ON 2-DIFFEOMORPHISMS WITH ONE-DIMENSIONAL BASIC SETS AND A FINITE NUMBER OF MODULI

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Abstract. This paper is a step towards the complete topological classification of Ω-stable diffeomorphisms on an orientable closed surface, aiming to give necessary and sufficient conditions for two such diffeomorphisms to be topologically conjugate without assuming that the diffeomorphisms are necessarily close to each other. In this paper we will establish such a classification within a certain class Ψ of Ω-stable diffeomorphisms defined below. To determine whether two diffeomorphisms from this class Ψ are topologically conjugate, we give (i) an algebraic description of the dynamics on their non-trivial basic sets, (ii) a geometric description of how invariant manifolds intersect, and (iii) define numerical invariants, called moduli, associated to orbits of tangency of stable and unstable manifolds of saddle periodic orbits. This description determines the scheme of a diffeomorphism, and we will show that two diffeomorphisms from Ψ are topologically conjugate if and only if their schemes agree.


Key words and phrases. A-diffeomorphism, moduli of stability, topological classification, expanding attractor.

1. Introduction and Formulation of the Results

The topological classification of structurally stable diffeomorphisms on closed orientable surfaces has made tremendous progress in the last 25 years, due to the work of C. Bonatti, V. Grines, R. Langevin, A. Zhirov, R. Plykin et al. (see for example [1], [3], [2], [9] for the history of the subject and more information). Any such classification naturally includes a description of its basic sets and a non-trivial description of how invariant manifolds of periodic points intersect. If invariant manifolds of saddles have tangencies, then the topological classification also involves expressions, called moduli, related to eigenvalues at saddle points, as was discovered by J. Palis [14].

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The first important step in the direction of a topological classification of \(\Omega\)-stable diffeomorphisms on orientable closed surfaces was made in the paper [11] by W. de Melo and S. J. van Strien, where they found necessary and sufficient conditions for \(\Omega\)-stable diffeomorphisms to have a finite number of moduli. (A diffeomorphism \(f\) is said to have a finite number of moduli if one can parametrise topological conjugacy classes of a neighbourhood of \(f\) by a finite number of parameters). Their result is local in the sense that it only considers the topological conjugacy of two diffeomorphisms which are sufficiently close to each other. To deal with the global situation, T. Mitryakova and O. Pochinka [12] partly generalised the previous result by construction a complete invariant for \(\Omega\)-stable diffeomorphisms for a certain class of \(\Omega\)-stable diffeomorphisms (with at most a finite number of periodic points) which can in general be “far” from each other.

Here we present the topological classification considering a wider class than in [12], within this class the existence of one-dimensional attractors and repellers is allowed. This class \(\Psi\) will be defined formally below.

Let \(M^2\) be an orientable closed surface and \(f : M^2 \to M^2\) be an A-diffeomorphism, i.e., an Axiom A diffeomorphism. By S. Smale [17], the non-wandering set \(\text{NW}(f)\) of \(f\) is represented as a finite union of disjoint closed invariant sets \(\Lambda_1, \ldots, \Lambda_k\), called basic sets, each of which contains a dense orbit. A basic set which consists of a periodic orbit will be called trivial and otherwise it is called non-trivial.

Let \(\Lambda\) be a one-dimensional basic set of \(f\). By R. Plykin [15], \(\Lambda\) is either an attractor or a repeller.

According to [5, Definition 3] a point \(p\) is called an \(s\)-boundary (\(u\)-boundary) point of attractor (repeller) \(\Lambda\), if one of the connected components of the set \(W^s_p \setminus p\) \((W^u_p \setminus p)\) is disjoint from \(\Lambda\); denote by \(\ell_p\) such a component (see Figure 1, where the construction of a DA-diffeomorphism is represented and where \(p_1, p_2\) are the \(s\)-boundary points).

Figure 1. DA-diffeomorphism
For definiteness we suppose that Λ is an attractor (all notions for repeller can be obtained by considering \( f^{-1} \)). By [6, Lemmas 2.4, 2.5], each \( s \)-boundary point is necessarily periodic and the set Λ has a non-empty and finite set of \( s \)-boundary points. We denote this set by \( P_\Lambda \).

**Definition 1** (Separable one-dimensional attractors). We say that a 1-dimensional attractor Λ of an \( A \)-diffeomorphism \( f \) is **separable** if a union \( Y_\Lambda \) of saddle and source trivial basic sets of the diffeomorphism \( f \) exists with the following properties:

1) \( \text{cl}(W^s_\Lambda) \setminus W^s_\Lambda = W^s_{\Lambda \Lambda} \);
2) \( \text{cl}(\ell_p) \setminus \ell_p = p \cup \alpha \) for every \( s \)-boundary point \( p \in P_\Lambda \), where \( \alpha \in Y_\Lambda \) is a source point;
3) for every saddle point \( \sigma \in Y_\Lambda \) the manifold \( W^s_\sigma \) does not contain heteroclinic points.

This definition is illustrated in Figure 3.

It follows from [7, Lemma 1, Lemma 2] that any one-dimensional basic set of a structural stable diffeomorphism \( f : M^2 \to M^2 \) is separable. We prove the following stronger result.

**Theorem 1.** If an \( \Omega \)-stable diffeomorphism \( f : M^2 \to M^2 \) has a finite number of moduli then any of its one-dimensional basic set is separable.

The proof of Theorem 1 is based on necessary and sufficient conditions, found in [11], under which a diffeomorphism of an orientable surface has a finite number of moduli of topological conjugacy, and described the structure of the neighborhood of such a diffeomorphism.

**Statement 1** (Criteria of a finite number moduli, [11]). Let \( f : M^2 \to M^2 \) be an \( \Omega \)-stable \( C^2 \)-diffeomorphism. Then \( f \) has a finite number moduli if and only if it satisfies the conditions below:

\[ \text{In fact the existence and finiteness of the set of boundary points without the term “boundary point” was proved by S. Newhouse and J. Palis in [13 Proposition 1].} \]
In these figures $\alpha, \omega, \sigma$ denote periodic points of source, sink and saddle type. In Figure (A) a separable one-dimensional attractor on the torus is shown by the dark curves and $\ell_p$ are the curves connecting $p_i$ to $\alpha_i$; In Figure (B) a situation is shown with a non-separable one-dimensional attractor on the pretzel by the dark curves; here the curves $\ell_p$ do not land on repelling fixed points, and so condition 2 in the definition of separable one-dimensional attractor is violated.

(1) if $x, y \in NW(f)$ are such that $W_x^u$ is not transverse to $W_y^s$ then the basic sets containing $x$ and $y$ are trivial;

(2) there is only a finite number of orbits of non-transverse intersections between stable and unstable manifolds and the contact between these manifolds along each of these orbits is of finite order;

(3) if $p, q$ are periodic points from trivial basic sets such that $W_p^u$ has an orbit of non-transverse intersection with $W_q^s$ then the number of orbits in $W_p^u$ (resp. in $W_q^s$) belonging to some unstable (resp. stable) manifolds of periodic saddle points of $f$ is finite;

(4) if $x$ is a point of non-transverse intersection of $W_p^u$ and $W_q^s$ then there exists an arc $\Sigma$ transverse to $W_x^u$ at $x$ such that no connected component of $\Sigma \setminus \{x\}$ contains points of both stable and unstable manifolds of saddles;

(5) if $W_p^u$ has a point of non-transverse intersection with $W_q^s$, and $W_q^u$ has a point of non-transverse intersection with $W_p^s$, then there is no saddle point of $f$ whose unstable manifold (resp. stable manifold) intersects $W_p^u$ (resp. $W_q^u$).
Definition 2 (The class \( \Psi \)). An orientation preserving \( \Omega \)-stable \( C^2 \)-diffeomorphism \( f: M^2 \to M^2 \) is called a diffeomorphism of class \( \Psi \) if it has a finite number of moduli and the following properties are satisfied:

1) each non-trivial basic set \( \Lambda \) of \( f \) is one-dimensional;
2) heteroclinic orbits can be contained in the stable or the unstable manifold of a periodic point of the trivial basic set of \( f \), but not in both.

![Figure 4. An example of a diffeomorphism \( f \) from \( \Psi \)](image)

Let \( f \in \Psi \) and \( W^u_x \) be not transverse to \( W^s_y \) for some saddle periodic points \( x, y \in NW(f) \). Set \( \Theta_{xy} = \frac{\ln |\lambda_x|}{\ln |\mu_y|} \), where \( \lambda_x \) is the eigenvalue of \( Df \) at \( x \) which is less than one by absolute value and \( \mu_y \) is the eigenvalue of \( Df \) at \( y \) which is greater than one by absolute value. Denote by \( \Psi^* \) the set of diffeomorphisms \( f \in \Psi \) such that \( \Theta_{xy} \) is an irrational for any such pair \( x, y \).

In Section 2 we introduce the notion of a scheme of diffeomorphism \( f \) containing

(i) an algebraic description of the dynamics on its non-trivial basic sets,
(ii) a geometric description of how invariant manifolds intersect,
(iii) numerical invariants, called moduli, associated to orbits of tangency of stable and unstable manifolds of saddle periodic orbits

and define an equivalence of two schemes.

The main result of this paper is the following theorem.

Theorem 2 (Classification with class \( \Psi \)). 1. If the schemes of diffeomorphisms \( f, f' \in \Psi \) are equivalent, then the diffeomorphisms are topologically conjugate.

