manipulation attempts lead to the same or worse choice for manipulating agents, the profile is marked as non-manipulable.
5. After all profiles are analyzed, manipulability indices are calculated.

It can be noticed that the computational complexity of the scheme is high. For each of the randomly generated 1,000,000 profiles all possible $n * (m\,! - 1)$ manipulation attempts should be checked, meaning we should calculate the new result of the aggregation procedure $C(P')$ for each attempt. Finally, the algorithm repeats for every n. Detailed estimations of the complexity of the algorithms of computer modelling for calculating manipulability indices can be found in Ivanov [12]. The estimations for the cases of up to 10,000 agents took 3 weeks on one personal computer.

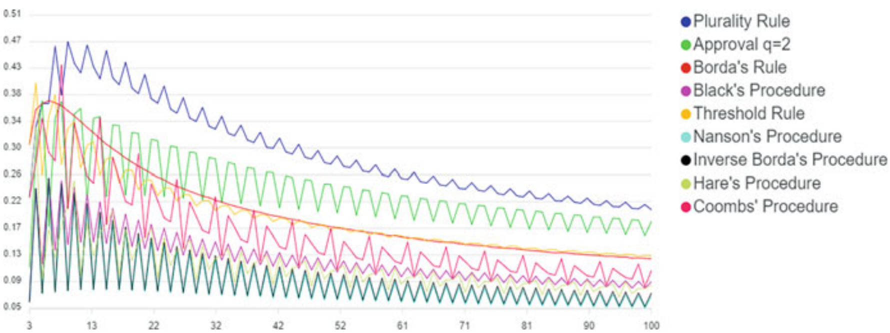
2.5 Results

We obtained the results for the case of $m = 3$ alternatives and for $n = 3 \ldots 10{,}000$ agents for the cases of Leximin, Leximax, Risk-averse and Risk-lover EP for Impartial Culture.

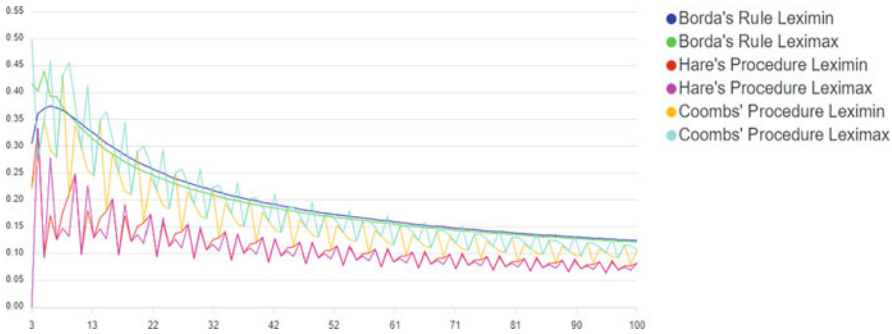
First, we provide the results for the case of $n = 3 \ldots 100$ agents. In Aleskerov et al. [3] the results for similar cases are provided, but with gaps: they calculated cases of n = 3 . . . 25, then n = 29, 30, 39, 40, etc. Picture 1 fills the missing picture for the cases of 31 . . . 38, 41 . . . 48, etc. agents.

The shapes of the lines suggest that there are certain periods in the values of NK-index. The idea that there are such periods in the values of NK-index was previously discussed in the literature, for example, in Aleskerov et al. [3] it is suggested that the period for most of the 9 discussed procedures is either 2 or 3. Having the chart for all n from 3 to 100, we can confirm that the periods for Black's procedure, Nanson's procedure, Inverse Borda's procedure are equal to 2, for Plurality rule, q-Approval q = 2 rule and Threshold rule periods are equal to 3, but we can suggest that the periods for Coomb's procedure and Hare's procedure are equal to 6, while NK-index for Borda's rule does not have any visible periods. This conclusion holds not only for Leximin EP, but also for other extended preferences as well. For example, Picture 2 illustrates the three rules (Borda's rule with no period, Coomb's rule and Hare's procedure with the period of 6) for both Leximin and Leximax EP.

