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Modular metric spaces, II: Application to superposition operators☆

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ABSTRACT

Applying the theory of modular metric spaces developed in the first part of this paper [V.V. Chistyakov, Modular metric spaces I Basic concepts, Nonlinear Anal. (2009) submitted for publication)] we define a metric semigroup and an abstract convex cone of functions of finite generalized variation in the approach of Schramm [M. Schramm, Functions of Φ-bounded variation and Riemann–Stieltjes integration, Trans. Amer. Math. Soc. 287 (1) (1985) 49–63], which are significantly larger as compared to the spaces of bounded variation in the sense of Jordan, Wiener–Young and Waterman. We present a complete description of generators of Lipschitz continuous, bounded and some other classes of superposition Nemytskii operators mapping in these semigroups and cones, which extends recent results by Matkowski and Miś [J. Matkowski, J. Mi, On a characterization of Lipschitzian operators of substitution in the space BV$\langle a, b\rangle$, Math. Nachr. 117 (1984) 155–159], Maligranda and Orlicz [L. Maligranda, W. Orlicz, On some properties of functions of generalized variation, Monatsh. Math 104 (1987) 53–65], Zawadzka [G. Zawadzka, On Lipschitzian operators of substitution in the space of set-valued functions of bounded variation, Rad. Mat. 6 (1990) 279–293] and Chistyakov [V.V. Chistyakov, Mappings of generalized variation and composition operators, J. Math. Sci. (New York) 110 (2) (2002) 2455–2466, V.V. Chistyakov, Lipschitzian Nemytskii operators in the cones of mappings of bounded Wiener φ-variation, Folia Math. 11 (1) (2004) 15–39].

4. Introduction to part II

This paper is a continuation of [1]. Its purpose is to present an exhausting description of Lipschitz continuous and some other classes of nonlinear superposition operators acting in modular metric spaces of functions of a real variable of finite generalized variation in the sense of Schramm [2] with values in metric semigroups and abstract cones. Part of the results of this paper were announced in [3] without proofs. Q1

Let I be a nonempty set (a closed interval $[a, b]$ in $\mathbb{R}$ in the sequel), $\mathbb{R}^I$ be the algebra of all functions $y : I \to \mathbb{R}$ mapping I into $\mathbb{R}$ equipped with the usual pointwise operations and $h : I \times \mathbb{R} \to \mathbb{R}$ be a given function. The *superposition* (*Nemytskii*) *operator* $\mathcal{H} = \mathcal{H}_h : \mathbb{R}^I \to \mathbb{R}^I$ generated by h is defined by

$$(\mathcal{H}y)(t) = h(t, y(t)), \quad t \in I, y \in \mathbb{R}^I.$$

The function h is said to be the *generator* of $\mathcal{H}$.

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Given an interval $I = [a, b]$, denote by $\mathrm{BV}(I)$ the subset of $\mathbb{R}^I$ of all functions $y : I \to \mathbb{R}$ of bounded Jordan variation

$$V_a^b(y) = \sup_{\pi} \sum_{i=1}^{m} |y(t_i) - y(t_{i-1})| < \infty,$$

where the supremum is taken over all partitions π of I, i.e., $\pi = \{t_i\}_{i=1}^m$ with $m \in \mathbb{N}$ and $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$. It is well known that $\mathrm{BV}(I)$ is a Banach algebra with respect to the norm $\|y\| = |y(a)| + V_a^b(y)$, $y \in \mathrm{BV}(I)$, and $\|y \cdot z\| \le 2\|y\| \cdot \|z\|$ for all $y, z \in \mathrm{BV}(I)$. The last inequality follows immediately from the following two inequalities (e.g., [4, Chapter 8, Section 3, Theorem 3]):

$$|y|_\infty \equiv \sup_{t\in I} |y(t)| \le \|y\| \quad \text{and} \quad V_a^b(y \cdot z) \le V_a^b(y)|z|_\infty + |y|_\infty V_a^b(z).$$

This property of the space $\mathrm{BV}(I)$ implies that if the generator $h : I \times \mathbb{R} \to \mathbb{R}$ of a superposition operator $\mathcal{H}$ is of the form $h(t, y) = x(t)y + h_0(t)$, $t \in I$, $y \in \mathbb{R}$, for some functions x and h_0 from $\mathrm{BV}(I)$, then $\mathcal{H}$ maps $\mathrm{BV}(I)$ into itself and satisfies a Lipschitz condition: there exists a constant $\eta \ge 0$ (one can set $\eta = 2\|x\|$) such that

$$\|\mathcal{H}y - \mathcal{H}z\| \le \eta\|y - z\| \quad \text{for all } y, z \in \mathrm{BV}(I). \tag{4.1}$$

Clearly, property (4.1) with $\eta < 1$ is closely connected with the solution of the functional equation $\mathcal{H}y = y$ by means of the Banach Contraction Theorem.

Conversely, Matkowski and Miś [5] showed that if the superposition operator $\mathcal{H}$ is generated by a function $h : I \times \mathbb{R} \to \mathbb{R}$, maps $\mathrm{BV}(I)$ into itself and satisfies a Lipschitz condition of the form (4.1) for some constant $\eta \ge 0$, then there exist two functions x and h_0 from $\mathrm{BV}(I)$, which are continuous from the left on $(a, b]$, such that

$$\lim_{s\to t-0} h(s, y) = x(t)y + h_0(t) \quad \text{for all } a < t \le b \text{ and } y \in \mathbb{R}. \tag{4.2}$$

We note that the representation of the form (4.2) for the function $h(t, y)$ (not for the left limit as in (4.2)) was found by Matkowski in [6,7] in the class $\mathrm{Lip}(I)$ of Lipschitz functions on I. Such a representation for the generators of Lipschitzian superposition operators does not hold in the algebra of all continuous functions $\mathrm{C}(I)$ on I with the uniform norm $|\cdot|_\infty$ or in the space $\mathrm{L}^p(I)$ of Lebesgue p-summable functions on I, $p \ge 1$, with the standard norm (cf. [6]; for example, one can set $h(t, y) = \sin y$ for all $t \in I$ and $y \in \mathbb{R}$). On the other hand, the representation of the form (4.2) is valid in many spaces of functions of one variable, with certain restrictions on the generalized variation if the functions are single-valued [8, Section 6.5], [9–15], or even multi-valued [16–24]. The representation of the form (4.2) also holds in the class of functions of several variables of bounded variation in the sense of Vitali–Hardy–Krause [25–30].

In this pape[illegible]plying the theory of modular metric spaces from [1[illegible] define a nonlinear space of functions of generalized bounded variation in the approach of Schramm [2], which is significantly larger than the spaces of functions of bounded variation in the sense of Jordan, Wiener–Young [18,31,32] and Waterman [33,34] (Sections 5.2, 5.3, 5.5 and 5.6). Then we characterize the generators of Lipschitzian and some other classes of superposition operators, which map in these modular metric spaces (Theorems 6.3, 6.5, 6.8, 6.14 and 6.16). Although the superposition operator is well studied in many classical functional spaces (cf. [8]), in spaces of functions of bounded generalized variation its properties (continuity, compactness, local Lipschitz continuity, differentiability, ...) are still not known (cf., however, [35–37]). In this respect our paper fills in certain gaps, which are concerned with the superposition operators in the BV context.

The enumeration of sections, subsections, assertions and formulas in this paper is a continuation of the enumeration adopted in [1].

5. Modular semigroups and cones of functions

5.1. The Φ-sequence

A sequence $\Phi = \{\varphi_i\}_{i=1}^\infty$ of continuous nondecreasing unbounded functions $\varphi_i : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$, each of which is such that $\varphi_i(u) = 0$ if and only if $u = 0$, satisfying the following two conditions:

$$\varphi_{i+1}(u) \le \varphi_i(u) \quad \text{for all } i \in \mathbb{N} \text{ and } u \in \mathbb{R}^+, \tag{5.1}$$

$$\sum_{i=1}^\infty \varphi_i(u) = \infty \quad \text{for all } u > 0, \tag{5.2}$$

is said to be a *Φ-sequence* in the terminology of [2] (note that each function φ_i is a φ-function). Such a sequence Φ is said to be *convex* if all functions φ_i from this sequence are convex (in this case each function φ_i is strictly increasing and has the continuous inverse function $\varphi_i^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$).

Examples of Φ-sequences $\Phi = \{\varphi_i\}_{i=1}^\infty$ can be given as sequences, in which $\varphi_i(u) = u$, $\varphi_i(u) = \varphi_1(u)$ or $\varphi_i(u) = u/\lambda_i$ for all $i \in \mathbb{N}$ and $u \in \mathbb{R}^+$, where $\{\lambda_i\}_{i=1}^\infty \subset (0, \infty)$ is a Waterman sequence, i.e., a nondecreasing sequence such that $\sum_{i=1}^\infty 1/\lambda_i = \infty$ (cf. [2,33,34]).

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5.2. Modular semigroup $\mathrm{BV}_\Phi(I; M)$

Let $I = [a, b]$ be a closed interval, $(M, d, +)$ be a metric semigroup with zero $0 \in M$ in the sense of [1, Section 2.14] (or an abstract convex cone in the sense of [1, Section 3.12]) and $\Phi = \{\varphi_i\}_{i=1}^{\infty}$ be a Φ-sequence. Then the set $X = M^I$ of all functions $x : I \to M$ with the operation $(x + y)(t) = x(t) + y(t)$ (and $(\alpha x)(t) = \alpha x(t)$, $\alpha \in \mathbb{R}^+$, respectively), where $t \in I$ and $x, y \in X$, is an Abelian semigroup with zero 0, so that $0(t) = 0 \in M$ for all $t \in I$ (in which the operation of multiplication by nonnegative numbers is defined satisfying the properties from [1, equalities (3.5)], respectively). Given $\lambda > 0$ and $x, y \in X$, we set

$$w_\lambda(x, y) \equiv w_\lambda^d(x, y) = \sup \sum_{i=1}^{m} \varphi_i\Big(\frac{1}{\lambda} d\big(x(b_i) + y(a_i), y(b_i) + x(a_i)\big)\Big),$$

where the supremum is taken over all $m \in \mathbb{N}$ and all nonordered collections of non-overlapping intervals $[a_k, b_k] \subset I$, $k = 1, \dots, m$.

Then w is a metric pseudomodular on X. Clearly, $w_\lambda(x, x) = 0$ and $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$. Now if $w_\lambda(x, y) = 0$ for all $\lambda > 0$, then, given $t, s \in I$, we have:

$$\varphi_1\Big(\frac{1}{\lambda} d\big(x(t) + y(s), y(t) + x(s)\big)\Big) \le w_\lambda(x, y) = 0,$$

and so, $d(x(t) + y(s), y(t) + x(s)) = 0$. By the property of the translation invariant metric d [1, inequality (2.3)], we find

$$|d(x(t), y(t)) - d(x(s), y(s))| \le d(x(t) + y(s), y(t) + x(s)), \tag{5.3}$$

and so, $d(x(t), y(t)) = \text{const} \in \mathbb{R}^+$ for all $t \in I$.