2. Diffeomorphisms \( f, f' \in \Psi^* \) are topologically conjugate if and only if their schemes are equivalent.

2. Descriptions of Diffeomorphisms from \( \Psi \)

2.1. An algebraic description of the dynamics on one-dimensional basic set. Now let \( \Lambda \) be a 1-dimensional attractor of an \( \Lambda \)-diffeomorphism \( f: M^2 \to M^2 \). From \( \Psi \) \( \Lambda \) is represented as a finite union of disjoint compact sets \( \Lambda_1, \ldots, \Lambda_k \), which are cyclically transformed into each other under the action of \( f \). Moreover,
cl(W'_x ∩ Λ_i) = Λ_i and cl(W'_x ∩ Λ_i) = Λ_i for any point x ∈ Λ_i. Every Λ_i is called periodic (or C-dense) component of the basic set Λ. In this section we suppose that the attractor Λ consists of only one periodic component. We will now associate to Λ a closed neighbourhood N_Λ.

Definition 3 (The bunch of an attractor). A bunch b of an attractor Λ is the union of the maximal number r_b of the unstable manifolds W_{p_1}, ..., W_{p_{r_b}} of the s-boundary points p_1, ..., p_{r_b} of the set Λ whose separatrices ℓ_{p_1}, ..., ℓ_{p_{r_b}} belong to the same connected component of the set W_Λ \ Λ. The number r_b is said to be the degree of the bunch.

Let δ ∈ {u, s} and x ∈ Λ. For points y, z ∈ W_xδ (y ≠ z), let

\[ \{y, z\}^δ, \{y, z\}^δ, \{y, z\}^δ, \{y, z\}^δ \]

denote the connected arcs on the manifold W_xδ with the boundary points y, z.

Denote by B_Λ the set of all bunches of Λ. From the definition of a bunch b ∈ B_Λ of degree r_b it follows that there is a sequence of points x_1, ..., x_{r_b} such that:

1) x_2j−1, x_2j belong to the different connected components of the set W_{p_j} \ p_j;
2) x_{2j+1} ∈ W_{x_{2j}}(we set x_{2r_b+1} = x_1)
3) \( (x_{2j}, x_{2j+1})^{s} ∩ Λ = ∅, j = 1, ..., r_b \).

For each j ∈ {1, ..., r_b} we pick a pair of points \( x_{2j−1}, x_{2j} \) and a simple curve \( ℓ_j \) with boundary points \( x_{2j−1}, x_{2j} \) such that:

1) \( (x_{2j}, x_{2j+1})^{s} ∩ Λ = ∅, j = 1, ..., r_b \);
2) the curve \( ℓ_j \) transversally intersects at a unique point the stable manifold of any point on the arc \( (x_{2j−1}, x_{2j})^{u} \);
3) \( L_b = \bigcup_{j=1}^{r_b} [ℓ_j ∪ (x_{2j}, x_{2j+1})^{s}] \) is a simple closed smooth curve and the set \( L_Λ = \bigcup_{b ∈ B_Λ} L_b \) is such that:

a) \( f(L_Λ) \cap L_Λ = ∅ \);
b) for every curve \( L_b, b ∈ B_Λ \), there is a curve in the set \( f(L_Λ) \) such that these two curves are the boundaries of an annulus \( K_b \);
c) the annuli \( \{K_b, b ∈ B_Λ \} \) are pairwise disjoint (see Figure 5).

Let \( N_Λ = Λ \cup \bigcup_{n≥1} f^n(\bigcup_{b ∈ B_Λ} K_b) \). By construction the annuli \( \{K_b, b ∈ B_Λ \} \) consist of the wandering points of the diffeomorphism \( f, N_Λ \) is a surface with non-empty boundary and \( N_Λ \) is a neighbourhood of the attractor Λ, which we call the support of \( N_Λ \).

Let \( p_Λ : U_{N_Λ} → N_Λ \) be the universal covering where \( U_{N_Λ} \) is a subset of Lobachevsky plane and let \( G_{N_Λ} \) be the group of its covering transformations. Set \( E_{N_Λ} = ∂U_{N_Λ} \). A lift \( f_Λ : U_{N_Λ} → U_{N_Λ} \) of \( f_Λ = f|_{N_Λ} \) with respect to \( p_Λ \) induces an automorphism \( T_{f_Λ} \) of the group \( G_{N_Λ} \) acting by the formula \( T_{f_Λ}(g) = f_Λg^{-1}f_Λ^{-1} \). \( T_{f_Λ}(g) = f_Λg^{-1}f_Λ^{-1} \).

Let \( Λ = p_Λ^{-1}(Λ) \). If \( x ∈ Λ \) then let \( x ∈ Λ \) denote a point in the preimage \( p_Λ^{-1}(x) \) and let \( W_x^u \) be the connected component of \( p^{-1}(W_x^u) \) containing \( x \). Let us choose a parametrisation \( ℝ \ni t → W_x^u(t) \) of \( W_x^u \) such that \( W_x^u(0) = x \). Then \( W_x^u(t) \) is a point

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2 Stable (unstable) separatrix of a hyperbolic periodic point p is a connected component of the set \( W_p^u \setminus p(W_p^u \setminus p) \).
on \( w^s_\bar{x} \) such that \( p_\Lambda(w^s_\bar{x}(t)) = W^s_{\bar{x}}(t) \) and \( W^{s+}_{\bar{x}}, W^{s-}_{\bar{x}} \) are the connected components of the curve \( W^s_{\bar{x}} \setminus \bar{x} \) for \( t > 0, t < 0 \) respectively. For points \( \bar{y}, \bar{z} \in W^s_{\bar{x}} (\bar{y} \neq \bar{z}) \), let \( [\bar{y}, \bar{z}]^s, [\bar{y}, \bar{z}]^s, (\bar{y}, \bar{z})^s, (\bar{y}, \bar{z})^s \) denote the connected arcs on the manifold \( w^s_{\bar{x}} \) with boundary points \( \bar{y}, \bar{z} \).

**Definition 4** (Asymptotic direction). We say that a curve \( w^s_{\bar{x}} \) has the asymptotic direction \( s_{\bar{x}} \bar{x} \) for \( t \to \nu \infty \), \( \nu \in \{-, +\} \) if \( \text{cl}(w^s_{\bar{x}}) \setminus w^s_{\bar{x}} \) is equal to \( \bar{x} \) and \( s_{\bar{x}} \bar{x} \in E_{N_{\Lambda}} = \partial \mathcal{U}_{N_{\Lambda}} \).

As before, let \( P_\Lambda \) be the set of \( s \)-boundary points of \( \Lambda \). For a boundary point \( p \in P_\Lambda \) denote by \( \ell^s_p \) a connected component of \( W^s_{\bar{x}} \setminus \bar{p} \) different from \( \ell_{\bar{p}} \) and by \( \ell^s_{\bar{p}} \) the connected component of \( w^s_{\bar{x}} \setminus \bar{p} \) for which \( p_\Lambda(\ell^s_{\bar{p}}) = \ell^s_{\bar{p}} \).

It was proved in [8] that for each point \( x \in (\Lambda \setminus W^s_{P_\Lambda}) \) the curve \( w^s_{\bar{x}} \) has two distinct boundary points (asymptotic directions) \( s^+_{\bar{x}}, s^-_{\bar{x}} \). Finally, for every point \( p \in P_\Lambda \) the curve \( w^s_{\bar{x}} \) has an asymptotic direction \( s^\infty_{\bar{p}} \).

**Statement 2** (Conjugacy on one-dimensional attractors, [8]). Let \( \Lambda, \Lambda' \) be attractors such that there is an automorphism \( \psi_\Lambda : G_{\mathcal{N}_{\Lambda}} \to G_{\mathcal{N}_{\Lambda'}} \) with property \( T^\Lambda_{f_\Lambda} = \psi_\Lambda T^\Lambda_{f_\Lambda} \psi_\Lambda^{-1} \). Then

1) \( \psi_\Lambda \) is uniquely induces a homeomorphism \( \psi^\Lambda : E_{\mathcal{N}_{\Lambda}} \to E_{\mathcal{N}_{\Lambda'}} \):
2) for every point \( \bar{x} \in \Lambda \) there exists a unique point \( \bar{x}' \in \Lambda' \) such that
\[
\psi^*_\Lambda(\text{cl}(w^s_\Lambda) \cap E_{\Lambda'}) = \text{cl}(w^s_{\Lambda'}) \cap E_{\Lambda'},
\]
for \( \delta \in \{u, s\} \) and the map \( \hat{\varphi}_\Lambda : \Lambda \to \Lambda' \), assigning \( \bar{x}' \) to \( \bar{x} \), is a homeomorphism;
3) \( \hat{\varphi}_\Lambda \) induces the homeomorphism
\[
\varphi_\Lambda = p_{\Lambda'} \circ \hat{\varphi}_\Lambda \circ \hat{\varphi}_\Lambda^{-1} : \Lambda \to \Lambda'.
\]

Conjugating \( f|_\Lambda \) with \( f'|_{\Lambda'} \) and possesses the property: if \( a, b \in W^s_a, x \in \Lambda \), then \( \varphi_\Lambda(a), \varphi_\Lambda(b) \in W^s_{\hat{\varphi}_\Lambda(x)} \).

It immediately follows from Statement 2 that each isomorphism \( \psi_\Lambda \) with property \( T_{\psi_\Lambda} = \psi_\Lambda T_{f|_{\Lambda'}} \psi_\Lambda^{-1} \) uniquely induces a one-to-one map
\[
\hat{\psi}_\Lambda : P_\Lambda \to P_{\Lambda'}.
\]

2.2. Moduli associated to diffeomorphisms from the class \( \Psi \). For two diffeomorphisms from \( \Psi \) to be topologically conjugate certain moduli conditions have to be satisfied. Let us define these conditions now. For \( f \in \Psi \) denote by \( \Omega_f \) the set of trivial basic sets of \( f \), \( \Omega_0^f, \Omega^1_f, \Omega^2_f \) its subsets consisting of the sinks, saddles and sources, accordingly. For a saddle point \( \sigma \in \Omega^1_f \) of a diffeomorphism \( f \in \Psi \) denote by \( k_\sigma \) the period of \( \sigma \) and \( \mu_\sigma \), \( \lambda_\sigma \) denote the eigenvalues of \( Df^k_\sigma \) which are greater and less than one by absolute value, accordingly \( (|\mu_\sigma| > 1 > |\lambda_\sigma| > 0) \).

