TWISTED GEOMETRIC SATAKE EQUIVALENCE

MICHAEL FINKELBERG AND SERGEY LYSENKO

Abstract. Let \( k \) be an algebraically closed field and \( O = k[[t]] \subset F = k((t)) \). For an almost simple algebraic group \( G \) we classify central extensions \( 1 \to \mathbb{G}_m \to E \to G(F) \to 1 \); any such extension splits canonically over \( G(O) \). Fix a positive integer \( N \) and a primitive character \( \zeta : \mu_N(k) \to \mathbb{Q}^* \) (under some assumption on the characteristic of \( k \)). Consider the category of \( G(O) \)-biinvariant perverse sheaves on \( E \) with \( \mathbb{G}_m \)-monodromy \( \zeta \). We show that this is a tensor category, which is tensor equivalent to the category of representations of a reductive group \( \check{G}_{E,N} \). We compute the root datum of \( \check{G}_{E,N} \).

Contents

1. Introduction 1
2. Main result 2

2.1. Notations 2
2.2. Convolution 6
2.3. Fusion 6
2.4. Main result 9
3. Classification of central extensions 10
3.1. Simply-connected case 10
3.2. Proof of Proposition 1 12
4. Proof of Theorem 1 13

4.1. Functors \( F_P' \) 13
4.2. Fibre functor 17
4.3. Structure of \( \check{G}_N \) 19
4.4. Rank one 20
References 21

1. INTRODUCTION

Let \( k \) be an algebraically closed field and \( O = k[[t]] \subset F = k((t)) \). For an almost simple algebraic group \( G \) we classify central extensions \( 1 \to \mathbb{G}_m \to E \to G(F) \to 1 \); any such extension splits canonically over \( G(O) \). Fix a positive integer \( N \) and a primitive character \( \zeta : \mu_N(k) \to \mathbb{Q}^* \) (under some assumption on the characteristic of \( k \)).
of \( k \). Consider the category of \( G(\mathbf{O}) \)-biinvariant perverse sheaves on \( E \) with \( \mathbb{G}_m \)-monodromy \( \zeta \). We show that this is a tensor category, which is tensor equivalent to the category of representations of a reductive group \( \check{G}_{E,N} \). We compute the root datum of \( \check{G}_{E,N} \) in Theorem 1. A list of examples is given after Theorem 1.

Our result has a natural place in the framework of “Langlands duality for quantum groups” [6]. Namely, if we take \( k = \mathbb{C} \), and \( q = \zeta(\exp(\pi i N)) \) in Conjecture 0.4 of loc. cit., then our category of \( \zeta \)-monodromic perverse sheaves naturally lies inside the twisted Whittaker sheaves \( \text{Whit}^\zeta(\text{Gr}_G) \), and corresponds under the equivalence of loc. cit. to the category of representations of the quantum Frobenius quotient of \( U_q(G) \).

From the physical point of view, our result is a manifestation of electric-magnetic duality for a rational parameter \( \Psi \), see [7], Section 11.3. Theorem 1 is a generalization of ([9], Theorem 3) and the classical geometric Satake equivalence [10]. It was probably known to experts for a few years, say it was suggested by an anonymous referee of [9] (compare also to [11]). Also, for \( G \) simply connected the root data of \( \check{G}_{E,N} \) appeared in Section 7 of [8]. Our result should follow essentially by comparing Lusztig’s results on quantum Frobenius homomorphism on the one hand, and Kazhdan-Lusztig-Kashiwara-Tanisaki-Arkhipov-Bezrukavnikov-Ginzburg description of representations of quantum groups at roots of unity in terms of perverse sheaves on affine Grassmani ans, on the other. Our goal is to provide a short self-contained proof, following the strategy of [10].

We are obliged to V. Drinfeld who explained to us the classification of central extensions of the loop groups for almost simple groups (not necessarily simply connected), see Proposition 1. We are also indebted to R. Bezrukavnikov and D. Gaitsgory for useful discussions. M.F. is grateful to University Paris 6 for hospitality and support; he was partially supported by the RFBR grant 09-01-00242 and the Science Foundation of the SU-HSE awards No.09-08-0008 and 09-09-0009.

2. Main result

The general reference for this section is [1] 4.5, 5.3 and, for more details, [10].

2.1. Notations. Let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \). Let \( G \) be an almost simple algebraic group over \( k \) with the simply connected cover \( G^{sc} \). We fix a Borel subgroup \( B \subset G \) and a maximal torus \( T \subset B \). We denote the preimage of \( T \) in \( G^{sc} \) by \( T^{sc} \). The Weyl group of \( G, T \) is denoted by \( W \). The weight and coweight lattices of \( T \) (resp. \( T^{sc} \)) are denoted by \( X^*(T) \) and \( X_*(T) \) (resp. \( X^*(T^{sc}) \) and \( X_*(T^{sc}) \)). The root system of \( T \subset B \subset G \) is denoted by \( \Delta \subset \Delta^{sc} \). The set of simple roots is \( \Delta^* = \{ \check{\alpha}_1, \ldots, \check{\alpha}_r \} \subset \Delta^* \). The sum of all positive roots is denoted \( 2\check{\rho} \). The coroot system of \( T \subset B \subset G \) is denoted by \( \check{\Delta} \subset \check{\Delta}^{sc} \). The set of simple coroots is \( \Pi_* = \{ \alpha_1, \ldots, \alpha_r \} \subset \check{\Delta}^* \). Denote by \( g \) the adjoint representation of \( G \).
There is a unique $W$-invariant bilinear pairing $(,): X_*(T^{sc}) \times X_*(T^{sc}) \to \mathbb{Z}$ such that $(\alpha, \alpha) = 2$ for a short coroot $\alpha$. It defines a linear map $\iota: X_*(T^{sc}) \to X^*(T^{sc})$ such that $(x, y) = \langle x, \iota(y) \rangle$ for any $x, y \in X_*(T^{sc})$. The map $\iota$ extends uniquely by linearity to the same named map $\iota: X_*(T) \to X^*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. The bilinear form $(,)$ extends uniquely by linearity to the same named bilinear form $(,): X_*(T) \times X_*(T) \to \mathbb{Q}.$

**Lemma 1.** For $\lambda \in X_*(T)$ we have $2\hat{h}u(\lambda) = \sum_{\alpha \in R^+} \langle \lambda, \check{\alpha} \rangle \hat{\alpha}$, where $\hat{h}$ is the dual Coxeter number of $G$.

**Proof** Write $\Phi_R(,,)$ for the canonical bilinear $W$-invariant linear form on $X^*(T)$ in the sense of ([2], §1, section 12). Formula (17) from *loc.cit.* reads

$$2\Phi_R(\check{\beta}, \check{\beta})^{-1} = \sum_{\alpha \in R^+} \langle \hat{\alpha}, \check{\beta} \rangle^2,$$

where $\check{\beta}$ is the root corresponding to a short coroot $\beta$. We must check that $\Phi_R(\check{\beta}, \check{\beta}) = \hat{h}^{-1}$. This is done case by case for all irreducible reduced root systems using the calculation of $\Phi_R$ performed in §4 of *loc.cit.*. □

Set $O = k[[t]]$, $F = k((t))$. The affine Grassmannian $\text{Gr}_G$ is an ind-scheme, the fpqc quotient $G(F)/G(O)$ (cf. [1]). Our conventions about $\mathbb{Z}/2\mathbb{Z}$-gradings are those of [9]. Recall that for free $O$-modules of finite type $V_1, V_2$ with an isomorphism $V_1(F) \cong V_2(F)$ one has the relative determinant $\text{det}(V_1 : V_2)$ ([9], Section 8.1), this is a $\mathbb{Z}/2\mathbb{Z}$-graded line given by

$$\text{det}(V_1 : V_2) = \text{det}(V_1/V) \otimes \text{det}(V_2/V)^{-1}$$

for any $O$-lattice $V \subset V_1 \cap V_2$.

Let $L$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of parity zero) line bundle on $\text{Gr}_G$ whose fibre at $gG(O)$ is $\text{det}(g(O)) : g(O)^9)$. Write $\text{Gr}_{G_G}$ for the punctured total space (that is, the total space with zero section removed) of the line bundle $L$. By abuse of notation, the restriction of $L$ under the map $G(F) \to G(O), g \mapsto gG(O)$ is again denoted by $L$. Write $E^a$ for the punctured total space of the line bundle $L$ on $G(F)$. Since $L$ is naturally a multiplicative $G_m$-torsor on $G(F)$ in the sense of ([3], Section 1.2), $E^a$ gives a central extension

$$1 \to \mathbb{G}_m \to E^a \to G(F) \to 1,$$

it splits canonically over $G(O)$.

Our first result extends the well-known classification of central extensions of $G^{sc}(F)$ by $G_m$ to the almost simple case.

**Proposition 1.** The isomorphism classes of central extensions of $G(F)$ by $G_m$ are naturally in bijection with integers $m \in \mathbb{Z}$ such that $m \mu(X_*(T)) \subset X^*(T)$. The corresponding central extension splits canonically over $G(O)$, and its group of automorphisms is $\text{Hom}(\pi_0(G(F)), G_m) \cong \text{Hom}(\pi_1(G), \mu_\infty(k))$. 
Let \( d > 0 \) be the smallest integer such that \( dt(X_s(T)) \subset X^*(T) \), this is a divisor of \( 2\hbar \). We pick and denote by \( E^c \) the corresponding central extension of \( G(\mathbf{F}) \) by \( \mathbb{G}_m \). Any central extension of \( G(\mathbf{F}) \) by \( \mathbb{G}_m \) is isomorphic to a multiple of \( E^c \). We also pick an isomorphism of central extensions of \( G(\mathbf{F}) \) by \( \mathbb{G}_m \) identifying \( E^a \) with the \((2\hbar/d)\)-th power of \( E^c \). (If \( G \) is simply-connected then \( d = 1 \), and the latter isomorphism is uniquely defined).

Fix a prime \( \ell \) different from \( p \), write \( P(S) \) (resp., \( D(S) \)) for the category of étale \( \mathbb{Q}_\ell \)-perverse sheaves (resp., the derived category of étale \( \mathbb{Q}_\ell \)-sheaves) on a \( k \)-scheme (or stack) \( S \). Since we are working over an algebraically closed field, we systematically ignore the Tate twists.

