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Marked order polytopes

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Contents

1. Object and subject of the research project

Definition 1.1. Let $\lambda_1 \leq \cdots \leq \lambda_s$ be a non-decreasing finite sequence of real numbers. Let us consider a triangular table

$$\begin{array}{ccccccccccccccccc}
\lambda_1 & & & \lambda_2 & & & \lambda_3 & & & \lambda_4 & & & \cdots & & & \lambda_s \\
& u_{1,1} & & & u_{1,2} & & & u_{1,3} & & & \cdots & & & u_{1,s-1} & & \\
& & u_{2,1} & & & u_{2,2} & & & \cdots & & & u_{2,s-2} & & & & \\
& & & \ddots & & \ddots & & \vdots & & \ddots & \iddots & & & & & & \\
& & & & u_{s-2,1} & & & & u_{s-2,2} & & & & & & & & \\
& & & & & & u_{s-1,1} & & & & & & & & & &
\end{array} \tag{1}$$

We will assume that every three symbols $a,\ b,\ c$ that form the triangle

$$\begin{array}{ccc} a & & c \\ & b & \end{array}$$

in the table (1) are connected by the double inequality $a \leq b \leq c$. Then our table gives a specific set of linear inequalities depending on λ_i which determines a certain convex polytope in $\mathbb{R}^{\frac{s(s-1)}{2}}$. Let us emphasize that the coordinates $u_{i,j}$ in our case are numbered by ordered pairs of integers $(i,\ j)$ where i runs from 1 to $s-1$ and j runs from 1 to $s-i$. This convex polytope is called the *Gelfand–Zetlin polytope corresponding to the sequence* $\lambda_1 \leq \cdots \leq \lambda_s$. Let us denote it by $GZ(\lambda_1 \ldots \lambda_s)$.

Example 1.4. Let us consider the polytope $GZ(1\,2\,3)$. According to Definition 1.1, the corresponding triangular table and the system of linear inequalities have the following form (3). It is easy to see that the first two rows of the system (3) define a parallelepiped from which $GZ(1\,2\,3)$ is cut out by the half-spaces defined by the inequalities: $u_{1,1,} \leq u_{2,1}$ and $u_{2,1} \leq u_{1,2}$. Let us construct the polytope $GZ(1\,2\,3)$ (see Fig. 1).

$$\begin{array}{ccccc} 1 & & 2 & & 3 \\ & u_{1,1} & & u_{1,2} & \\ & & u_{2,1} & & \end{array} \Leftrightarrow \begin{cases} 1 \leq u_{1,1} \leq 2, \\ 2 \leq u_{1,2} \leq 3, \\ u_{1,1} \leq u_{2,1} \leq u_{1,2}. \end{cases} \tag{3}$$

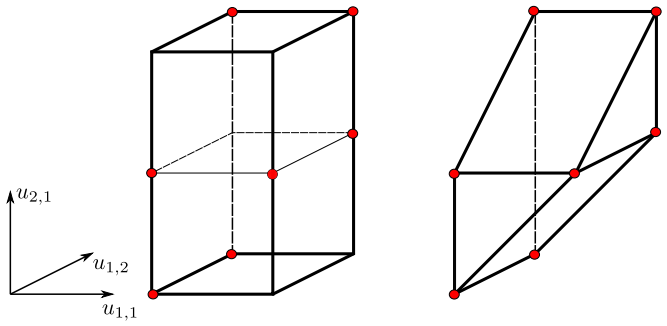

Fig. 1. Polytope $GZ(1\,2\,3)$, cut out from the parallelepiped.

Let us give one more definition. Let $\mathcal{P} = (P, \preceq)$ be a finite partially ordered set (in the sequel called a *poset*) and P^* be a subset of P containing all its extremal elements. Let $\mathcal{P}^* = (P^*, \preceq)$ denote the subposet of $\mathcal{P}$ defined by P^* (in the sequel called the *subposet of marked elements*), and let λ be an order-preserving function $\mathcal{P}^* \to \mathbb{R}$. The vector space $\mathbb{R}^P$ of functions $P \to \mathbb{R}$ is isomorphic to the coordinate space $\mathbb{R}^{\#P}$; the value of the function $x \in \mathbb{R}^P$ on an element $p \in P$ is identified with the p-th coordinate x_p of the corresponding point.

Definition 2.11. *A marked order polytope* $O(\mathcal{P}, \lambda)$ is a set of points $x \in \mathbb{R}^P$ such that $x_p \leq x_q$ if $p \preceq q$ and $x_a = \lambda(a)$ if $a \in P^*$.

Definition 2.14. We will say that an element q covers an element p in a poset $\mathcal{P} = (P, \preceq)$ and write $p \prec\cdot q$ if p and q are distinct, $p \preceq q$ and the condition $p \preceq r \preceq q$ is not satisfied for any $r \in P$ distinct from p and q.

Example 2.15. Let $P = \{r, p, q, s, t\}$. We define the relation $\preceq$ by the covering relations $r \prec\cdot p \prec\cdot q \prec\cdot s$ and $p \prec\cdot t$ (Fig. 6a). We set P^* to be $\{r, s, t\}$. We define the function $\lambda : \mathcal{P}^* \to \mathbb{R}$ by the equalities $\lambda(r) = 0$, $\lambda(s) = 2$ and $\lambda(t) = 1$ (Fig. 6b). Let us write out the system of inequalities defining the polytope $O(\mathcal{P}, \lambda)$ and construct its projection onto the plane Ox_px_q (Fig. 6c).