For \( 0 < |\lambda| < 1 < |\mu| \) denote by \( f_{\mu, \lambda} : \mathbb{R}^2 \to \mathbb{R}^2 \) linear diffeomorphism given by the formula
\[
f_{\mu, \lambda}(x, y) = (\mu x, \lambda y).
\]

Set
\[
U_{\mu, \lambda} = \{(x, y) \in \mathbb{R}^2 : |x||y|^{-\log_\lambda \mu} \leq 1\}.
\]

Notice that the set \( U_{\mu, \lambda} \) is \( f_{\mu, \lambda} \)-invariant and possesses two \( f_{\mu, \lambda} \)-invariant foliations \( \mathcal{F}^s = \bigcup_{c \in \mathbb{R}} \{(x, y) \in U_{\mu, \lambda} : x = c\} \) and \( \mathcal{F}^u = \bigcup_{c \in \mathbb{R}} \{(x, y) \in U_{\mu, \lambda} : y = c\} \).

**Definition 5** (\( C^1 \) linearization). A saddle point \( \sigma \in \Omega^1_\Lambda \) and the foliations \( \mathcal{F}^s = \psi_{\sigma}^{-1}(\mathcal{F}^s), \mathcal{F}^u = \psi_{\sigma}^{-1}(\mathcal{F}^u) \) is \( C^2 \)-smooth.

The existence of a linearizable neighborhood for any saddle point of a diffeomorphism \( f \) from \( \Psi \) (or indeed any \( C^2 \) diffeomorphism) is well-known, see for example [10] Chapter 5].

Denote by \( \mathcal{A} \) the set of points at which one-sided heteroclinic tangencies of invariant manifolds of saddle points of the diffeomorphism \( f \) there is. For each \( a \in \mathcal{A} \) denote by \( \sigma^a_\Lambda \in \Omega^1_\Lambda, \sigma^a_\sigma \in \Omega^1_\sigma \) the saddle points such that \( a \in W^s_{\sigma^a_\Lambda} \cap W^u_{\sigma^a_\sigma} \). Set \( \mu_a = \mu_{\sigma^a_\Sigma} \) and \( \lambda_a = \lambda_{\sigma^a_\Sigma} \).

Denote by \( U_a \) the connected component of \( U_{\sigma^a_\Lambda} \cap U_{\sigma^a_\sigma} \) containing the point \( a \). For any point \( z \in U_a \) let us set \( z^s = \psi_{\sigma^a_\Lambda}(z) = (z^s_1, z^s_2) \) and \( z^u = \psi_{\sigma^a_\sigma}(z) = (z^u_1, z^u_2) \). Set
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Figure 6. Linearizable neighborhood $U_{\sigma}$

Figure 7. The transition map $g_a$

$g_a = \psi_{\sigma_a^\alpha}(\psi_{\sigma_a^\alpha}|_{U_a})^{-1} : \psi_{\sigma_a^\alpha}(U_a) \to \psi_{\sigma_a^\alpha}(U_a)$ (see Figure 7) and write the map $g_a$ in the coordinate form

$$g_a(x, y) = (\xi_a(x, y), \eta_a(x, y)).$$

Set

$$\tau_a = \frac{\partial \eta_a}{\partial x}(a^s).$$
Set
\[ H_a = A \cap W^s_{\sigma_a^u} \cap W^u_{\sigma_a^u}. \]

**Statement 3** (Moduli for homeomorphisms if \( h \) manifolds.

2.3. Geometric description of the intersection pattern of invariant manifolds. Denote by \( L^s, L^u \) the sets of non-trivial attractors respectively repellers. Set \( \mathcal{L} = L^s \cup L^u \). As before let \( \Omega^0, \Omega^1, \Omega^2 \) be the sets of sinks, saddles and sources from the trivial basic set \( \Omega_f \). We let \( \Omega^{1u} \) be the set of saddle points \( p \in \Omega^1 \) for which there is either a saddle point \( q \in (\Omega^1 \setminus p) \) such that \( W^u_p \cap W^s_q \neq \emptyset \) or a set \( \Lambda \in \mathcal{L}^s \) such that \( W^u_p \cap W^s_q \neq \emptyset \). Next define \( \Omega^{1u} = \Omega^1 \setminus \Omega^{1u} \). Note that the definitions of the sets \( \Omega^{1s} \) and \( \Omega^{1u} \) are not symmetric, but, by the class \( \Psi \) assumptions, if \( p \in \Omega^{1u} \) then there exists no saddle point \( q \) for which \( W^u_p \cap W^s_q \neq \emptyset \) for some saddle point \( q \) and that there exists no set \( \Lambda \in \mathcal{L}^u \) such that \( W^u_p \cap W^u_q \neq \emptyset \).

Let us set
\[ A_f = W^u_{\Omega_{f1}^u} \cup \Omega^0 \cup L^s, \quad R_f = W^s_{\Omega_{f1}^s} \cup \Omega^2 \cup L^u. \]

By construction the set \( A_f \) is an attractor and \( R_f \) is a repeller of \( f \), see Figure 8.

Set
\[ V_f = M^2 \setminus (A_f \cup R_f). \]

Let \( k_f \) be a minimal natural number such that each separatrix of saddle and boundary points is invariant with respect to \( f^{k_f} \). By construction the orbit space \( \hat{V}_f = V_f / f^{k_f} \) of the action of the diffeomorphism \( f^{k_f} \) on \( V_f \) consists of a finite number of copies of the two-dimensional torus, and the natural projection \( p_f : V_f \to \hat{V}_f \) is a covering (see, for example [12, Lemma 2.1]). In Figure 9 this construction is illustrated for the diffeomorphism shown in Figure 8. Set
\[ \hat{\phi}_f = p_f f p_f^{-1} : \hat{V}_f \to \hat{V}_f. \]
Figure 8. Attractor $A_f$ and repeller $R_f$ for a diffeomorphism $f \in \Psi$

Figure 9. The orbit space $\hat{V}_f$ for the diffeomorphism $f$ from Figure 8 with the projections of the separatrices
For each point $\sigma \in \Omega^1$ set $\hat{\gamma}_\sigma = p_f(W^s_\sigma \setminus \sigma)$ and for each point $\sigma \in \Omega^1$ set $\hat{\gamma}_\sigma = p_f(W^u_\sigma \setminus \sigma)$. By the construction $\hat{\gamma}_\sigma$ is a pair of circles. Set

$$\hat{\Gamma}_f = \bigcup_{\sigma \in \Omega^1} \hat{\gamma}_\sigma, \quad \hat{\Gamma}^u_f = \bigcup_{\sigma \in \Omega^1} \hat{\gamma}_\sigma.$$  

For each non-trivial basic set $\Lambda$ denote by $k_\Lambda$ the number of periodic components of $\Lambda$ and by $\alpha_\Lambda$ the restriction of $f^{k_\Lambda}$ on a periodic component of $\Lambda$. Set $\bar{\ell}_p = p_f(\ell_p)$, $p \in P_\Lambda$, $\ell_p = \bigcup_{p \in \Lambda} \bar{\ell}_p$ and

$$\hat{L}^s_f = \bigcup_{\Lambda \in \mathcal{E}^s} \hat{\ell}_\Lambda, \quad \hat{L}^u_f = \bigcup_{\Lambda \in \mathcal{E}^u} \hat{\ell}_\Lambda.$$ 

Set $\hat{H}_\Lambda = p_f(H_\Lambda)$, $\lambda_{\hat{H}_\Lambda} = \lambda_\Lambda$ and $\mu_{\hat{H}_\Lambda} = \mu_\Lambda$ for $a \in A$. Denote by $\hat{H}_f$ the union of all sets $\hat{H}_\Lambda$. For a connected component $\hat{T}$ of $\hat{V}_f$ let us set $\hat{H}_f = \hat{H}_f \cap \hat{T}$. If the set $\hat{H}_f$ is not empty let us choose a simple closed curve $\hat{\beta}_f$ which intersects each curve from $\hat{T} \cap \hat{\Gamma}^s_f$ at exactly one point not being from $\hat{H}_f$ (such curve exists as $\hat{T} \cap \hat{\Gamma}^s_f$ is a set of disjoint non-contractile curves). Denote by $\beta_f$ a connected component of the preimage $p_f^{-1}(\hat{\beta}_f)$ and by $K_f$ an annulus on $V_f$ situated between $\beta_f$ and $f^{k_f}(\beta_f)$. For an oriented path $\hat{\nu} \subset \hat{T}$ from a point $\hat{x}$ to a point $\hat{y}$ there is a unique lift $\nu \subset V_f$ with the start point $x = p_f^{-1}(\hat{x}) \cap K_f$ (see, for example, [10]). Then the end point of $\nu$ is situated in $f^{k_f}(K_f)$ for some integer $k_f$.