Fix a positive integer \( N \) and a primitive character \( \zeta : \mu_N(k) \to \mathbb{Q}_\ell \), we assume that \( p \) does not divide \( 2\hbar N/d \). For the map \( s_N : \mathbb{G}_m \to \mathbb{G}_m, x \mapsto x^N \) let \( \mathcal{L}^\zeta \) denote the direct summand of \( s_N^*\mathbb{Q}_\ell \) on which \( \mu_N(k) \) acts by \( \zeta \). If \( S \) is a scheme with a \( \mathbb{G}_m \)-action, we say that a perverse sheaf \( K \) on \( S \) has \( \mathbb{G}_m \)-monodromy \( \zeta \) if it is equipped with a \((\mathbb{G}_m, \mathcal{L}^\zeta)\)-equivariant structure.

Let \( \text{Perv}_{G,N} \) denote the category of \( G(O) \)-equivariant \( \mathbb{Q}_\ell \)-perverse sheaves on \( E^c/G(O) \) with \( \mathbb{G}_m \)-monodromy \( \zeta \).

**Remark 1.** If \( S \) is a scheme and \( L \) is a line bundle on \( S \), let \( \tilde{L} \) be the total space of the punctured line bundle \( L \). Let \( f : \tilde{L} \to \tilde{L}^m \) be the map over \( S \) sending \( l \) to \( l^\otimes m \). Assume that \( p \) does not divide \( N/d \). Let \( \chi : \mu_N(k) \to \mathbb{Q}_\ell^\times \) be a character. Then the functor \( K \mapsto f^*K \) is an equivalence between the categories of \( \chi \)-monodromic perverse sheaves on \( \tilde{L}^m \) and \( \chi^m \)-monodromic perverse sheaves on \( \tilde{L} \).

Pick a primitive character \( \zeta_a : \mu_{2\hbar N/d}(k) \to \mathbb{Q}_\ell^\times \) satisfying \( \zeta_a^{2\hbar/d} = \zeta \). By Remark 1, \( \text{Perv}_{G,N} \) identifies with the category of \( G(O) \)-equivariant perverse sheaves on \( \text{Gra}_G \) with \( \mathbb{G}_m \)-monodromy \( \zeta_a \). Set

\[
\text{Perv}_{G,N} = \text{Perv}_{G,N}[-1] \subset D(\text{Gra}_G)
\]

For the associativity and commutativity constraints we are going to introduce later to be natural (and avoid some sign problems), one has to work with \( \text{Perv}_{G,N} \) rather than \( \text{Perv}_{G,N} \). Let \( \text{Gr}_G \) be the stack quotient of \( \text{Gra}_G \) by \( \mathbb{G}_m \), where \( x \in \mathbb{G}_m \) acts as multiplication by \( x^{2\hbar N/d} \). One may think of \( \text{Perv}_{G,N} \) as the category of certain perverse sheaves on \( \text{Gr}_G \).

We have a natural embedding \( X_s(T) \subset \text{Gr}_G \) as the set of \( T \)-fixed points. Two coweights \( \lambda, \mu \in X_s(T) \subset \text{Gr}_G \) lie in the same \( G(O) \)-orbit iff \( \lambda \in W(\mu) \). Thus the set of \( G(O) \)-orbits in \( \text{Gr}_G \) coincides with the set of Weyl group orbits \( X_s(T)/W \), or equivalently, with the set of dominant coweights \( X_s^+(T) \subset X_s(T) \). The orbit corresponding to \( \lambda \in X_s^+(T) \) will be denoted by \( \text{Gr}_G^\lambda \). The \( G \)-orbit of \( \lambda \) is isomorphic to a partial flag variety \( \mathcal{B}^\lambda = G/P^\lambda \) where \( P^\lambda \) is a parabolic subgroup whose Levi has the Weyl group \( W^\lambda \subset W \) coinciding with the stabilizer of \( \lambda \) in \( W \). Write \( \text{Gra}_G^\lambda \) for the preimage of \( \text{Gr}_G^\lambda \) under \( \text{Gra}_G \to \text{Gr}_G \).
Let $\text{Aut}(O)$ denote the group ind-scheme over $k$ such that, for any $k$-algebra $R$, $\text{Aut}(O)(R)$ is the automorphism group of the topological $R$-algebra $R \otimes O$ ([1], 2.6.5). Write $\text{Aut}^0(O)$ for the reduced part of $\text{Aut}(O)$. The group scheme $\text{Aut}^0(O)$ acts naturally on the exact sequence (1) acting trivially on $\mathbb{G}_m$ and preserving $G(O)$.

The action of the loop rotation group $\mathbb{G}_m \subset \text{Aut}^0(O)$ contracts $\text{Gr}^\lambda$ to $\mathcal{B}^\lambda \subset \text{Gr}_\lambda$, and realizes $\text{Gr}^\lambda$ as a composition of affine fibrations over $\mathcal{B}^\lambda$. We denote the projection $\text{Gr}^\lambda \to \mathcal{B}^\lambda$ by $\varpi_\lambda$.

If $\bar{\nu} \in X^*(T^\text{sc})$ is orthogonal to all coroots $\alpha$ satisfying $\langle \lambda, \bar{\alpha} \rangle = 0$ then write $\mathcal{O}(\bar{\nu})$ for the corresponding $G^\text{sc}$-equivariant line bundle on $\mathcal{B}^\lambda$. It is canonically trivialized at $1 \in \mathcal{B}^\lambda$. If $\lambda \in X_+(T)$ then $\nu(\lambda) \in X^*(T^\text{sc})$ gives rise to the line bundle $\mathcal{O}(\nu(\lambda))$ on $\mathcal{B}^\lambda$.

For a free $O$-module $E$ write $E_x$ for its geometric fibre. Write $\Omega$ for the completed module of relative differentials of $O$ over $k$.

For a root $\bar{\alpha}$ write $g^{\bar{\alpha}} \subset g$ for the corresponding root subspace. If $\bar{\alpha} = \sum_{i=1}^r a_i \bar{\alpha}_i$ we have $g^{\bar{\alpha}} \simeq \bigotimes_{i=1}^r (g^{\bar{\alpha}_i})^{a_i}$ canonically. In particular, $g^{-\bar{\alpha}}$ is identified with the dual of $g^{\bar{\alpha}}$ via the Killing form.

Pick a trivialization $\phi_i : g^{\bar{\alpha}_i} \to k$ for each simple root $\bar{\alpha}_i \in \Pi^*$. Set $\Phi = \{ \phi_i \}_{i=1}^r$.

Lemma 2. Let $\lambda \in X_+(T)$. The family $\Phi$ yields a uniquely defined $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\mathcal{L} \big|_{\text{Gr}^{\lambda}_G} \simeq \Omega^\vee_{\lambda, \gamma}(\lambda, \lambda) \otimes \varpi^\ast_\lambda \mathcal{O}(2\bar{\nu}(\lambda))$$

Set $X^{\ast +} (\bar{T}_N) = \{ \lambda \in X^+_+(T) \mid d\nu(\lambda) \in N X^* (T) \}$. By Lemma 2, for $\lambda \in X^{\ast +} (\bar{T}_N)$ the scheme $\text{Gra}^\lambda_G$ admits a $G(O)$-equivariant local system with $\mathbb{G}_m$-monodromy $\zeta_a$ iff $\lambda \in X^{\ast +} (\bar{T}_N)$.

Remark 2. In ([1], Section 4.4.9, p. 166, formula (217)) an extension of $G(F)$ by $\mathbb{G}_m$ has been constructed whose square identifies with (1). So, $d$ is a divisor of $\bar{h}$. Another way to see this is to note that, by Lemma 1, for $\lambda \in X_+(T)$ we have $\bar{\nu}(\lambda) = \sum_{\bar{\alpha} \in R^{\ast +}} \langle \lambda, \bar{\alpha} \rangle \bar{\alpha} \in X^*(T)$. Here $R^{\ast +}$ denotes the set of positive roots.

Write $\Omega^\vee_{\lambda}(O)$ for the groupoid of square roots of $\Omega$. For $E \in \Omega^\vee_{\lambda}(O)$ and $\lambda \in X^{\ast +} (\bar{T}_N)$ define the line bundle $\mathcal{L}_{\lambda, E}$ on $\text{Gr}^\lambda_G$ as

$$\mathcal{L}_{\lambda, E} = E_{\lambda}^\vee \otimes \varpi^\ast_\lambda \mathcal{O}(\frac{d}{N} \nu(\lambda))$$

It is equipped with an isomorphism $\mathcal{L}_{\lambda, E}^{2\bar{h}N/d} \simeq \mathcal{L} \big|_{\text{Gr}^{\lambda}_G}$. Write $\mathcal{L}_{\lambda, E}$ for the punctured total space of $\mathcal{L}_{\lambda, E}$. Denote by

$$p_\lambda : \mathcal{L}_{\lambda, E} \to \text{Gra}^\lambda_G$$

the map over $\text{Gr}^\lambda_G$ sending $x$ to $x^{2\bar{h}N/d}$. 

(2)
For \( \lambda \in X^{\ast+}(\tilde{T}_N) \) we define \( A^\lambda_\xi \in \mathcal{Perv}_{G,N} \) as the intermediate extension of \( E^\lambda_\xi[-1 + \dim \Gamma^\lambda_G] \). Here \( E^\lambda_\xi \) is the local system on \( \Gamma^\lambda_G \) with \( \mathbb{G}_m \)-monodromy \( \zeta_\lambda \) equipped with an isomorphism \( p^\lambda_\xi E^\lambda_\xi \cong \mathcal{O}_\xi \). Both \( E^\lambda_\xi \) and \( A^\lambda_\xi \) are defined up to a unique isomorphism. The irreducible objects of \( \mathcal{Perv}_{G,N} \) in the degrees of the same parity, and derive s from this that \( \mathcal{Perv}_{G,N} \) is semi-simple.

### 2.2. Convolution

Consider the automorphism \( \tau \) of \( E^a \times E^a \) sending \((g,h)\) to \((g,gh)\). Let \( G(O) \times G(O) \times \mathbb{G}_m \) act on \( E^a \times E^a \) in such a way that \((\alpha, \beta, b) \in G(O) \times G(O) \times \mathbb{G}_m \) send \((g,h)\) to \((g\beta^{-1}b^{-1}, \beta bh)\). Write \( E^a \times_{G(O) \times \mathbb{G}_m} \Gamma_{G} \) for the quotient of \( E^a \times E^a \) under this free action. Then \( \tau \) induces an isomorphism

\[
\tau: E^a \times_{G(O) \times \mathbb{G}_m} \Gamma_{G} \cong \Gamma_{G} \times \Gamma_{G}
\]

sending \((g,gG(O))\) to \((gG(O),ghG(O))\), where \( \bar{g} \) is the image of \( g \in E^a \) in \( G(F) \).