Let us verify that $w_{\lambda+\mu}(x, y) \le w_\lambda(x, z) + w_\mu(y, z)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$. Let $m \in \mathbb{N}$ and $\{[a_k, b_k]\}_{k=1}^m$ be a nonordered collection of non-overlapping subintervals of I. By the inequality (2.3) from [1] and the translation invariance of d, for any $i \in \{1, \dots, m\}$, we have (cf. the estimate $C_i \le A_i + B_i$ from [1, Section 2.15]):

$$\begin{aligned}\frac{d(x(b_i) + y(a_i), y(b_i) + x(a_i))}{\lambda + \mu} &\le \frac{\lambda}{\lambda + \mu} \cdot \frac{d(x(b_i) + z(a_i), z(b_i) + x(a_i))}{\lambda} + \frac{\mu}{\lambda + \mu} \cdot \frac{d(y(b_i) + z(a_i), z(b_i) + y(a_i))}{\mu} \\ &\equiv \frac{\lambda}{\lambda + \mu} \cdot A_i + \frac{\mu}{\lambda + \mu} \cdot B_i, \end{aligned} \tag{5.4}$$

whence, the monotonicity of φ_i implies

$$\begin{aligned}&\varphi_i\Big(\frac{1}{\lambda + \mu} d\big(x(b_i) + y(a_i), y(b_i) + x(a_i)\big)\Big) \le \varphi_i\Big(\frac{\lambda}{\lambda + \mu} \cdot A_i + \frac{\mu}{\lambda + \mu} \cdot B_i\Big) \\ &\quad \le \max\{\varphi_i(A_i), \varphi_i(B_i)\} \le \varphi_i(A_i) + \varphi_i(B_i) \\ &\quad = \varphi_i\Big(\frac{1}{\lambda} d\big(x(b_i) + z(a_i), z(b_i) + x(a_i)\big)\Big) + \varphi_i\Big(\frac{1}{\mu} d\big(y(b_i) + z(a_i), z(b_i) + y(a_i)\big)\Big). \end{aligned} \tag{5.5}$$

Summing over $i = 1, \dots, m$ and taking the supremum over all collections of subintervals as above, we arrive at the desired inequality.

Note that, for any $x, y \in X$, the function

$$\lambda \mapsto w_\lambda(x, y) \text{ is continuous from the right on } (0, \infty); \tag{5.6}$$

this is established as the corresponding fact from [1, Section 2.15]. Moreover, the pseudomodular w on X is translation invariant in the sense of [1, equality (3.7)], and if the quadruple $(M, d, +, \cdot)$ is an abstract convex cone, then w is in addition homogeneous in the sense of [1, equality (3.8)].

We set $\mathrm{BV}_\Phi(I; M) = X_w$, where $X_w = X_w^\circ(0)$ is the modular space from [1, Section 2.5] and $0 \in X = M^I$. By virtue of Theorem 3.14(a) from [1], the pair $(\mathrm{BV}_\Phi(I; M), +)$ is an Abelian semigroup with pseudometric d_w° from [1, Theorem 2.6], which is translation invariant. Let us show that

$$\mathrm{d}_M^\circ(x, y) = d(x(a), y(a)) + d_w^\circ(x, y), \quad x, y \in \mathrm{BV}_\Phi(I; M),$$

is a metric on $\mathrm{BV}_\Phi(I; M)$; it suffices to check only that if $\mathrm{d}_M^\circ(x, y) = 0$, then $x = y$. In fact, by the definition of $d_w^\circ(x, y)$, for any number $\lambda > d_w^\circ(x, y)$ and all $t, s \in I$, setting $\delta(t, s) = d(x(t) + y(s), y(t) + x(s))$ we have:

$$\varphi_1\big(\delta(t, s)/\lambda\big) \le w_\lambda(x, y) \le \lambda. \tag{5.7}$$

Let $\varphi_{1+}^{-1}(v) = \max\{u \in \mathbb{R}^+ : \varphi_1(u) \le v\} = \max \varphi_1^{-1}(\{v\}) \in \mathbb{R}^+$, $v \in \mathbb{R}^+$, be the right inverse function for φ_1; in particular, φ_{1+}^{-1} is nondecreasing, vanishes at zero only, and the following (in)equalities hold: $u \le \varphi_{1+}^{-1}(\varphi_1(u))$ and $\varphi_1(\varphi_{1+}^{-1}(v)) = v$ for all $u, v \in \mathbb{R}^+$. Applying the former inequality to (5.7), we find

$$\delta(t, s)/\lambda \le \varphi_{1+}^{-1}(\varphi_1(\delta(t, s)/\lambda)) \le \varphi_{1+}^{-1}(\lambda)$$

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or $\delta(t,s) \le \lambda\varphi_{1+}^{-1}(\lambda)$ for all $\lambda > d_w^\circ(x,y)$, whence

$$d(x(t)+y(s), y(t)+x(s)) = \delta(t,s) \le d_w^\circ(x,y)\cdot\varphi_{1+}^{-1}\big(d_w^\circ(x,y)\big).$$

Taking into account (5.3), it follows that

$$d(x(t),y(t)) \le d(x(a),y(a)) + d_w^\circ(x,y)\cdot\varphi_{1+}^{-1}\big(d_w^\circ(x,y)\big), \quad t\in I.$$

In what follows the sequence Φ is assumed to be *convex*.

5.3. *The modular semigroup and cone* $\mathrm{BV}_\Phi(I;M)$ *with convex* Φ

If, under the assumptions of Section 5.2, the sequence Φ is convex, then, as it seen from (5.5), w is a convex metric pseudomodular on X, and so, by the equality (3.2) and Theorem3.14(a) from [1], $\mathrm{BV}_\Phi(I;M) = X_w^*$ is an Abelian semigroup with pseudometric d_w^* from [1, Theorem 3.6], which is translation invariant. Put

$$\mathrm{d}_M(x,y) = d(x(a),y(a)) + d_w^*(x,y), \quad x,y\in \mathrm{BV}_\Phi(I;M). \tag{5.8}$$

Then the triple $(\mathrm{BV}_\Phi(I;M), \mathrm{d}_M, +)$ is a metric semigroup with zero 0. It suffices to verify that if $\mathrm{d}_M(x,y)=0$, then $x=y$ in $\mathrm{BV}_\Phi(I;M)$: in fact, by the definition of $d_w^*(x,y)$, given $\lambda > d_w^*(x,y)$ and $t, s\in I$, we have

$$\varphi_1\Big(\frac{1}{\lambda}d(x(t)+y(s), y(t)+x(s))\Big) \le w_\lambda(x,y) \le 1,$$

implying $d(x(t)+y(s), y(t)+x(s)) \le \varphi_1^{-1}(1)\lambda$, and so,

$$d(x(t)+y(s), y(t)+x(s)) \le \varphi_1^{-1}(1) d_w^*(x,y). \tag{5.9}$$

It follows from (5.3) that

$$d(x(t),y(t)) \le d(x(a),y(a)) + \varphi_1^{-1}(1)d_w^*(x,y) \tag{5.10}$$

$$\le \max\{1,\varphi_1^{-1}(1)\}\mathrm{d}_M(x,y), \quad t\in I. \tag{5.11}$$

If Φ is convex and $(M,d,+,\cdot)$ is an abstract convex cone, then, by virtue of Theorem 3.14 from [1], we find that the quadruple $(\mathrm{BV}_\Phi(I;M), d_w^*, +, \cdot)$ is an abstract convex pseudocone and the quadruple $(\mathrm{BV}_\Phi(I;M), \mathrm{d}_M, +, \cdot)$ is an abstract convex cone.

In the sequel the space $\mathrm{BV}_\Phi(I;M)$ will be considered for a convex Φ-sequence Φ and equipped with metric (5.8).

The space $\mathrm{BV}_\Phi(I;M)$ for $M=\mathbb{R}$ was initially defined in [2], and so, in the general case it is called the *space of functions of Φ-bounded variation in the sense of Schramm*. The *generalized Φ-variation* $V_\Phi(x)$ of a function $x\in\mathrm{BV}_\Phi(I;M)$ is the quantity $V_\Phi(x) = d_w^*(x,0)$.

Lemma 5.4. *If* (M,d) *is complete, then* $(\mathrm{BV}_\Phi(I;M), \mathrm{d}_M)$ *is complete as well.*

Proof. Let $\{x_j\}\subset \mathrm{BV}_\Phi(I;M)$ be a Cauchy sequence, i.e.,

$$\mathrm{d}_M(x_j,x_k) = d(x_j(a),x_k(a)) + d_w^*(x_j,x_k) \to 0 \quad \text{as } j,\ k\to\infty.$$

Then (5.11) implies that $\{x_j(t)\}$ is a Cauchy sequence in M for all $t\in I$, and so, by the completeness of M, there exists a function $x\in M^I$ such that $d(x_j(t),x(t))\to 0$ as $j\to\infty$ for all $t\in I$. It follows that

$$d_w^*(x_j,x) \le \liminf_{k\to\infty} d_w^*(x_j,x_k) \le \lim_{k\to\infty}\mathrm{d}_M(x_j,x_k) < \infty, \quad j\in\mathbb{N}, \tag{5.12}$$

where the first inequality will be established below (cf. (5.14)). Since $\{x_j\}$ is a Cauchy sequence, then it follows from (5.12) that

$$\limsup_{j\to\infty} d_w^*(x_j,x) \le \lim_{j\to\infty}\lim_{k\to\infty}\mathrm{d}_M(x_j,x_k) = 0,$$

and so, $\mathrm{d}_M(x_j,x)\to 0$ as $j\to\infty$. By the triangle inequality,

$$|d_w^*(x_j,0) - d_w^*(x_k,0)| \le d_w^*(x_j,x_k)\to 0 \quad \text{as } j,\ k\to\infty,$$

and so, $\{d_w^*(x_j,0)\}$ is a Cauchy sequence in $\mathbb{R}$. Then it is bounded and convergent, and it remains to note that $x\in\mathrm{BV}_\Phi(I;M)$: this follows from (cf. (5.14))

$$d_w^*(x,0) \le \liminf_{j\to\infty} d_w^*(x_j,0) = \lim_{j\to\infty} d_w^*(x_j,0) < \infty.$$

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In order to prove the first inequality in (5.12), we show that if $\{x_k\}, \{y_k\} \subset \mathrm{BV}_\Phi(I; M)$, $x, y \in M^I$, $d(x_k(t), x(t)) \to 0$ and $d(y_k(t), y(t)) \to 0$ as $k \to \infty$ for all $t \in I$, then

$$w_\lambda(x, y) \le \liminf_{k\to\infty} w_\lambda(x_k, y_k) \quad \text{for all } \lambda > 0 \tag{5.13}$$

and

$$d_w^*(x, y) \le \liminf_{k\to\infty} d_w^*(x_k, y_k). \tag{5.14}$$

First, we establish (5.13). From the pointwise convergence of x_k to x and y_k to y and property 2.5 from [1] we find

$$d(x_k(t) + y_k(s), y_k(t) + x_k(s)) \to d(x(t) + y(s), y(t) + x(s)) \quad \text{as } k \to \infty$$

for all $t, s \in I$. The definition of w implies that if $m \in \mathbb{N}$ and $\{[a_i, b_i]\}_{i=1}^m$ is a non-ordered collection of non-overlapping intervals in I, then, given $k \in \mathbb{N}$ and $\lambda > 0$, we have:

$$\sum_{i=1}^m \varphi_i\Big(\frac{1}{\lambda} d\big(x_k(b_i) + y_k(a_i), y_k(b_i) + x_k(a_i)\big)\Big) \le w_\lambda(x_k, y_k).$$

Inequality (5.13) follows if we pass to the inferior limit as $k \to \infty$, take into account the continuity of each function φ_i, and then in the resulting inequality take the supremum over all $m \in \mathbb{N}$ and intervals $\{[a_i, b_i]\}_{i=1}^m$ as above.