According to Definition 2.11,

$$O(\mathcal{P}, \lambda) = \begin{cases} 0 = x_r \leq x_p \leq x_q \leq x_s = 2, \\ x_p \leq x_t = 1. \end{cases}$$

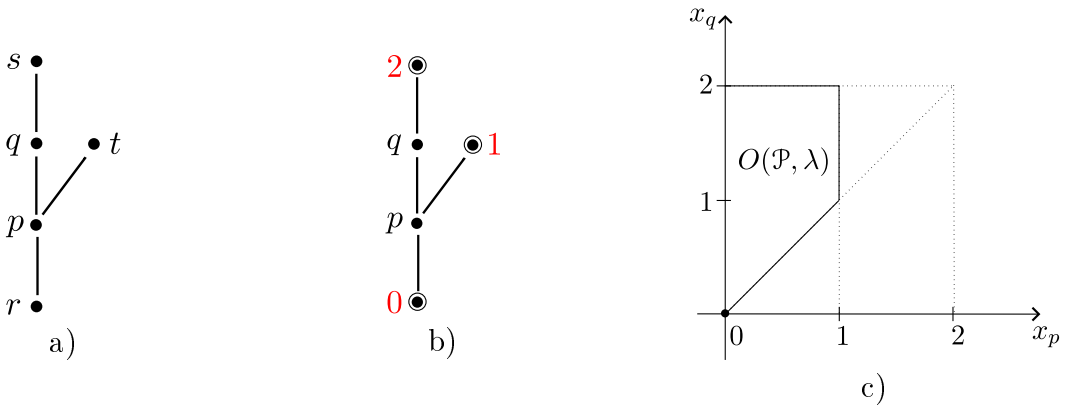

FIG. 6. a) The Hasse diagram of the poset $\mathcal{P}$ from Example 2.15; b) the same Hasse diagram with elements of P^* emphasized and the values of λ shown; c) the projection of $O(\mathcal{P}, \lambda)$ onto the coordinate plane Ox_px_q.

Remark 2.13. It is easy to see that every Gelfand–Zetlin polytope is a marked order polytope.

2. The purpose and objectives of the research project

We study the combinatorics of marked order polytopes. In particular, we would like to describe the f-vector of such a polytope (i.e., the finite sequence $(f_0, f_1, ..., f_n)$, where

f_i is the number of i-dimensional faces and n is the dimension of the polytope). The corresponding generating function $f(t) = f_0 + f_1 \cdot t + \cdots + f_n \cdot t^n$ is called the *f-polynomial*.

In some cases, it will be more convenient for us to reason in terms of the *h-polynomial*. The latter is uniquely reconstructed from the *f-polynomial*, namely:

$$h(s) := f(s-1) = \sum_{i=0}^{n} h_i s^i, \quad h_i = \sum_{k=i}^{n} f_k (-1)^{k-i} \binom{k}{i}. \tag{2}$$

The finite sequence $(h_0, h_1, \ldots, h_n)$ is called the *h-vector*.

3. Relevance of the research project

The Gelfand–Zetlin polytopes have their origins in the representation theory of the group $GL(n)$ (see [13, §5.2] and [5]) and are closely related to the Schubert calculus and the topology of complete flag varieties (see [13] and [9]). For example, the cohomology ring of the complete flag variety $Fl(n)$ is isomorphic to the Khovanskii–Pukhlikov ring of the Gelfand–Zetlin polytope $GZ(\lambda_1 \ldots \lambda_n)$, where $\lambda_1 < \cdots < \lambda_n$ (K. Kaveh; see [13, §5.1]), and the multiplication in this ring is described in terms of the intersection of faces of the polytope $GZ(\lambda_1 \ldots \lambda_n)$ (see [9] and [13, §5.4]). Thus, the combinatorics of Gelfand–Zetlin polytopes is of some interest.

The concept of a marked order polytope was first introduced by F. Ardila, T. Blim, and D. Salazar [2] and they were the first to point out that every Gelfand–Zetlin polytope can be regarded as a marked order polytope. Therefore, the study of the combinatorics of marked order polytopes will certainly be useful, in particular, for studying the combinatorial properties of Gelfand–Zetlin polytopes and some other polytopes that arise in combinatorics, representation theory and algebraic geometry.

4. General description of the research area and the place of the obtained results in the general context of the research area

Let $\mathcal{P} = (P, \preceq)$ be a finite poset with least and greatest elements $\hat{0}$ and $\hat{1}$. *The order polytope* $O(\mathcal{P})$ lies in the finite-dimensional vector space $\mathbb{R}^P$ of all mappings x from P to the line. As a subset of this space it is defined by the inequalities $x(p) \leq x(q)$ for all p, q in P such that $p \preceq q$ and by the equalities $x(\hat{0}) = 0$, $x(\hat{1}) = 1$. The order polytope was defined and studied by Geisinger in 1981. In his paper [4] he described the face lattice of $O(\mathcal{P})$ and the canonical triangulation of this polytope. These results can also be found in the work [14] of R. Stanley, where they are presented as a collection of already known facts.

F. Ardila, T. Blim, and D. Salazar [2] considered a common generalization of the polytope $O(\mathcal{P})$ and Gelfand–Zetlin polytopes. It is constructed from a poset $\mathcal{P}$ endowed with marked elements (including all extremal elements of $\mathcal{P}$) and an order-preserving function λ on these elements. Thus, the notion of a *marked order polytope* $O(\mathcal{P}, \lambda)$ appeared (see Definition 2.11).