Next, we would like to verify the hypothesis from Aleskerov et al. [3] that both Leximin EP and Risk-averse EP show the same values of NK-index, and both Leximax and Risk-lover EP show the same values of NK-index for most rules. Such a result was presented in Aleskerov et al. [3] for the cases of $n \leq 100$ agents. We studied the cases of $100 < n \leq 10{,}000$ agents, and found out that for 5 procedures (Plurality, q-Approval with q = 2, Borda, Threshold and Coombs' procedures) it is true, but for other 4 procedures (Nanson, Inverse Borda, Black and Hare's procedures) there are differences in the values of NK-index between Leximin EP and Risk-averse EP, Leximax EP and Risk-lover EP.



Picture 1 NK-index, Leximin, 3–100 agents



Picture 2 Leximin and Leximax EP for Borda's rule, Coombs and Hare's procedures

Picture 3 NK-index for Hare's procedure for four types of extended preferences

For example, Picture 3 shows the results for Hare's procedure for all four types of extended preferences, and it can be noticed, that, for example, for the case of 77 agents NK-index is the same, but for 78 agents NK-index is different for all four extended preferences.

The next idea is to evaluate the difference in NK-index for four extended preferences. For small numbers of agents we can notice that the difference in NK-index may be large, for example, NK-index for Hare's procedure for 18 agents for Risk-lover EP is equal to 0.12, while for Risk-averse EP NK = 0.23, almost twice higher. Let us compare the differences for larger numbers of agents. For n = 90: NK = 0.0788 for Risk-lover and NK = 0.0954 for Risk-averse (21% higher); for n = 990: NK = 0.0253 for Risk-lover and NK = 0.0265 for Risk-averse (4.7% higher); for n = 9990: NK = 0.008043 for Risk-lover and NK = 0.008136 for Risk-averse (1.1% higher). The difference between values of the NK-index for different extended preferences decreases with growing number of agents, but still exists.

From Aleskerov et al. [3] and Aleskerov et al. [4] we know that there are 3 aggregation procedures which are the least manipulable for most cases for $n = 3 \ldots 100$ agents: Hare's procedure, Nanson's procedure and Inverse Borda's procedure. Let us compare the values of NK-index for these rules for the cases of large number of agents (Picture 4)

It can be noticed, that for large number of agents Nanson's procedure shows the smallest values of NK-index, i.e., is the least manipulable. Inverse Borda's procedure is more manipable than Nanson's procedure, but less manipulable than Hare's procedure.

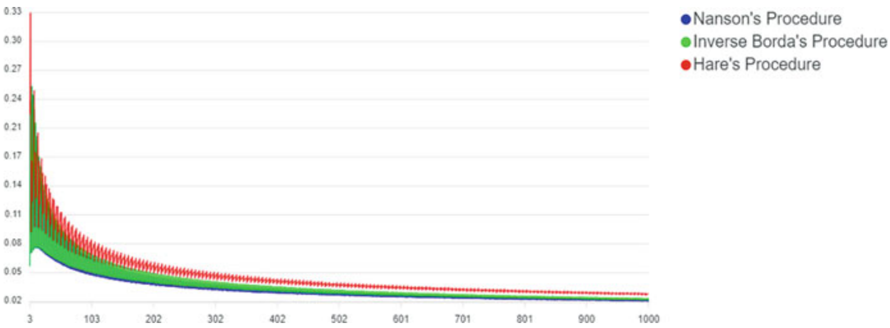
In order to have a better look at the difference between manipulability of the aggregation procedure, we suggest the following chart (Picture 5)

Horizontal axis stands for the number of agents, while the vertical axis stands for the ratio between a certain rule and the least manipulable rule (Nanson's procedure).