In order to prove inequality (5.14), it suffices to assume that the quantity $\lambda = \liminf_{k\to\infty} d_w^*(x_k, y_k)$ is finite. Then $d_w^*(x_{k_j}, y_{k_j}) \to \lambda$ as $j \to \infty$ over some subsequence $\{k_j\}_{j=1}^\infty$ of $\{k\}_{k=1}^\infty$. It follows that, given $\varepsilon > 0$, there exists a number $j_0 = j_0(\varepsilon)$ such that $d_w^*(x_{k_j}, y_{k_j}) < \lambda + \varepsilon$ for all $j \ge j_0$. Then the definition of d_w^* implies $w_{\lambda+\varepsilon}(x_{k_j}, y_{k_j}) \le 1$ for all $j \ge j_0$. Since x_{k_j} and y_{k_j} converge pointwise on I to x and y as $j \to \infty$, respectively, then, by (5.13), we find

$$w_{\lambda+\varepsilon}(x, y) \le \liminf_{j\to\infty} w_{\lambda+\varepsilon}(x_{k_j}, y_{k_j}) \le 1,$$

and this means that $d_w^*(x, y) \le \lambda + \varepsilon$ for all $\varepsilon > 0$, and so, (5.14) follows. □

Note that Lemma 5.4 holds also for the space $\mathrm{BV}_\Phi(I; M)$ with metric d_M° when Φ and w are not necessarily convex.

5.5. Spaces of Lipschitzian and additive operators

(a) Let (N, d) and (M, d) be two metric spaces (with different metrics d, in general). Recall that an operator $T : N \to M$ is said to be *Lipschitzian* (or *Lipschitz continuous*), which is written as $T \in \mathrm{Lip}(N; M)$, if its (*least*) *Lipschitz constant* given by

$$\mathcal{L}(T) = \sup\{d(Ty, Tz)/d(y, z) : y, z \in N,\ y \ne z\}$$

is finite.

(b) If $(M, d, +)$ is a metric semigroup (or an abstract convex cone), then the set $\mathrm{Lip}(N; M)$ is an Abelian semigroup with respect to the pointwise addition operation $(T + S)y = Ty + Sy$ for all $y \in N$, because, by virtue of (2.4) from [1], we have $\mathcal{L}(T + S) \le \mathcal{L}(T) + \mathcal{L}(S)$ (in $\mathrm{Lip}(N; M)$ the pointwise operation of multiplication by numbers $\alpha \in \mathbb{R}^+$ is defined as $(\alpha T)y = \alpha(Ty)$, $y \in N$, so that $\mathcal{L}(\alpha T) = \alpha\mathcal{L}(T)$, respectively), where $T, S \in \mathrm{Lip}(N; M)$. In this case, given $\lambda > 0$ and $T, S \in \mathrm{Lip}(N; M)$, we set

$$W_\lambda(T, S) = \frac{1}{\lambda} \sup\left\{\frac{d(Ty + Sz, Sy + Tz)}{d(y, z)} : y, z \in N,\ y \ne z\right\}$$

and

$$D(T, S) \equiv W_1(T, S) = \lambda W_\lambda(T, S).$$

Note that, by virtue of (2.3) and (2.4) from [1], we have:

$$|\mathcal{L}(T) - \mathcal{L}(S)| \le D(T, S) \le \mathcal{L}(T) + \mathcal{L}(S), \tag{5.15}$$

$$|d(Ty, Sy) - D(Tz, Sz)| \le d(Ty + Sz, Sy + Tz) \le D(T, S)d(y, z), \quad y, z \in N. \tag{5.16}$$

The function W is a convex metric pseudomodular on $\mathrm{Lip}(N; M)$; the convexity of W follows from the inequality similar to (5.4): if $R \in \mathrm{Lip}(N; M)$, then

$$d(Ty + Sz, Sy + Tz) \le d(Ty + Rz, Ry + Tz) + d(Sy + Rz, Ry + Sz), \quad y, z \in N.$$

Also, W satisfies an analogue of condition (3.7) from [1] (and condition (3.8) from [1] if M is an abstract convex cone). So, by [1, Theorem 3.14] and (5.15), the space

$$\big(\mathrm{Lip}(N; M)\big)_W^*(C) = \mathrm{Lip}(N; M),$$

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where $C : N \to M$ is an arbitrary constant operator (zero in the sequel), is a pseudometric semigroup (or an abstract convex pseudocone if M is an abstract convex cone, respectively) with pseudometric

$$d^*_W(T, S) = D(T, S), \quad T,\ S \in \mathrm{Lip}(N; M),$$

and $\mathcal{L}(T) = D(T, C)$. The definition of pseudometric D on the space of Lipschitzian operators $\mathrm{Lip}(N; M)$ was proposed in [21].

The following counterpart of inequality (5.14) holds: if $\{T_k\}, \{S_k\} \subset \mathrm{Lip}(N; M)$ and operators $T,\ S : N \to M$ are such that $d(T_k y, Ty) \to 0$ and $d(S_k y, Sy) \to 0$ as $k \to \infty$ for all $y \in N$, then

$$D(T, S) \le \liminf_{k\to\infty} D(T_k, S_k). \tag{5.17}$$

(c) Let $(N, d, +)$ be a metric semigroup and $(M, d, +)$ be a metric semigroup (or an abstract convex cone). Denote by

$$\mathrm{Add}(N; M) = \{T : N \to M \mid T(y + z) = Ty + Tz\ M \text{ for all } y,\ z \in N\}$$

the Abelian semigroup of all additive operators mapping N into M and equipped with the pointwise operations of addition (and multiplication by nonnegative numbers, respectively).

(d) Now let $(N, d, +)$ and $(M, d, +)$ be two metric semigroups with (different, in general) zeros 0. Note that if $T \in \mathrm{Add}(N; M)$, then $T(0) = 0$: in fact, $T(0) = T(0 + 0) = T(0) + T(0)$, and so, $d(0, T(0)) = d(T(0), T(0) + T(0)) = 0$. The zero in $\mathrm{Add}(N; M)$ and $\mathrm{Lip}(N; M)$ is the operator $0 : N \to M$ such that $0y = 0 \in M$ for all $y \in N$.

In what follows we will need the following two Abelian semigroups with zeros:

$$\mathrm{Lip}_0(N; M) = \{T \in \mathrm{Lip}(N; M) : T(0) = 0\},$$
$$\mathrm{L}(N; M) = \mathrm{Lip}(N; M) \cap \mathrm{Add}(N; M).$$

Clearly, $\mathrm{L}(N; M) \subset \mathrm{Lip}_0(N; M) \subset \mathrm{Lip}(N; M)$. Moreover, if $T,\ S \in \mathrm{Lip}_0(N; M)$ and $y,\ z \in N$, then (5.16) implies

$$d(Ty, Sy) \le D(T, S)d(y, 0), \tag{5.18}$$
$$d(Ty, Tz) \le D(T, 0)d(y, z), \tag{5.19}$$

and so, W is a convex modular on $\mathrm{Lip}_0(N; M)$ and $\mathrm{L}(N; M)$, and D is a translation invariant metric on these spaces (which is homogeneous if M is an abstract convex cone). Moreover, $\mathcal{L}(T) = D(T, 0)$ if $T \in \mathrm{Lip}_0(N; M)$, and if (M, d) is complete, then, by virtue of (5.16) and (5.17), $\mathrm{Lip}_0(N; M)$ and $\mathrm{L}(N; M)$ endowed with metric D are complete spaces as well.

(e) Finally, note that if $(N, d, +, \cdot)$ and $(M, d, +, \cdot)$ are two abstract convex cones, then any additive continuous operator $T : N \to M$ also have the following property: $T(\alpha y) = \alpha Ty$ for all $\alpha \in \mathbb{R}^+$ and $y \in N$ (cf., e.g., the text preceding Section 3 in [27]).

5.6. *The spaces* $\mathrm{BV}_\Phi(I; \mathrm{Lip}_0(N; M))$ *and* $\mathrm{BV}_\Phi(I; \mathrm{L}(N; M))$

Suppose that $(N, d, +)$ and $(M, d, +)$ are two metric semigroups with zeros (or M is an abstract convex cone) and Φ is a convex Φ-sequence. Replacing M by $\mathrm{Lip}_0(N; M)$ or $\mathrm{L}(N; M)$ in Section 5.3 and taking into account Section 5.5(d) we find that on the metric semigroup (abstract convex cone, respectively) $\mathrm{BV}_\Phi(I; \mathrm{Lip}_0(N; M))$ or $\mathrm{BV}_\Phi(I; \mathrm{L}(N; M))$ the following translation invariant (and homogeneous if M is an abstract convex cone) metric is well defined (cf. (5.8)):

$$\mathbb{D}_{N,M}(x, y) = D(x(a), y(a)) + D^*_w(x, y), \quad x,\ y \in \mathrm{BV}_\Phi(I; \mathrm{Lip}_0(N; M)),$$

where $D^*_w(x, y) = \inf\{\lambda > 0 : w^D_\lambda(x, y) \le 1\}$ and

$$w^D_\lambda(x, y) = \sup \sum_{i=1}^{m} \varphi_i\Big(\frac{1}{\lambda} D\big(x(b_i) + y(a_i), y(b_i) + x(a_i)\big)\Big),$$

the supremum being taken over the same collection $\{[a_i, b_i]\}_{i=1}^m$ as in Section 5.2. In particular, we have analogues of inequalities (5.9) and (5.10), where d is replaced by D.

6. Superposition operators on spaces BV_Φ

6.1. *Nonautonomous superposition operator* $\mathcal{H}$

Let I, M and N be nonempty sets and M^I be the set of all functions mapping I into M. Given a function $h : I \times N \to M$, the operator $\mathcal{H} : N^I \to M^I$, defined by $(\mathcal{H}y)(t) = h(t, y(t))$ for all $t \in I$ and $y \in N^I$, is said to be the (*nonautonomous*) *superposition* (*Nemytskii*) *operator* with the *generator* h.

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6.2. Assumptions

Throughout Section 6 we suppose that $I = [a, b]$ is a closed interval in $\mathbb{R}$, $a < b$, $(N, d, +)$ and $(M, d, +)$ are two metric semigroups with zeros and $\Phi = \{\varphi_i\}_{i=1}^{\infty}$ is a convex Φ-sequence.

Theorem 6.3. *If $x \in \mathrm{BV}_\Phi(I; \mathrm{Lip}_0(N; M))$ and $y \in \mathrm{BV}_\Phi(I; N)$, then the function $xy : I \to M$, given by $(xy)(t) = x(t)y(t)$ for all $t \in I$, belongs to the metric semigroup $\mathrm{BV}_\Phi(I; M)$, and the following inequality holds:*

$$\mathrm{d}_M(xy, 0) \le \gamma(\Phi)\mathbb{D}_{N,M}(x, 0)\mathrm{d}_N(y, 0), \quad \text{where } \gamma(\Phi) = \max\{1, 2\varphi_1^{-1}(1)\}. \tag{6.1}$$

Proof. By definition (5.8), we have:

$$\mathrm{d}_M(xy, 0) = d((xy)(a), 0) + d_w^*(xy, 0).$$

Since $x(a) \in \mathrm{Lip}_0(N; M)$, by virtue of (5.19), we find

$$d((xy)(a), 0) = d(x(a)y(a), x(a)(0)) \le D(x(a), 0)d(y(a), 0) = A_0B_0, \tag{6.2}$$

where we have set $A_0 = D(x(a), 0)$, $B_0 = d(y(a), 0)$. It remains to show that

$$d_w^*(xy, 0) \le A\lambda + \mu B, \tag{6.3}$$

where $\lambda = d_w^*(y, 0)$, $\mu = D_w^*(x, 0)$ and, taking into account (5.10),

$$B = \sup_{t\in I} d(y(t), 0) \le d(y(a), 0) + \varphi_1^{-1}(1)d_w^*(y, 0) = B_0 + \varphi_1^{-1}(1)\lambda \tag{6.4}$$

and (similarly, replacing d by D in (5.10))

$$A = \sup_{t\in I} D(x(t), 0) \le D(x(a), 0) + \varphi_1^{-1}(1)D_w^*(x, 0) = A_0 + \varphi_1^{-1}(1)\mu. \tag{6.5}$$

In fact, it follows from (5.8) that $\mathrm{d}_N(y, 0) = B_0 + \lambda$ and $\mathbb{D}_{N,M}(x, 0) = A_0 + \mu$, and so, (6.2)–(6.5) imply:

$$\begin{aligned} \mathrm{d}_M(xy, 0) &\le A_0B_0 + A\lambda + \mu B \\ &\le A_0B_0 + (A_0 + \varphi_1^{-1}(1)\mu)\lambda + \mu(B_0 + \varphi_1^{-1}(1)\lambda) \\ &\le \gamma(\Phi)(A_0 + \mu)(B_0 + \lambda), \end{aligned} \tag{6.6}$$

and the inequality (6.1) follows.