C. Pegel [11], generalizing Geisinger's approach, gave a combinatorial description of the face lattice of the polytope $O(\mathcal{P}, \lambda)$. In the same paper he defined *regular* posets and

proved that their covering relations are in one-to-one correspondence with the facets of corresponding marked order polytopes.

We would like to describe the f-vector of a marked order polytope. Of course, one can approach this problem by relying on the description of the face lattice of $O(\mathcal{P}, \lambda)$ given by K. Pegel [11]. We propose another, less straightforward approach.

For an arbitrary (polytopal) subdivision $\mathcal{K}$ of an arbitrary convex polytope M we construct a cochain complex $C^*_{\mathcal{K}}$ (over $\mathbb{Z}_2$) such that the dimensions of its cohomology groups coincide with the components of the f-vector of M (see Lemma 2.7). In the case of marked order polytopes, this construction can be carried through to a computation of the f-vector. Namely, every marked order polytope $O(\mathcal{P}, \lambda)$ has a certain standard *cubosimplicial* subdivision $\mathcal{K}_{\mathcal{P},\lambda}$ (each element of this subdivision is a product of simplices), which generalizes the "canonical" Stanley triangulation of the order polytope $O(\mathcal{P})$. For the Gelfand–Zetlin polytope, this cubosimplicial subdivision was constructed by A. Postnikov [12, proof of Theorem 15.1], and in the general case by R. Liu, K. Meszaros, and E. Saint-Dizier [10, proof of Theorem 3.4 (see also §5.2)]. One can also extract from these papers a combinatorial description of the facets of the subdivision $\mathcal{K}_{\mathcal{P},\lambda}$. We describe the faces *of all* dimensions of the subdivision $\mathcal{K}_{\mathcal{P},\lambda}$ in purely combinatorial terms: they correspond to some chains of embedded ideals of $\mathcal{P}$ (see Proposition 2.24 and Corollary 2.30). Using this description of $\mathcal{K}_{\mathcal{P},\lambda}$, we give a combinatorial description of the cochain complex $C^*_{\mathcal{K}_{\mathcal{P},\lambda}}$. The resulting combinatorial description of $C^*_{\mathcal{K}_{\mathcal{P},\lambda}}$ enables us to compute the cohomology groups of this cochain complex, whose dimensions, according to Lemma 2.7, coincide with the components of the f-vector of $O(\mathcal{P}, \lambda)$. Thus, we obtain a new method of computing the f-vector of a marked order polytope $O(\mathcal{P}, \lambda)$ (see Theorem 2.41).

The combinatorics of Gelfand–Zetlin polytopes is studied in particular in [M1, 1, 6, 7]. In the papers [M1, 1, 6] a Gelfand–Zetlin polytope is understood in the sense of Definition 1.1. In [6] a recurrence relation on the number of vertices of Gelfand–Zetlin polytopes was found. In the papers of An, Cho, Kim [1] and, independently, E. Melikhova [M1] a recurrence relation on the f-polynomial of Gelfand–Zetlin polytopes was found.

The approach of the authors of [1] differs significantly from that used in [M1]. Our proof of the Theorem 1.17 is based on geometric properties of the linear projection of the Gelfand–Zetlin polytope onto a cube. It can be said that it develops the methods and ideas of [6].

The authors of [1] describe the face lattice of a Gelfand–Zetlin polytope in terms of a certain directed planar graph called a *ladder diagram*. More precisely, An, Cho, Kim associate a certain ladder diagram with each class of combinatorially equivalent Gelfand–Zetlin polytopes and prove that their face lattices are isomorphic. (The authors call some special subgraphs of a ladder diagram the "faces" of a ladder diagram.) Using only the combinatorial properties of ladder diagrams, the authors derive a recurrence relation that describes the f-polynomial of the ladder diagram in terms of the f-polynomials of the ladder diagrams of lower dimensions.

It is also worth mentioning the work [7], where the definition of the Gelfand–Zetlin polytope includes an additional condition: the sum of the coordinates in each row of the table (1) must be equal to a given number. That article is devoted to the study of the vertices of such polytopes.

5. Results submitted for the defense

(a) We derive a recurrence relation for the f-polynomial of Gelfand–Zetlin polytopes (Theorem 1.17).

(b) We obtain explicit formulas for the components of the h-vector of the Gelfand–Zetlin polytopes corresponding to sequences of the form $(1, 2, \dots, 2, 3)$ (Theorem 1.19).

(c) We obtain formulas involving the Vieta–Fibonacci polynomials for the h-polynomials of the Gelfand–Zetlin polytopes corresponding to sequences of the form $(1, 2, 3, \dots, 3)$ and $(2, 2, 3, \dots, 3)$ (Theorem 1.21 and Remark 1.22); we also obtain closed formulas for the generating functions of these h-polynomials (Corollary 1.24).

(d) Given an arbitrary convex polytope and its arbitrary (polytopal) subdivision, we construct a cochain complex (over $\mathbb{Z}_2$) such that the dimensions of its cohomology groups coincide with the components of the f-vector of the original polytope (Lemma 2.7).

(e) In the case of a marked order polytope and its standard cubosimplicial subdivision we give a purely combinatorial description of the cochain complex in (d) (Theorem 2.41).

(f) We obtain a combinatorial description of the standard cubosimplicial subdivision of a marked order polytope, which was known earlier in geometric terms (Proposition 2.24 and Corollary 2.30).