Let $\hat{a}_1, \hat{a}_2 \in \hat{H}_f$. If $\hat{a}_1$ and $\hat{a}_2$ belong to the same connected component of $\hat{\Gamma}_f^s$ then denote by $\hat{v}_{\hat{a}_1, \hat{a}_2}$ a directed curve connecting the points $\hat{a}_1$ with $\hat{a}_2$ which is the part of curve from $\hat{\Gamma}_f^s$ oriented along the stable manifold. If $\hat{a}_1$ and $\hat{a}_2$ belong to different connected components $\hat{\gamma}_1^s, \hat{\gamma}_2^s$ of $\hat{\Gamma}_f^s$ then set $\hat{z}_1 = \hat{\gamma}_1^s \cap \hat{\beta}_f$, $\hat{z}_1 = \hat{\gamma}_1^s \cap \hat{\beta}_f$ and denote by $\hat{v}_{\hat{a}_1, \hat{a}_2}$ a directed curve connecting the points $\hat{a}_1$ with $\hat{a}_2$ consisting of a part of $\hat{\gamma}_1^s$ oriented opposite the stable manifold, a part of curve $\hat{\beta}_f$ connecting $\hat{z}_1$ with $\hat{z}_2$ and a part of $\hat{\gamma}_2^s$ oriented along the stable manifold.

For each point $b \in (H_f \cap \hat{K}_f)$ let us calculate $\tau_b$ and set $\tau_b = \tau_b$ for $b = p_f(b)$. For $\hat{H}_\Lambda$ is from $\hat{H}_f$ let us set

$$\tau_{\hat{H}_\Lambda} = \{ \tau_{\hat{b}}, \hat{b} \in \hat{H}_\Lambda \} \quad \text{and} \quad \hat{C}_{\hat{H}_\Lambda} = \{ \lambda_{\hat{H}_\Lambda}, \mu_{\hat{H}_\Lambda}, \tau_{\hat{H}_\Lambda} \}.$$ 

Set

$$\hat{C}_f = \{ \hat{C}_{\hat{H}_\Lambda}, \hat{H}_\Lambda \subset \hat{H}_f \}.$$ 

**Definition 6** (The scheme of a diffeomorphism). We call the set

$$S_f = (\hat{V}_f, \phi_f, \hat{\Gamma}_f^s, \hat{\Gamma}_f^u, \hat{C}_f, \hat{L}_f^s, \hat{L}_f^u)$$

a scheme of the diffeomorphism $f \in \Psi$.

**Definition 7** (Equivalence of schemes). The schemes

$$S_f = (\hat{V}_f, \phi_f, \hat{\Gamma}_f^s, \hat{\Gamma}_f^u, \hat{C}_f, \hat{L}_f^s, \hat{L}_f^u) \quad \text{and} \quad S_{f'} = (\hat{V}_{f'}, \phi_{f'}, \hat{\Gamma}_{f'}^s, \hat{\Gamma}_{f'}^u, \hat{C}_{f'}, \hat{L}_{f'}^s, \hat{L}_{f'}^u)$$

of diffeomorphisms $f, f' \in \Psi$, respectively, are said to be equivalent if there exists an orientation-preserving homeomorphism $\hat{\psi} : \hat{V}_f \to \hat{V}_{f'}$ such that:
transverse. Periodic points of the basic set \( \Lambda' \) then the point \( f_p \) of its points are the wandering points of the diffeomorphism and for every non-trivial basic set \( \Lambda' \) of moduli. Let us prove that the three conditions of Definition 1 hold.

1. We now prove that a 1-dimensional attractor of an \( A \)-diffeomorphism \( f : M^2 \to M^2 \) with a finite number moduli is separable.

Let \( \Lambda \) be an attractor of an \( A \)-diffeomorphism \( f : M^2 \to M^2 \) with a finite number of moduli. Let us prove the three conditions of Definition [2] hold.

1) To prove item 1 of Definition [1] it suffices to prove that \( W^s_{\Lambda'} \cap W^u_{\Lambda'} = \emptyset \) holds for every non-trivial basic set \( \Lambda' \) distinct from \( \Lambda \). Suppose the contrary: there are points \( x \in \Lambda, x' \in \Lambda' \) such that \( W^s_{\Lambda'} \cap W^u_{\Lambda'} = \emptyset \). Since stable manifolds of the points of \( \Lambda \) (unstable manifolds of the points of \( \Lambda' \)) are \( C^1 \)-close on compact sets, without loss of generality one can assume that the manifold \( W^s_{\Lambda} \) contains no \( s \)-boundary periodic points of the basic set \( \Lambda \) and the manifold \( W^u_{\Lambda'} \) contains no \( u \)-boundary periodic points of the basic set \( \Lambda' \). By Statement [4] the intersection \( W^s_{\Lambda} \cap W^u_{\Lambda'} \) is transverse.

Let \( y \in (W^s_{\Lambda} \cap W^u_{\Lambda'}) \). As \( \Lambda \) and \( \Lambda' \) have local structure of the product on interval by Cantor set then the point \( y \) belongs to an adjacent interval \( (a, b)^* \subset W^s_{\Lambda} \) which consists of the wandering points of the diffeomorphism \( f \) and such that \( a, b \in \Lambda \) and \( W^s_{\Lambda} \cap W^u_{\Lambda'} = \emptyset \). Denote by \( L_a \) the connected component of the set \( W^s_{\Lambda} \cap (W^u_{\Lambda'} \setminus b) \) disjoint from the point \( p_a \). Then the curve \( l_{ab} = L_a \cup L_b \cup [a, b]^* \) bounds a domain \( D_{ab} \). This domain is a continuous immersion of the open disk into the manifold \( M^2 \), all of its points are the wandering points of the diffeomorphism \( f \) and the curve \( l_{ab} \) is the boundary of \( D_{ab} \) which is accessible from inside.
Denote by $W_{y}^{u*}$ the connected component of the set $W_{y}^{u} \setminus y$ disjoint from the point $x'$. The transversality condition implies $W_{y}^{u*} \cap D_{ab} \neq \emptyset$. On the other hand the component $W_{y}^{u*}$ contains a set which is dense in the periodic component of the set $\mathcal{N}$. Therefore there are points in $W_{y}^{u*}$ disjoint from the domain $D_{ab}$. Then there is a point $y' \in (a, b)^{s}$ distinct from the point $y$ and such that the arc $(y, y')^{u} \subset W_{x'}^{u}$ belongs to the domain $D_{xu}$. Since for any point $\tilde{a} \in L_{p}\tilde{a}$ there is a unique point $\tilde{b} \in L_{p}\tilde{a}$ such that $\tilde{a} \in W^{s}_{\tilde{x}}, \tilde{x} \in \Lambda$ and $(\tilde{a}, \tilde{b})^{s} \subset D_{ab}$ it follows that there is a point $\tilde{x}$ for which the arc $(\tilde{a}, \tilde{b})^{s}$ is tangent to the arc $(y, y')^{u}$ and this contradicts the transversality condition.

2) To prove item 2 of Definition 1 it suffices to show that for every $s$-boundary point $p$ of the basic set $\Lambda$ there is no saddle point $\sigma$ from the trivial basic set of the diffeomorphism $f$ such that $W_{\sigma}^{u} \cap \ell_{p} \neq \emptyset$. If we assume the contrary then similarly to the proof of item 1 we come to a contradiction to the transversality condition.

3) Assuming the contrary in this case we come to a contradiction to the transversality condition as well. □

4. Proof of the Classification Theorem

It follows from the geometrical construction of the schemes and Statement 3 that diffeomorphisms $f, f' \in \Psi^{s}$ are topologically conjugate then their schemes are equivalent. Let us show that if the schemes of diffeomorphisms $f, f' \in \Psi$ are equivalent, then the diffeomorphisms are topologically conjugate.

Proof of Theorem 2. Let

\[ S_{f} = (\hat{V}_{f}, \hat{\phi}_{f}, \hat{\Gamma}_{f}^{s}, \hat{\Gamma}_{f}, \hat{C}_{f}, \hat{L}_{f}^{s}, \hat{L}_{f}) \quad \text{and} \quad S_{f'} = (\hat{V}_{f'}, \hat{\phi}_{f'}, \hat{\Gamma}_{f'}^{s}, \hat{\Gamma}_{f'}, \hat{C}_{f'}, \hat{L}_{f'}^{s}, \hat{L}_{f'}) \]

be schemes of diffeomorphisms $f, f' \in \Psi$, respectively, for which there exists an orientation-preserving homeomorphism $\hat{\varphi}: \hat{V}_{f} \to \hat{V}_{f'}$ with the properties 1–7 of Definition 2. We divide the construction of a conjugating homeomorphism $h: M^{2} \to M^{2}$ such that $hf = f'h$ in to steps.

Step 1. The existence of the homeomorphism $\varphi: \hat{V}_{f} \to \hat{V}_{f'}$ with property $\hat{\varphi}\hat{\phi}_{f'} = \hat{\phi}_{f}\hat{\varphi}$ implies that there exists a homeomorphism $\varphi: V_{f} \to V_{f'}$ that conjugates the restriction of the diffeomorphism $f$ to $V_{f}$ with the restriction of the diffeomorphism $f'$ to $V_{f'}$ and is such that $\varphi = p_{f'}\varphi_{f}^{-1}$. For each point $b \in (H_{a} \cap K_{f})$ let us denote by $n(b)$ an integer such that $\varphi(b) \in f^{n(b)}(K_{\hat{\varphi}(\hat{f})})$. Due to condition 5 in Definition 2, we can suppose that $\varphi$ is chosen such that $a_{1} \in (H_{a} \cap K_{f})$ and $a_{2} \in (H_{a} \cap K_{f'})$ then $n(a_{2}) - n(a_{1}) = m_{a_{1}, a_{2}}$. So we have a conjugating homeomorphism on the set $M^{2} \setminus (A_{f} \cup R_{f})$.