Let us establish (6.3). Given $t, s \in I$, by virtue of (5.19) and (5.18), we have:

$$\begin{aligned} d((xy)(t), (xy)(s)) &\le d(x(t)y(t), x(t)y(s)) + d(x(t)y(s), x(s)y(s)) \\ &\le D(x(t), 0)d(y(t), y(s)) + D(x(t), x(s))d(y(s), 0) \\ &\le Ad(y(t), y(s)) + D(x(t), x(s))B. \end{aligned} \tag{6.7}$$

First, suppose that $AB \ne 0$ and $\lambda\mu \ne 0$. Applying (6.7) and the convexity of functions φ_i, for any collection of non-overlapping intervals $\{[a_i, b_i]\}_{i=1}^m$ from I, we find

$$\begin{aligned} \sum_{i=1}^m \varphi_i\Big(\frac{1}{A\lambda + \mu B} d\big((xy)(b_i), (xy)(a_i)\big)\Big) &\le \sum_{i=1}^m \varphi_i\Big(\frac{A\lambda}{A\lambda + \mu B} \cdot \frac{d(y(b_i), y(a_i))}{\lambda} + \frac{\mu B}{A\lambda + \mu B} \cdot \frac{D(x(b_i), x(a_i))}{\mu}\Big) \\ &\le \frac{A\lambda}{A\lambda + \mu B} \sum_{i=1}^m \varphi_i\Big(\frac{1}{\lambda} d(y(b_i), y(a_i))\Big) + \frac{\mu B}{A\lambda + \mu B} \sum_{i=1}^m \varphi_i\Big(\frac{1}{\mu} D(x(b_i), x(a_i))\Big) \\ &\le \frac{A\lambda}{A\lambda + \mu B} w_\lambda^d(y, 0) + \frac{\mu B}{A\lambda + \mu B} w_\mu^D(x, 0), \end{aligned} \tag{6.8}$$

whence

$$w_{A\lambda+\mu B}^d(xy, 0) \le \frac{A\lambda}{A\lambda + \mu B} w_\lambda^d(y, 0) + \frac{\mu B}{A\lambda + \mu B} w_\mu^D(x, 0). \tag{6.9}$$

Taking into account the continuity from the right (5.6) of functions $\xi \mapsto w_\xi^d(y, 0)$ and $\eta \mapsto w_\eta^D(x, 0)$ on $(0, \infty)$ and applying [1, Theorem 3.8(c)], we get $w_\lambda^d(y, 0) \le 1$ and $w_\mu^D(x, 0) \le 1$. Then (6.9) implies $w_{A\lambda+\mu B}^d(xy, 0) \le 1$, and so, we obtain the inclusion $xy \in \mathrm{BV}_\Phi(I; M)$ and the inequality (6.3), which are consequences of the definition of the space $\mathrm{BV}_\Phi(I; M) = X_w^*(0)$ from Section 5.3 and metric d_w^* from [1, Theorem 3.6].

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Suppose that $AB = 0$. Since D is a metric on $\mathrm{Lip}_0(N; M)$ and d is a metric on N, then $x = 0$ in $(\mathrm{Lip}_0(N; M))^I$ or $y = 0$ in N^I, and so, the left and right hand sides of inequalities (6.7) and (6.3) are equal to zero. Now let $AB \neq 0$. If $\lambda = 0$, then, by (5.9), the function $y : I \to N$ is constant, and so, we find from (6.7) that

$$d((xy)(t), (xy)(s)) \le D(x(t), x(s))B, \quad t, s \in I. \tag{6.10}$$

If $\mu = 0$, then (by similar arguments) the function $x : I \to \mathrm{Lip}_0(N; M)$ is constant, and so, by (6.10), the function $xy : I \to M$ is also constant. Therefore, $w^d_\xi(xy, 0) = 0$ for all $\xi > 0$, whence $d^*_w(xy, 0) = 0$. Now if $\mu \neq 0$, then, by virtue of (6.10), instead of inequality (6.8) we get:

$$\sum_{i=1}^m \varphi_i\Big(\frac{1}{\mu B} d\big((xy)(b_i), (xy)(a_i)\big)\Big) \le \sum_{i=1}^m \varphi_i\Big(\frac{1}{\mu} D(x(b_i), x(a_i))\Big) \le w^D_\mu(x, 0),$$

implying $w^d_{\mu B}(xy, 0) \le w^D_\mu(x, 0) \le 1$, the last inequality being obtained, as above, from (5.6) and [1, Theorem 3.8(c)]. It follows that $d^*_w(xy, 0) \le \mu B$, and (6.3) follows. Finally, supposing that $\lambda \neq 0$ and $\mu = 0$ and arguing as above we find that the function x is constant, (6.7) implies the inequality

$$d((xy)(t), (xy)(s)) \le Ad(y(t), y(s)), \quad t, s \in I,$$

from which we get $w^d_{A\lambda}(xy, 0) \le w^d_\lambda(y, 0) \le 1$, and so, $d^*_w(xy, 0) \le A\lambda$. $\square$

Remark 6.4. Clearly, Theorem 6.3 remains valid if we replace the condition $x \in \mathrm{BV}_\Phi(I; \mathrm{Lip}_0(N; M))$ by a more strict condition $x \in \mathrm{BV}_\Phi(I; \mathrm{L}(N; M))$. In this case if $N = M = \mathbb{R}$, then Theorem 6.3 and Lemma 5.4 show that $\mathrm{BV}_\Phi(I; \mathbb{R})$ is a Banach algebra. Thus, Theorem 6.3 refines and generalizes the results from [10,18,38,31,39,32,40,2,23].

Theorem 6.5. *If, under the assumptions 6.2, $x \in \mathrm{BV}_\Phi(I; \mathrm{L}(N; M))$, $h_0 \in \mathrm{BV}_\Phi(I; M)$ and the generator $h : I \times N \to M$ of the superposition operator $\mathcal{H} : N^I \to M^I$ is of the form $h(t, y) = x(t)y + h_0(t)$ for all $t \in I$ and $y \in N$, then $\mathcal{H}$ maps $\mathrm{BV}_\Phi(I; N)$ into $\mathrm{BV}_\Phi(I; M)$ and is Lipschitzian (and also additive if $h_0 = 0$) and $\mathcal{L}(\mathcal{H}) \le \gamma(\Phi)\mathbb{D}_{N,M}(x, 0)$.*

Proof. First, we show that $\mathcal{H}$ maps from $\mathrm{BV}_\Phi(I; N)$ into $\mathrm{BV}_\Phi(I; M)$. Let $y \in \mathrm{BV}_\Phi(I; N)$. Then $(\mathcal{H}y)(t) = h(t, y(t)) = x(t)y(t) + h_0(t)$, $t \in I$, i.e., $\mathcal{H}y = xy + h_0$. By Theorem 6.3, $xy \in \mathrm{BV}_\Phi(I; M)$, and so, $\mathcal{H}y \in \mathrm{BV}_\Phi(I; M)$ and, by virtue of the inequality (2.4) from [1], $d^*_w(\mathcal{H}y, 0) \le d^*_w(xy, 0) + d^*_w(h_0, 0)$ and $\mathrm{d}_M(\mathcal{H}y, 0) \le \mathrm{d}_M(xy, 0) + \mathrm{d}_M(h_0, 0)$.

Now, let us prove that $\mathcal{H}$ is Lipschitzian. Let $y, z \in \mathrm{BV}_\Phi(I; N)$. In view of (5.8) and translation invariance of d_M, we have to estimate the quantity:

$$\mathrm{d}_M(\mathcal{H}y, \mathcal{H}z) = \mathrm{d}_M(xy, xz) = d((xy)(a), (xz)(a)) + d^*_w(xy, xz).$$

By (5.19), the first term is estimated as

$$d((xy)(a), (xz)(a)) = d(x(a)y(a), x(a)z(a)) \le D(x(a), 0)d(y(a), z(a)) = A_0B_0$$

with $A_0 = D(x(a), 0)$ and $B_0 = d(y(a), z(a))$. It remains to show that

$$d^*_w(xy, xz) \le A\lambda + \mu B, \tag{6.11}$$

where $\lambda = d^*_w(y, z)$, $\mu = D^*_w(x, 0)$, $A = \sup_{t\in I} D(x(t), 0)$ is defined and estimated in (6.5) and, by virtue of (5.10),

$$B = \sup_{t\in I} d(y(t), z(t)) \le d(y(a), z(a)) + \varphi_1^{-1}(1)d^*_w(y, z) = B_0 + \varphi_1^{-1}(1)\lambda.$$

In fact, in accordance with (5.8) we have $\mathrm{d}_N(y, z) = B_0 + \lambda$ and $\mathbb{D}_{N,M}(x, 0) = A_0 + \mu$, and arguing as in (6.6), we arrive at the estimate for $\mathcal{L}(\mathcal{H})$ from Theorem 6.5, because

$$\mathrm{d}_M(\mathcal{H}y, \mathcal{H}z) \le A_0B_0 + A\lambda + \mu B \le \gamma(\Phi)(A_0 + \mu)(B_0 + \lambda).$$

Now we establish (6.11). Given $t, s \in I$, by the additivity property of operators $x(t)$, we have the equality:

$$\begin{aligned}&[(xy)(t) + (xz)(s)] + [x(t)(z(t) + y(s))] + [x(s)y(s) + x(t)z(s)]\\ &\quad = [(xz)(t) + (xy)(s)] + [x(t)(y(t) + z(s))] + [x(t)y(s) + x(s)z(s)].\end{aligned}$$

Denote by ℓ_k (by r_k) the k-th term in the square bracket on the left (right, respectively) hand side of this equality, $k = 1, 2, 3$, so that $\ell_1 + \ell_2 + \ell_3 = r_1 + r_2 + r_3$. Then it follows from [1, inequality (2.4)] that

$$\begin{aligned}d(\ell_1, r_1) &= d(\ell_1 + \ell_2 + \ell_3, r_1 + \ell_2 + \ell_3) = d(r_1 + r_2 + r_3, r_1 + \ell_2 + \ell_3)\\ &= d(r_2 + r_3, \ell_2 + \ell_3) \le d(r_2, \ell_2) + d(r_3, \ell_3),\end{aligned} \tag{6.12}$$

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which, by virtue of (5.19) and (5.16), can be rewritten as

$$\begin{aligned} d((xy)(t)+(xz)(s),(xz)(t)+(xy)(s)) &\le d\big(x(t)(y(t)+z(s)),x(t)(z(t)+y(s))\big)\\ &\quad +d\big(x(t)y(s)+x(s)z(s),x(s)y(s)+x(t)z(s)\big)\\ &\le D(x(t),0)d(y(t)+z(s),z(t)+y(s))+D(x(t),x(s))d(y(s),z(s))\\ &\le Ad(y(t)+z(s),z(t)+y(s))+D(x(t),x(s))B. \end{aligned} \tag{6.13}$$

Suppose that $AB \neq 0$ and $\lambda\mu \neq 0$. It follows from (6.13) and the convexity of φ_i that, given a collection of non-overlapping intervals $\{[a_i,b_i]\}_{i=1}^m$ from I,

$$\begin{aligned} &\sum_{i=1}^m \varphi_i\Big(\frac{1}{A\lambda+\mu B}d\big((xy)(b_i)+(xz)(a_i),(xz)(b_i)+(xy)(a_i)\big)\Big)\\ &\quad\le \frac{A\lambda}{A\lambda+\mu B}\sum_{i=1}^m\varphi_i\Big(\frac{1}{\lambda}d(y(b_i)+z(a_i),z(b_i)+y(a_i))\Big)+\frac{\mu B}{A\lambda+\mu B}\sum_{i=1}^m\varphi_i\Big(\frac{1}{\mu}D(x(b_i),x(a_i))\Big)\\ &\quad\le \frac{A\lambda}{A\lambda+\mu B}\,w_\lambda^d(y,z)+\frac{\mu B}{A\lambda+\mu B}\,w_\mu^D(x,0), \end{aligned}$$

whence

$$w_{A\lambda+\mu B}^d(xy,xz) \le \frac{A\lambda}{A\lambda+\mu B}\,w_\lambda^d(y,z)+\frac{\mu B}{A\lambda+\mu B}\,w_\mu^D(x,0). \tag{6.14}$$

By (5.6), functions $\xi \mapsto w_\xi^d(y,z)$ and $\eta \mapsto w_\eta^D(x,0)$ are continuous from the right on $(0,\infty)$, and so, by [1, Theorem 3.8(c)], we find $w_\lambda^d(y,z) \le 1$ and $w_\mu^D(x,0) \le 1$. Then the estimate (6.14) implies $w_{A\lambda+\mu B}^d(xy,xz) \le 1$, whence the inequality (6.11) follows.