6. Methods

In Chapter 1 of the thesis, we develop the ideas and methods of the paper [6]. Namely, we use *combinatorial and geometric properties of a linear projection* of the Gelfand–Zetlin polytope onto a cube to derive a geometric version of the recurrence relation for its f-polynomial (see Corollary 1.15). In order to obtain the final version of the recurrence relation (see Theorem 1.17) we use what is known as the *"bijection method"* in combinatorics. Namely, we establish a one-to-one correspondence between the set of barycenters of all faces of the aforementioned cube and the set of all monomials of a certain polynomial. Upon investigating specific series of Gelfand–Zetlin polytopes, we use the *h-polynomial*, related to the f-polynomial by a variable change, which significantly simplifies the algebraic computations. Finally, we use the *method of generating functions* to deal with sequences of the h-polynomials of Gelfand–Zetlin polytopes of specific series.

In Chapter 2 of the thesis we use certain methods of *homology theory*, which can be motivated by their connection with the spectral sequence of a filtration. By using these methods we are able to reduce the computation of the components of the f-vector of an arbitrary marked order polytope to a computation of the dimensions of the cohomology groups (over $\mathbb{Z}_2$) of a certain cochain complex. In our study of marked order polytopes we use, firstly, their description in [10] in terms of *cubosimplicial polytopes*, each of them being quite unsophisticated marked order polytope (namely, a product of simplices); and secondly, our interpretation of that description in terms of *chains of the poset of order ideals* of the original poset.

7. Statements of main results

7.1. **Recurrence relation for the f-polynomial of Gelfand–Zetlin polytopes.** In this section we formulate the recurrence relation mentioned above, using the geometric properties of the linear projection onto a cube. This projection was constructed in [6]. The following property of this projection is the most important for us: the image of every face of the Gelfand–Zetlin polytope is a face of the cube, and the full preimage of each interior point of every face of the cube is combinatorially equivalent to some Gelfand–Zetlin polytope. The resulting recurrence relation (Theorem 1.17) describes the f-polynomial of the Gelfand–Zetlin polytope in terms of the f-polynomials of the full preimages of the barycenters of the faces of the cube under the projection being considered. The geometric part of the proof of Theorem 1.17 is contained in Lemmas 1.7, 1.11 and 1.15.

More details now.

Hereafter we use the multiplicative notation for a sequence $\lambda_1 \leq \cdots \leq \lambda_s$ with repeats: the exponent equals the number of occurrences of the base. (In symbols: $\lambda_1^{i_1} \ldots \lambda_k^{i_k}$ stands for the sequence $\lambda_1 = \cdots = \lambda_1 < \lambda_2 = \cdots = \lambda_2 < \cdots < \lambda_k = \cdots = \lambda_k$.) In the sequel it will be convenient to use the polytope $GZ(1^{i_1}\, 2^{i_2} \ldots k^{i_k})$ as a canonical representative of its combinatorial class.

Following [6], let us consider the linear projection π of the polytope $GZ(1^{i_1}\, 2^{i_2} \ldots k^{i_k})$ onto the cube C defined by the inequalities

$$1 \leq u_1 \leq 2 \leq u_2 \leq 3 \cdots \leq k-1 \leq u_{k-1} \leq k. \tag{4}$$

Lemma 1.7. *Let G be a face of the polytope $GZ(1^{i_1}\, 2^{i_2} \ldots k^{i_k})$. Then $\pi(G)$ is a face of the cube C.*

Let A be a (nonempty) face of the cube C; denote by $\widehat{A}$ the barycenter of A. It is not hard to see that there is a one-to-one correspondence between the set of all barycenters of the faces of C and the set of all monomials of the polynomial $p := \prod_{j=1}^{k-1}(x_j + x_{j+0.5} + x_{j+1})$. Let us explain this correspondence. Indeed, in order to fix the barycenter $\widehat{A}$ of a face of the cube C, one must decide which one of the identities $u_j = j$, $u_j = j + 0.5$, $u_j = j + 1$ is fulfilled for every $j \in \{1, 2, \ldots, k-1\}$. Similarly, to fix a monomial in p one must

decide which summand of the expression $(x_j+x_{j+0.5}+x_{j+1})$ contributes to the monomial for every $j \in \{1, 2, \ldots, k-1\}$.

Let $\widehat{A}$ be the barycenter corresponding to the monomial

$$x_1^{\alpha_1} x_{1.5}^{\alpha_{1.5}} \cdots x_{k-0.5}^{\alpha_{k-0.5}} x_k^{\alpha_k}. \tag{6}$$

Then it can be shown that $\pi^{-1}(\widehat{A})$ is combinatorially equivalent to

$$GZ(1^{i_1-1+\alpha_1}\ 1.5^{\alpha_{1.5}}\ \ldots\ (k-0.5)^{\alpha_{k-0.5}}\ k^{i_k-1+\alpha_k}). \tag{7}$$

Now we know that the preimages of the barycenters of the faces of C are Gelfand—Zetlin polytopes, moreover, for every preimage we know the corresponding non-decreasing sequence of real numbers

$$1^{i_1-1+\alpha_1}\ 1.5^{\alpha_{1.5}}\ \ldots\ (k-0.5)^{\alpha_{k-0.5}}\ k^{i_k-1+\alpha_k}.$$

Let G be a face of the polytope $GZ(1^{i_1} \ldots k^{i_k})$. We denote by A_G the face $\pi(G)$ of the cube C (here we use Lemma 1.7). Then $\widehat{A_G}$ is the barycenter of the face A_G. The intersection of the polytope $\pi^{-1}(\widehat{A_G})$ and the face G is a face of this polytope which we denote by E_G.

