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COMPUTER SCIENCE

The Choice of Several Best Objects with Respect to a Partial Preference Relation

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INTRODUCTION

In the literature on decision making, the problem of choosing a unique best object is largely considered; ordering objects according to preference is given much less attention. However, in practice, the problem of choosing several (a given number $l > 1$) of best objects often arises. An example is the choice of several best investment projects among all competing bids. Moreover, it may happen that not all of the objects are comparable with respect to preference; this is typical of, e.g., multicriteria choice problems. The current state-of-the-art in the mathematical theory of the choice of l best objects is surveyed in [1]. It was demonstrated in [1] that, although the basic definitions of l-optimal (l-maximal) and l-undominated (l-maximal) objects were introduced as early as in 1978 [2] and some important properties of these objects were found in [3], the level of development of the theory was, obviously, insufficient, and the directions of further development were outlined. In this paper, we present the main results obtained in the course of implementation of this program.

1. OPTIMAL AND UNDOMINATED SETS OF OBJECTS

We consider the problem of choosing l best objects in a finite set of objects X with $|X| = n > l > 1$. On the set X, the relation R of nonstrict preference of a decision making person (DMP) is defined: xRy means that the object x is no less preferred than y. The relation R generates a (strict) preference relation P, an indifference relation I, and an incomparability relation N: xIy if xRy and yRx; xPy if xRy but not yRx; xNy if neither xRy nor yRx. We assume that R is a quasi-order (i.e., a reflexive transitive relation), but only partial, or disconnected (the relation N is nonempty).

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Consider a family L of l-object sets. This family is finite; it contains C_n^l sets. We denote sets from L by $A = \{a^1, a^2, \ldots, a^l\}$ or likewise. Let Π be the set of permutations on $\{1, 2, \ldots, l\}$. By the rearrangement of a set $A = \{a^1, a^2, \ldots, a^l\}$ corresponding to $\pi \in \Pi$ we understand the l-tuple $\pi(A) = \langle a^{\pi(1)}, a^{\pi(2)}, \ldots, a^{\pi(l)}\rangle$. For example, if $l = 3$ and $\pi = (3, 1, 2)$, then $\pi(A) = \langle a^3, a^1, a^2\rangle$.

Since the order of objects does not matter, in accordance with the essence of the problem under consideration, we define a binary relation R^l on the set L, which determines when one of the two tuples in a pair can be assumed to be no less preferred than the other.

Definition 1. The relation AR^lB holds if and only if there exist permutations $\pi, \rho \in \Pi$ for which

$$a^{\pi(i)}Rb^{\rho(i)}, \quad i = 1, 2, \ldots, l. \tag{1}$$

Definition 1 is equivalent to each of the following definitions:

$$AR^lB \Leftrightarrow \exists\pi \in \Pi\colon a^iRb^{\pi(i)}, \quad i = 1, 2, \ldots, l, \tag{1'}$$

$$AR^lB \Leftrightarrow \exists\pi \in \Pi\colon a^{\pi(i)}Rb^i, \quad i = 1, 2, \ldots, l. \tag{1''}$$

The nonstrict preference relation R^l is a quasi-order and generates the (strict) preference relation ((partial) order) P^l and the indifference relation (equivalence) I^l on L. Note that AI^lB if, for each i, R can be replaced by I in (1) (or in (1') or (1'')), and AP^lB if, for at least one i, R can be replaced by P in (1) (or in (1') or (1'')).

Definition 2. A set A is said to be strictly optimal if, for any set B different from A, AP^lB.

Definition 3. A set A is said to be optimal if, for any set B, AR^lB.

Definition 4. A set A is said to be undominated if BP^lA for no set B.

Let $P^l(L)$, $R^l(L)$, and $Q^l(L)$ be the sets of strictly optimal, optimal, and undominated sets, respectively. We have

$$P^l(L) \subseteq R^l(L) \subseteq Q^l(L). \tag{2}$$

If a strictly optimal set exists, then it is unique (exhausts the set $P^l(L)$; in this case, both inclusions in (2) become equalities), and the required l best objects are determined uniquely as well: these are all objects from such a set. Suppose that there is no strictly optimal set but an optimal set exists (then, the second (right) inclusion in (2) is an equality). Even if this set is nonunique, all optimal sets are equivalent (with respect to I^l). Therefore, any of them can be considered best, and the objects from any optimal set can be taken for the best objects.

Finally, even if optimal sets do not exist either, then the finiteness of L implies the existence of undominated sets. Moreover, the family of all such sets is externally stable in the sense that, for any $B \in L \backslash Q^l(L)$, there exists an $A \in Q^l(L)$ for which AP^lB. Therefore, only undominated sets are candidates for optimal sets, and only objects from such sets are candidates for the best ones.

2. l-OPTIMAL AND l-UNDOMINATED OBJECTS

Since the task of forming optimal or undominated sets directly from Definitions 2–4 is very burdensome even for not very large sets of objects X, there arises the problem of searching for constructive methods for accomplishing this task. In this connection, the following notions turn out to be useful.

Definition 5. An object x is said to be strictly l-optimal if xPy for all but fewer than l objects y different from x.

Definition 6. An object x is said to be l-optimal if xRy for all but fewer than l objects y.

Definition 7. An object x is said to be l-undominated if the number of objects y for which yPx is smaller than l.

If the equivalence I is the equality relation, then the notions of strictly l-optimal and l-optimal objects coincide.

