Вступительное задание по Высшей математике (английский)
НИУ "Высшая школа экономики"
Факультет компьютерных наук
Магистерская программа
"Наука о данных"

Демо

Прочитайте фрагмент статьи и ответьте на вопросы.
Правильный ответ на каждый вопрос – 10 баллов, неправильный – 0.

We can resolve the tension by incorporating an automatic management of the *stage of definition*, the rings A such that we're considering A-valued points, into our language. Such a language is provided by the internal language of the big Zariski topos. It allows for the Fermat scheme to be given by the naive expression

$$\{(x, y, z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$$

and for projective n-space to be given by either of the expressions

the set of lines through the origin in $(\underline{\mathbb{A}}^1)^{n+1}$ or

$$\{[x_0 : \cdots : x_n] \mid x_i \neq 0 \text{ for some } i\}.$$

This is not a specialized trick to give short descriptions of some schemes: Like with the internal universe of any topos, the full power of intuitionistic logic is available to reason about the objects constructed in this way.

We can thus add an approach to the list of ways of giving a rigorous foundation to algebraic geometry, the synthetic approach which layers scheme theory not upon a classical set theory, but rather directly encodes schemes as sets and morphisms of schemes as maps of sets in the nonclassical universe provided by the big Zariski topos of a base scheme. We can therefore use a simple, element-based language to talk about schemes.

This is similar to synthetic approaches to other fields of mathematics, such as differential geometry [81], domain theory [69], computability theory [16], and more recently and very successfully homotopy theory [136] and related subjects [115, 116, 113]. The synthetic approaches allow in each case to encode the objects of study directly as (nonclassical) sets, with geometric, domain-theoretic, computability-theoretic, or homotopic structure being automatically provided for.

The implicit algebro-geometric structure has visible consequences on the internal universe of the big Zariski topos and endows it with a distinctive algebraic flavor. For instance, the statement "*any* map $\underline{\mathbb{A}}^1 \to \underline{\mathbb{A}}^1$ is a polynomial function" holds from the internal point of view. This is also a property which sets the internal universe of the big Zariski topos apart from the toposes studied in synthetic differential geometry.

If one is content with building upon classical scheme theory, the big Zariski topos $\mathrm{Zar}(X)$ of a base scheme X can be constructed as the topos of sheaves on the Grothendieck site Sch/X of X-schemes.[3] Explicitly, an object of $\mathrm{Zar}(X)$ is a functor $F : (\mathrm{Sch}/X)^{\mathrm{op}} \to \mathrm{Set}$ satisfying the gluing condition with respect to Zariski coverings: If $T = \bigcup_i U_i$ is a cover of an X-scheme T by open subsets, the diagram

$$F(T) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{j,k} F(U_j \cap U_k)$$

should be an equalizer diagram. A premier example of an object of $\mathrm{Zar}(X)$ is the functor $\underline{Y}$ of points associated to an X-scheme Y, mapping an X-scheme T to $\mathrm{Hom}_X(T, Y)$. It satisfies the gluing condition since one can glue morphisms of

2. The internal language of a sheaf topos

At its heart, the internal language of a topos provides a coherent way of translating any mentions of set-theoretical elements to *generalized elements*, carefully keeping track of and adapting the stage of definition. We want to illustrate this with a simple example before giving the formal definition.

A map $f : X \to Y$ of sets is injective if and only if

$$\forall x, x' \in X.\ f(x) = f(x') \Longrightarrow x = x'. \tag{1}$$

This condition can not only be interpreted in Set, but in any category $\mathcal{C}$ whose objects are structured sets and whose morphisms are maps between the underlying sets. If we want to go beyond such kind of categories, we have to restate the condition in purely category-theoretic language:

$$\forall (1 \xrightarrow{x} X), (1 \xrightarrow{x'} X).\ f \circ x = f \circ x' \Longrightarrow x = x'. \tag{2}$$

This condition makes sense in all categories which contain a terminal object 1, and is equivalent to condition (1) in the case $\mathcal{C} = \text{Set}$. This has a deeper reason: The one-element set $1 = \{\star\}$ is a *separator* of Set, that is objects of Set are uniquely determined by their *global elements*, morphisms from the terminal object.

However, in categories in which the terminal object is not a separator, condition (2) is not very meaningful. This is for instance the case if $\mathcal{C}$ is the category of vector spaces over a field or if $\mathcal{C}$ is the category $\text{Sh}(X)$ of set-valued sheaves on a topological space X. Global elements of a sheaf $\mathcal{F}$ are in natural one-to-one correspondence with global sections $s \in \mathcal{F}(X)$ (hence the name), whereby condition (2) only states that f is *injective on global sections*. Since many interesting sheaves admit no or only few global sections, this statement is typically not very substantial.

A basic tenet of category theory is therefore to not only refer to global elements $1 \to X$, but also to *generalized elements* $A \to X$, where A ranges over all objects. The domain A is called the *stage of definition* in this context. Bearing this principle in mind, a better translation of the injectivity condition is the statement

$$\forall \text{objects } A \text{ in } \mathcal{C}.\ \forall (A \xrightarrow{x} X), (A \xrightarrow{x'} X) \text{ in } \mathcal{C}.\ \ f \circ x = f \circ x' \ \Longrightarrow\ x = x'. \tag{3}$$

This statement expresses that f is a monomorphism and therefore correctly captures the structural essence of injectivity.