Lemma 1.11. *The map from the set of all nonempty faces of the polytope $GZ(1^{i_1} \ldots k^{i_k})$ into the set of all nonempty faces of all polytopes of the form $\pi^{-1}(\widehat{A})$ that sends a face G into the face E_G is a bijection, which does not preserve the dimension of the face but changes it as follows:*

$$\dim G = \dim E_G + \dim A_G. \tag{9}$$

Corollary 1.15. *One can restore the f-polynomial of the polytope $GZ(1^{i_1} \ldots k^{i_k})$ from the f-polynomials of the polytopes $\pi^{-1}(\widehat{A})$ with the aid of the equation*

$$f_{GZ}(t) = \sum_{A \in L(C)} t^{\dim A} f_{\pi^{-1}(\widehat{A})}(t). \tag{10}$$

Recall that we have enumerated the barycenters of the faces of the cube C by using the monomials of the polynomial $p := \prod_{j=1}^{k-1}(x_j+x_{j+0.5}+x_{j+1})$. We define the coefficients $c_{\alpha_1,\alpha_{1.5},\ldots,\alpha_{k-0.5},\alpha_k}$ by means of the equality

$$p = \sum_{\alpha_1,\alpha_{1.5},\ldots,\alpha_{k-0.5},\alpha_k} c_{\alpha_1,\alpha_{1.5},\ldots,\alpha_{k-0.5},\alpha_k}\, x_1^{\alpha_1} x_{1.5}^{\alpha_{1.5}} \cdots x_{k-0.5}^{\alpha_{k-0.5}} x_k^{\alpha_k}. \tag{11}$$

We will use multi-indices $\alpha = (\alpha_1, \alpha_{1.5}, \ldots, \alpha_{k-0.5}, \alpha_k)$ and $i = (i_1, i_2, \ldots, i_k)$ to write the final formula. We shall denote by $GZ[\alpha, i]$ the Gelfand–Zetlin polytope that is combinatorially equivalent to $\pi^{-1}(\widehat{A}_\alpha)$, which we have already described explicitly by the formula (7).

Theorem 1.17. *The following recurrence relation for the f-polynomial of the Gelfand–Zetlin polytope $GZ(1^{i_1} \ldots k^{i_k})$ holds:*

$$f_{GZ(1^{i_1}\dots k^{i_k})}(t) = \sum_{\alpha} c_\alpha \cdot t^{\dim(A_\alpha)} \cdot f_{GZ[\alpha,i]}(t). \tag{12}$$

7.2. **Applications of the recurrence relation.** In this section we apply the obtained recurrence relation to investigate the Gelfand–Zetlin polytopes corresponding to non-decreasing sequences of the form (12^k3), (123^k) and (223^k). For the polytopes $GZ(12^k3)$ we will see that the polynomial $h_{GZ(12^k3)}$ is *unimodal*: its components first increase and then decrease. Moreover, if we exclude from consideration the component of the highest dimension, then every two adjacent components differ by one (see Theorem 1.19). For the polytopes $GZ(123^k)$ and $GZ(223^k)$ the result is Theorem 1.21, which gives fairly compact formulas for their h-polinomials; and also Corollary 1.24, which derives from these formulas the generating functions of the sequences $\{h_{GZ(123^k)}\}$ and $\{h_{GZ(223^k)}\}$.

Here are the precise statements.

Theorem 1.19. *The components of the h-vector of the $(2k+1)$-dimensional polytope $GZ(1\,2^k\,3)$ are given by the formulas*

$$h_{2k+1} = 1,$$
$$h_i = \begin{cases} i+1, & \text{if } 0 \le i \le k+1; \\ 2k+3-i, & \text{if } k+1 < i \le 2k. \end{cases} \tag{17}$$

Theorem 1.21. *Let the sequence $\{\Phi_k\}$ be defined by the recurrence relation*

$$\Phi_{k+1} = (s^2+s)\Phi_k - s^2\Phi_{k-1},$$

with initial conditions $\Phi_0 = 0$, $\Phi_1 = 1$ and $\Phi_{-1} = -\frac{1}{s^2}$. Then

A) the h-polynomial of the polytope $GZ(1\,2\,3^k)$ is

$$h_{GZ(1\,2\,3^k)}(s) = \sum_{j=0}^{k} \frac{s^{j+2}-1}{s-1}\Phi_{k-j+1}; \tag{24}$$

B) the h-polynomial of the polytope $GZ(2^2\,3^k)$ is

$$h_{GZ(2^2\,3^k)}(s) = \sum_{j=0}^{k} s^{j+2}\Phi_{k-j} + \frac{s^{k+1}-1}{s-1}. \tag{25}$$

Remark 1.22. It is easy to see that $\Phi_k(s) = s^{k-1}V_k(s+1)$, where $V_k(x)$ is the so-called Vieta–Fibonacci polynomial, defined by the recurrence relation $V_{k+1}(x) = xV_k(x) - V_{k-1}(x)$ with initial conditions $V_0(x) = 0$ and $V_1(x) = 1$ [8], [15, Theorem VII]. It should be noted that $V_k(x) = U_{n-1}(x/2)$, where U_n is the Chebyshev polynomial of the second kind. An explicit formula for V_k is also known (see [8]):

$$V_k(x) = \sum_{i=0}^{\lfloor\frac{k-1}{2}\rfloor} (-1)^i \binom{k-i-1}{i} x^{k-2i-1}.$$