Set \( m \) be the composition of \( \tau \) with the projection to \( \Gamma_{G} \). Let \( p_G: E^a \to \Gamma_{G} \) be the map \( h \mapsto hG(O) \). Similarly to \((\cite{10}, \cite{9})\), we get a diagram

\[
\begin{array}{ccc}
\Gamma_{G} \times \Gamma_{G} & \xleftarrow{q_G \times \text{id}} & E^a \times \Gamma_{G} \\
\downarrow{p_G} & & \downarrow{q_G} \\
\Gamma_{G} \times \Gamma_{G} & \to & E^a \times_{G(O) \times \mathbb{G}_m} \Gamma_{G} \\
\downarrow{m} & & \downarrow{m} \\
\Gamma_{G} & = & \Gamma_{G} \\
\end{array}
\]

where \( q_G \) is the quotient map under the action of \( G(O) \times \mathbb{G}_m \).

For \( K_1, K_2 \in \mathcal{Perv}_{G,N} \) define the convolution product \( K_1 * K_2 \in D(\Gamma_{G}) \) by \( K_1 * K_2 = m_1K \), where \( K[1] \) is a perverse sheaf on \( E^a \times_{G(O) \times \mathbb{G}_m} \Gamma_{G} \). This is a unit object in \( \mathcal{Perv}_{G,N} \) as in \((\cite{1}, 3.5.2)\). It fits into an exact sequence \( 1 \to \mu_2 \to \text{Aut}_2(O) \to \text{Aut}(O) \to 1 \), and \( \text{Aut}_2(O) \) is connected. Write \( \text{Aut}_2(O) \) for the preimage of \( \text{Aut}(O) \) in \( \text{Aut}_2(O) \).
The map (2) is \( \text{Aut}_V^0(\mathcal{O}) \)-equivariant, so the action of \( \text{Aut}_V^0(\mathcal{O}) \) on \( \text{Gra}_G \) lifts to a \( \text{Aut}_V^0(\mathcal{O}) \)-equivariant structure on each \( A^\lambda_{\mathcal{E}} \in \text{Perv}_{G,N} \). The corresponding \( \text{Aut}_V^0(\mathcal{O}) \)-equivariant structure on each \( A^\lambda_{\mathcal{E}} \) is unique, as the action of \( \text{Aut}_V^0(\mathcal{O}) \) on \( \text{Gra}_G^\lambda \) factors through a smooth connected quotient group of finite type. Here \( \text{Gra}_G^\lambda \) is the preimage of \( \text{Gra}_G^\lambda \) under the projection \( \text{Gra}_G \rightarrow \text{Gr}_G \).

Let \( X \) be a smooth connected projective curve over \( k \). For a closed \( x \in X \) let \( O_x \) be the completed local ring of \( X \) at \( x \), and \( F_x \) its fraction field. Write \( \mathcal{F}_G^0 \) for the trivial \( G \)-torsor on a scheme (or stack). Write \( \text{Gr}_{G,x} = G(F_X)/G(O_x) \) for the corresponding affine Grassmannian. Then \( \text{Gr}_{G,x} \) identifies canonically with the ind-scheme classifying a \( G \)-torsor \( \mathcal{F}_G \) on \( X \) together with a trivialization \( \nu : \mathcal{F}_G \cong \mathcal{F}_G^0 |_{X-x} \).

For \( m \geq 1 \) write \( \text{Gr}_{G,X}^m \) for the ind-scheme classifying \( (x_1, \ldots, x_m) \in X^m \), a \( G \)-torsor \( \mathcal{F}_G \) on \( X \), and a trivialization \( \mathcal{F}_G \cong \mathcal{F}_G^0 |_{X-\cup x_i} \).

Let \( G_X^m \) be the group scheme over \( X^m \) classifying \( \{(x_1, \ldots, x_m) \in X^m, \mu\} \), where \( \mu \) is an automorphism of \( \mathcal{F}_G^0 \) restricted to the formal neighborhood of \( D = \cup_{i} x_i \). The fibre of \( G_X^m \) over \( (x_1, \ldots, x_m) \in X^m \) is \( \prod_i G(O_{x_i}) \) with \( \{y_1, \ldots, y_s\} = \{x_1, \ldots, x_m\} \) and \( y_i \) pairwise distinct.

Let \( \mathfrak{L}_{X}^m \) be the \((\mathbb{Z}/2\mathbb{Z}\)-graded purely of parity zero) line bundle on \( \text{Gr}_{G,X}^m \) whose fibre is \( \det R\Gamma(X, \mathfrak{g}_{\mathfrak{T}_G}) \otimes \det R\Gamma(X, \mathfrak{g}_{\mathfrak{T}_G})^{-1} \). Here for a \( G \)-module \( V \) and a \( G \)-torsor \( \mathcal{F}_G \) on a base \( S \) we write \( V_{\mathfrak{T}_G} \) for the induced vector bundle on \( S \).

**Lemma 4.** For a \( k \)-point \( (x_1, \ldots, x_m, \mathcal{F}_G) \) of \( \text{Gr}_{G,X}^m \) let \( \{y_1, \ldots, y_s\} = \{x_1, \ldots, x_m\} \) with \( y_i \) pairwise distinct. The fibre of \( \mathfrak{L}_{X}^m \) at this \( k \)-point is canonically isomorphic as \( \mathbb{Z}/2\mathbb{Z} \)-graded to

\[
\bigotimes_{i=1}^{s} \det(\mathfrak{g}(O_{y_i}) : \mathfrak{g}_{\mathfrak{T}_G}(O_{y_i}))
\]

Write \( \text{Gra}_{G,X}^m \) for the punctured total space of \( \mathfrak{L}_{X}^m \). The group scheme \( G_X^m \) acts naturally on \( \text{Gra}_{G,X}^m \) and \( \text{Gr}_{G,X}^m \), and the projection \( \text{Gr}_{G,X}^m \rightarrow \text{Gr}_{G,X}^m \) is \( G_X^m \)-equivariant. Let \( \text{Perv}_{G,N,X}^m \) be the category of \( G_X^m \)-equivariant perverse sheaves on \( \text{Gra}_{G,X}^m \) with \( G_m \)-monodromy \( \zeta_\alpha \). Set

\[
\text{Perv}_{G,N,X}^m = \text{Perv}_{G,N,X}^m[-m-1] \subset D(\text{Gr}_{G,X}^m)
\]

For \( x \in X \) write \( D_x = \text{Spec } O_x \), \( D_x^* = \text{Spec } F_x \). Consider the diagram, where the left and right square is cartesian

\[
\begin{array}{ccc}
\text{Gra}_{G,X} \times \text{Gra}_{G,X} & \xrightarrow{\tilde{p}_{G,X}} & \tilde{C}_{G,X} \xrightarrow{\tilde{q}_{G,X}} \tilde{\text{Conv}}_{G,X} \xrightarrow{\tilde{m}_{X}} \text{Gra}_{G,X}^2 \\
\downarrow & & \downarrow & & \downarrow \\
\text{Gra}_{G,X} \times \text{Gr}_{G,X} & \xrightarrow{p_{G,X}} & C_{G,X} \xrightarrow{q_{G,X}} \text{Conv}_{G,X} \xrightarrow{m_{X}} \text{Gr}_{G,X}^2 \\
\end{array}
\]

Here the low row is the convolution diagram from [10]. Namely, \( C_{G,X} \) is the ind-scheme classifying collections:

\[
\begin{array}{l}
(3)
\begin{cases}
x_1, x_2 \in X, \ G \ - \text{torsors } \mathcal{F}_G^i, \mathcal{F}_G^0 \text{ on } X \text{ with } \mu_i : \mathcal{F}_G^i \cong \mathcal{F}_G^0 |_{X-x_i}, \\
\mu_1 : \mathcal{F}_G^1 \cong \mathcal{F}_G^0 |_{D_{x_2}}
\end{cases}
\end{array}
\]
The map $p_{G,X}$ forgets $\mu_1$. The ind-scheme Conv$_{G,X}$ classifies collections:

$$\begin{cases} x_1, x_2 \in X, G - \text{torsors } F^1_G, F_G \text{ on } X, \\
\text{isomorphisms } \nu_1 : F^1_G \supseteq F^0_G |_{X-x_1} \text{ and } \eta : F^1_G \supseteq F_G |_{X-x_2} \end{cases}$$

The map $m_X$ sends this collection to $(x_1, x_2, F_G)$ together with the trivialization $\eta \circ \nu_1^{-1} : F^0_G \supseteq F_G |_{X-x_1-x_2}$.

The map $q_{G,X}$ sends (3) to (4), where $F_G$ is obtained by gluing $F^1_G$ on $X - x_2$ and $F^2_G$ on $D_{x_2}$ using their identification over $D_{x_2}^*$ via $\nu_2^{-1} \circ \mu_1$.

The canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$q^*_{G,X}m^*_X \mathcal{L}_{X^2} \supseteq \tilde{p}^*_{G,X}(\mathcal{L}_X \boxtimes \mathcal{L}_X)$$

allows to define $\tilde{q}_{G,X}$, it sends (3) together with $v_i \in \det(g(O_{x_i}) : g_{F^1_G}(O_{x_i}))$ for $i = 1, 2$ to the image of (3) under $q_{G,X}$ together with $v_1 \otimes v_2$. Here we used the isomorphism

$$\det(g(O_{x_1}) : g_{F^1_G}(O_{x_1})) \otimes \det(g(O_{x_2}) : g_{F^1_G}(O_{x_2})) \supseteq \det(g(O_{x_1}) : g_{F^1_G}(O_{x_1}) \otimes \det(g_{F^2_G}(O_{x_2}) : g_{F^2_G}(O_{x_2}))$$

given by $\mu_1$ and $g_{F^1_G}(O_{x_2}) \supseteq g_{F^2_G}(O_{x_2})$, so (5) is the fibre of $\mathcal{L}_{X^2}$ over $F_G$.

For $K_1, K_2 \in \text{Perv}_{G,N,X}$ there is a (defined up to a unique isomorphism) perverse sheaf $K_{12}[3]$ on Conv$_{G,X}$ with $\tilde{q}^*_{G,X}K_{12} \supseteq \tilde{p}^*_{G,X}(K_1 \boxtimes K_2)$. Moreover, $K_{12}$ has $G_m$-monodromy $\zeta_a$. We then let

$$K_1 *_X K_2 = \tilde{m}_X!K_{12}$$

Let $U \subset X^2$ be the complement to the diagonal. Let $j : \text{Gra}_{G,X^2}(U) \hookrightarrow \text{Gra}_{G,X^2}$ be the preimage of $U$. Recall that $m_X$ is stratified small, an isomorphism over the preimage of $U$ (cf. [10]), so the same holds for $\tilde{m}_X$. Thus, $(K_1 *_X K_2)[3]$ is a perverse sheaf, the intermediate extension from $\text{Gra}_{G,X^2}(U)$. Clearly, $K_1 *_X K_2$ is $G_{X^2}$-equivariant over $\text{Gra}_{G,X^2}(U)$, and this property is preserved under the intermediate extension. So, $K_1 *_X K_2 \in \text{Perv}_{G,N,X^2}$.