Due to condition 2 in Definition 2, for any point $\sigma \in \Omega^{1, \delta}$, $\delta \in \{u, s\}$ there exists a point $\sigma' \in \Omega^{s, \delta}$ such that $\varphi(W_{\sigma}^{s} \setminus \sigma) = W_{\sigma'}^{s} \setminus \sigma'$. Let us extend $\varphi$ on the set $\Omega^{s}$ by setting $\varphi(\sigma) = \sigma'$. Due to condition 1 in Definition 2 $\varphi$ conjugates $f|_{W_{\sigma}^{s, \delta}}$ with $f'|_{W_{\sigma'}^{s, \delta}}$.

Due to condition 6 in Definition 2, for any basic set $\Lambda \in \mathcal{L}^{s}$ there exists a basic set $\Lambda' \in \mathcal{L}^{s}$ such that $\varphi(\ell_{\Lambda}) = \ell_{\Lambda'}$. Let us extend $\varphi$ by continuity on the set $\bigcup_{\Lambda \in \mathcal{L}} P_{\Lambda}$.
of boundary points of non-trivial basic sets. Due to condition 1 in Definition 7, we can define a homeomorphism \( \phi \) such that for a connected component \( E \) of \( U_\sigma \) containing \( \ell_\sigma^s \) and a connected component \( E' \) of \( U_{\sigma'} \) containing \( \ell_{\sigma'}^s \), the following condition holds \( \phi(E \setminus W_{\sigma}^s) \cap E' \neq \emptyset \). Let us define a homeomorphism \( \phi_{E} : \ell_\sigma^s \to \ell_{\sigma'}^s \) by the following way. For a point \( t \in \ell_\sigma^s \) such that \( t^u = (0, t_{1}^u) \) let us set \( \phi_{E}(t) = t' \) where \( t'^u = (0, t_{1}^u) \).

Let \( \ell_\sigma^s \) be a stable separatrix of \( \sigma \). Denote by \( s_{\sigma} \) the stable separatrix of \( \sigma' = \phi(\sigma) \) such that \( \phi(s_{\sigma}) = s_{\sigma'} \) and \( \phi(E) \cap E' \neq \emptyset \). Let us define a homeomorphism \( \phi_{s} : \ell_\sigma^s \to \ell_{\sigma'}^s \) by the following way. For a point \( x \in \ell_\sigma^s \) such that \( x \in \ell_{f^k(\sigma)}^s \) we can define a homeomorphism \( \phi_{s} : \ell_\sigma^s \to \ell_{\sigma'}^s \) by the formula

\[
\phi_{s}(x) = f^{k}(\phi_{s}(f^{-k}(x)))
\]

for each \( x \in \ell_{f^k(\sigma)}^s \). Doing a similar construction for all saddle periodic orbits of the set \( \Omega^{1} \) we get the sought conjugating homeomorphism \( \phi_{\Omega^{1} \phi} \).

Now let \( \sigma \in \Omega^{1} \) and \( \sigma' = \phi(\sigma) \). Set

\[
\rho_{\sigma} = \frac{\ln |\lambda_{\sigma}|}{\ln |\mu_{\sigma}|}.
\]

Similar to the construction above for corresponding separatrices \( \ell_\sigma^s, \ell_{\sigma'}^s \) we define a homeomorphism \( \phi_{\sigma} : \ell_\sigma^s \to \ell_{\sigma'}^s \) by the following way. For a point \( t \in \ell_{\sigma'}^s \) such that \( t^s = (t_{1}^s, 0) \) let us set \( \phi_{\sigma}(t) = t' \) where \( t'^s = (t_{1}^s, 0) \),

\[
|t'^s| = c_{\sigma} |t_{1}^s|^{\rho_{\sigma}},
\]

where \( c_{\sigma} = \frac{[\lambda_{\sigma}, \mu_{\sigma}]^{n(\sigma)} |\gamma_{\sigma}|}{|\gamma_{\sigma}|^{n(\sigma)}} \) if there is a point \( \hat{a} \in \hat{E}_{\sigma} \cap \hat{f} \) and equals 1 in the opposite case. As above it is possible to verify that \( \phi_{\sigma} \) conjugates the diffeomorphisms \( f^{k}\ell_{\sigma}^s \) and \( f^{k}\ell_{\sigma'}^s \). For each \( k = 0, \ldots, k_{\sigma} \) we can define a homeomorphism \( \phi_{s}(x) = f^{k}(\phi_{s}(f^{-k}(x))) \) for each \( x \in \ell_{f^k(\sigma)}^s \). Doing a similar construction for all saddle periodic orbits of the set \( \Omega^{1} \) we get the sought conjugating homeomorphism \( \phi_{\Omega^{1}} \).

Step 3. In this step we construct a homeomorphism \( \phi_{L} : L^{s} \to L_{\sigma}^{s} \) which conjugates \( f|\ell_{\sigma}^{s} \) with \( f'\ell_{\sigma}^{s} \). Let us construct \( \phi_{L} \), the construction of \( \phi_{L} \) is similar.

Let \( \Lambda \) be a one-dimensional attractor of \( f \) and \( L \) be one from \( k_{\Lambda} \) periodic components of \( \Lambda \). Then \( L \) is a one-dimensional attractor of the diffeomorphism \( g = f^{k_{\Lambda}} \).
with the unique periodic component. Denote by $L'$ a periodic component of $\Lambda'$ such that $\varphi$ sends the boundary points of $L$ to the boundary points of $L'$. Then $L'$ is a one-dimensional attractor of the homeomorphism $g' = f^k \cdot \psi$ with the unique periodic component. Due to conditions 1) and 6) in Definition 7, $k_\Lambda = k_{\Lambda'}$. Due to condition 7) in Definition 7, there is an isomorphism $\psi_\Lambda$ conjugating $T_{gL}$ with $T_{g'\cdot \psi L}$, for some $g_L, g'_L$ and such that $\varphi(p) = \psi_\Lambda(p)$ for each point $p$ from the set $P_L$ of the boundary points of $L$. Statement 2 implies that there is a homeomorphism $\varphi_L : L \to L'$ such that $\hat{g}_L' \circ \varphi_L|_L = \varphi_L g_L|_L$. Set

$$\varphi_L = p_{N_L} \circ \tilde{\varphi}_L \circ p^{-1}_{N_L} : L \to L'.$$

Then $\varphi_L g_L|_L = g'_L \circ \varphi_L|_L$ and $\varphi_L(p) = \varphi(p)$ for each $p \in P_L$. Define $\varphi_\Lambda : \Lambda \to \Lambda'$ by the formula $\varphi_\Lambda(v) = f^k(\varphi_L(f^{-k}(v)))$ where $f^k(v) \in L$ for $v \in \Lambda$. Doing a similar construction for all attractors of the set $\mathcal{L}^s$ we get the sought conjugating homeomorphism $\varphi_\Lambda$.

**Step 4.** In this step we modify the homeomorphism $\varphi|_{W^s_{\mathcal{L}} \setminus \Lambda}$ by replacing it with $\tilde{h}_{\mathcal{L}^s} : W^s_{\mathcal{L}} \setminus \Lambda \to W^u_{\mathcal{L}} \setminus \mathcal{L}^u$ which extends continuously to the set $\mathcal{L}^s (\mathcal{L}^u)$ by the mapping $\varphi_\mathcal{L} (\varphi_\mathcal{L})$. We construct $\tilde{h}_{\mathcal{L}^s}$ (construction of $\tilde{h}_{\mathcal{L}^u}$ is similar).

Let $\Lambda$ be a one-dimensional attractor of $f$. We modify the homeomorphism $\varphi|_{W^s_\Lambda \setminus \Lambda}$ by replacing it with the homeomorphism $\tilde{h}_\Lambda : W^s_\Lambda \setminus \Lambda \to W^u_\Lambda \setminus \Lambda'$, which extends continuously to $\Lambda$ by the mapping $\varphi_\Lambda$, and which conjugates $f|_{W^s_\Lambda \setminus \Lambda}$ with $f'|_{W^s_\Lambda \setminus \Lambda}$.

Let $L$ be a periodic component of $\Lambda$ as in Step above. Denote by $B_L$ the set of all bunches of $L$ and will use further the denotations of Section 2.3. Let us consider the closed curve $L_0$ of the bunch $b \in B_L$ and enumerate the separatrices $l_1, \ldots, l_m$ of all saddle points and boundary points which intersect $L_0$ due to some orientation on $L_0$. Set $b' = \varphi_L(b)$. Due to items 2 and 6 of Definition 7 we have that the separatrices $\varphi(l_1), \ldots, \varphi(l_m)$ intersect $L_{b'}$ in order. If $\varphi_L|_{P_L} = \varphi|_{P_L}$ then for each $j \in \{1, \ldots, r_0\}$ there is a homeomorphism $h_j : [x_{2j}, x_{2j+1}] \to [\varphi_L(x_{2j}), \varphi_L(x_{2j+1})]$ such that $h_j([x_{2j}, x_{2j+1}] \cap l_\mu) = [\varphi_L(x_{2j}), \varphi_L(x_{2j+1})] \cap \varphi(l_\mu)$ for each $\mu \in \{1, \ldots, m\}$ and $h_j(x_{2j}) = \varphi_L(x_{2j})$. Set $I_0^j = \bigcup_{j=1}^{r_0} [x_{2j}, x_{2j+1}]$,

$I_B^j = \bigcup_{j=1}^{r_0} [\varphi_L(x_{2j}), \varphi_L(x_{2j+1})]$ and denote by $h^j : I_B^j \to I_B^j$ a homeomorphism which composed from $h^j_j$, $j \in \{1, \ldots, r_0\}$. Set $I_B^j = \bigcup_{j \in B_L} I_B^j$, $I_B^j = \bigcup_{j \in B_L} I_B^j$ and denote by $h^j : I_B^j \to I_B^j$ a homeomorphism which composed from $h^j_j$, $b \in B_L$.