If $AB = 0$, then, since D is a metric on $\mathrm{L}(N;M)$ and d is a metric on N, then $x = 0$ in $\mathrm{L}(N;M)^I$ or $y = z$ in N^I, and so, the left and right hand sides in (6.13) and (6.11) are equal to zero. Now let $AB \neq 0$. If $\lambda = 0$, then, by virtue of (5.9), $d(y(t)+z(s),z(t)+y(s)) = 0$, and so, (6.13) implies

$$d\big((xy)(t)+(xz)(s),(xz)(t)+(xy)(s)\big) \le D(x(t),x(s))B, \quad t,\ s\in I. \tag{6.15}$$

If $\mu = 0$, then it follows from (5.9), where d is replaced by D, that the function $x : I \to \mathrm{L}(N;M)$ is constant, so that, for all $t,\ s \in I$, the left hand side in (6.15) is equal to zero. Then the definition of the pseudomodular w implies $w_\eta^d(xy,xz) = 0$ for all $\eta > 0$, and so, $d_w^*(xy,xz) = 0$. Now if $\mu \neq 0$, then it follows from (6.15) that $w_{\mu B}^d(xy,xz) \le w_\mu^D(x,0) \le 1$ (instead of (6.14)), and so, $d_w^*(xy,xz) \le \mu B$. Finally, assumimg that $\lambda \neq 0$ and $\mu = 0$ we find that the function x is constant, and inequality (6.13) implies $w_{A\lambda}^d(xy,xz) \le w_\lambda^d(y,z) \le 1$, and so, $d_w^*(xy,xz) \le A\lambda$.

Now if $h_0 = 0$, the additivity of the operator $\mathcal{H}$ follows from the additivity of operators $x(t)$: in fact, if $t \in I$ and $y,\ z \in \mathrm{BV}_\Phi(I;N)$, then

$$\begin{aligned} \mathcal{H}(y+z)(t) &= x(t)(y+z)(t) = x(t)(y(t)+z(t)) = x(t)y(t)+x(t)z(t)\\ &= (\mathcal{H}y)(t)+(\mathcal{H}z)(t) = (\mathcal{H}y+\mathcal{H}z)(t). \quad \square \end{aligned}$$

Remark 6.6. Although the proofs of Theorems 6.3 and 6.5 follow the same universal scheme, generally the former theorem is not a consequence of the latter theorem (formally with $z = 0$): in fact, the assumption in Theorem 6.5 that all the operators $x(t)$ are *additive* is much more strict than condition $x(t) \in \mathrm{Lip}_0(N;M)$ in Theorem 6.3 and, moreover, this assumption is essential for the validity of Theorem 6.5 (cf. Theorem 6.14 and Remark 6.15(c) below). However, in a weaker formulation, when $x \in \mathrm{BV}_\Phi(I;\mathrm{L}(N;M))$, Theorem 6.3 follows from Theorem 6.5 if $z(t) = 0$ in N for all $t \in I$. Theorem 6.5 generalizes the results from [10,18] when the sequence Φ is constant.

6.7. A variant of the superposition operator

Let I, N and M be three nonempty sets and M^N denotes the set of all maps from N into M. Given a function $g : I\times M^N \to M$, define the operator $\mathcal{G} : (M^N)^I \to M^I$ by $(\mathcal{G}x)(t) = g(t,x(t))$ for all $t \in I$ and $x : I \to M^N$. The operator $\mathcal{G}$ is a superposition operator with generator g in the sense of definition 6.1 if we note that the set N in definition 6.1 is replaced here by M^N.

Theorem 6.8. *If, under the assumptions 6.2, $y \in \mathrm{BV}_\Phi(I;N)$, $h_0 \in \mathrm{BV}_\Phi(I;M)$ and the generator $g : I\times M^N \to M$ of the superposition operator $\mathcal{G} : (M^N)^I \to M^I$ is of the form $g(t,x) = xy(t)+h_0(t)$ for all $t \in I$ and $x \in M^N$, then $\mathcal{G}$ maps the metric semigroup $\mathrm{BV}_\Phi(I;\mathrm{Lip}_0(N;M))$ into the metric semigroup $\mathrm{BV}_\Phi(I;M)$ and is Lipschitzian (and also additive if $h_0 = 0$) and $\mathcal{L}(\mathcal{G}) \le \gamma(\Phi)\,d_N(y,0)$.*

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Proof. The scheme of proof is the same as that of Theorem 6.5 (and Theorem 6.3). So, we expose only the main ingradients. Let $x \in \mathrm{BV}_\Phi(I; \mathrm{Lip}_0(N; M))$. Given $t \in I$, we have $(\mathcal{G}x)(t) = x(t)y(t) + h_0(t)$, and so, $\mathcal{G}x = xy + h_0$. By Theorem 6.3, $xy \in \mathrm{BV}_\Phi(I; M)$, and so, $\mathcal{G}x \in \mathrm{BV}_\Phi(I; M)$.

In order to show that $\mathcal{G}$ is Lipschitzian, we fix $x, \bar{x} \in \mathrm{BV}_\Phi(I; \mathrm{Lip}_0(N; M))$. Noting that, by virtue of (5.8),

$$\mathrm{d}_M(\mathcal{G}x, \mathcal{G}\bar{x}) = \mathrm{d}_M(xy, \bar{x}y) = d((xy)(a), (\bar{x}y)(a)) + d_w^*(xy, \bar{x}y)$$

and, by virtue of (5.18),

$$d((xy)(a), (\bar{x}y)(a)) = d(x(a)y(a), \bar{x}(a)y(a)) \le D(x(a), \bar{x}(a))d(y(a), 0) = A_0B_0$$

with $A_0 = D(x(a), \bar{x}(a))$ and $B_0 = d(y(a), 0)$, it remains to show that

$$d_w^*(xy, \bar{x}y) \le A\lambda + \mu B, \tag{6.16}$$

where $\lambda = d_w^*(y, 0)$, $\mu = D_w^*(x, \bar{x})$, $B = \sup_{t\in I} d(y(t), 0)$ is defined and estimated in (6.4) and, by virtue of (5.10) with d replaced by D,

$$A = \sup_{t\in I} D(x(t), \bar{x}(t)) \le D(x(a), \bar{x}(a)) + \varphi_1^{-1}(1)D_w^*(x, \bar{x}) = A_0 + \varphi_1^{-1}(1)\mu.$$

In fact, definition (5.8) implies $\mathrm{d}_N(y, 0) = B_0 + \lambda$ and $\mathbb{D}_{N,M}(x, \bar{x}) = A_0 + \mu$, and so, (6.16) and the estimates for A and B give (as in (6.6)):

$$\mathrm{d}_M(xy, \bar{x}y) \le A_0B_0 + A\lambda + \mu B \le \gamma(\Phi)(A_0 + \mu)(B_0 + \lambda),$$

whence the desired estimate of the Lipschitz constant $\mathcal{L}(\mathcal{G})$ of $\mathcal{G}$ follows.

In order to establish (6.16), note that, given $t, s \in I$, we have:

$$[(xy)(t) + (\bar{x}y)(s)] + [\bar{x}(s)y(t) + x(s)y(s)] + [(\bar{x}(t) + x(s))y(t)]$$
$$= [(\bar{x}y)(t) + (xy)(s)] + [x(s)y(t) + \bar{x}(s)y(s)] + [(x(t) + \bar{x}(s))y(t)].$$

Applying inequality (6.12), the arguments preceding it and inequalities (5.16) and (5.18), we get:

$$\begin{aligned} d((xy)(t) + (\bar{x}y)(s), (\bar{x}y)(t) + (xy)(s)) &\le d\big(x(s)y(t) + \bar{x}(s)y(s), \bar{x}(s)y(t) + x(s)y(s)\big) \\ &\quad + d\big((x(t) + \bar{x}(s))y(t), (\bar{x}(t) + x(s))y(t)\big) \\ &\le D(x(s), \bar{x}(s))d(y(t), y(s)) + D(x(t) + \bar{x}(s), \bar{x}(t) + x(s))d(y(t), 0) \\ &\le Ad(y(t), y(s)) + D(x(t) + \bar{x}(s), \bar{x}(t) + x(s))B. \end{aligned}$$

In the case when $AB \ne 0$ and $\lambda\mu \ne 0$ we deduce the estimate

$$w^d_{A\lambda+\mu B}(xy, \bar{x}y) \le \frac{A\lambda}{A\lambda + \mu B}\, w^d_\lambda(y, 0) + \frac{\mu B}{A\lambda + \mu B}\, w^D_\mu(x, \bar{x}) \le 1$$

(note that $w^d_\lambda(y, 0) \le 1$ and $w^D_\mu(x, \bar{x}) \le 1$), from which (6.16) follows. The cases when $AB = 0$ or $\lambda\mu = 0$ are considered in the same way as in the proofs of Theorems 6.3 and 6.5. The additivity of $\mathcal{G}$ for $h_0 = 0$ follows from the definition of the addition operation in $\mathrm{Lip}_0(N; M)$. □

As it will be shown below, Theorems 6.5 and 6.8 almost completely characterize Lipschitzian superposition operators $\mathcal{H}$ and $\mathcal{G}$ between spaces of the form $\mathrm{BV}_\Phi(I; \cdot)$ (more precisely, see definition 6.11 and Theorems 6.14 and 6.16 below). For this, we need four more lemmas (Lemmas 6.9, 6.10, 6.12 and 6.13).