Let $\Omega_X$ be the canonical line bundle on $X$. Write $\Omega^+_X(X)$ for the groupoid of square roots of $\Omega_X$. For $E_X \in \Omega^+_X(X)$ let $\hat{X}_2 \rightarrow X$ be the Aut$_0^0(O)$-torsor whose fibre over $x$ is the scheme of isomorphisms between $(E_x, O_x)$ and $(E, O)$. Then $Gr_{G,X} \supseteq \hat{X}_2 \times_{\text{Aut}_O^0(O)} Gr_G$ (cf. [1], 5.3.11), and similarly $Gr_{G,X} \supseteq \hat{X}_2 \times_{\text{Aut}_O^0(O)} Gr_G$. Since any $K \in \text{Perv}_{G,N}$ is Aut$_0^0(O)$-equivariant (in a unique way), we get a fully faithful functor

$$\tau^0 : \text{Perv}_{G,N} \rightarrow \text{Perv}_{G,N,X}$$

sending $K$ to the descent of $\overline{Q}_\ell \boxtimes K$ under $\hat{X}_2 \times \text{Gra}_G \rightarrow \text{Gra}_X$. 
Let $i : \text{Gra}_{G,X} \rightarrow \text{Gra}_{G,X^2}$ be the preimage of the diagonal in $X^2$. For $F_i \in \mathbb{P}erv_{G,N}$ letting $K_i = \tau^0 F_i$ define

$$K_{12} \mid_U := K_{12} \mid_{\text{Gra}_{G,X^2}(U)}$$

as above (it is placed in perverse degree 3). We get $K_1 \ast_X K_2 \simeq j_{!*}(K_{12} \mid_U)$ and $\tau^0(F_1 \ast F_2) \simeq i^*(K_1 \ast_X K_2)$. So, the involution $\sigma$ of $\text{Gra}_{G,X^2}$ interchanging $x_i$ yields

$$\tau^0(F_1 \ast F_2) \simeq i^* j_{!*}(K_{12} \mid_U) \simeq i^* j_{!*}(K_{21} \mid_U) \simeq \tau^0(F_2 \ast F_1),$$

because $\sigma^*(K_{12} \mid_U) \simeq K_{21} \mid_U$ canonically. As in ([1], 5.3.13-5.3.17) one shows that the associativity and commutativity constraints are compatible. Thus, $\mathbb{P}erv_{G,N}$ is a symmetric monoidal category.

The idea to use $\tau^0$ instead of $\tau^0[1]$ in the above definition of the commutativity constraint goes back to ([1], 5.3.17), this is a way to avoid sign problems.

**Remark 3.** Write $\mathbb{P}erv_{G(O)}(\text{Gra}_G)$ for the category of $G(O)$-equivariant perverse sheaves on $\text{Gra}_G$. Let $\ast$ be the covariant self-functor on $\mathbb{P}erv_{G(O)}(\text{Gra}_G)$ induced by the map $E^a \rightarrow E^a$, $e \mapsto e^{-1}$. Then $K \mapsto K^\vee := \mathbb{D}(\ast K)[-2]$ is a contravariant functor $\mathbb{P}erv_{G,N} \rightarrow \mathbb{P}erv_{G,N}$. As in ([9], Remark 6), one checks that for $K_i \in \mathbb{P}erv_{G,N}$ we have canonically $\text{RHom}(K_1 \ast K_2, K_3) \simeq \text{RHom}(K_1, K_3 \ast K_2^\vee)$. So, $K_3 \ast K_2^\vee$ represents the internal $\text{Hom}(K_2, K_3)$ in the sense of the tensor structure on $\mathbb{P}erv_{G,N}$. Besides, $\ast(K_1 \ast K_2) \simeq (\ast K_2) \ast (\ast K_1)$ canonically.

**2.4. Main result.** In Section 4.2 below we introduce a tensor category $\mathbb{P}erv^2_{G,N}$ obtained from $\mathbb{P}erv_{G,N}$ by some modification of the commutativity constraint. Set

$$X^*(\tilde{T}_N) = \{ \nu \in X^*(T) \mid d\nu(\nu) \in N X^*(T) \}$$

Let $\tilde{T}_N = \text{Spec} k[X^*(\tilde{T}_N)]$ be the torus whose weight lattice is $X^*(\tilde{T}_N)$. The natural inclusion $X^*(T) \subset X_*(\tilde{T}_N)$ allows to see each root $\tilde{\alpha} \in R^*$ as a coweight of $\tilde{T}_N$. For $a \in \mathbb{Q}, a > 0$ written as $a = a_1/a_2$ with $a_1 \in \mathbb{N}$ prime to each other say that $a_2$ is the denominator of $a$. Recall that $p$ does not divide $2h N/d$.

**Theorem 1.** There is a connected semi-simple group $\hat{G}_N$ and a canonical equivalence of tensor categories

$$\mathbb{P}erv^2_{G,N} \simeq \text{Rep}(\hat{G}_N)$$

There is a canonical inclusion $\tilde{T}_N \subset \hat{G}_N$ whose image is a maximal torus in $\hat{G}_N$. The Weyl groups of $G$ and $\hat{G}_N$ viewed as subgroups of $\text{Aut}(X^*(\tilde{T}_N))$ are the same. Our choice of a Borel subgroup $T \subset B \subset G$ yields a Borel subgroup $\tilde{T}_N \subset \tilde{B}_N \subset \hat{G}_N$. The corresponding simple roots (resp., coroots) of $(\hat{G}_N, \tilde{T}_N)$ are $\delta_i \alpha_i$ (resp., $\alpha_i/\delta_i$), where $\delta_i$ is the denominator of $\frac{d(\alpha_i, \alpha_i)}{2N}$.

**Examples.** (If $G$ is simply-connected then $d = 1$.)

- $G = \text{SL}_2$ then $\hat{G}_N \simeq \text{SL}_2$ for $N$ even, and $\hat{G}_N \simeq \text{PSL}_2$ for $N$ odd.
\[ G = PSL_2 \text{ then } d = 2, \text{ and } \hat{G}_N \cong SL_2 \text{ for } N \text{ odd, } \hat{G}_N \cong PSL_2 \text{ for } N \text{ even.} \]

\[ G = Sp_{2n} \text{ then } \hat{G}_N \cong SO_{2n+1} \text{ for } N \text{ odd, and } \hat{G}_N \cong Sp_{2n} \text{ for } N \text{ even. For } N = 2 \text{ this has been also proved in [9].} \]

\[ G = Spin_{2n+1} \text{ with } n \geq 2 \text{ then } \hat{G}_N \cong \begin{cases} \frac{Sp_{2n}}{\{\pm 1\}}, & N \text{ odd} \\ Spin_{2n+1}, & N \text{ even and } nN/2 \text{ even} \\ SO_{2n+1}, & N \text{ even and } nN/2 \text{ odd} \end{cases} \]

\[ G \text{ is simply-connected of type } E_6, \text{ its center identifies with } \mathbb{Z}/3\mathbb{Z} \text{ and} \]

\[ \hat{G}_N \cong \begin{cases} \text{adjoint of type } E_6, & 3 \nmid N \\ \text{simply-connected of type } E_6, & 3 \mid N \end{cases} \]

\[ G \text{ is simply-connected of type } E_7, \text{ its center identifies with } \mathbb{Z}/2\mathbb{Z} \text{ and} \]

\[ \hat{G}_N \cong \begin{cases} \text{simply-connected of type } E_7, & N \text{ even} \\ \text{adjoint of type } E_7, & N \text{ odd} \end{cases} \]

Remark 4. The case when \( p \) divides \( 2\hat{h}/d \), but \( p \) does not divide \( N \) can also be treated. In this case one can pick a character \( \zeta_a : \mu_{2hN/d}(k) \to \overline{\mathbb{Q}}_\ell^* \) satisfying \( \zeta_a^{2h/d} = \zeta \) and define \( \text{Perv}_T^{G,N} \) in the same way. But the category \( \text{Perv}_{T,G,N} \) will have more objects, we excluded this case to simplify the proof.

3. Classification of central extensions

3.1. Simply-connected case. In this subsection we remind the classification of central extensions of \( G^{sc}(F) \) by \( \mathbb{G}_m \) in relation with [3].

By [3], the central extensions of \( G^{sc} \) by (the sheaf version of) \( K_2 \) are classified by integer-valued \( W \)-invariant quadratic forms on \( X_s(T^{sc}) \) (and have no automorphisms). Let \( Q \) be the unique \( \mathbb{Z} \)-valued quadratic form on \( X_s(T^{sc}) \) satisfying \( Q(\alpha) = 1 \) for a short coroot \( \alpha \). So, \( (\lambda_1, \lambda_2) = Q(\lambda_1 + \lambda_2) - Q(\lambda_1) - Q(\lambda_2) \) for \( \lambda_i \in X_s(T^{sc}) \). Let

\[ 1 \to K_2 \to E_Q \to G^{sc} \to 1 \tag{7} \]

denote the central extension corresponding to \( Q \).

Write \( v(f) \) for the valuation of \( f \in F^* \). Write \((\cdot, \cdot)_{st} \) for the tame symbol given by

\[ (f, g)_{st} = (-1)^{v(f)v(g)}(g^{v(f)} f^{-v(g)})(0) \]
for \( f, g \in F^* \). We may view it as a map \( K_2(F) \to k^* \). Taking the \( F \)-valued points of (7) and further the push-forward by the tame symbol, one gets a central extension

\[
1 \to k^* \to \tilde{G} \to G^{sc}(F) \to 1
\]

(8)

For \( \theta \in \pi_1(G) \) write \( Gr_G^\theta \) for the connected component of \( Gr_G \) that contains \( t^\lambda G(O) \) for \( \lambda \in X_*(T) \) whose image in \( \pi_1(G) \) equals \( \theta \). The natural map \( Gr_G^{sc} \to Gr_G^0 \) is an isomorphism.