Corollary 1.24. *A) The generating function of the sequence* $\left(h_{GZ(1\,2\,3^k)}(s)\right)_{k=0}^{\infty}$ *can be described by the following closed formula:*

$$H_{GZ(1\,2\,3^k)}(s,z)=\sum_{k=0}^{\infty}h_{GZ(1\,2\,3^k)}(s)z^k=\frac{-sz+s+1}{(1-z)(1-sz)(s^2z^2-(s^2+s)z+1)}. \tag{32}$$

B) The generating function of the sequence $\left(h_{GZ(2^2\,3^k)}(s)\right)_{k=0}^{\infty}$ *can be described by the following closed formula:*

$$H_{GZ(2^2\,3^k)}(s,z)=\sum_{k=0}^{\infty}h_{GZ(2^2\,3^k)}(s)z^k=\frac{-sz+1}{(1-z)(1-sz)(s^2z^2-(s^2+s)z+1)}. \tag{33}$$

7.3. **The cochain complex that computes the f-vector of a polytope.** In this section, given an arbitrary convex polytope M and its arbitrary (polytopal) subdivision (see Definition 2.4), we construct a cochain complex with coefficients in the field $\mathbb{Z}_2$ such that the dimension of its nth cohomology group coincides with the nth component of the f-vector of M.

Definition 2.1. *A polytopal complex* $\mathcal{K}$ is a finite collection of non-empty polytopes in some Euclidean space such that:

(1) every face[1] of a polytope from $\mathcal{K}$ again belongs to $\mathcal{K}$;
(2) the intersection of every two polytopes $M_1, M_2 \in \mathcal{K}$ is either empty or is a face of each of them.

The point set $|\mathcal{K}| = \bigcup_{M\in\mathcal{K}} M$ is called the *body* of the polytopal complex $\mathcal{K}$.

Definition 2.2. *The complex* $\mathcal{K}(M)$ *of a polytope* M is the polytopal complex formed by all faces of M.

Definition 2.3. A polytopal complex $\mathcal{K}'$ is called a *subdivision* of a polytopal complex $\mathcal{K}$ if $|\mathcal{K}'| = |\mathcal{K}|$ and each polytope of $\mathcal{K}'$ is contained in some polytope of $\mathcal{K}$.

Definition 2.4. *A (polytopal) subdivision* of a convex polytope M is a polytopal complex $\mathcal{K}$ whose body coincides with M. (The condition that each polytope of $\mathcal{K}$ is contained in some face of M is automatically satisfied in this case.)

Definition 2.5. A subset of a polytopal complex is called a *subcomplex* if it is itself a polytopal complex.

Let M be an arbitrary convex polytope, $\mathcal{K}'$ be some subdivision of the complex $\mathcal{K}(M)$. Let us note that $\mathcal{K}'$ naturally endows the polytope M with the structure of a finite CW complex.

Let A be an arbitrary face of M. Consider the subcomplex $\mathcal{A}'$ of $\mathcal{K}'$ consisting of all polytopes $S \in \mathcal{K}'$ that lie entirely in A. Also consider the subcomplex $\partial\mathcal{A}'$ of $\mathcal{A}'$ consisting of all polytopes $T \in \mathcal{A}'$ that lie entirely in ∂A.

[1] We do not consider the empty set to be a face.

Let $C^*(\mathcal{A}', \partial\mathcal{A}'; \mathbb{Z}_2)$ be the complex of relative cellular cochains with coefficients in $\mathbb{Z}_2$ of the pair of CW complexes defined by the pair of polytopal complexes $(\mathcal{A}', \partial\mathcal{A}')$. Let us take the direct sum of such cochain complexes over all faces of M:

Definition 2.6.

$$C^*_{\mathcal{K}'} := \bigoplus_{A \in \mathcal{K}(M)} C^*(\mathcal{A}', \partial\mathcal{A}'; \mathbb{Z}_2). \tag{34}$$

It turns out that the dimensions of the cohomology groups $C^*_{\mathcal{K}'}$ coincide with the corresponding components of the f-vector of the original polytope, that is, Lemma 2.7 holds.

Lemma 2.7. $\dim H^n(C^*_{\mathcal{K}'}) = f_n(M)$.

Remark 2.8. Nothing prevents us from considering the dual approach, with chain groups instead of cochain groups. In this case, $C^{\mathcal{K}'}_* = \bigoplus_{A \in \mathcal{K}(M)} C_*(\mathcal{A}', \partial\mathcal{A}'; \mathbb{Z}_2)$ and $f_n(M) = \dim H_n(C^{\mathcal{K}'}_*)$.

Remark 2.9. M. Kazaryan observed that the proposed approach to computing the f-vector of a polytope M can be motivated by the close relation of this approach with the initial terms of the spectral sequence associated with the filtration of M by its skeletons. Namely, it is easy to see that $C^*_{\mathcal{K}'} \simeq \bigoplus_{i=0}^m C^*(\mathcal{K}'_i, \mathcal{K}'_{i-1})$, where $\mathcal{K}'_i$ is the subcomplex of $\mathcal{K}'$ that is a subdivision of the i-dimensional skeleton of the polytope M, and m is the dimension of M. On the other hand, the filtration $\emptyset = \mathcal{K}'_{-1} \subset \mathcal{K}'_0 \subset \cdots \subset \mathcal{K}'_m = \mathcal{K}'$ of the complex $\mathcal{K}'$ is associated with the cohomology spectral sequence for which $E_0^{p,q} = C^{p+q}(\mathcal{K}'_p, \mathcal{K}'_{p-1})$ and $E_1^{p,q} = H^{p+q}(\mathcal{K}'_p, \mathcal{K}'_{p-1})$ (see, for example, [3]). Thus, $C^n_{\mathcal{K}'} \simeq \bigoplus_{i=0}^m E_0^{i,n-i}$ and consequently $H^n(C^*_{\mathcal{K}'}) \simeq \bigoplus_{i=0}^m E_1^{i,n-i}$. In other words, the components of the f-vector of M are the sums of the dimensions of the groups on the nth diagonal of the page E_1 of the spectral sequence.