Denote by $y_{2j-1}, y_{2j} \in W^u_{\mathcal{L}}$ the intersection points of $f(I_B^j)$ with $W^u_{\mathcal{L}}$ such that $p_j \notin [x_{2j}, y_{2j}]$ (see Figure 10). Set $I_B^j = \bigcup_{j=1}^{r_0} [x_{2j}, y_{2j}]$, $I_B^j = \bigcup_{b \in B_L} I_B^j$, and $I_B^j = h_L(I_B^j)$. Set $h^{i_j} = \varphi_L|_{I_B^j} : I_B^j \to I_B^j$. Let $\mathcal{H}_L (\mathcal{H}_L')$ be the closure of the set $W^u_{\mathcal{L}} \setminus L (W^u_{\mathcal{L}} \setminus L')$. Let us construct on $\mathcal{H}_L$ a pair of transverse one-dimensional foliation $F_{\mathcal{H}}^L$, $F_{\mathcal{H}}^L$ with the following properties:
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Figure 10. Illustration to the Step 4

a) each leaf of $F^s_L$ is a connected component of the intersection stable manifold of a point from $L$ with $\Pi_L$;
b) each leaf of $F^u_L$ is a segment $[x, y]$ with the boundary points $x, y$ such that $x \in [x_{2j}, x_{2j+1}]$, $y \in [y_{2j}, y_{2j+1}]$;
c) if $[x, y]$ belongs to $F^u_L$ then $[g^{-1}(y), g(x)]$ also belongs to $\mathcal{O}$;
d) each connected component of intersection of the separatrices of the saddle points with $\Pi_L$ is a leaf of $F^u_L$.

For each point $z \in I^u_L$ denote by $F^s_L,z$ a leaf of the foliation $F^s_L$ passing through the point $z$. For each point $x \in I^s_L$ denote by $F^u_L,x$ a leaf of the foliation $F^u_L$ passing through the point $x$. Let us construct similar foliations $F^s_L', F^u_L'$ on $\Pi_L'$ and define a homeomorphism $h_{\Pi_L}: \Pi_L \to \Pi_L'$ by the formula

$$h_{\Pi_L}(F^s_L,z \cap F^u_L,x) = F^s_L,h_{\Pi_L}(z) \cap F^u_L,h_{\Pi_L}(x).$$

Notice that $\Pi_L \setminus B_L$ is a fundamental domain of $f$ restriction on $W^s_{\Lambda} \setminus \Lambda$. Then for each point $w \in (W^s_{\Lambda} \setminus \Lambda)$ there is $k \in \mathbb{Z}$ such that $f^k(w) \in \Pi_L$. As $h_{\Pi_L}$ conjugates $g|_{\Pi_L}$ with $g'|_{\Pi_L'}$ then we can extend $h_{\Pi_L}$ up to $\tilde{h}_{\Lambda}: W^s_{\Lambda} \setminus \Lambda \to W^s_{\Lambda'} \setminus \Lambda'$ conjugating $f$ and $f'$ by the formula

$$\tilde{h}_{\Lambda}(w) = f^k(h_{\Pi_L}(f^{-k}(w))).$$

Doing a similar construction for all attractors of the set $\mathcal{L}^s$ we get the sought conjugating homeomorphism $\tilde{h}_{\mathcal{L}^s}$.

Denote by $\varphi_1: \hat{V}_f \to \hat{V}_f'$ a homeomorphism given by the formula

$$\varphi_1(z) = \begin{cases} 
\tilde{h}_{\mathcal{L}^s}(z), & z \in (W^s_{\mathcal{L}^s} \setminus \mathcal{L}^s); \\
\tilde{h}_{\mathcal{L}^u}(z), & z \in (W^u_{\mathcal{L}^s} \setminus \mathcal{L}^u); \\
\varphi(z), & z \in V_f \setminus (W^s_{\mathcal{L}^s} \cap W^u_{\mathcal{L}^u}). 
\end{cases}$$

Set $\varphi_1 = p_f' \varphi_1 p_f^{-1}: \hat{V}_f \to \hat{V}_f'$. 
STEP 5. In this step we modify the homeomorphism \( \varphi_1|W^s_{\bar{\mathcal{L}}} \setminus \mathcal{L}^s \) (\( \varphi_1|W^u_{\bar{\mathcal{L}}} \setminus \mathcal{L}^u \)) by replacing it with \( h_{\mathcal{L}}: W^s_{\bar{\mathcal{L}}} \setminus \mathcal{L}^s \to W^s_{\bar{\mathcal{L}}} \setminus \mathcal{L}^s \) (\( h_{\mathcal{L}}: W^u_{\bar{\mathcal{L}}} \setminus \mathcal{L}^u \to W^u_{\bar{\mathcal{L}}} \setminus \mathcal{L}^u \)), which extends to the set \( \mathcal{L}^s \) (\( \mathcal{L}^u \)) by \( \varphi_{\mathcal{L}} \) (\( \varphi_{\mathcal{L}} \)), and which extends to the set \( \text{cl}(W^s_{\bar{\mathcal{L}}} \setminus (\mathcal{L}^s \cup \mathcal{L}^u)) \) (\( \text{cl}(W^u_{\bar{\mathcal{L}}} \setminus (\mathcal{L}^u \cup \mathcal{L}^s)) \)) by the mapping \( \varphi_{\Omega_1} \) (\( \varphi_{\Omega_1} \)).

By Theorem 1 each non-trivial attractor of the diffeomorphism \( f \) is separable. Then there is a set \( \Sigma^u \subset \Omega^u \) such that \( \text{cl}(W^s_{\bar{\mathcal{L}}} \setminus (\mathcal{L}^s \cup \mathcal{L}^u)) = W^s_{\bar{\mathcal{L}}} \). Let \( \sigma \in \Sigma^u \).

Set \( h^s_{\sigma} = \varphi_{\Omega_1}|_{W^s_{\bar{\mathcal{L}}}}: W^s_{\bar{\mathcal{L}}} \to W^s_{\bar{\mathcal{L}}} \), and \( h^u_{\sigma} = \varphi_1|_{W^u_{\bar{\mathcal{L}}}}: W^u_{\bar{\mathcal{L}}} \to W^u_{\bar{\mathcal{L}}} \). In an \( f^{k^s} \)-invariant neighborhood \( \mathcal{N}_\sigma \) of \( \sigma \) let us construct a pair of transverse \( f^{k^s} \)-invariant foliations \( G^s_\sigma, G^u_\sigma \) with the following properties:

a) \( W^s_\sigma \in G^s_\sigma, W^u_\sigma \in G^u_\sigma \),

b) if \( W^s_\sigma \cap W^u_\sigma = \emptyset \) for some periodic component \( L \) of a non-trivial attractor then each connected component of \( W^s_\sigma \cap \mathcal{N}_\sigma, x \in L \) is a leaf of \( G^s_\sigma \).

For each point \( z_\sigma \in W^s_\sigma \) denote by \( G^s_{\sigma, z_\sigma} \) a leaf of the foliation \( G^s_\sigma \) passing through the point \( z_\sigma \). For each point \( z_\sigma \in W^s_\sigma \) denote by \( G^u_{\sigma, z_\sigma} \) a leaf of the foliation \( G^u_\sigma \) passing through the point \( z_\sigma \). Let us construct similar foliations \( G^s_{\sigma, f^s}, G^u_{\sigma, f^s} \) on \( \mathcal{N}_\sigma \) and define a homeomorphism \( h_{\mathcal{N}_\sigma} : \mathcal{N}_\sigma \to \mathcal{N}_\sigma \) by the formula

\[
h_{\mathcal{N}_\sigma}(G^s_{\sigma, z_\sigma} \cap G^u_{\sigma, z_\sigma}) = G^s_{\sigma, f^s(z_\sigma)} \cap G^u_{\sigma, f^u(z_\sigma)}.
\]

Then in some tubular neighbourhood \( N(\tilde{\gamma}_\sigma) \) of \( \tilde{\gamma}_\sigma \) a map \( \tilde{h}_{\mathcal{N}_\sigma} \) is well-defined by the formula \( \tilde{h}_{\mathcal{N}_\sigma} = p_f h_{\mathcal{N}_\sigma} p_f^{-1} \). Choose a tubular neighbourhood \( \tilde{N}(\tilde{\gamma}_\sigma) \) of \( \tilde{\gamma}_\sigma \) such that \( N(\tilde{\gamma}_\sigma) \subset \tilde{N}(\tilde{\gamma}_\sigma), \tilde{h}_{\mathcal{N}_\sigma}(N(\tilde{\gamma}_\sigma)) \subset \varphi_1(\tilde{N}(\tilde{\gamma}_\sigma)) \) and the set \( Q = \text{cl}(\tilde{N}(\tilde{\gamma}_\sigma) \setminus N(\tilde{\gamma}_\sigma)) \) is a two-dimensional annulus. Then there is a homeomorphism \( \tilde{\varphi}_{\tilde{Q}} : \tilde{Q} \to \tilde{Q}' \) such that \( \tilde{\varphi}_{\tilde{Q}}|_{\tilde{N}(\tilde{\gamma}_\sigma)} = \tilde{h}_{\mathcal{N}_\sigma} \) and \( \tilde{\varphi}_{\tilde{Q}}|_{\tilde{N}(\tilde{\gamma}_\sigma)} = \varphi_1 \). As the homeomorphisms \( \varphi_1 \) and \( h_{\mathcal{N}_\sigma} \) send leaves of the foliation \( W^s_{\bar{\mathcal{L}}}, x \in \mathcal{L}^s \), to leaves of the foliation \( W^s_{\bar{\mathcal{L}}}, x' \in \mathcal{L}^s \), and are coincide on \( W^u_{\bar{\mathcal{L}}} \) then we can construct \( \tilde{\varphi}_{\tilde{Q}} \) such that its lift sends leaves of the foliation \( W^s_{\bar{\mathcal{L}}}, x \in \mathcal{L}^s \), to leaves of the foliation \( W^s_{\bar{\mathcal{L}}}, x' \in \mathcal{L}^s \).