From [5] one knows that there is a line bundle \( \mathcal{L} \) on \( Gr_G \) generating the Picard group \( \text{Pic}(Gr_G^\theta) \isom \mathbb{Z} \) of each connected component \( Gr_G^\theta \) of \( Gr_G \), and an isomorphism \( \mathcal{L}^{2h} \isom \mathcal{L} \). Write \( G_Q \) for the punctured total space of the inverse image of \( \mathcal{L} \) under \( G^{sc}(F) \to Gr_G, x \mapsto xG(O) \). It can be seen as the Mumford extension

\[
1 \to G_m \to \tilde{G} \to G^{sc}(F) \to 1,
\]

(9)

that is, the ind-scheme classifying pairs \((g \in G^{sc}(F), \sigma)\), where \( \sigma : g^*\mathcal{L} \isom \mathcal{L} \) is an isomorphism over \( Gr_G^0 \). The central extension (9) splits canonically over \( G^{sc}(O) \). Any central extension of \( G^{sc}(F) \) by \( G_m \) is a multiple of (9) and has no automorphisms. In the rest of this section we prove the following.

**Lemma 5.** Passing to \( k \)-points in (9) one gets a central extension isomorphic to (8).

For a central extension \( 1 \to A \to E \to H \to 1 \) we write \((.,.)_c : H \times H \to A \) for the corresponding commutator given by

\[
(h_1, h_2)_c = \bar{h}_1 \bar{h}_2 \bar{h}_1^{-1} \bar{h}_2^{-1},
\]

where \( \bar{h}_i \) is any lifting of \( h_i \) to \( E \). If \( H \) is abelian then the commutator \((h_1, h_2)_c \) depends only on the isomorphism class of the central extension.

Note that \( T(F) = X_*(T) \otimes \mathbb{Z} \mathbb{F}^* \). For \( f_i \in \mathbb{F}^* \) and \( \lambda_i \in X_*(T) \) the commutator for the central extension (1) is given by

\[
(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)^{2\bar{h}(\lambda_1, \lambda_2)}
\]

Indeed, for \( \lambda \in X_*(T) \), \( f \in \mathbb{F}^* \) the fibre of \( \mathcal{L} \) at \( \lambda \otimes f \) identifies as \( \mathbb{Z}/2\mathbb{Z} \)-graded line with

\[
\otimes_{\alpha \in R^*} \det(g^\alpha(O) : f^{(\lambda, \alpha)} g^\alpha(O))
\]

and, using Lemma 1, it suffices to calculate the commutator of the canonical central extension of \( G_m(F) \) by \( G_m \) given by the relative determinant. But the latter commutator is given by the tame symbol (cf. [3], 12.13, p. 82). We learn that the commutator for the central extension (9) is given on the torus by

\[
(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)^{\bar{h}(\lambda_1, \lambda_2)}
\]

for \( \lambda_i \in X_*(T^{sc}), f_i \in \mathbb{F}^* \).
We will check that the commutators corresponding to (9) and to (8) are the same on $T^{sc}(\mathbf{F})$. The commutator for (8) can be calculated using, for example, ([3], Proposition 11.11, p. 77). Namely, consider first the case of $G^{sc} = \text{SL}_2$. In this case identify $T^{sc}$ with $\mathbb{G}_m$ via the positive coroot $\alpha : \mathbb{G}_m \cong T^{sc}$ then the commutator for (8) becomes

$$(f_1, f_2)_c = (f_1, f_2)^2_{st}$$

Indeed, for $h_i \in T^{sc}$ consider in the notation of ([3], formula (11.1.4) on p. 73) Steinberg’s cocycle $c(h_1, h_2) \in K_2(F)$. The image of $c(h_1, h_2)$ under the tame symbol $K_2(F) \to k^*$ equals $(h_1, h_2)_{st}$. So, the commutator $(f_1, f_2)_c$ is the image of $c(f_1, f_2)_c$ under the tame symbol $K_2(F) \to k^*$. For $G^{sc} = \text{SL}_2$ our assertion follows.

The general case can be reduced to the case $G^{sc} = \text{SL}_2$ by restricting to the $\text{SL}_2$-subgroups $\tilde{S}_\alpha \subset G$ corresponding to the roots $\tilde{\alpha}$ as in ([3], Section 11.2, p. 74). We are done.

Remark 5. For $\lambda \in X_*(T^{sc})$ we have $(\lambda, \lambda) \in 2\mathbb{Z}$. Indeed, $(\lambda, \lambda) = Q(2\lambda) - 2Q(\lambda) = 2Q(\lambda)$.

3.2. Proof of Proposition 1. The idea of the argument below was communicated to us by Drinfeld.

For $m \in \mathbb{Z}$ write $\tilde{G}_{mq}$ for the $m$-th multiple of the central extension $\tilde{G}_Q$. There is a canonical action $\delta_0$ of $G(F)$ on the exact sequence

$$(10) \quad 1 \to \mathbb{G}_m \to \tilde{G}_{mq} \to G^{sc}(F) \to 1$$

such that $G(F)$ acts trivially on $\mathbb{G}_m$ and by conjugation on $G^{sc}(F)$. Indeed, we know that the extension (9) comes from the canonical extension (7), so that the automorphisms of $G^{sc}$ act on it. Write $\delta_0 : G(F) \times \tilde{G}_{mq} \to \tilde{G}_{mq}$ for the action map.

If $\lambda \in X_*(T^{sc}), \mu \in X_*(T), f, g \in F^*$ then $\mu \otimes g \in T(F) \subset G(F)$ acts on the fibre of $\tilde{G}_{mq}$ over $\lambda \otimes f \in T^{sc}(F)$ via $\delta_0$ as a multiplication by

$$(g, f)_{st}^m(\mu, \lambda)$$

This is a kind of ‘explanation’ of the fact that the form $(\cdot, \cdot)$ initially defined on $X_*(T^{sc})$ extends by linearity to a form $(\cdot, \cdot) : X_*(T) \times X_*(T^{sc}) \to \mathbb{Z}$ taking values in $\mathbb{Z}$ and not just in $\mathbb{Q}$.

3.2.1. The isomorphism classes of central extensions

$$(11) \quad 1 \to \mathbb{G}_m \to \mathcal{E}_b \to T(F) \to 1$$
are classified by symmetric bilinear forms \((.,.)_b : X_*(T) \times X_*(T) \to \mathbb{Z}\), namely for the corresponding extension (11) we have

\[(\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c = (f_1, f_2)^{(\lambda_1, \lambda_2)}_b\]

for \(f_i \in F^*, \lambda_i \in X_*(T)\).

The group of automorphisms of the central extension (11) is \(\text{Hom}(T(F), \mathbb{G}_m)\). Since \(T(F)\) is abelian, the commutator \((\lambda_1 \otimes f_1, \lambda_2 \otimes f_2)_c\) is invariant under these automorphisms. The extension (11) admits a (non unique) splitting over \(T(O)\).

3.2.2. The group \(T\) acts on \(G_{sc}\) by conjugation, let \(\bar{G} = G_{sc} \ltimes T\) denote the corresponding semi-direct product. The map \(G_{sc} \ltimes T \to G, (g, t) \mapsto \bar{g}t\), where \(\bar{g}\) is the image of \(g \in G_{sc}\) in \(G\), yields an exact sequence \(1 \to T_{sc} \to \bar{G} \to G \to 1\). Hence, an exact sequence

\[(12) \quad 1 \to T_{sc}(F) \to \bar{G}(F) \to G(F) \to 1\]

The category of central extensions of \(G(F)\) by \(\mathbb{G}_m\) is equivalent to the category of pairs: a central extension

\[(13) \quad 1 \to \mathbb{G}_m \to ? \to \bar{G}(F) \to 1\]

together with a splitting of its pull-back under \(T_{sc}(F) \to \bar{G}(F)\).

By a slight (we have to drop off the assumption of being of finite type) generalization of ([3], Construction 1.7), the category of central extensions (13) is equivalent to the category of triples: central extensions (10) and (11) together with an action \(\delta\) of \(T(F)\) on \(\bar{G}_{mQ}\) extending the action of \(T(F)\) on \(G_{sc}(F)\) by conjugation. Write \(\delta : T(F) \times \bar{G}_{mQ} \to \bar{G}_{mQ}\) for the action map.

Since (10) has no automorphisms, \(\delta\) coincides with the restriction of \(\delta_0\) to \(T(F) \times \bar{G}_{mQ}\). Write

\[(14) \quad 1 \to \mathbb{G}_m \to \mathbb{G}_{mQ}^T \to T_{sc}(F) \to 1\]

for the restriction of (10) to \(T_{sc}(F)\). We conclude that the category of central extensions of \(G(F)\) by \(\mathbb{G}_m\) is equivalent to the category of pairs: a central extension (11) together with an isomorphism of its restriction to \(T_{sc}(F)\) with (14). Clearly, the corresponding form \((.,.)_b : X_*(T) \times X_*(T) \to \mathbb{Z}\) is given by \((\lambda_1, \lambda_2)_b = m(\lambda_1, \lambda_2)\). Proposition 1 follows.

4. Proof of Theorem 1

4.1. Functors \(F'_p\). Let \(P\) be a parabolic subgroup of \(G\) containing \(B\). Let \(M \subset P\) be a Levi subgroup containing \(T\). Write

\[1 \to \mathbb{G}_m \to E_M^a \to M(F) \to 1\]

for the restriction of (1) to \(M(F)\), it is equipped with an action of \(\text{Aut}^0(O)\) and a section over \(M(O)\) coming from the corresponding objects for (1). Let \(\text{Perv}_{M,G,N}\)
denote the category of $M(O)$-equivariant $\overline{\mathbb{T}}$-perverse sheaves on $E^a_M/M(O)$ with $\mathbb{G}_m$-monodromy $\zeta_a$. Set

$$\mathbb{Perv}_{M,G,N} = \mathbb{Perv}_{M,G,N}[-1] \subset D(E^a_M/M(O))$$

Write $\mathfrak{J}_{M,G}$ for the restriction of $\mathfrak{J}$ under $Gr_M \to Gr_G$, we equip it with the action of $\text{Aut}^0(O)$ coming from that on $\mathfrak{J}$.

Write $Gr_M$ for the affine grassmanian for $M$. The connected components of $Gr_M$ are indexed by $\pi_1(M)$. For $\theta \in \pi_1(M)$ write $Gr^\theta_M$ for the connected component of $Gr_M$ containing $t^\lambda M(O)$ for any coweight $\lambda$ whose image in $\pi_1(M)$ is $\theta$. The diagram $M \leftarrow P \leftarrow G$ yields the following diagram of affine grassmanians

$$Gr_M \xleftarrow{t_p} Gr_P \xrightarrow{s_p} Gr_G$$

The map $t_p$ yields a bijection between the connected components of $Gr_P$ and those of $Gr_M$. Let $Gr^\theta_P$ be the connected component of $Gr_P$ such that $t_p$ restricts to a map $t_p^\theta : Gr^\theta_P \to Gr^\theta_M$. Write $s_p^\theta : Gr^\theta_P \to Gr_G$ for the restriction of $s_p$. The restriction of $s_p^\theta$ to $(Gr^\theta_P)_{red}$ is a closed immersion.