Lemma 6.9. *If Φ is a convex Φ-sequence and $\zeta : I = [a, b] \to \mathbb{R}$ is a monotone function, then $w_\lambda(\zeta, 0) = \varphi_1(|\zeta(b) - \zeta(a)|/\lambda)$ for all $\lambda > 0$ and $d_w^*(\zeta, 0) = |\zeta(b) - \zeta(a)|/\varphi_1^{-1}(1)$.*

Proof. By the definition of $w_\lambda(\zeta, 0)$, we have: $\varphi_1(|\zeta(b) - \zeta(a)|/\lambda) \le w_\lambda(\zeta, 0)$ for all $\lambda > 0$. Note that the convex function φ_1 is strictly increasing and superadditive: $\varphi_1(u) + \varphi_1(v) \le \varphi_1(u + v)$, $u, v \in \mathbb{R}^+$. It follows that if $\{[a_k, b_k]\}_{k=1}^m$ is a collection of non-overlapping intervals in I such that $a \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m \le b$, then, by virtue of (5.1), for each permutation $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$, we get:

$$\begin{aligned} \sum_{i=1}^m \varphi_{\sigma(i)}\Big(\frac{|\zeta(b_i) - \zeta(a_i)|}{\lambda}\Big) &\le \sum_{i=1}^m \varphi_1\Big(\frac{|\zeta(b_i) - \zeta(a_i)|}{\lambda}\Big) \\ &\le \varphi_1\Big(\frac{1}{\lambda}\sum_{i=1}^m |\zeta(b_i) - \zeta(a_i)|\Big) \le \varphi_1\Big(\frac{|\zeta(b) - \zeta(a)|}{\lambda}\Big), \end{aligned}$$

whence the arbitrariness of intervals $\{[a_k, b_k]\}_{k=1}^m$ implies

$$w_\lambda(\zeta, 0) = \varphi_1(|\zeta(b) - \zeta(a)|/\lambda), \quad \lambda > 0.$$

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If $\zeta(b) = \zeta(a)$, then ζ is constant, $w_\lambda(\zeta, 0) = 0$ for all $\lambda > 0$ and $d_w^*(\zeta, 0) = 0$. Now if $\zeta(b) \neq \zeta(a)$, then $w_\lambda(\zeta, 0) = 1$ if and only if $\lambda = |\zeta(b) - \zeta(a)|/\varphi_1^{-1}(1)$, and so, by [1, Theorem 3.8(b)], $d_w^*(\zeta, 0) = \lambda$. □

Lemma 6.10. *Let $(M, d, +)$ be a complete metric semigroup with zero and $\Phi = \{\varphi_i\}_{i=1}^\infty$ be a convex Φ-sequence. Then, given a function $x \in \mathrm{BV}_\Phi(I; M)$, there exist the limit from the left $x(t-0) = \lim_{s\to t-0} x(s) \in M$ at each point $a < t \le b$ and the limit from the right $x(t+0) = \lim_{s\to t+0} x(s) \in M$ at each point $a \le t < b$, and the set of all points of discontinuity of x on I is at most countable.*

If $M = \mathbb{R}$, this assertion is established in [2, p. 51, 2-nd paragraph]. In the general case it can be established in a straightforward way or follows from more general results in [41, Lemma 3 and example 7 in Section 3], [42, Section 6.1] and [43, Theorem 3].

6.11. *The left regularization*

Let $(M, d, +)$ be a complete metric semigroup with zero and Φ be a convex Φ-sequence. Given $x \in \mathrm{BV}_\Phi(I; M)$, define its *left regularization* $x^- : I \to M$ by

$$x^-(t) = x(t-0) \quad \text{if } a < t \le b, \quad \text{and} \quad x^-(a) = x^-(a+0) = x(a+0),$$

where the limits in the equalities above are calculated in the space M, and so, by Lemma 6.10, the function $x^- : I \to M$ is well defined. Denote by $\mathrm{BV}_\Phi^-(I; M)$ the set of all functions x from $\mathrm{BV}_\Phi(I; M)$, which are continuous from the left on $(a, b]$ (i.e., $x^-(t) = x(t)$ for all $t \in (a, b]$). Under the assumptions above, we have

Lemma 6.12. *If $x \in \mathrm{BV}_\Phi(I; M)$, then $x^- \in \mathrm{BV}_\Phi^-(I; M)$, and the following inequality holds: $d_w^*(x^-, 0) \le d_w^*(x, 0)$.*

Proof. 1. Let us show that the function x^- is continuous from the left at each point $a < t \le b$. By Lemma 6.10, the set of points of continuity of x is dense in $I = [a, b]$, and so, there exists a sequence $\{s_n\}_{n=1}^\infty \subset (a, t)$ of points of continuity of x such that $s_n \to t$ as $n \to \infty$. It follows that

$$\lim_{s\to t-0} x^-(s) = \lim_{n\to\infty} x^-(s_n) = \lim_{n\to\infty} x(s_n) = \lim_{s\to t-0} x(s) = x^-(t) \quad \text{in } M.$$

2. Let us prove that $x^- \in \mathrm{BV}_\Phi(I; M)$ and $d_w^*(x^-, 0) \le d_w^*(x, 0)$. We may suppose that $\lambda = d_w^*(x, 0) > 0$ (otherwise, by virtue of (5.9), the function x is constant). It follows from (5.6) and [1, Theorem 3.8(c)] that $w_\lambda^d(x, 0) \le 1$. Let $\{t_n\}_{n\in Q} \subset (a, b]$ be the set of points, where the function x is discontinuous from the left and Q is an at most countable set. With no loss of generality we assume that $Q = \mathbb{N}$.

2a. Define the function $x_1 : I \to M$ by $x_1(t) = x(t)$ if $t \neq t_1$ and $x_1(t_1) = x^-(t_1)$. Let us show that $x_1 \in \mathrm{BV}_\Phi(I; M)$ and $d_w^*(x_1, 0) \le \lambda$. Fix $\varepsilon > 0$ arbitrarily. Let $m \in \mathbb{N}$, $\{[a_i, b_i]\}_{i=1}^m$ be a collection of non-overlapping intervals from I such that $a \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m \le b$ and $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ be a permutation. Set

$$S_i(x) = \varphi_{\sigma(i)}\left(\frac{1}{\lambda} d(x(b_i), x(a_i))\right) \quad \text{if } i \in \{1, \ldots, m\}, \quad \text{and} \quad S(x) = \sum_{i=1}^m S_i(x).$$

From the definition of $w_\lambda^d(x, 0)$ we find $S(x) \le w_\lambda^d(x, 0) \le 1$. We have to estimate the quantity $S(x_1)$. If $t_1 \notin \{a_i\}_{i=1}^m \cup \{b_i\}_{i=1}^m$, then $S(x_1) = S(x) \le 1$; otherwise, we have either (A) $t_1 = a_j$ for some $j \in \{1, \ldots, m\}$ or (B) $t_1 = b_j$ for some $j \in \{1, \ldots, m\}$. In the case (A) we set $b_0 = a$. We have three possibilities: (A1) $j \ge 1$ and $b_{j-1} \neq a_j$; (A2) $j = 1$ and $b_0 = a_1$; (A3) $j \ge 2$ and $b_{j-1} = a_j$. In the cases (A1) and (A2) we find that $S_i(x_1) = S_i(x)$ for all $i \in \{1, \ldots, m\}$, $i \neq j$, and, by the definition of $x^-(a_j)$ and the continuity of metric d and function $\varphi_{\sigma(j)}$, there exists a point a_j' such that $a_j' \in (b_{j-1}, a_j)$ in the case (A1) or $a_j' \in (b_{j-1}, b_j)$ in the case (A2) and

$$S_j(x_1) = \varphi_{\sigma(j)}\left(\frac{1}{\lambda} d(x(b_j), x^-(a_j))\right) \le \varphi_{\sigma(j)}\left(\frac{1}{\lambda} d(x(b_j), x(a_j'))\right) + \varepsilon. \tag{6.17}$$

Since, under the assumptions (A1) or (A2), the intervals $[a_1, b_1], \ldots, [a_{j-1}, b_{j-1}], [a_j', b_j], [a_{j+1}, b_{j+1}], \ldots, [a_m, b_m]$ are still nonoverlapping, this implies

$$S(x_1) = \sum_{i=1}^{j-1} S_i(x) + S_j(x_1) + \sum_{i=j+1}^m S_i(x) \le w_\lambda^d(x, 0) + \varepsilon \le 1 + \varepsilon.$$

If possibility (A3) holds, then $S_i(x_1) = S_i(x)$ for all $i \in \{1, \ldots, m\}$, $i \neq j-1$, $i \neq j$, and applying the definition of $x^-(a_j)$ and continuity of d, $\varphi_{\sigma(j-1)}$ and $\varphi_{\sigma(j)}$, we find a point $a_j' \in (a_{j-1}, a_j)$ such that (6.17) holds and

$$S_{j-1}(x_1) = \varphi_{\sigma(j-1)}\left(\frac{1}{\lambda} d(x^-(a_j), x(a_{j-1}))\right) \le \varphi_{\sigma(j-1)}\left(\frac{1}{\lambda} d(x(a_j'), x(a_{j-1}))\right) + \varepsilon.$$

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Since $[a_1, b_1], \ldots, [a_{j-2}, b_{j-2}], [a_{j-1}, a'_j], [a'_j, b_j], [a_{j+1}, b_{j+1}], \ldots, [a_m, b_m]$ are non-overlapping intervals, we get:

$$S(x_1) = \sum_{i=1}^{j-2} S_i(x) + S_{j-1}(x_1) + S_j(x_1) + \sum_{i=j+1}^{m} S_i(x) \le w_\lambda^d(x, 0) + 2\varepsilon \le 1 + 2\varepsilon.$$

In the case (B) we also have three possibilities: (B1) $m = 1$; (B2) $m \ge 2$, $1 \le j \le m-1$ and $a_{j+1} \ne b_j$; (B3) $m \ge 2$, $1 \le j \le m-1$ and $a_{j+1} = b_j$, and they are considered similarly. Thus, given a collection of intervals $\{[a_i, b_i]\}_{i=1}^m$ as above, we have $S(x_1) \le 1 + 2\varepsilon$. Taking the supremum over all these collections we find $w_\lambda^d(x_1, 0) \le 1 + 2\varepsilon$. By the arbitrariness of $\varepsilon > 0$, we get $w_\lambda^d(x_1, 0) \le 1$, and so, $d_w^*(x_1, 0) \le \lambda$.

2b. Given $n \in \mathbb{N}$, we define the function $x_n : I \to M$ by $x_n(t) = x(t)$ if $t \in I \setminus \{t_1, \ldots, t_n\}$, and $x_n(t_i) = x^-(t_i)$ for all $i = 1, \ldots, n$, and note that x_n and x_{n-1} are different only at the point $t = t_n$, for which $x_n(t) = x_{n-1}(t)$ if $t \ne t_n$, and $x_n(t_n) = (x_{n-1})^-(t_n)$. The arguments of step 2a imply that $x_n \in \mathrm{BV}_\Phi(I; M)$ and

$$d_w^*(x_n, 0) \le d_w^*(x_{n-1}, 0) \le \cdots \le d_w^*(x_1, 0) \le \lambda \quad \text{for all } n \in \mathbb{N}.$$

Now we define the function $x_\infty : I \to M$ by $x_\infty(t) = x(t)$ if $t \notin \{t_n\}_{n=1}^\infty$, and $x_\infty(t_n) = x^-(t_n)$ for all $n \in \mathbb{N}$. Since x_n converges to x_∞ pointwise on I as $n \to \infty$, applying (5.14) we get:

$$d_w^*(x_\infty, 0) \le \liminf_{n\to\infty} d_w^*(x_n, 0) \le \lambda.$$

2c. Finally, noting that $x^-(t) = x_\infty(t)$ if $t \ne a$, and $x^-(a) = x_\infty(a+0)$, that is, x^- and x_∞ are different only at the point $t = a$, we conclude from step 2a that $d_w^*(x^-, 0) \le d_w^*(x_\infty, 0) \le \lambda$. □

Lemma 6.13 *([22, Theorem 1 and Corollary 2]). Let $(N, +)$ be an Abelian semigroup with zero and division by 2 and $(M, d, +, \cdot)$ be a complete abstract convex cone. Then the operator $T : N \to M$ satisfies the Jensen functional equation*

$$2\,T\Big(\frac{y+z}{2}\Big) = Ty + Tz \text{ in } M \quad \text{for all } y,\ z \in N$$

if and only if there exist a unique additive operator $A \in \mathrm{Add}(N; M)$ and an element $h_0 \in M$ such that $Ty = Ay + h_0$ in M for all $y \in N$.