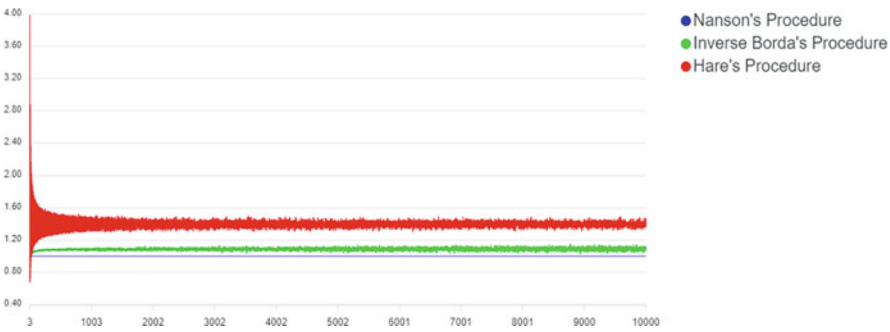
The line for the Nanson's procedure (blue line) is given as a reference line (NK-index for Nanson's procedure divided by NK-index for Nanson's procedure is always equal to 1), because it allows to see that other procedures (e.g., Hare's procedure) may be less manipulable than Nanson's procedure only for small numbers of agents. According to our calculations, for the cases of $n \leq 26$ agents Hare's procedure is sometimes less manipulable than Nanson's procedure, but for all cases of $n > 26$ agents Nanson's procedure is the least manipulable for Leximin EP.

Next, we analyze the results for Leximax EP (Picture 6)

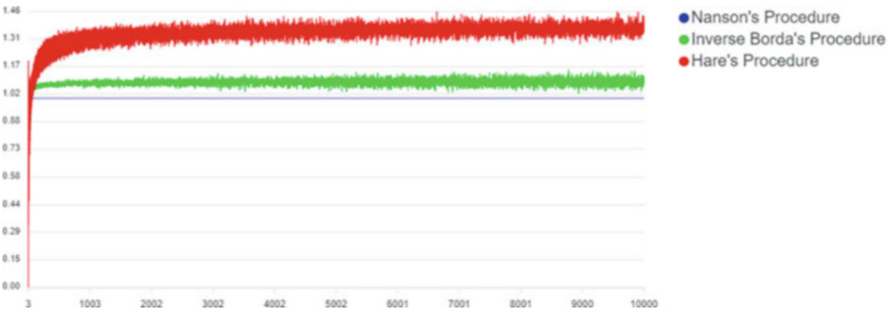
For the case of Leximax EP, for small numbers of agents Hare's procedure is often less manipulable than Nanson's procedure. For example, for the cases of



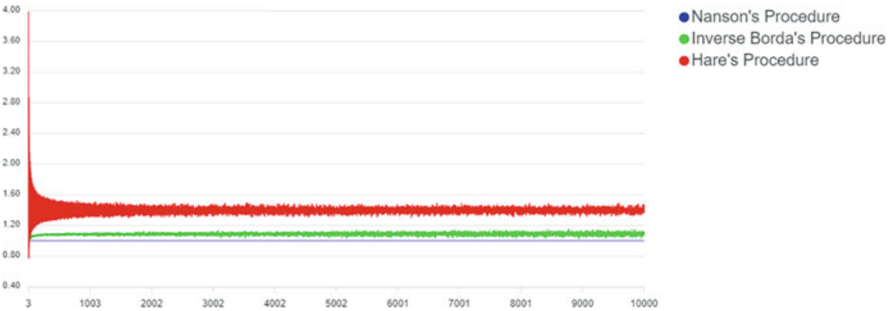
Picture 4 Nanson's, Inverse Borda's and Hare's procedures for Leximin EP



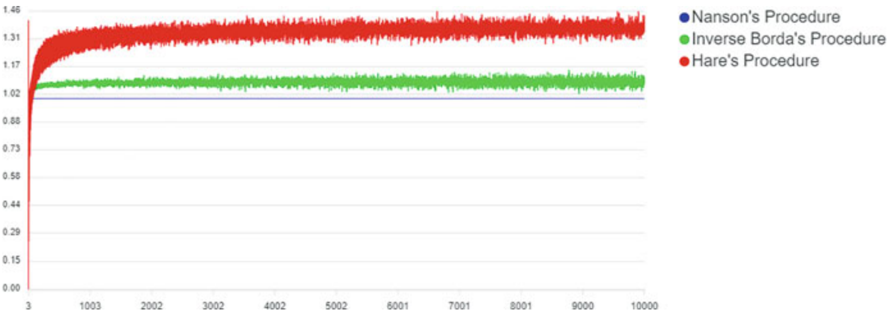
Picture 5 NK-index ratios to Nanson's procedure NK-index, Leximin EP