The section $M \leftarrow P$ yields a section $r_P : Gr_M \to Gr_P$ of $t_P$. By abuse of notation, we write

$$\text{Gra}_M \xleftarrow{r_P} \text{Gra}_P \xrightarrow{s_P} \text{Gra}_G$$

for the diagram obtained from $Gr_M \xleftarrow{t_P} Gr_P \xrightarrow{s_P} Gr_G$ by the base change $\text{Gra}_G \to Gr_G$. Clearly, $t_P$ lifts naturally to a map $t_P : \text{Gra}_P \to \text{Gra}_M$. Define the functor

$$F'_P : \mathbb{Perv}_{G,N} \to D(\text{Gra}_M)$$

by $F'_P(K) = t_p s_P^\theta K$. Write $\text{Gra}^\theta_M$ (resp., $\text{Gra}_P^\theta$) for the connected component of $\text{Gra}_M$ (resp., $\text{Gra}_P$) over $Gr^\theta_M$ (resp., $Gr^\theta_P$). Write

$$\mathbb{Perv}_{M,G,N} \subset \mathbb{Perv}_{M,G,N}$$

for the full subcategory of objects that vanish off $\text{Gra}^\theta_M$. Set

$$\mathbb{Perv}'_{M,G,N} = \bigoplus_{\theta \in \pi_1(M)} \mathbb{Perv}^\theta_{M,G,N}[\theta, 2\hat{\rho}_M - 2\hat{\rho}]$$

As in ([1], 5.3.29) one shows that $F'_P$ sends $\mathbb{Perv}_{G,N}$ to $\mathbb{Perv}'_{M,G,N}$ (cf. also [9], appendix A.4).

The above construction applied to the Borel subgroup yields a functor $F'_B : \mathbb{Perv}_{G,N} \to \mathbb{Perv}'_{T,G,N}$.

Let $B(M) \subset M$ be a Borel subgroup containing $T$ such that the preimage of $B(M)$ under $P \to M$ equals $B$. The functor $F'_B(M) : \mathbb{Perv}^T_{M,G,N} \to D(\text{Gra}_T)$ is defined as follows. As above, the inclusions $T \subset B(M) \subset M$ yield a diagram

$$Gr_T \xleftarrow{t_B(M)} Gr_{B(M)} \xrightarrow{s_B(M)} Gr_M$$

Write

$$\text{Gra}_T \xleftarrow{r_B(M)} \text{Gra}_{B(M)} \xrightarrow{s_B(M)} \text{Gra}_M$$
for the diagram obtained from (15) by the base change $\text{Gra}_M \to \text{Gr}_M$. The
projection $B(M) \to T$ yields $t_{B(M)} : \text{Gr}_{B(M)} \to \text{Gr}_T$, which lifts naturally to $t_{B(M)} : \text{Gra}_{B(M)} \to \text{Gra}_T$. For $K \in \text{Perv}'_{M,G,N}$ set
\[ F'_{B(M)}(K) = (t_{B(M)})!s^*_B(M)K \]
As in ([1], 5.3.29), one shows that $F'_{B(M)}$ is a functor
\[ F'_{B(M)} : \text{Perv}'_{M,G,N} \to \text{Perv}'_{T,G,N} \]
By base change, we have canonically
\[ F'_{B(M)} \circ F'_P \simeq F'_B \]

4.1.1. Write $X^{+M}(T) \subset X_*(T)$ for the coweights of $T$ dominant for $M$. For
$\lambda \in X^{+M}_*(T)$ denote by $\text{Gr}_M^\lambda$ the $M(O)$-orbit through $t^\lambda M(O)$. Let $\text{Gra}_M^\lambda$ be the
preimage of $\text{Gr}_M^\lambda$ under $\text{Gra}_M \to \text{Gr}_M$. The $M$-orbit on $\text{Gr}_M$ through $t^\lambda M(O)$
is isomorphic to a partial flag variety $B^\lambda_M = M/P^\lambda_M$, where the Levi subgroup
of $P^\lambda_M$ has the Weyl group coinciding with the stabilizer of $\lambda$ in $W_M$. Write
$\tilde{\omega}_{M,\lambda} : \text{Gr}_M^\lambda \to B^\lambda_M$ for the projection. As in Lemma 2, one gets a
$\mathbb{Z}/2\mathbb{Z}$-graded isomorphism
\[ \mathcal{L}_{M,G} \big|_{\text{Gr}_M^\lambda} \simeq \Omega_e^{h(\lambda,\lambda)} \otimes \tilde{\omega}_{M,\lambda}^* \mathcal{O}(2\tilde{h}(\lambda)), \]
here the line bundles $\mathcal{O}(\tilde{\nu})$ on $B^\lambda_M$ are defined as in Section 2.1. Set
\[ X^{+M}_*(\tilde{T}_N) = \{ \lambda \in X^{+M}_*(T) \mid d(\lambda) \in NX^*_N \} \]
As for $G$ itself, for $\lambda \in X^{+M}_*(T)$ the scheme $\text{Gra}_M^\lambda$ admits a $M(O)$-equivariant
local system with $\mathbb{G}_m$-monodromy $\zeta_a$ iff $\lambda \in X^{+M}_M(\tilde{T}_N)$. For $\lambda \in X^{+M}_M(\tilde{T}_N)$ denote
by $A^\lambda_{M,E}$ the irreducible object of $\text{Perv}_{M,G,N}$ defined as in Section 2.1.

4.1.2. More tensor structures. One equips $\text{Perv}_{M,G,N}$ and $\text{Perv}'_{M,G,N}$ with a con-
volution product as in Section 2.2. Let us define the commutativity constraint
on these categories via fusion.
Recall the line bundles $\mathcal{L}_{X^m}$ on $\text{Gr}_{G,X^m}$ from Section 2.3. For the convenience
of the reader we remind the factorization structure on these line bundles, which
allowed to do fusion for $\text{Perv}_{G,N}$.
For a surjective map of finite sets $\alpha : J \to I$ one has a cartesian square
\[
\begin{array}{ccc}
\text{Gr}_{G,X^I} & \xrightarrow{\Delta^\alpha} & \text{Gr}_{G,X^J} \\
\downarrow & & \downarrow \\
X^I & \xrightarrow{\Delta^\alpha} & X^J,
\end{array}
\]
where $\Delta^\alpha$ is the corresponding diagonal. We have canonically $(\tilde{\Delta}^\alpha)^* \mathcal{L}_{X^J} \simeq \mathcal{L}_{X^I}$.
Write $\nu^\alpha : U^\alpha \hookrightarrow X^J$ for the open subscheme given by the condition that the divisors $D_i$ do not intersect pairwise, where $D_i = \sum_{j \in J, \alpha(j) = i} x_j$ for $(x_j) \in X^J$.

We have a cartesian square

$$
\begin{array}{ccc}
\prod_{i \in I} \text{Gr}_{G,X^i} & \xrightarrow{\nu^\alpha} & \text{Gr}_{G,X^J} \\
\downarrow & & \downarrow \\
U^\alpha & \xrightarrow{\nu^\alpha} & X^J
\end{array}
$$

We have canonically

$$(\tilde{\nu}^\alpha)^* \mathfrak{L}_{X^J} \cong (\mathfrak{L}_{\bigoplus_{i \in I} X^{\nu^\alpha(i)}}) |_{U^\alpha}$$

Let $\mathfrak{L}_{M,G,X^m}$ be the restriction of $\mathfrak{L}_{X^m}$ under the map $\text{Gr}_{M,X^m} \rightarrow \text{Gr}_{G,X^m}$ induced by $M \hookrightarrow G$. The collection $\{\mathfrak{L}_{M,G,X^m}\}$ is endowed with the induced factorization structure. Write $\text{Gra}_{M,G,X^m}$ for the punctured line bundle of $\mathfrak{L}_{M,G,X^m}$.

Let $M_{X^m}$ be the group scheme over $X^m$ classifying $\{(x_1, \ldots, x_m) \in X^m, \mu\}$, where $\mu$ is an automorphism of $\mathcal{F}_X^0$ restricted to the formal neighborhood of $x_1 \cup \ldots \cup x_m$ in $X$. The group scheme $M_{X^m}$ acts naturally on $\text{Gra}_{M,G,X^m}$. Write $\text{Perv}_{M,G,N,X^m}$ for the category of $M_{X^m}$-equivariant perverse sheaves on $\text{Gra}_{M,G,X^m}$ with $G_m$-monodromy $\zeta_a$. Set

$$\text{Perv}_{M,G,N,X^m} = \text{Perv}_{M,G,N,X^m}[-m-1]$$

Let $\text{Aut}_0^0(O)$ act on $\text{Gra}_M$ via its quotient $\text{Aut}_0^0(O)$. Then every object of $\text{Perv}_{M,G,N}$ admits a unique $\text{Aut}_0^0(O)$-equivariant structure. Note that $\text{Gra}_{M,X} \xrightarrow{\sim} \hat{X}_2 \times_{\text{Aut}_0^0(O)} \text{Gra}_M$. As in Section 2.3, we get a fully faithful functor

$$\tau^0 : \text{Perv}_{M,G,N} \rightarrow \text{Perv}_{M,G,N,X}$$

Now we define the commutativity constraint on $\text{Perv}_{M,G,N}$ and $\text{Perv}'_{M,G,N}$ using the above factorization structure as in Section 2.3. As in ([1], 5.3.16) one checks that $\text{Perv}_{M,G,N}$ and $\text{Perv}'_{M,G,N}$ are symmetric monoidal categories.

**Lemma 6.** The functors $F'_P$, $F'_{B(M)}$, $F'_B$ are tensor functors, and (16) is an isomorphism of tensor functors.

