MODELING OF BEHAVIOR AND INTELLIGENCE

Partial Orders and Jordan Normal Form

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Abstract—Consideration is given to the relation between the structure of the acyclic binary relation and the adjacency matrix of its corresponding graph. In this case, the existing methods for studying the binary relations and their corresponding graphs in terms of the spectrum, that is, the set of eigenvalues, of the adjacency matrix are inapplicable because for the acyclic relations this matrix is nilpotent and its spectrum is identically zero. Therefore, a more refined characteristic of the matrix is required. The present paper considers the Jordan normal form (JNF) as such.

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1. NECESSARY DEFINITIONS AND EXISTING RESULTS

The definitions concerning the binary relations follow [4].

Let $U = \{u_1, \ldots, u_N\}$ be a finite set.

By the binary relation on $U$ is meant the subset $P \subset U \times U$. The property of $(a, b) \in P$ is denoted by $aPb$. The relation $P$ is representable as an oriented graph with the sets of vertices, $U$, and arcs, $P$, the vertices $a$ and $b$ are connected if and only if $aPb$.

For the element $x \in U$, the sets $Px = \{y \in U \mid yPx\}$ and $xP = \{y \in U \mid xPy\}$ will be called its upper and lower cones, respectively.

For the relation $P$, the adjacency matrix $M = (m_{ij})$ is a square $N \times N$ matrix where $m_{ij} = 1$ if $u_iPu_j$ and $m_{ij} = 0$, otherwise.

We denote by $\overline{U}$ the vector space made up by the formal linear combinations of the elements of $U$: $\overline{U} = \left\{\sum_{i=1}^{N} a_iu_i\right\}$. The elements of $U$ considered as those of $\overline{U}$ make up the basis $\overline{U}$. The linear operator whose matrix is equal to $M$ in the basis $U$ is denoted by $L_P$. It acts on the elements of the basis $U$ as follows:

$$L_P(u) = \sum_{v \in uP} v.$$

The relation $P^c = \{(x, y) \in U \times U \mid (x, y) \notin P\}$ is called the complementary relation to the binary relation $P$.

The sequence of different elements $u_1, \ldots, u_n$ of the set $U$ is called the path if $u_1Pu_2, u_2Pu_3, \ldots, u_{n-1}Pu_n$, and the cone if additionally $u_1Pu_n$. By the length of a path (cone) is meant the number of its arcs, that is, the length of the path $u_1, \ldots, u_n$ is equal to $n - 1$, and that of the cone $u_1, \ldots, u_n$, to $n$.

The binary relation $P$ is called

— reflexive if $\forall x \in U \ (x, x) \in P$;
— antireflexive if $\forall x \in U \ (x, x) \notin P$;
—asymmetrical if \( \forall x, y \in U \ (x, y) \in P \Rightarrow (y, x) \notin P \);
—acyclic AR if it has no cycles;
—transitive if \( \forall x, y, z \in U \ xPy \text{ and } yPz \Rightarrow xPz \);
—negative transitive if \( P^c \) transitive;
—complete if \( \forall x, y \in U \ xPy \text{ or } yPx \);
—preorder if it is transitive and reflexive;
—partial order PO if it is transitive and asymmetrical;
—interval order IO if \( P \) is the partial order and \( \forall x, y, z, t \in U \ xPy \text{ and } zPt \Rightarrow xPt \text{ or } zPy \);
—semiorder SO if \( P \) is the interval order and satisfied is the semitransitivity condition \( \forall x, y, z, t \in U \ xPy \text{ and } yPz \Rightarrow xPt \text{ or } tPz \);
—weak order WO if \( P \) is asymmetrical and negative transitive;
—linear order LO if \( P \) is connected weak order.

These classes are correlated as follows:

\[ LO \subset WO \subset SO \subset IO \subset PO \subset AR. \]

We introduce equivalent formulations that are more convenient for the proofs to follow.

**Proposition 1** (see [1, 4]). The following conditions are equivalent:

(i) \( P \) is a linear order;
(ii) there exists a numeration \( U = \{ u_1, \ldots, u_N \} \) of the elements of \( U \) such that \( u_i Pu_j \Leftrightarrow i < j \).

**Proposition 2** (see [1, 4]). The following conditions are equivalent:

(i) \( P \) is a weak order;
(ii) there exists a decomposition of \( U \) in nonempty subsets \( U_1, \ldots, U_m \) such that \( xPy \Leftrightarrow x \in U_i, \text{ } y \in U_j, \text{ } i < j \).

For the binary relation \( P \), we define the binary relation \( T \): \( xTy \Leftrightarrow xP \supset yP \text{ and } Px \subset Py \). Automatically, \( T \) is a preorder.

**Proposition 3** (see [1, 4]). The following conditions are equivalent:

(i) \( P \) is a semiorder;
(ii) \( T \) is a complete preorder.

Let \( A \) be a nilpotent operator in the \( n \)-dimensional space \( V \) and \( X = \{ x_1, \ldots, x_n \} \), the basis of \( V \).

The basis \( X \) is called the Jordan basis of the operator \( A \) if in it \( A \) is as follows:

\[
\begin{align*}
Ax_1 &= x_2, \\
Ax_{n_1+1} &= x_{n_1+2}, \\
& \vdots \\
Ax_{n_1+n_k-1+1} &= x_{n_1+\ldots+n_k-1+2}, \\
& \ldots \\
Ax_{n_1+\ldots+n_k-1} &= x_{n_1+\ldots+n_k}, \\
Ax_{n_1+\ldots+n_k} &= 0,
\end{align*}
\]  

(1)

where \( n_1, \ldots, n_k \) are some natural numbers, \( n_1 + \ldots + n_k = n \).