7.4. **Marked order polytope and its cubosimplicial subdivision.** As we already mentioned, every marked order polytope $O(\mathcal{P}, \lambda)$ (see Definition 2.11) has a certain canonical *cubosimplicial* subdivision $\mathcal{K}_{\mathcal{P},\lambda}$ (see Definition 2.16) coming from the structure of the poset $\mathcal{P}$. In this section we give a combinatorial description of $\mathcal{K}_{\mathcal{P},\lambda}$.

Definition 2.16. A polytopal complex in which each polytope is a product of simplices is called a *cubosimplicial complex*.

Definition 2.17. An *order ideal* of a poset $\mathcal{P} = (P, \preceq)$ is a subset $I \subset P$ such that if $x \in I$ and $y \preceq x$, then $y \in I$.

Let us note that the order ideals of the poset $\mathcal{P}$ also form a poset by inclusion.

Let $\mathcal{P}$, $\mathcal{P}^*$, and λ be respectively a poset, a subposet of marked elements, and an order-preserving function from $\mathcal{P}^*$ to $\mathbb{R}$ as in the definition of $O(\mathcal{P}, \lambda)$.

Definition 2.19. A chain L (i.e. a finite subset linearly ordered by inclusion) in the poset of order ideals of a poset $\mathcal{P}$ will be called λ-*admissible* or simply *admissible* (when

λ is clear from the context) if it has the form $\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_m \subsetneq I_{m+1} = P$ and satisfies the condition

$$\begin{aligned} &\text{each } \lambda((I_i \setminus I_{i-1}) \cap P^*) \text{ is empty or equal to } \{t_i\} \text{ for some } t_i \in \mathbb{R}, \\ &\text{where } t_i < t_j \text{ provided that } i < j \text{ and both } t_i \text{ and } t_j \text{ are defined.} \end{aligned} \tag{39}$$

Let L be a chain of the form $\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_m \subsetneq I_{m+1} = P$ in the poset of order ideals of a poset $\mathcal{P}$.

Definition 2.20. Let us denote by $F_L = F_L(\lambda)$ the set of points $x \in \mathbb{R}^P$ for which three conditions are satisfied:

$$\begin{aligned} &\bullet\ x|_{P^*} = \lambda, \\ &\bullet\ \text{function } x : P \to \mathbb{R} \text{ is constant on the sets } I_1 \setminus I_0, \ldots, I_{m+1} \setminus I_m, \\ &\bullet\ x(I_1) \le x(I_2 \setminus I_1) \le \cdots \le x(I_m \setminus I_{m-1}) \le x(P \setminus I_m). \end{aligned} \tag{40}$$

Lemma 2.21. *The correspondence $L \mapsto F_L$ between admissible chains of the poset of order ideals of a poset $\mathcal{P}$ and non-empty sets of the form F_L is one-to-one.*

Proposition 2.24. *The collection*

$$\mathcal{K}_{\mathcal{P},\lambda} := \{F_L \mid L \text{ is a } \lambda\text{-admissible chain of the poset of order ideals of } \mathcal{P}\}$$

is a polytopal complex.

Corollary 2.30. *The polytopal complex $\mathcal{K}_{\mathcal{P},\lambda}$ is a subdivision of the polytope $O(\mathcal{P}, \lambda)$.*

Remark 2.31. The polytopal complex $\mathcal{K}_{\mathcal{P},\lambda}$ is cubosimplicial, since each of the polytopes $F_{\widetilde{L}_\sigma} = O(\mathcal{P}_\sigma, \lambda)$ is a product of simplices.

Remark 2.32. The subdivision $\mathcal{K}_{\mathcal{P},\lambda}$ of the marked order polytope can be described geometrically [10, proof of Theorem 3.4].

7.5. **Computing the f-vector of the polytope $O(\mathcal{P}, \lambda)$.** In this section we describe the cochain complex $C^*_{\mathcal{K}_{\mathcal{P},\lambda}}$ in purely combinatorial terms, i.e., directly in terms of the poset $\mathcal{P}$ and the order-preserving map λ, without reference to any polytopes.

Definition 2.34. The *dimension* of a λ-admissible chain $L = \Big(\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_m \subsetneq I_{m+1} = P\Big)$ is the number of those set differences of consecutive ideals $I_i \setminus I_{i-1}$ that do not intersect P^*. Notation: $\dim L$.

It is easy to see that $\dim F_L = \dim L$.

Definition 2.35. The *dimension* of the pair $(\mathcal{P}, \lambda)$ is the maximum of the dimensions of λ-admissible chains in the poset of order ideals of $\mathcal{P}$. Notation: $\dim(\mathcal{P}, \lambda)$.

It is easy to see that $\dim(\mathcal{P}, \lambda) = \dim O(\mathcal{P}, \lambda)$.