**Proof** We will only check that $F'_P$ is a tensor functor, the rest is similar.

1) Let us show that $F'_P$ is compatible with the convolution. Let $\text{Gr}_{P,X^m}$ be the ind-scheme classifying $(x_1, \ldots, x_m) \in X^m$, a $P$-torsor $\mathcal{F}_P$ on $X$, and a trivialization $\mathcal{F}_P \cong \mathcal{F}_P |_{X \setminus \cup_i x_i}$. Write $\text{Gra}_{P,X^m}$ for the ind-scheme obtained from $\text{Gr}_{G,X^m}$ by the base change $\text{Gr}_{P,X^m} \rightarrow \text{Gr}_{G,X^m}$. As in Section 4.1, we get a diagram

$$
\text{Gra}_{M,X^m} \xleftarrow{t_{P,X^m}} \text{Gra}_{P,X^m} \xrightarrow{s_{P,X^m}} \text{Gra}_{G,X^m}
$$

and a functor

$$F'_P : \text{D}(\text{Gra}_{G,X^m}) \rightarrow \text{D}(\text{Gra}_{M,G,X^m})$$
given by $F'_{P,X^2}(K) = (t_{P,X^m})^*F_{P,X^m}$. For $i = 1, 2$ let $F_i \in \mathbb{P}erv_{G,N}$ and $K_i = \tau^0 F_i$. Recall that $U \subset X^2$ is the complement to the diagonal. We have a natural diagram, where both squares are cartesian

\[
\begin{array}{ccc}
(Gra_{G,X} \times Gra_{G,X}) \mid_U & \xrightarrow{\nu_{G,U}} & Gra_{G,X^2} \\
\uparrow & & \uparrow s_{P,X^2} \\
(Gra_{P,X} \times Gra_{P,X}) \mid_U & \rightarrow & Gra_{P,X^2} \\
\downarrow & & \downarrow t_{P,X^2} \\
(Gra_{M,X} \times Gra_{M,X}) \mid_U & \xrightarrow{\nu_{M,U}} & Gra_{M,X^2},
\end{array}
\]

and the maps $\nu_{G,U}$ and $\nu_{M,U}$ come from the above factorization structures. As in ([9], Proposition 14), one shows that $F'_{P,X^2}(K_1 \ast_X K_2)$ is the Goresky-MacPherson extension from $Gra_{M,X^2} \mid_U$. Now the isomorphism

\[\nu_{M,U}^*F'_{P,X^2}(K_1 \ast_X K_2) \cong \tau^0(F'_P(F_1)) \boxtimes \tau^0(F'_P(F_2))\]

yields an isomorphism

\[\epsilon_{12} : F'_{P,X^2}(K_1 \ast_X K_2) \cong \tau^0(F'_P(F_1)) \ast_X \tau^0(F'_P(F_2))\]

Restricting it to the diagonal in $X$ one gets

\[\tau^0(F'_P(F_1 \ast F_2)) \cong \tau^0(F'_P(F_1) \ast F'_P(F_2))\]

2) Let us check that $F'_P$ is compatible with the commutativity constraints. Recall that $\sigma$ is the involution of $X^2$ permuting the two coordinates. One has a commutative diagram, where the vertical arrows are canonical isomorphisms

\[
\begin{array}{ccc}
\sigma^*F'_{P,X^2}(K_1 \ast_X K_2) & \xrightarrow{\sigma^*\epsilon_{12}} & \sigma^*(\tau^0(F'_P(F_1)) \ast_X \tau^0(F'_P(F_2))) \\
\downarrow & & \downarrow \\
F'_{P,X^2}(K_2 \ast_X K_1) & \xrightarrow{\epsilon_{21}} & \tau^0(F'_P(F_2)) \ast_X \tau^0(F'_P(F_1))
\end{array}
\]

Restricting it to the diagonal, one gets the desired compatibility isomorphism. □

4.2. **Fibre functor.** In Section 4.1 we introduced the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of parity zero) line bundle $\mathcal{L}_{T,G}$ on $Gr_T$. The action of $T(\mathcal{O})$ on this line bundle comes from the action of $T(\mathcal{O})$ on $E^T_{Z}$ by left multiplication. The fibre of $\mathcal{L}_{T,G}$ at $\nu \in X*(T)$ is $T(\mathcal{O})$-equivariantly isomorphic to $\Omega^b_{\nu} \mathcal{L}_{\nu}$, where $T(\mathcal{O})$ acts on the latter space via $T(\mathcal{O}) \to T \xrightarrow{2h_{\nu}(\nu)} G_m$. Recall the torus $\tilde{T}_N$ introduced in Section 2.4. For $\nu \in X*(T)$ the orbit $\text{Gra}_{\nu}^u$ supports a nonzero object of $\mathbb{P}erv_{T,G,N}$ iff $\nu \in X^*(\tilde{T}_N)$.

For $\nu \in X^*(\tilde{T}_N)$ consider the map $a_{\nu} : E^d_{\nu}/N \to \{0\} \to \Omega^b_{\nu} \mathcal{L}_{\nu} - \{0\}$ sending $x$ to $x^{2h_{\nu}/d}$. For $K \in \mathbb{P}erv_{T,G,N}$ the complex $a_{\nu}^*K$ is a constant sheaf placed in degree zero, so we view it as a vector space denoted $F^\nu_T(K)$. Then

\[F_T = \bigoplus_{\nu \in X^*(\tilde{T}_N)} F^\nu_T\]
is a fibre functor \( \text{Perv}_{T,G,N} \to \text{Vect} \). Let \( \tilde{T}_N = \text{Spec} \ k[X^*(\tilde{T}_N)] \) be the torus whose weight lattice is \( X^*(\tilde{T}_N) \). By ([4], Theorem 2.1), we get

\[
\text{Perv}_{T,G,N} \cong \text{Rep}(\tilde{T}_N)
\]

For \( \nu \in X^*(\tilde{T}_N) \) write \( F'_{B(M)}^{\nu} \) for the functor \( F'_{B(M)} \) followed by restriction to \( \text{Gra}^{\nu}_T \). Write \( F'_{M} : \text{Perv}_{M,G,N} \to \text{Vect} \) for the functor

\[
F'_{M} F'_{B(M)}^{\nu}([\nu, 2\tilde{\rho}_M])
\]

For \( \nu \in X^*(\tilde{T}_N) \) any \( x \in \mathbf{C}_e^{d(\nu,\nu)/N} \) yields a section \( a_{B(M),\nu} : \text{Gr}^\nu_{B(M)} \to \text{Gra}^{\nu}_T \) of the projection \( \text{Gra}^{\nu}_{B(M)} \to \text{Gr}^\nu_{B(M)} \) sending \( x \) to \( x^{2h/N} \).

**Lemma 7.** If \( \nu \in X^*(\tilde{T}_N), \lambda \in X^*_M(\tilde{T}_N) \) then \( F'_{M}(A^\lambda_{M,E}) \) has a canonical components base consisting of those connected components of

\[
\text{Gr}_{B(M)}^{\nu} \cap \text{Gr}_{M}^{\lambda}
\]

over which the (shifted) local system \( a^*_{B(M),\nu} \text{A}^\lambda_{M,E} \) is constant. In particular, \( F_{M}^{\lambda}(\text{A}^\lambda_{E}) \cong \mathbb{Q}_e \) canonically, and \( F^{w(\lambda)}(\text{A}^\lambda_{E}) \cong \mathbb{Q}_e \) for \( w \in W_M \).

**Proof** The first claim is proved as in ([10], Proposition 3.10). If \( \lambda \in X^*_M(\tilde{T}_N) \) then \( \text{Gr}_{B(M)}^{\rho(\lambda)} \cap \text{Gr}_{M}^{\lambda} \) is an affine space ([10], proof of Theorem 3.2), and the local system \( a^*_{B(M),\lambda} \text{A}^\lambda_{M,E} \) is constant. \( \square \)

Consider the following \( \mathbb{Z}/2\mathbb{Z} \)-grading on \( \text{Perv}^{\nu}_{M,G,N} \). For \( \theta \in \pi_1(M) \) call an object of \( \text{Perv}^{\theta}_{M,G,N}([\theta, 2\tilde{\rho}_M - 2\tilde{\rho}]) \) of parity \( \langle \theta, 2\tilde{\rho} \rangle \) mod 2. Write \( E^{\theta}_{M} \) for the connected component of \( E^{\theta}_{M} \) such that

\[
E^{\theta}_{M}/M(\text{O}) = \text{Gra}^{\theta}_{M}
\]

The product in \( E^{\theta}_{M} \) is compatible with this \( \mathbb{Z}/2\mathbb{Z} \)-grading of \( \pi_1(M) \), so the \( \mathbb{Z}/2\mathbb{Z} \)-grading we get on \( \text{Perv}^{\nu}_{M,G,N} \) is compatible with the tensor structure. In particular, for \( M = G \) we get a \( \mathbb{Z}/2\mathbb{Z} \)-grading on \( \text{Perv}_{G,N} \). If \( (\text{Gra}^{\theta}_{G})_{\text{red}} \) is contained in the connected component \( \text{Gra}^{\theta}_{G} \) of \( \text{Gra}^{\theta}_{C} \) then \( \theta \) is the image of \( \theta \) in \( \pi_1(G) \). So, the functors \( F'_{T} \) and \( F'_{B(M)} \) are compatible with these gradings.

Write \( \text{Vect}^{\epsilon} \) for the tensor category of \( \mathbb{Z}/2\mathbb{Z} \)-graded vector spaces. Let \( \text{Perv}^{\epsilon}_{M,G,N} \) be the category of even objects in \( \text{Perv}^{\epsilon}_{M,G,N} \otimes \text{Vect}^{\epsilon} \). Let \( \text{Perv}^{\epsilon}_{G,N} \) be the category of even objects in \( 
\text{Perv}^{\epsilon}_{G,N} \otimes \text{Vect}^{\epsilon} \). We get a canonical equivalence of tensor categories \( sh : \text{Perv}^{\epsilon}_{T,G,N} \cong \text{Perv}^{\epsilon}_{T,G,N} \). The functors \( F'_{B(M)}^{\nu}, F'_{p}, F'_{B} \) yield tensor functors

\[
\text{Perv}^{\epsilon}_{G,N} \xrightarrow{F'_{B}} \text{Perv}^{\epsilon}_{M,G,N} \xrightarrow{F'_{B(M)}} \text{Perv}^{\epsilon}_{T,G,N}
\]

whose composition is \( F'_{B} \). Write \( F^{\epsilon} : \text{Perv}^{\epsilon}_{G,N} \to \text{Vect} \) for \( F_{T} \circ sh \circ F_{B}^{\epsilon} \). By Lemma 7, \( F^{\epsilon} \) does not annihilate a nonzero object, so it is faithful. By Remark 3,
\[\text{Lemma 8.} \text{ If } \lambda, \mu \in X^{x+}(\tilde{T}_N) \text{ then } A^{x+}_E \text{ appears in } A^x_E \ast A^\mu_E \text{ with multiplicity one.}\]