Picture 6 NK-index ratios to Nanson's procedure NK-index, Leximax EP



Picture 7 NK-index ratios to Nanson's procedure NK-index, Risk-averse EP



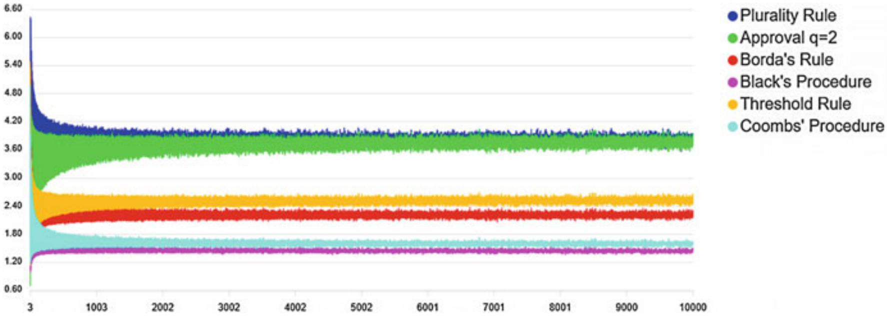
Picture 8 NK-index ratios to Nanson's procedure NK-index, Risk-lover EP

3 . . . 68 agents Hare's procedure is less manipulable in 34 out of 66 cases. However, Nanson's procedure is the least manipulable for all cases of $n > 68$ agents.

Picture 7 shows the case of Risk-averse EP.

Again, for small n Hare's procedure sometimes is the least manipulable, but Nanson's procedure is the least manipulable for all cases of $n > 20$ agents.

Picture 8 shows the case of Risk-lover EP.



Picture 9 NK-index ratios to Nanson's procedure NK-index, Leximin EP (remaining 6 procedures)

Table 1 Ratios between NK-index for a given procedure to Nanson's procedure's NK-index

Procedure	Leximin	Leximax	Risk-averse	Risk-lover
Plurality rule	3.81	3.69	3.81	3.69
q-approval, q = 2	3.70	3.65	3.70	3.65
Borda's rule	2.22	2.14	2.22	2.14
Black's procedure	1.45	1.4	1.45	1.40
Threshold rule	2.50	2.43	2.50	2.43
Inverse Borda's rule	1.09	1.08	1.09	1.08
Hare's procedure	1.39	1.35	1.40	1.34
Coomb's procedure	1.61	1.61	1.61	1.61

We see the same situation: for the cases of $n \leq 74$ Hare's procedure sometimes is the least manipulable, but for $n > 74$ Nanson's procedure is the least manipulable.

The next interesting result is that the ratio between the values of NK-index for aggregation procedures converges with growing n. In Pictures 5, 6, 7, and 8 we saw that result for Hare's procedure and Inverse Borda's procedure, Picture 9 shows the rest of the considered procedures (Plurality rule, Borda's rule, Black's procedure, Threshold rule, q-Approval rule with q = 2, Coombs procedure) in the same type of the chart: ratio of their NK-index to the NK-index of Nanson's procedure.

In Table 1 we provide summarized results as the average ratios between each procedure's NK index and Nanson's procedure's NK index.

For example, Plurality rule has in average 3.81 times larger values of NK-index than Nanson's procedure for the same n.

It can be noticed that the second least manipulable procedure is Inverse Borda's procedure which is in average 1.08–1.09 times more manipulable than Nanson's procedure. Such close results may be explained by similar mechanisms of the procedures: in both procedures Borda's count is calculated, and in Nanson's procedure alternatives with less than average count are eliminated, while in Inverse Borda's procedure only the alternative with the lowest Borda's count is eliminated.

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