Definition 2.36. Let us say that a λ-admissible chain L_k is obtained by *I_k-refinement* from a λ-admissible chain $L = \Big(\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_m \subsetneq I_{m+1} = P\Big)$ if L_k is obtained from L by adding a unique ideal J_k such that $I_{k-1} \subsetneq J_k \subsetneq I_k$.

Lemma 2.37. *In this case* $\dim L_k = \dim L + 1$.

Corollary 2.38. *Let* L *and* L' *be* λ*-admissible chains,* $L \subset L'$*, and* $\dim L' = \dim L + 1$*. Then* L' *is obtained from* L *by some* I_k*-refinement.*

Definition 2.40. Let us say that λ-admissible chains L' and L'' are I_k-*conjugate* if they are obtained by I_k-refinement from some λ-admissible chain L and the equalities $J_{k+1} \setminus J_k = J'_k \setminus J'_{k-1}$ and $J_k \setminus J_{k-1} = J'_{k+1} \setminus J'_k$ hold.[2]

Let $C^i(\mathcal{P}, \lambda)$ be a vector space over $\mathbb{Z}_2$ with basis consisting of all λ-admissible chains of dimension i in the poset of order ideals of $\mathcal{P}$. For a λ-admissible chain $L = \Big(\emptyset = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_m = P\Big)$ let us define $\delta(L) = \Sigma_1(L) + \cdots + \Sigma_m(L)$, where $\Sigma_k(L)$ is the formal sum of all I_k-refinements of L that have an I_k-conjugate. Note that $\delta(L) \in C^{i+1}(\mathcal{P}, \lambda)$, since every I_k-refinement increases the dimension of a λ-admissible chain by one (see Lemma 2.37). Extending δ by linearity, we obtain linear maps $\delta : C^i(\mathcal{P}, \lambda) \to C^{i+1}(\mathcal{P}, \lambda)$. Put

$$C^*(\mathcal{P}, \lambda) = \Big(C^0(\mathcal{P}, \lambda) \xrightarrow{\delta} \ldots \xrightarrow{\delta} C^n(\mathcal{P}, \lambda)\Big), \text{ where } n = \dim(\mathcal{P}, \lambda).$$

Theorem 2.41. $C^*(\mathcal{P}, \lambda)$ *is a cochain complex (i.e.* $\delta \circ \delta = 0$*) and is isomorphic to the complex* $C^*_{\mathcal{K}_{\mathcal{P},\lambda}}$.

8. Theoretical and practical significance of the work

This research is of theoretical character. The study of the combinatorics of marked order polytopes might be useful in problems of algebraic geometry, representation theory, and in various areas of enumerative combinatorics related to the study of finite partially ordered sets.

9. Validation of the results

Conference talks:

(1) *On the number of faces of Gelfand–Zetlin polytopes*, 60th Scientific Conference of Moscow Institute of Physics and Technology, Dolgoprudny, Russia, November 2017

(2) *On the number of faces of Gelfand–Zetlin polytopes*, XV International Conference "Algebra, Number Theory and Discrete Geometry: Contemporary Problems and Applications" dedicated to the centenary of the birth of Professor N. M. Korobov, Tula, Russia, May 2018

(3) *On the number of faces of Gelfand–Zetlin polytopes*, Algorithmic and Enumerative Combinatorics Summer School, poster session, Hagenberg, Austria, July 2018

[2] In other words, the difference of ideals $I_k \setminus I_{k-1}$ is partitioned into two pieces ($I_k \setminus I_{k-1} = A \cup B$), which, when adjoined in different orders, result in the formation of L' and L''. In symbols, $J_k = I_{k-1} \cup A = I_k \setminus B$ and $J'_k = I_{k-1} \cup B = I_k \setminus A$.

(4) *On the number of faces of marked ordered polytopes*, XIV Belarusian Mathematical Conference, Minsk, Belarus, November 2024

Seminar talks:

(1) *On the number of faces of Gelfand–Zetlin polytopes*, Master students' seminar led by S. A. Loktev, Math Faculty of National Research University Higher School of Economics, Moscow, February 2017

(2) *On the number of faces of Gelfand–Zetlin polytopes*, Combinatorics Seminar led by M. N. Vyalyi and S. P. Tarasov, Dorodnicyn Computing Centre of RAS, Moscow, March 2017

(3) *On the number of faces of Gelfand–Zetlin polytopes*, Geometry Seminar led by N. P. Dolbilin and N. G. Moshchevitin, Lomonosov Moscow State University, Moscow, May 2017

(4) *f-Vector of marked order polytopes*, Online Seminar "Polytopes and polynomials" led by A. I. Esterov, November 2021

(5) *f-Vector of marked order polytopes*, Session of graduate students' reports at the Seminar of International Laboratory of Cluster Geometry led by M. E. Kazarian and S. K. Lando, October 2023

10. Publications by the author on the subject of the thesis

[M1] E. V. Melikhova, *On the number of faces of Gelfand–Zetlin polytopes*, Algebra i analiz **33** (2021), no. 3, 169–190 (Russian); English transl., *On the number of faces of Gelfand–Zetlin polytopes*, St. Petersburg Math. J. **33** (2022), no. 3, 553–568.

[M2] E. V. Melikhova, *On the number of faces of marked order polytopes*, Matem. zametki SVFU (accepted for publication).

11. Personal contribution

All results submitted for the defense were obtained by the author personally, and all publications of the author on the subject of the thesis were written without co-authors.

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