**Proof** Write \(\tilde{E}^{a,\lambda} (\text{resp., } E^{a,\lambda})\) for the preimage of \(\text{Gra}_G^{\lambda}\) (resp., \(\text{Gra}_G^\lambda\)) under \(E^a \to \text{Gra}_G, x \mapsto xG(O)\). Write \(m^{x,\mu} : E^{a,\lambda} \times_{G(O) \times G_m} \text{Gra}_G^\mu \to \text{Gra}_G^{x+} \mu\) for the convolution diagram as in Section 2.2. If \(W\) is the preimage of \(\text{Gra}_G^{x+} \mu\) under \(m^{x,\mu}\), then \(m^{x,\mu} : W \to \text{Gra}_G^{x+} \mu\) is an isomorphism, and \(W\) is open in \(E^{a,\lambda} \times_{G(O) \times G_m} \text{Gra}_G^\mu\). □

Clearly, if \(X^{x+}(\tilde{T}_N)\) is a \(Z_+\)-span of \(\lambda_1, \ldots, \lambda_r\) then \(\bigoplus_r A^{x,\lambda}_E\) is a tensor generator for \(\text{Perv}_{G,N}\). So, \(\tilde{G}_N\) is algebraic by ([4], Proposition 2.20). By Lemma 7, for \(\mu \in X^{x+}(\tilde{T}_N)\) and \(w \in W\) the weight \(w(\mu)\) of \(\tilde{T}_N\) appears in \(F^x(A^x_E)\). So, \(\tilde{T}_N\) is closed in \(\tilde{G}_N\) by ([4], Proposition 2.21). By Lemma 8, there is no tensor subcategory of \(\text{Perv}_{G,N}\) whose objects are direct sums of finitely many fixed irreducible objects, so \(\tilde{G}_N\) is connected by ([4], Corollary 2.22). Since \(\text{Perv}_{G,N}\) is semisimple, \(\tilde{G}_N\) is reductive by ([4], Proposition 2.23). We will use the following.

**Lemma 9.** Let \(G\) be a connected reductive group with a maximal torus \(T \subset G\). Let \(\tilde{\Lambda}^+\) be a subsemigroup in the group \(\Lambda^+\) of weights of \(T\). Assume that we are given a bijection \(\nu \mapsto V^{\nu}\) between \(\tilde{\Lambda}^+\) and the set of irreducible representations of \(G\) such that the following holds:

- if \(\nu \in \tilde{\Lambda}^+\) then the \(\nu\)-weight space \(L^{\nu}\) of \(T\) in \(V^{\nu}\) is of dimension one;
- if \(\nu_1, \nu_2 \in \tilde{\Lambda}^+\) then \(V^{\nu_1+\nu_2}\) appears with multiplicity one in \(V^{\nu_1} \otimes V^{\nu_2}\), and the subspace \(L^{\nu_1} \otimes L^{\nu_2} \subset V^{\nu_1} \otimes V^{\nu_2}\) coincides with the image of \(L^{\nu_1+\nu_2} \hookrightarrow V^{\nu_1+\nu_2} \hookrightarrow V^{\nu_1} \otimes V^{\nu_2}\).

Then there is a unique Borel subgroup \(T \subset B \subset G\) such that \(\tilde{\Lambda}^+\) is the set of dominant weights for \(B\). □

Write \(V^{\nu}\) for the irreducible representation of \(\tilde{G}_N\) corresponding to \(A^{\nu}_E\) via (18).
Lemma 10. The torus \( \tilde{T}_N \) is maximal in \( \tilde{G}_N \). There is a unique Borel subgroup \( \tilde{T}_N \subset \tilde{B}_N \subset \tilde{G}_N \) whose set of dominant weights coincides with \( X^{\ast+}(\tilde{T}_N) \).

Proof. Let \( T' \subset \tilde{G}_N \) be a maximal torus containing \( \tilde{T}_N \). By Lemma 7, for each \( \nu \in X^{\ast+}(\tilde{T}_N) \) there is a unique character \( \nu' \) of \( T' \) such that the composition \( \tilde{T}_N \to T' \xrightarrow{\nu'} \mathbb{G}_m \) is \( \nu \), and the \( T' \)-weight \( \nu' \) appears in \( V^{\nu} \). Clearly, \( \nu \mapsto \nu' \) is a homomorphism of semigroups, and we can apply Lemma 9. Since \( \nu \mapsto \nu' \) is a bijection between \( X^{\ast+}(\tilde{T}_N) \) and the dominant weights of \( \tilde{B}_N \), \( \tilde{T}_N \) is maximal. \( \square \)

Applying similar arguments for \( M \), one checks that \( \tilde{M}_N \) is reductive, and \( \tilde{T}_N \to \tilde{M}_N \to \tilde{G}_N \) are closed immersions, so \( \tilde{M}_N \) is a Levi subgroup of \( \tilde{G}_N \).

4.4. Rank one. Let \( M \) be the subminimal Levi subgroup of \( G \) corresponding to the simple root \( \check{\alpha}_i \). As in Lemma 10, there is a unique Borel subgroup \( \tilde{T}_N \subset \tilde{B}(M)_N \subset \tilde{M}_N \) whose set of dominant weights is \( X^{\ast+}_M(\tilde{T}_N) \). View \( \check{\alpha}_i \) as a coweight of \( \tilde{T}_N \). Then

\[
\{ \check{\nu} \in X_*(\tilde{T}_N) \mid \langle \lambda, \check{\nu} \rangle \geq 0 \text{ for all } \lambda \in X^{\ast+}_M(\tilde{T}_N) \}
\]

is a \( \mathbb{Z}_+ \)-span of a multiple of \( \check{\alpha}_i \). So, \( \tilde{M}_N \) is of semisimple rank one, and its unique simple coroot is of the form \( \check{\alpha}_i/\kappa_i \) for some \( \kappa_i \in \mathbb{Q} \), \( \kappa_i > 0 \).

Take any \( \lambda \in X^{\ast+}_M(\tilde{T}_N) \) with \( \langle \lambda, \check{\alpha}_i \rangle > 0 \). Write \( s_i \in W \) for the simple reflection corresponding to \( \check{\alpha}_i \). By Lemma 7, \( F^\lambda_M(A^\lambda_{M,E}) \) and \( F^{s_i(\lambda)}_M(A^\lambda_{M,E}) \) do not vanish, so \( \lambda - s_i(\lambda) \) is a multiple of the positive root of \( \tilde{M}_N \). So, this positive root is \( \kappa_i \check{\alpha}_i \).

We learn that the simple reflection for \( (\tilde{M}_N, \tilde{T}_N) \) acts on \( X^*(\tilde{T}_N) \) as \( s_i \). So, the Weyl groups of \( G \) and of \( \tilde{G}_N \), viewed as subgroups of \( \text{Aut}(X^*(\tilde{T}_N)) \) are the same.

We must show that \( \kappa_i = \delta_i \).

Recall that the scheme \( \text{Gr}^{\nu}_{\tilde{B}(M)} \cap \text{Gr}^\lambda_M \) is non empty iff

\[
\nu = \lambda, \lambda - \alpha_i, \lambda - 2\alpha_i, \ldots, \lambda - \langle \lambda, \check{\alpha}_i \rangle \alpha_i
\]

For \( 0 < k < \langle \lambda, \check{\alpha}_i \rangle \) and \( \nu = \lambda - k\alpha_i \) one has

\[
\text{Gr}^{\nu}_{\tilde{B}(M)} \cap \text{Gr}^\lambda_M \cong \mathbb{G}_m \times A^{(\lambda, \check{\alpha}_i) - k - 1}
\]

Write \( M_0 \) for the derived group of \( M \), let \( T_0 \subset M_0 \) be the maximal torus such that \( T_0 \subset T \). Consider the central extension

\[
1 \to \mathbb{G}_m \to E^\alpha_{M_0} \to M_0(\mathbb{F}) \to 1
\]

obtained by pulling back of \( 1 \to \mathbb{G}_m \to E^\alpha_M \to M(\mathbb{F}) \to 1 \) via \( M_0(\mathbb{F}) \to M(\mathbb{F}) \).

It corresponds to the restriction of the bilinear form \( 2\check{h}_i \) under

\[
X_*(T_0) \times X_*(T_0) \subset X_*(T) \times X_*(T)
\]

So, \( E^\alpha_{M_0}/M_0(\mathbb{O}) \to \text{Gr}^\lambda_{M_0} \) is isomorphic to the punctured total space of \( \mathcal{L}^{h(\alpha_i, \alpha_i)}_{M_0} \), where \( \mathcal{L}_{M_0} \) is an ample generator of the Picard group of (each connected component of) \( \text{Gr}_{M_0} \).
Assume that $\lambda = a\alpha_i$ with $a > 0, a \in \mathbb{Z}$ such that $\lambda \in X^*(\hat{T}_N)$. Let $\nu = b\alpha_i$ with $b \in \mathbb{Z}$ satisfy $-\lambda < \nu < \lambda$.

Write $U \subset M(F)$ for the 1-parameter unipotent subgroup corresponding to the affine root space $t^{-a+b}\mathfrak{g}_{\alpha_i}$. Let $Y$ be the closure of the $U$-orbit through $t^\nu M(O)$ in $\text{Gr}_M$. It is a $T$-stable subscheme $Y \cong \mathbb{P}^1$, the $T$-fixed points in $Y$ are $t^\nu M(O)$ and $t^{-\lambda}M(O)$. The restriction of $L_{M_0}$ to $Y$ identifies with $\mathcal{O}_{\mathbb{P}^1}(a+b)$. The section $a_{B(M),\nu} : \text{Gr}^\nu_{B(M)} \to \text{Gr}^\nu_{A(M)}$ viewed as a section of the line bundle $\mathcal{L}^\beta_{M_0}$ over $Y$ will vanish only at $t^{-\lambda}M(O)$ with multiplicity $(a+b)\hat{h}(\alpha_i, \alpha_i)$. So, the local system $a_{B(M),\nu}^*\mathcal{A}_{M,E}^\lambda$ will have the $\mathbb{G}_m$-monodromy $\zeta_{(a+b)\hat{h}(\alpha_i, \alpha_i)}$. This local system is trivial iff

$$(a+b)\hat{h}(\alpha_i, \alpha_i) \in \frac{2\hat{h}N}{d}\mathbb{Z}$$

We may assume that $\frac{da}{2N}(\alpha_i, \alpha_i) \in \mathbb{Z}$. Then the above condition is equivalent to $b \in \frac{2N}{d(\alpha_i, \alpha_i)}\mathbb{Z}$. The smallest positive integer $b$ satisfying this condition is $\delta_i$. So, $\kappa_i = \delta_i$. Theorem 1 is proved.

References


S.L.: Institut Élie Cartan Nancy (Mathématiques), Université Henri Poincaré Nancy 1, B.P. 239, F-54506 Vandœuvre-lès-Nancy Cedex, France
Sergey.Lysenko@iecn.u-nancy.fr