Unlike this manual translation guided by trial and error and categorical philosophy, the internal language provides a purely mechanical translation scheme. It is fully formal, can be analyzed rigorously, works smoothly with arbitrarily convoluted statements, and most importantly can be trusted to support *reasoning*: If a statement formulated in a naive element-based language intuitionistically implies a further such statement, then the translation of the former implies the translation of the latter.

The power of the internal language doesn't unfold in basic situations like with the example above, where one can easily translate statements and even proofs by hand.

2.1. Internal statements. Let X be a topological space. Later, X will be the underlying space of a scheme. The meaning of internal statements is given by a set of rules, the *Kripke–Joyal semantics* of the topos of sheaves on X.

Definition 2.1. The meaning of

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and formulas φ over U is given by the rules listed in Table 1, recursively in the structure of φ. In a *formula over* U there may appear sheaves defined on U as domains of quantifications, U-sections of sheaves as terms, and morphisms of sheaves on U as function symbols. If $V \subseteq U$ is an open subset, then formulas over U can be pulled back to formulas over V. The symbols “$\top$” and “$\bot$” denote truth and falsehood, respectively. The universal and existential quantifiers come in two flavors: for bounded and unbounded quantification. The translation of $U \models \neg\varphi$ does not have to be separately defined, since negation can be expressed using other symbols: $\neg\varphi :\equiv (\varphi \Rightarrow \bot)$. If we want to emphasize the particular topos, we write

$$\mathrm{Sh}(X) \models \varphi \quad :\Longleftrightarrow \quad X \models \varphi.$$

Remark 2.2. The last two rules in Table 1, concerning *unbounded quantification*, are not part of the classical Kripke–Joyal semantics. They are part of Mike Shulman's stack semantics [122], a slight but important extension. They are needed so that we can formulate universal properties in the internal language. (Prior work in the same direction include the topos models explored by Pitts [109, Section 3] and, in the context of set theory, work by Awodey, Butz, Simpson, and Streicher [11], which was carried out independently and published after Shulman's paper.)

Example 2.3. Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X. Then α is a monomorphism of sheaves if and only if, from the internal perspective, α is simply an injective map:

$$X \models \ulcorner \alpha \text{ is injective} \urcorner$$

$$\Longleftrightarrow X \models \forall s\!:\!\mathcal{F}.\ \forall t\!:\!\mathcal{F}.\ \alpha(s) = \alpha(t) \Rightarrow s = t$$

$$\Longleftrightarrow \text{for all open } U \subseteq X, \text{ sections } s \in \Gamma(U, \mathcal{F})\text{:}$$
$$\text{for all open } V \subseteq U, \text{ sections } t \in \Gamma(V, \mathcal{F})\text{:}$$
$$V \models \alpha(s) = \alpha(t) \Rightarrow s = t$$

$$\Longleftrightarrow \text{for all open } U \subseteq X, \text{ sections } s \in \Gamma(U, \mathcal{F})\text{:}$$
$$\text{for all open } V \subseteq U, \text{ sections } t \in \Gamma(V, \mathcal{F})\text{:}$$

Proposition 2.5 (Soundness of the internal language). *If a formula φ implies a further formula ψ in intuitionistic logic, then $U \models \varphi$ implies $U \models \psi$.*

Proof. Proof by induction on the structure of formal intuitionistic proofs; we are to show that any inference rule of intuitionistic logic is satisfied by the Kripke–Joyal semantics. For instance, there is the following rule governing disjunction:

> If $\varphi \vee \psi$ holds, and both φ and ψ imply a further formula χ, then χ holds.

So we are to prove that if $U \models \varphi \vee \psi$, $U \models (\varphi \Rightarrow \chi)$, and $U \models (\psi \Rightarrow \chi)$, then $U \models \chi$. This is done as follows: By assumption, there exists a covering $U = \bigcup_i U_i$ such that for each index i, $U_i \models \varphi$ or $U_i \models \psi$. Again by assumption, we may conclude that $U_i \models \chi$ for each i. The statement follows because of the locality of the internal language.

A complete list of which rules are to prove is in Appendix 24. □

In particular, if a formula ψ has an unconditional intuitionistic proof, then $U \models \psi$.

The restriction to intuitionistic logic is really necessary at this point. We will encounter many examples of classically equivalent internal statements whose translations using the Kripke–Joyal semantics are wildly different. To anticipate just one example, the statement

$$X \models \ulcorner \mathcal{F} \text{ is finite free} \urcorner,$$

referring to a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, means that $\mathcal{F}$ is finite locally free. The statement

$$X \models \neg\neg(\ulcorner \mathcal{F} \text{ is finite free} \urcorner)$$

instead means that $\mathcal{F}$ is finite locally free on a dense open subset of X.

In particular, our treatment of modal operators to understand spreading of properties from points to neighborhoods depends on having the ability to make finer distinctions – distinctions which are not visible in classical logic. In Section 2.4 there is a discussion of what the restriction to intuitionistic logic amounts to in practice.

Because of the multitude of quantifiers, literal translations of internal statements can sometimes get slightly unwieldy. There are simplification rules for certain often-occurring special cases:

Вопросы:

1. Как переводится слово tenet, использованное в статье?
2. Что сказано про мощь внутреннего языка?
3. На что влияет неявная алгебро-геометрическая структура и имеет видимые последствия?
4. В каком разделе обсуждается, к чему на практике сводится ограничение на интуиционистскую логику?
5. Про какое условие в статье сказано, что в категориях, где терминальный объект не является сепаратором, это условие не имеет особого смысла?