Denote by \( \tilde{\varphi}_{\tilde{Q}} : \tilde{V}_f \to \tilde{V}_f \) a homeomorphism given by the formula

\[
\tilde{\varphi}_{\tilde{Q}}(\tilde{z}) = \begin{cases} 
\tilde{\varphi}_{\tilde{Q}}(\tilde{h}_{\mathcal{N}_\sigma}(\varphi_{f^s}^{-k}(\tilde{z}))), & \tilde{z} \in \tilde{\varphi}_{\tilde{Q}}(\tilde{N}(\tilde{\gamma}_\sigma)); \\
\tilde{\varphi}_{\tilde{Q}}(\tilde{\varphi}_{\tilde{Q}}^{-1}(\tilde{Q})), & \tilde{z} \in \tilde{\varphi}_{\tilde{Q}}(\tilde{Q}); \\
\tilde{\varphi}_{\tilde{Q}}(\tilde{z}), & \tilde{z} \in (\tilde{V}_f \setminus \tilde{N}(\tilde{\gamma}_\sigma)).
\end{cases}
\]

Denote by \( \varphi_{\tilde{Q}} \) a lift of \( \tilde{\varphi}_{\tilde{Q}} \) coinciding with \( \varphi_1 \) on \( V_f \setminus p_f^{-1}(\tilde{N}(\tilde{\gamma}_\sigma)) \). Doing in series a similar construction for all saddle periodic orbits of the set \( \Sigma^u \) we get a homeomorphism \( \varphi_{\Sigma^u} : V_f \to V_f \). Also we construct a homeomorphism \( \varphi_{\Sigma^s} : V_f \to V_f \).

Denote by \( \varphi_2 : V_f \to V_f \) a homeomorphism given by the formula

\[
\varphi_2(z) = \begin{cases} 
\varphi_{\Sigma^s}(z), & \tilde{z} \in (W^s_{\bar{\mathcal{L}}} \setminus \mathcal{L}^s); \\
\varphi_{\Sigma^u}(z), & \tilde{z} \in (W^u_{\bar{\mathcal{L}}} \setminus \mathcal{L}^u); \\
\varphi_1(z), & \tilde{z} \in V_f \setminus (W^s_{\bar{\mathcal{L}}} \cap W^u_{\bar{\mathcal{L}}}).
\end{cases}
\]

Set \( \varphi_2 = p_f \varphi_2 p_f^{-1} : \tilde{V}_f \to \tilde{V}_f \).
Step 6. Let $\sigma \in \Omega^1$. For a point $x \in U_\sigma$ denote by $F^u_{\sigma,x} (F^s_{\sigma,x})$ the unique leaf of $F^u_{\sigma} (F^s_{\sigma})$ that passes through the point $x$. Define projections $\pi^u_\sigma: U_\sigma \to W^u_\sigma$ ($\pi^s_\sigma: U_\sigma \to W^s_\sigma$) along the leaves of the foliation $F^u_{\sigma} (F^s_{\sigma})$ as follows: $\pi^u_\sigma(x) = F^u_{\sigma,x} \cap W^u_\sigma$ ($\pi^s_\sigma(x) = F^s_{\sigma,x} \cap W^s_\sigma$).

Let $a \in W^u_\sigma \cap W^s_\sigma$ be a point of one-sided tangency and $a' = \varphi(a)$. Set $l_a = \psi^{-1}_{\sigma, a} (\{(x, y) \in U_{\mu_\sigma} : x = a_u^\mu, y = a_s^\sigma\}) \cap U_a$, $l_{a'} = \psi^{-1}_{\sigma,a'} (\{(x, y) \in U_{\mu_\sigma} : x = a_u^\mu, y = a_s^\sigma\}) \cap U_{a'}$. Set $L_A = \bigcup_{a \in A} l_a$ and $L_{A'} = \bigcup_{a' \in A'} l_{a'}$. In this step we construct a homeomorphism $\varphi_{L_A}: L_A \to L_{A'}$ which conjugates $f|_{L_A}$ with $f'|_{L_{A'}}$. This homeomorphism extends continuously to the set $W^u_{\Omega^1,a}$ by the mapping $\varphi_{\Omega^1,a}$, and to the set $W^u_{\Omega^1,b}$ by the mapping $\varphi_{\Omega^1,b}$.

Define a homeomorphism $\varphi_{l_a}: l_a \to l_{a'}$ by the formula

$$\varphi_{l_a}(z) = z' = ((\pi^u_{\sigma, a})^{-1}(\pi^u_{\sigma, a'}(z))) \cap l_{a'}.$$

Set $L_a = \bigcup_{n \in \mathbb{Z}} f^{kn}(l_a)$ and $L_{a'} = \bigcup_{n \in \mathbb{Z}} f^{kn}(l_{a'})$, where $k$ is the period of unstable separatrix containing $a$. Define a homeomorphism $\varphi_{L_a}: L_a \to L_{a'}$ by the formula $\varphi_{L_a}(z) = z' = f^{kn}(\varphi_{l_a}(f^{-kn}(z)))$ for each point $z \in f^{kn}(l_a)$. Set $E_a = W^u_{\sigma, a} \cup W^s_{\sigma, a} \cup L_a$, $E_{a'} = W^u_{\sigma, a'} \cup W^s_{\sigma, a'} \cup L_{a'}$. Denote by $\varphi_a: E_a \to E_{a'}$ a map, coinciding with $\varphi_{l_a}$ on $W^u_{\sigma, a}$, with $\varphi_{l_a}$ on $W^s_{\sigma, a}$ and with $\varphi_{l_a}$ on $L_a$. Using condition 2 of Definition 7 it is possible to verify that $\varphi_a$ is a homeomorphism (see [12] for details).

Denote by $A \subset A$ a set of such points that any two from their are not belonging to the same orbit of $f$ and $\bigcup f^n(A) = A$. Set $A' = \varphi(A), E_A = \bigcup_{a \in A} E_a$, and $E_{A'} = \bigcup_{a' \in A'} E_{a'}$. Let us define a map $\varphi_A: E_A \to E_{A'}$ as coinciding with $\varphi_a$ on each set $E_a$. Using condition 4 of Definition 7 it is possible to verify that $\varphi_A$ is a homeomorphism (see [12] for details).

Step 7. In the neighborhood $U_{a}$ of a point $a \in A$ define foliations $F^u_a$ and $F^s_a$ by the following way. The leaves of $F^u_a$ are coincide with the leaves of $F^u_{\sigma, a} \cap U_a$. In the neighborhood $\psi_{\sigma, a}(U_a)$ the curve $\psi_{\sigma, a}(W^s_{\sigma, a})$ has the equation $q(x) = Q(x - a_u^{\mu}) + o((x - a_u^{\mu})) \to 0$ for $x \to a_u^{\mu}$. Set $F^s_a = \psi^{-1}_{\sigma,a}(\{(x, y) \in U_{\mu_\sigma, x, y} : y = q(x + c)\}) \cap U_a$. Hence, in a neighborhood $U_a$ the leaves of $F^u_a$ are transverse to the leaves of $F^s_a$ on the set $U_a \setminus l_a$ and have tangency along the curve $l_a$. Set $U_A = \bigcup_{a \in A} U_a, U_A = \bigcup_{a \in A} f^n(U_A)$, $F^u_A = \bigcup_{a \in A} F^u_a, F^s_A = \bigcup_{a \in A} f^n(F^s_a)$, $F_A = \bigcup_{a \in A} F^u_a$ and $F_A = \bigcup_{n \in \mathbb{Z}} f^n(F_A)$. The similar foliations $F^u_{A'}$ and $F^s_{A'}$ let us construct in the neighbourhood $U_{a'}$ of the set $A'$.

Let $d$ be a point of the heteroclinic intersection of the manifolds $W^u_{\sigma, a} \cap W^s_{\sigma, a}$ not belonging to the set $A$. Denote $U_d$ a connected component of the set $U_{\sigma, a} \cap U_{\mu_\sigma}$ which contains $d$. Define foliations $F^u_d$ and $F^s_d$ by the following way: $F^u_d = F^u_{\sigma, a} \cap U_d$ and $F^s_d = F^s_{\sigma, a} \cap U_d$. Let $D (D')$ be the set of all heteroclinic points of $f (f')$ not belonging to $A, D \subset D'$ the set of points such that any two of them do not belong to the same orbit of the diffeomorphism $f$ and $\bigcup_{n \in \mathbb{Z}} f^n(D) = D$. Set $U_D = \bigcup_{d \in D} U_d$,
Figure 11. Construction of foliations

\[ U_D = \bigcup_{n \in \mathbb{Z}} f^n(U_D), \quad F_D^u = \bigcup_{d \in D} f^n(U_D), \quad F_D^s = \bigcup_{d \in D} f^n(F_D^s), \quad \text{and } F_D^s = \bigcup_{n \in \mathbb{Z}} f^n(F_D^s). \]

Construct the similar foliation \( F_D^u \) and \( F_D^s \) in the neighbourhood \( U_D' \) of the set \( D' = \varphi(D) \).