Note that particular cases of this lemma when M is the family of all nonempty compact convex subsets of a real normed space equipped with the Hausdorff metric [44] were established in [45, Theorem 2] for $N = [0, \infty)$ and [46, Theorem 5.6] for a convex cone N in a normed space.

Theorem 6.14. *Let $(N, d, +, \cdot)$ and $(M, d, +, \cdot)$ be two abstract convex cones, where M is complete, $h : I \times N \to M$ be the generator of the superposition operator $\mathcal{H}$ and $\Phi = \{\varphi_i\}_{i=1}^\infty$ be a convex Φ-sequence. If the operator $\mathcal{H}$ maps $\mathrm{BV}_\Phi(I; N)$ into $\mathrm{BV}_\Phi(I; M)$ and is Lipschitzian, then the family of functions $\{h(t, \cdot)\}_{t\in I} \subset \mathrm{Lip}(N; M)$ is uniformly Lipschizian, and there exist two functions $x : I \to \mathrm{L}(N; M)$ and $h_0 : I \to M$ such that $x(\cdot)y,\ h_0 \in \mathrm{BV}_\Phi^-(I; M)$ for all $y \in N$, and Matkowski's representation holds:*

$$h^-(t, y) = x(t)y + h_0(t) \quad \text{for all } t \in I \text{ and } y \in N, \tag{6.18}$$

where $x(\cdot)y : I \to M$ is given by $t \mapsto x(t)y$, and $h^-(t, y)$ is the left regularization of the function $s \mapsto h(s, y)$ at the point $s = t$ for each fixed $y \in N$.

Proof. For the sake of clarity we divide the proof into four steps.

1. *Common part.* Since $\mathcal{H}$ is Lipschitzian, then there exists a constant $\eta \ge 0$ such that $\mathrm{d}_M(\mathcal{H}\bar{y}, \mathcal{H}\bar{z}) \le \eta\, \mathrm{d}_N(\bar{y}, \bar{z})$ for all $\bar{y}, \bar{z} \in \mathrm{BV}_\Phi(I; N)$. If $\eta = 0$, then $\mathcal{H}\bar{y} = \mathcal{H}\bar{z}$, and so, if we define functions $\bar{y}$ and $\bar{z}$ to be constants $\bar{y}(t) = y$ and $\bar{z}(t) = z$, $t \in I$, where $y, z \in N$, then, by the definition of $\mathcal{H}$, we find $h(t, y) = h(t, z)$ for all $t \in I$ and $y, z \in N$. Thus, the function h does not depend on the second variable, and so, if we set $h_0(t) = h^-(t)$ for $t \in I$, then we obtain the representation (6.18) with $x = 0 : I \to \mathrm{L}(N; M)$.

In what follows we assume that $\eta > 0$. Since $\mathcal{H}$ is Lipschitzian, the definition of d_M (cf. (5.8)) implies, in particular, the inequality $d_w^*(\mathcal{H}\bar{y}, \mathcal{H}\bar{z}) \le \eta\, \mathrm{d}_N(\bar{y}, \bar{z})$ for all $\bar{y}, \bar{z} \in \mathrm{BV}_\Phi(I; N)$. Setting $\lambda = \eta\, \mathrm{d}_N(\bar{y}, \bar{z}) > 0$ for $\bar{y} \ne \bar{z}$, by virtue of (5.6) and [1, Theorem 3.8(d)], this inequality can be equivalently rewritten as $w_\lambda^d(\mathcal{H}\bar{y}, \mathcal{H}\bar{z}) \le 1$. It follows from the definition of w and $\mathcal{H}$ that, given $n \in \mathbb{N}$ and $a \le a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \le b$, we have:

$$\sum_{i=1}^{n} \varphi_{\sigma(i)}\left(\frac{d\big(h(b_i, \bar{y}(b_i)) + h(a_i, \bar{z}(a_i)), h(b_i, \bar{z}(b_i)) + h(a_i, \bar{y}(a_i))\big)}{\eta\, \mathrm{d}_N(\bar{y}, \bar{z})}\right) \le 1 \tag{6.19}$$

for any permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

Given $\alpha, \beta \in \mathbb{R}, \alpha < \beta$, consider auxiliary functions $\zeta_{\alpha,\beta} \in \mathrm{Lip}(\mathbb{R}; [0, 1])$ given by

$$\zeta_{\alpha,\beta}(t) = \begin{cases} 0 & \text{if } t \le \alpha, \\ (t-\alpha)/(\beta-\alpha) & \text{if } \alpha \le t \le \beta, \\ 1 & \text{if } t \ge \beta. \end{cases}$$

2. Let us show that $\{h(t, \cdot)\}_{t \in I} \subset M^N$ is a uniformly Lipschitzian family. In this step M can be any metric semigroup. Let $y, z \in N$ be arbitrary elements, $y \ne z$. First, we assume that $a < t \le b$. In the inequality (6.19) we set $n = 1, a_1 = a, b_1 = t$, $\bar{y}(s) = \zeta_{a,t}(s)y$ and $\bar{z}(s) = \zeta_{a,t}(s)z$ for all $s \in I$, so that $\bar{y}(a) = \bar{z}(a) = 0, \bar{y}(t) = y$ and $\bar{z}(t) = z$. By virtue of [1, equality (3.6)], given $s, r \in I$, we have:

$$d(\bar{y}(s) + \bar{z}(r), \bar{z}(s) + \bar{y}(r)) = |\zeta_{a,t}(s) - \zeta_{a,t}(r)|\, d(y, z),$$

and so, from the definition of w^d from Section 5.2 and Lemma 6.9, for $\lambda > 0$, we find

$$w^d_\lambda(\bar{y}, \bar{z}) = w_\lambda\big(d(y, z)\zeta_{a,t}(\cdot)\big) = \varphi_1(d(y, z)/\lambda) = 1$$

if and only if $\lambda = d(y, z)/\varphi_1^{-1}(1)$. It follows from [1, Theorem 3.8(b)] that $d^*_w(\bar{y}, \bar{z}) = d(y, z)/\varphi_1^{-1}(1)$. Since $\mathsf{d}_N(\bar{y}, \bar{z}) = d^*_w(\bar{y}, \bar{z})$, (6.19) implies

$$\varphi_1\left(\frac{d\big(h(t, \bar{y}(t)) + h(a, 0), h(t, \bar{z}(t)) + h(a, 0)\big)}{\eta\,(d(y, z)/\varphi_1^{-1}(1))}\right) \le 1,$$

whence

$$d(h(t, y), h(t, z)) \le \eta\, d(y, z) \quad \text{for all } y, z \in N.$$

Now, let $t = a$. In the inequality (6.19) we set $n = 1, a_1 = a, b_1 = b, \bar{y}(s) = (1 - \zeta_{a,b}(s))y$ and $\bar{z}(s) = (1 - \zeta_{a,b}(s))z$ for all $s \in I$, so that $\bar{y}(a) = y, \bar{z}(a) = z$ and $\bar{y}(b) = \bar{z}(b) = 0$. Also, we get, as above, $d^*_w(\bar{y}, \bar{z}) = d(y, z)/\varphi_1^{-1}(1)$, and so,

$$\mathsf{d}_N(\bar{y}, \bar{z}) = d(\bar{y}(a), \bar{z}(a)) + d^*_w(\bar{y}, \bar{z}) = \left(1 + \frac{1}{\varphi_1^{-1}(1)}\right) d(y, z).$$

It follows from (6.19) that

$$\varphi_1\left(\frac{d\big(h(b, 0) + h(a, z), h(b, 0) + h(a, y)\big)}{\eta\,(1 + \varphi_1^{-1}(1))(d(y, z)/\varphi_1^{-1}(1))}\right) \le 1,$$

and so,

$$d(h(a, z), h(a, y)) \le \eta\,(1 + \varphi_1^{-1}(1)) d(y, z) \quad \text{for all } y, z \in N.$$

By the definition of h^- and Lemma 6.10, we also get

$$d(h^-(t, y), h^-(t, z)) \le \eta\,(1 + \varphi_1^{-1}(1)) d(y, z), \quad t \in I, y, z \in N. \tag{6.20}$$

3. Now we establish the representation (6.18). First, suppose that $a < t \le b$, $n \in \mathbb{N}$ and $a < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < t$. Define the function $\zeta_n \in \mathrm{Lip}(I; [0, 1])$ as follows:

$$\zeta_n(s) = \begin{cases} 0 & \text{if } a \le s \le a_1, \\ \zeta_{a_i,b_i}(s) & \text{if } a_i \le s \le b_i, \quad i = 1, \dots, n, \\ 1 - \zeta_{b_i,a_{i+1}}(s) & \text{if } b_i \le s \le a_{i+1}, \quad i = 1, \dots, n-1, \\ 1 & \text{if } b_n \le s \le b. \end{cases}$$

Also, given $y, z \in N, y \ne z$, define two functions $\bar{y}, \bar{z} : I \to N$ by

$$\bar{y}(s) = \frac{1}{2}(1 + \zeta_n(s))y + \frac{1}{2}(1 - \zeta_n(s))z, \quad s \in I,$$

$$\bar{z}(s) = \frac{1}{2}\zeta_n(s)y + \frac{1}{2}(2 - \zeta_n(s))z, \quad s \in I.$$

Then $\bar{y}, \bar{z} \in \mathrm{BV}_\Phi(I; N)$. In fact, by the translation invariance and homogeneity of d and [1, equality (3.6)], given $s, r \in I$, we have:

$$d(\bar{y}(s), \bar{y}(r)) = d(\bar{z}(s), \bar{z}(r)) \le \frac{d(y, 0) + d(z, 0)}{2}\,|\zeta_n(s) - \zeta_n(r)|,$$

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and so, by virtue of (5.1),

$$w_\lambda^d(\bar{y},0) \quad \text{and} \quad w_\lambda^d(\bar{z},0) \le \sum_{i=1}^{2n+1} \varphi_i\Big(\frac{d(y,0)+d(z,0)}{2\lambda}\Big) < \infty, \quad \lambda > 0.$$

Noting that $d(\bar{y}(s)+\bar{z}(r),\bar{z}(s)+\bar{y}(r)) = 0$ for all $s,\ r \in I$, we find $w_\lambda^d(\bar{y},\bar{z}) = 0$ for all $\lambda > 0$, implying $d_w^*(\bar{y},\bar{z}) = 0$, and so,

$$\mathsf{d}_N(\bar{y},\bar{z}) = d(\bar{y}(a),\bar{z}(a)) + d_w^*(\bar{y},\bar{z}) = d(y,z)/2.$$

Note also that $\bar{y}(b_i) = y, \bar{z}(b_i) = (y+z)/2, \bar{y}(a_i) = (y+z)/2$ and $\bar{z}(a_i) = z$. Consequently, inequality (6.19) yields

$$\sum_{i=1}^{n} \varphi_{\sigma(i)}\bigg(\frac{d\big(h(b_i,y)+h(a_i,z),\, h(b_i,(y+z)/2)+h(a_i,(y+z)/2)\big)}{\eta\,(d(y,z)/2)}\bigg) \le 1. \tag{6.21}$$