Let $P_l(X)$, $R_l(X)$, and $Q_l(X)$ be the sets of strictly l-optimal, l-optimal, and l-undominated objects, respectively. Definitions 5–7 directly imply that

$$P_l(X) \subseteq R_l(X) \subseteq Q_l(X);$$

$$\text{if } k < l, \text{ than } P_k(X) \subseteq P_l(X), \quad R_k(X) \subseteq R_l(X),$$

$$Q_k(X) \subseteq Q_l(X).$$

Theorem 1. *The following assertions are valid.*

(1.1) *The number of strictly l-optimal objects does not exceed l.*

(1.2) *The number of I-equivalence classes in $R_l(X)$ does not exceed l.*

(1.3) *The set of l-undominated objects is nonempty and externally l-stable, that is,*

(1.3.1) *if $x \notin Q_l(X)$, then $Q_l(X)$ contains at least l objects y for which yPx;*

(1.3.2) *if $x^1, x^2, \ldots, x^s \notin Q_l(X)$ and $x^{s+1}, \ldots, x^l \in Q_l(X)$, where $s \le l$, then $Q_l(X)$ contains s pairwise different objects $x^{01}, x^{02}, \ldots, x^{0s}$ different from $x^{s+1}, x^{s+2}, \ldots, x^l$ and such that $x^{01}Px^1, \ldots, x^{0s}Px^s$.*

(1.4) *The sets $P_l(X)$, $R_l(X)$, and $Q_l(X)$ are upper closed for R, that is,*

(1.4.1) *if $x \in P_l(X)$ and yRx, then $y \in P_l(X)$;*

(1.4.2) *if $x \in R_l(X)$ and yRx, then $y \in R_l(X)$;*

(1.4.3) *if $x \in Q_l(X)$ and yRx, then $y \in Q_l(X)$.*

(1.5) *Objects from different sets are related as follows:*

(1.5.1) *if $x \in Q_l(X)$ and $y \notin Q_l(X)$, then xNy or xPy;*

(1.5.2) *if $x \in R_l(X)$ and $y \notin Q_l(X)$, then xPy;*

(1.5.3) *if $x \in R_l(X)$ and $y \notin Q_l(X) \backslash R_l(X)$, then xNy or xPy;*

(1.5.4) *if $x \in P_l(X)$ and $y \in R_l(X) \backslash P_l(X)$, then xNy or xPy.*

(1.6) *The equalities $|Q_l(X)| = l$, $|P_l(X)| = l$, and $Q_l(X) = P_l(X)$ are equivalent.*

(1.7) *If the equivalence I is the equality relation, then*

(1.7.1) $R_l(X) = P_l(X)$;

(1.7.2) *the number of variants in $R_l(X)$ is at most l;*

(1.7.3) *if $|R_l(X)| = l$, then $R_l(X) = Q_l(X)$.*

(1.8) *If the quasi-order R is connected (complete, i.e., $N = \varnothing$), then $R_l(X) = Q_l(X)$.*

(1.9) *For $k < l$, the following assertions hold:*

(1.9.1) *if $P_k(X) \subset P_l(X)$, then $|P_k(X)| \le l - 1$;*

(1.9.2) *if $R_k(X) \subset R_l(X)$, then $|P_l(X) \cup R_k(X)| \le l - 1$.*

3. THE APPLICATION OF l-OPTIMAL AND l-UNDOMINATED OBJECTS TO THE FORMATION OF BEST l-OBJECT SETS

Theorem 2. *The following assertions hold.*

(2.1) *The existence and properties of a strictly optimal set:*

(2.1.1) *a strictly optimal set contains only strictly l-optimal objects;*

(2.1.2) *a strictly optimal set exists if and only if $|P_l(X)| = l$.*

(2.2) *The existence and properties of optimal sets:*

(2.2.1) *an optimal set contains only l-optimal objects;*

(2.2.2) *an optimal set contains all strictly l-optimal objects;*

(2.2.3) *if $|R_l(X)| = l$, then an optimal set exists if and only if $R_l(X) = Q_l(X)$; in this case, all l-optimal objects constitute an optimal set and $R^l(L) = P^l(L)$;*

(2.2.4) *if $|R_l(X)| > l$, then an optimal set exists if and only if $R_l(X) = Q_l(X)$ and $R_l(X)$ contains an object least* *with respect to R* (*i.e., an object* x_* *such that* xRx_* *for any* $x \in R_l(X)$);

(2.2.5) *if the equivalence I is the equality relation, then* $|R_l(X)| = l$ *is a necessary and sufficient condition for the existence of an optimal set; in this case, the optimal set consists of all l-optimal objects and* $R^l(L) = P^l(L)$;

(2.2.6) *if* $k < l$, $R_k(X) \subset R_l(X)$, *and an object* $x \in R_k(X)$ *is not contained in a set* $A \in L$, *then this set is not optimal.*

(2.3) *The existence and properties of undominated sets:*

(2.3.1) *the family of undominated sets is nonempty and externally stable;*

(2.3.2) *each undominated set contains only l-undominated objects;*

(2.3.3) *each undominated set contains all strictly l-optimal objects;*

(2.3.4) *if* $x \notin A$, $y \in A$, *and xPy, then the set A is not undominated.*

Theorems 1 and 2 suggest the following recommendations on the choice of l best objects. First, all l-dominated (i.e., not being l-undominated) objects must be eliminated.

If there are precisely l strictly l-best objects, then these objects form a strictly optimal set, and only such objects can be considered best.

If there are more than l objects in $R_l(X)$ and the condition in (2.2.4) holds, then we must remove all objects not being least from $R_l(X)$ (there are at most $l - 1$ such objects) and augment the remaining set to a set of cardinality l with any least objects (all of them are in the indifference relation). If this condition does not hold, then there exists no optimal set, and we must determine objects which are surely to be included in the set of l best objects. These are all objects from the set $P_l(X) \cup R_k(X)$, where k is the largest number $k < l$ for which $R_k(X) \subset R_l(X)$. There are at most $l - 1$ such objects. The remaining objects are only candidates for the l best objects.

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