Set \( L_A = \bigcup_{a \in A} \). For a point \( \sigma \in \Omega_f \) define foliations \( F_s^\sigma \) and \( F_u^\sigma \) transverse to each other everywhere except \( U_A \) by the following way. The foliation \( F_s^\sigma (F_u^\sigma) \) coincides with \( F_s^A (F_u^A) \) on \( U_A \cap U_A \), coincides with \( F_s^D (F_u^D) \) on \( U_A \cap U_D \) and coincides with \( F_s^D (F_u^D) \) out of the set \( U_A \cup U_D \) (see figure 11). Denote by \( \tilde{\pi}_s^\sigma: U_\sigma \to W_\sigma \) (\( \tilde{\pi}_u^\sigma: U_\sigma \to W_\sigma \)) a projection along the leaves of the foliation \( F_s^\sigma (F_u^\sigma) \).

Construct similarly the foliation \( F_s^\sigma (F_u^\sigma) \) and define the projection \( \tilde{\pi}_s^\sigma: U_\sigma \to W_\sigma \) (\( \tilde{\pi}_u^\sigma: U_\sigma \to W_\sigma \)) in the neighbourhood \( U_\sigma \). Denote by \( \tilde{\pi}_s^\sigma, \tilde{\pi}_u^\sigma, \tilde{\pi}_s^\sigma, \tilde{\pi}_u^\sigma, \sigma \in \Omega^1 \), accordingly.

Step 8. For each point \( a \in A \) let us define a homeomorphism \( \varphi_{U_a}: U_a \to U_a' \) by the following way. Denote by \( U_a^+ \) and \( U_a^- \) the connected components of \( U_a \setminus l_a \) following a rule that any point \( z = (z_u^a, 0) \in U_a \) belongs to \( U_a^+ \) if \( z_u^a > a_u^a \) and belongs to \( U_a^- \) if \( z_u^a < a_u^a \). Similarly denote the connected components of \( U_a \setminus l_a' \). Define a homeomorphism \( \varphi_{U_a^+}: U_a^+ \to U_a'^+ \) by the following way: for a point \( z \in U_a^+ \) set \( \varphi_{U_a^+}(z) = z' \), where \( z' \in U_a'^+ \) is the intersection point of the leaves \( (\tilde{\pi}_s^\sigma_{u_a'})^{-1}(\varphi_{U_a^+}(\tilde{\pi}_s^\sigma_{u_a'}(z))) \) and \( (\tilde{\pi}_u^\sigma_{u_a'})^{-1}(\varphi_{U_a^+}(\tilde{\pi}_u^\sigma_{u_a'}(z))) \). In the similar way let us define
a homeomorphism $\varphi_{U^-}: U^- \to U^-$. Set
\[
\varphi_{U_a}(z) = \begin{cases} 
\varphi_{U_a}^+(z), & z \in U_a^+; \\
\varphi_{U_a}^-(z), & z \in U_a^-; \\
\varphi_{a}, & z \in I_a.
\end{cases}
\]

Define a homeomorphism $\varphi_{U_a}: U_a \to U_a'$ as coinciding with $\varphi_{U_a}$ for each $a \in A$

For each point $d \in D$ define a homeomorphism $\varphi_{U_d}: U_d \to U_d'$ by the following way: $\varphi_{U_d}(z)$ is the intersection point of the leaves $(\tilde{\pi}_{a_d}^a)^{-1}(\varphi_{U_a}^+(\tilde{\pi}_{a_d}^a(z)))$ and $(\tilde{\pi}_{a_d}^{-1} \varphi_{U_d}^+(\tilde{\pi}_{a_d}^{-1}(\varphi_{U_d}(z))))$ belonging to $U_d$. Let us define a homeomorphism $\varphi_{U_d}: U_d \to U_d'$ as a homeomorphism coinciding with $\varphi_{U_d}$ for each $d \in D$.

For $\delta \in \{u, s\}$ denote by $\varphi_{U_\Omega}^\delta: W_{U_\Omega}^\delta \to W_{U_\Omega}^\delta$ a homeomorphism conjugating the diffeomorphisms $f|_{W_{U_\Omega}^\delta}$, $f'|_{W_{U_\Omega}^\delta}$, coinciding with the homeomorphism $\varphi_{U_\Omega}$ on $W_{U_\Omega}^\delta \cap U_\Omega$, coinciding with the homeomorphism $\varphi_{U_\Omega}$ on $W_{U_\Omega}^\delta \cap U_\Omega$ and coinciding with the homeomorphism $\varphi$ out of some neighborhood of the set $W_{U_\Omega}^\delta \cap (U_\Omega \cup U_D)$. Denote by $\varphi_{U_\Omega}^\delta: W_{U_\Omega}^\delta \to W_{U_\Omega}^\delta$ a homeomorphism composed from $\varphi_{U_\Omega}^u$ and $\varphi_{U_\Omega}^s$.

Denote by $\varphi_{U_\Omega}^\delta: W_{U_\Omega}^\delta \to W_{U_\Omega}^\delta$ a homeomorphism composed from $\varphi_{U_\Omega}^u$ and $\varphi_{U_\Omega}^s$.

Set $U_\Omega = \bigcup_{\sigma \in \Omega^1} U_\sigma$ and $U_{\Omega_1} = \bigcup_{\sigma \in \Omega^1} U_\sigma$. Define a homeomorphism $\varphi_{U_\Omega}: U_\Omega \to U_{\Omega_1}$ as a homeomorphism conjugating the diffeomorphisms $f|_{U_\Omega}$, $f'|_{U_{\Omega_1}}$, coinciding with the homeomorphism $\varphi_{U_\Omega}$ on $U_\Omega \cap U_\Omega$, coinciding with the homeomorphism $\varphi_{U_\Omega}$ on $U_\Omega \cap U_D$ and such that for a $z \in (U_\sigma \setminus (U_\Lambda \cup U_D))$, $\varphi_{U_\Omega}(z)$ is the intersection point of the leaves $(\tilde{\pi}_{a_d}^a)^{-1}(\varphi_{U_a}^+(\tilde{\pi}_{a_d}^a(z)))$ and $(\tilde{\pi}_{a_d}^{-1} \varphi_{U_d}^+(\tilde{\pi}_{a_d}^{-1}(\varphi_{U_d}(z))))$.

**Step 9.** For any $t \in (0, 1)$ set $U_{\Omega, t}^\delta = \{ (x, y) \in \mathbb{R}^2 : |x||y| \leq \log_t x \leq t \}$. For any $\sigma \in \Omega^1$ set $U_{\sigma}^\delta = \psi_{\sigma}^{-1}(U_{\sigma_a, \lambda_{\sigma}})$ and $U_{\sigma}^\delta = \bigcup_{\sigma \in \Omega^1} U_{\sigma}^\delta$.

Let us choose a value $t_0 \in (0, 1)$ such that $\varphi_{U_\Omega}(U_{\Omega_1, t_0}) \subset (\varphi(U_\Omega) \cup W_{U_\Omega, u} \cup W_{U_\Omega, s})$. Set $Q = U_{\Omega} \setminus \text{int} U_{\Omega}^\delta$, $R = \partial U_{\Omega_1}$, $R_0 = \partial U_{\Omega_1}^\delta$, $Q' = \varphi(U_\Omega) \setminus \text{int} \varphi_{U_\Omega}(U_{\Omega_1}^\delta)$, $R' = \varphi(\partial U_\Omega)$, $R_0 = \varphi_{U_\Omega}(\partial U_{\Omega_1}^\delta)$, $\hat{Q} = p_f(Q)$, $\hat{Q}' = p_f'(Q')$ and
\[
\varphi_{\Omega_1} = p_f \varphi_{U_\Omega}(p_f|_{R_0})^{-1}: R_0 \to \hat{R}_0.
\]

By the construction the sets $\hat{Q}$, $\hat{Q}'$ have the same number of the connected components each of them is homeomorphic to the standard two-dimensional annulus (see Figure 12 where the set $\hat{Q}$ is coloured). Then there is a homeomorphism $\hat{\varphi}_Q: \hat{Q} \to \hat{Q}'$ such that $\hat{\varphi}_Q|_{\hat{R}} = \varphi$ and $\hat{\varphi}_Q|_{R_0} = \varphi_{U_\Omega}$.

Denote by $\varphi_Q: Q \to Q'$ a lift of the homeomorphism $\hat{\varphi}_Q$ coinciding with $\varphi$ on $\partial U_{\Omega_1}$. Define a homeomorphism $\varphi_3: V_f \to V_{f'}$ by the formula:
\[
\varphi_3(x) = \begin{cases} 
\varphi_{U_\Omega}(x), & x \in U_{\Omega_1}^\delta; \\
\varphi_Q(x), & x \in Q; \\
\varphi(x), & x \in M^2 \setminus U_{\Omega_1}.
\end{cases}
\]
Let us define a homeomorphism $h : M^2 \setminus (\Omega^0 \cup \Omega^2) \to M^2 \setminus (\Omega^0 \cup \Omega^2)$ by the formula:

$$
    h(x) = \begin{cases} 
        \varphi_2(x), & x \in (W^s_{L^u} \setminus L^s \cup W^u_{L^s \setminus L^u}); \\
        \varphi_3(x), & x \in (V_f \setminus (W^s_{L^u} \cup W^u_{L^s \setminus L^u})); \\
        \varphi_{L^s}(x), & x \in L^s; \\
        \varphi_{L^u}(x), & x \in L^u; \\
        \varphi_{\Omega^u}^*(x), & x \in W^u_{\Omega^u}; \\
        \varphi_{\Omega^s}^*(x), & x \in W^s_{\Omega^s}. 
    \end{cases}
$$

Then, to obtain a desired homeomorphism, it suffices to extend the homeomorphism $h$ continuously to the set $\Omega^0 \cup \Omega^2$.

**References**


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