Since constant functions from I into N lie in $\mathrm{BV}_\Phi(I;N)$ and the operator $\mathcal{H}$ maps $\mathrm{BV}_\Phi(I;N)$ into $\mathrm{BV}_\Phi(I;M)$, then $h(\cdot,u) = \mathcal{H}u \in \mathrm{BV}_\Phi(I;M)$ for all $u \in N$, and so, by Lemma 6.12, $h^-(\cdot,u) \in \mathrm{BV}_\Phi^-(I;M)$ for all $u \in N$. Taking into account the completeness of M, the definition of the left regularization $h^-(\cdot,u)$, the continuity of the operation of addition $+$ in M and the continuity of functions φ_i and passing to the limit as $a_1 \to t-0$ in (6.21), we find

$$\sum_{i=1}^{n} \varphi_{\sigma(i)}\bigg(\frac{d\big(h^-(t,y)+h^-(t,z),\, h^-(t,(y+z)/2)+h^-(t,(y+z)/2)\big)}{\eta\,(d(y,z)/2)}\bigg) \le 1.$$

Due to the arbitrariness of n, it follows from (5.2) that

$$d\Big(h^-(t,y)+h^-(t,z),\, h^-\Big(t,\frac{y+z}{2}\Big)+h^-\Big(t,\frac{y+z}{2}\Big)\Big) = 0$$

for all $t \in (a,b]$. By the definition of $h^-(a,u)$, the last equality holds also at the point $t = a$. Since d is a metric on M and M is an abstract convex cone, the last inequality implies

$$h^-(t,y)+h^-(t,z) = h^-\Big(t,\frac{y+z}{2}\Big)+h^-\Big(t,\frac{y+z}{2}\Big) = 2h^-\Big(t,\frac{y+z}{2}\Big).$$

Thus, for each $t \in I$, the operator $h^-(t,\cdot) : N \to M$ satisfies the following Jensen functional equation:

$$2h^-\Big(t,\frac{y+z}{2}\Big) = h^-(t,y)+h^-(t,z) \quad \text{for all } y,\ z \in N.$$

By Lemma 6.13, for each $t \in I$ there exist an additive operator $x(t) : N \to M$ (so that $x(t)(y+z) = x(t)y + x(t)z$ for all $y,\ z \in N$) and an element $h_0(t) \in M$ such that the equality in (6.18) holds for all $y \in N$. Since $x(t)(0) = 0$, then it follows from (6.18) that $h_0(t) = h^-(t,0)$ for all $t \in I$, and so, the function h_0 lies in $\mathrm{BV}_\Phi^-(I;M)$. Equality (6.18) and inequality (6.20) imply that if $y,\ z \in N$ and $t \in I$, then

$$\begin{aligned} d(x(t)y, x(t)z) &= d(x(t)y + h_0(t), x(t)z + h_0(t)) \\ &= d(h^-(t,y), h^-(t,z)) \\ &\le \eta\,(1+\varphi_1^{-1}(1))d(y,z), \end{aligned}$$

whence $x(t) \in \mathrm{L}(N;M)$, and so, $x : I \to \mathrm{L}(N;M)$.

4. It remains to show that $x(\cdot)y \in \mathrm{BV}_\Phi^-(I;M)$ for all $y \in N$. Applying [1, inequality (2.3)] and (6.18), given $t,\ s \in I$, we have:

$$\begin{aligned} d(x(t)y, x(s)y) &\le d(x(t)y + h_0(t), x(s)y + h_0(s)) + d(h_0(t), h_0(s)) \\ &= d(h^-(t,y), h^-(s,y)) + d(h_0(t), h_0(s)). \end{aligned} \tag{6.22}$$

Noting that $h^-(\cdot,y),\ h_0 \in \mathrm{BV}_\Phi^-(I;M)$, we set $\lambda = d_w^*(h^-(\cdot,y),0)$ and $\mu = d_w^*(h_0,0)$ and with no loss of generality assume that $\lambda\cdot\mu \ne 0$. Since, by (6.22), we have

$$\frac{d(x(t)y, x(s)y)}{\lambda+\mu} \le \frac{\lambda}{\lambda+\mu}\cdot\frac{d(h^-(t,y),h^-(s,y))}{\lambda} + \frac{\mu}{\lambda+\mu}\cdot\frac{d(h_0(t),h_0(s))}{\mu},$$

then, by virtue of the convexity of functions φ_i, (5.6) and Theorem 3.8(c) from [1], we find

$$w_{\lambda+\mu}^d(x(\cdot)y,0) \le \frac{\lambda}{\lambda+\mu}\, w_\lambda^d(h^-(\cdot,y),0) + \frac{\mu}{\lambda+\mu}\, w_\mu^d(h_0,0) \le 1.$$

Therefore, $x(\cdot)y \in \mathrm{BV}_\Phi(I;M)$ and $d_w^*(x(\cdot)y,0) \le \lambda+\mu$ for all $y \in N$. The continuity from the left of $x(\cdot)y$ on $(a,b]$ is a consequence of inequality (6.22): in fact, given $a < t \le b$, we pass to the limit as $s \to t-0$ in (6.22) and note that both terms at the right hand side of (6.22) tend to zero. We conclude that $x(\cdot)y \in \mathrm{BV}_\Phi^-(I;M)$ for all $y \in N$. □

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6.15. Remarks

(a) Theorem 6.14 contains as particular cases the results of [16, Theorem 5.5], [10, Theorem 7], [18, Theorem 4], [12, Theorem 2], [5, Theorem 1] and [24, Theorem 1]. The representation of the form $h^-(t, y) = x(t)y + h_0(t)$ for the generators of Lipschitzian superposition operators were found by Matkowski [6,7] in the space of Lipschitz functions on I and Lipschitz maps on a normed space and by Matkowski and Miś [5] in the case when $N = M = \mathbb{R}$ and $\varphi_i(u) = u$ for all $i \in \mathbb{N}$. The idea to apply the Jensen functional equation in the space of Lipschitzian operators goes back to the papers of A. and W. Smajdor [21,22]. Theorem 6.14 remains true if we replace the left regularization of the function $t \mapsto h(t, y)$ by its right regularization. However, the regularization $h^-(t, y)$ in that theorem cannot be replaced simply by $h(t, y)$—the corresponding example in the case when $N = M = \mathbb{R}$ and $\varphi_i(u) = u$ was constructed in [5, p. 157].

(b) Let $\mathcal{B}(I; N) \subset \mathrm{BV}_\Phi(I; N)$ be the set of all functions satisfying the condition: given $y, z \in N$, $n \in \mathbb{N}$ and $a < a_1 < b_1 < \cdots < a_n < b_n < b$, the function $I \ni s \mapsto \zeta_n(s)y + z \in N$ belongs to $\mathcal{B}(I; N)$, where ζ_n is defined in the proof of Theorem 6.14. Let us endow the set $\mathcal{B}(I; N)$ with metric d_N from $\mathrm{BV}_\Phi(I; N)$. Then the conclusion of Theorem 6.14 remains true if we replace the condition $\mathcal{H} \in \mathrm{Lip}\big(\mathrm{BV}_\Phi(I; N); \mathrm{BV}_\Phi(I; M)\big)$ in it by the condition $\mathcal{H} \in \mathrm{Lip}\big(\mathcal{B}(I; N); \mathrm{BV}_\Phi(I; M)\big)$.

(c) The next remark (d) is interesting in connection with generalizations of the Banach Contraction Theorem. Let (X, d) be a complete metric space, $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ be a function and $T : X \to X$ be an operator such that $d(Tx, Ty) \le \eta\,(d(x, y))$ for all $x, y \in X$. Then T admits a unique fixed point $x_* \in X$ such that $Tx_* = x_*$ and $\lim_{n\to\infty} d(T^n x, x_*) = 0$ for all $x \in X$, where T^n designates the n-th iteration of the operator T, provided at least one of the following three conditions hold: (1) η is nondecreasing, continuous from the right on $\mathbb{R}^+$ and $\eta(u) < u$ for all $u > 0$ [47, Theorem 1]; (2) η is upper semicontinuous from the right on $\mathbb{R}^+$ and $\eta(u) < u$ for all $u > 0$ [48]; (3) η is nondecreasing on $\mathbb{R}^+$ and $\eta^n(u) \to 0$ as $n \to \infty$ for all $u > 0$, where η^n is the n-th iteration of η ([49, Theorem 1.2]). For more information on the equivalence of conditions (1), (2) and (3) we refer to [50, Theorem 1].

(d) Suppose that, under the assumptions of Theorem 6.14, the function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is such that $\eta(0) = 0$ and $\eta(u) > 0$ for all $u > 0$, and the superposition operator $\mathcal{H}$ maps $\mathrm{BV}_\Phi(I; N)$ into $\mathrm{BV}_\Phi(I; M)$ and satisfies the condition:

$$\mathrm{d}_M(\mathcal{H}y, \mathcal{H}z) \le \eta\,(\mathrm{d}_N(y, z)) \quad \text{for all } y, z \in \mathrm{BV}_\Phi(I; N).$$

Then (cf. step 2 in the proof of Theorem 6.14), given $t \in I$ and $y, z \in N$, if $\alpha = \varphi_1^{-1}(1)$, $\beta = 1/\alpha$ and $\gamma = 1 + (1/\alpha)$, then

$$d(h(t, y), h(t, z)) \le \alpha \max\{\eta(\beta d(y, z)), \eta(\gamma d(y, z))\} \equiv \eta_*(d(y, z))$$

and a similar estimate holds also for h^- in place of h, and there exist functions $x : I \to \mathrm{Add}(N; M)$ and $h_0 : I \to M$, for which $x(\cdot)y,\ h_0 \in \mathrm{BV}_\Phi^-(I; M)$ for all $y \in N$, such that the representation (6.18) holds. If η is continuous (or bounded), then, for each $t \in I$, the operator $x(t) : N \to M$ is continuous (bounded, respectively) as well: by virtue of (6.18), we have

$$d(x(t)y, x(t)z) = d(h^-(t, y), h^-(t, z)) \le \eta_*(d(y, z)), \quad y, z \in N.$$

Moreover, if $\liminf_{u\to\infty} \eta(u)/u = 0$, then $x = 0$ in the representation (6.18), so that $h^-(t, y) = h_0(t)$ for all $t \in I$ and $y \in N$: in fact, the additivity property of $x(t)$, for each rational number $\lambda > 0$, implies (cf. Section 5.5(e))

$$\lambda d(x(t)y, 0) = d(\lambda x(t)y, 0) = d(x(t)(\lambda y), x(t)(0)) \le \eta_*(\lambda d(y, 0)),$$

whence $x(t)y = 0$ for all $y \in N$, and so, $x(t) = 0$ for all $t \in I$.

Replacing N by $\mathrm{L}(N; M)$ in Theorem 6.14, we get the following theorem, which is partially converse to Theorem 6.8.

Theorem 6.16. *Let $(N, d, +, \cdot)$ and $(M, d, +, \cdot)$ be two abstract convex cones, where M is complete, $g : I \times \mathrm{L}(N; M) \to M$ be the generator of the superposition operator $\mathcal{G} : \mathrm{L}(N; M)^I \to M^I$ and Φ be a convex Φ-sequence. If $\mathcal{G}$ maps $\mathrm{BV}_\Phi(I; \mathrm{L}(N; M))$ into $\mathrm{BV}_\Phi(I; M)$ and is Lipschitzian, then there exist two functions $Y : I \to \mathrm{L}(\mathrm{L}(N; M); M)$ and $h_0 : I \to M$, for which $Y(\cdot)x,\ h_0 \in \mathrm{BV}_\Phi^-(I; M)$ for all $x \in \mathrm{L}(N; M)$, such that $g^-(t, x) = Y(t)x + h_0(t)$ for all $t \in I$ and $x \in \mathrm{L}(N; M)$, where $Y(\cdot)x : I \to M$ is given by $t \mapsto Y(t)x$ and $g^-(t, x)$ is the left regularization of the function $s \mapsto g(s, x)$ at the point $s = t$ for each fixed $x \in \mathrm{L}(N; M)$.*

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