The square submatrices obtained from \( A \) by selecting the rows and columns corresponding to the subsets of \( X \): \( \{ x_1, \ldots, x_{n_1} \}, \{ x_{n_1+1}, \ldots, x_{n_1+n_2} \}, \ldots, \{ x_{n_1+\ldots+n_{k-1}+1}, \ldots, x_{n_1+\ldots+n_k} \} \), are called the Jordan cells, the numbers \( n_i \), the lengths of the Jordan cells.

The vectors \( x_{n_1+\ldots+n_j} \) (\( j = 1, \ldots, k \)) are called the eigenvectors of \( A \), the rest of the elements of \( X \), the attached vectors: \( x \in X \) is the attached vector of the first order if \( Ax \) is the eigenvector of \( A \), and attached vector of the \( m \)th order if \( Ax \) is the attached vector of the \( m-1 \)st order.
We call $x_1, x_{n_1+1}, \ldots, x_{n_1+n_2+\ldots+n_k+1}$ the senior vectors of the corresponding Jordan cells. The senior vector of the $j$th cell will be the eigenvector for $A$ if $n_j = 1$ and, if $n_j > 1$, the attached vector of the vector of order $n_j - 1$ that is maximum possible for this cell.

The matrix of the operator $A$ in the basis $X$ is called the Jordan normal form (JNF) of $A$. Also, the operator $A$ will be said to be reducible to the JNF in the basis $X$.

The proofs of Propositions 4–6, sometimes in somewhat different formulations, can be found in [2,3].

**Proposition 4.** The nilpotent operator in the space over any field is reducible to the JNF. If $X_1$ and $X_2$ are two different Jordan bases, then the lengths of their Jordan cells differ in permutation.

We assume below that the Jordan cells are arranged in the decreasing order, which is always possible by numeration of the vectors in the Jordan basis.

Since the JNF is defined by decomposition of $n$ into a sum of more than one natural addend, the Young scheme (Fig. 1) can be conveniently correlated with it. The Jordan cells correspond to the columns, that is, the JNF of this scheme has 2 cells of lengths 8, 3 cells of lengths 4, and by one cell of lengths 1, 5, and 6.

By the defect of the operator $L$ in the finite-dimensional space $V$ is called the difference $\dim V - \dim \text{Im} L$, that is, the dimensionality of the null space.

**Proposition 5.** The defect of the nilpotent operator is equal to the number of its Jordan cells.

By the nilpotence index of the operator $L$ is meant a number $k$ such that $A^{k-1} \neq 0$, but $A^k = 0$.

**Proposition 6.** The nilpotence index of the nilpotent operator is equal to the length of the maximal of its Jordan cells.

**Proposition 7.** The vector of the Jordan basis of the operator $L$ is senior one if and only if it is not contained in $\text{Im} L$.

The last fact follows from $\text{Im} L$ which is a linear hull of all elements of $U$, except for the senior ones (see (1)).

2. MAIN RESULTS

**Proposition 8** (see, for example, Theorem 13.1 in [5]). The sum $M^k$ of the elements is equal to the number of paths of length $k$ in $P$.

**Proposition 9.** The operator corresponding to the acyclic binary relation $P$ is nilpotent. Its index of nilpotency is one greater than the length of the longest path in $P$.

**Proof.** The acyclic relation has no paths passing twice through the same vertex. Therefore, the path length is at most $|U| - 1$. Let $k$ be the length of the longest path. Then, according to the above
statement, the sum of the elements $M^{k+1}$ is 0. But the elements of this matrix are nonnegative integers; therefore, all of them are 0. Consequently, $M^{k+1} = 0$ and, therefore, $L_P^{k+1} = 0$.

On the other hand, the sum of the coefficients of $M^k$ is other than zero. Therefore, $M^k$ and $L_P^k$ are nonzero as well.

We denote by $J_P$ the JNF of the operator $L_P$ of the acyclic relation $P$ and obtain the following corollaries to Proposition 9.

**Corollary 1.** The length of the maximal path in the acyclic relation $P$ is one less than that of the maximal of the Jordan cells $J_P$.

**Corollary 2.** Let there be a path of the length $|U| - 1$ in the acyclic relation $P$. Then, $J_P$ contains a single Jordan cell.

**Proposition 10.** Let $P$ be a partial order. Then, the following conditions are equivalent:

(i) $P$ is a linear order;

(ii) $J_P$ has exactly one Jordan cell.

**Proof.** (i) $\Rightarrow$ (ii) follows immediately from Corollary 2.

(ii) $\Rightarrow$ (i). Let $J_P$ have exactly one Jordan cell. Then, $P$ has a path of length $|U| - 1$, that is, there exists a numeration $U = \{u_1, \ldots, u_N\}$ of the elements of $U$ such that $u_1 Pu_2, \ldots, u_{N-1} Pu_N$. From transitivity we obtain $u_i Pu_j$ for $i < j$, which is one of the definitions of the linear order.

**Theorem 1.** Let $P$ be a weak order. Then, $J_P$ has at most one cell of length greater than 1. The length of this cell is equal to the number of sets in the decomposition of $U$ introduced in the Proposition 2.

**Proof.** We make use of the equivalent definition of the weak order (Proposition 2): there exists a decomposition of $U$ into nonempty subsets $U_1, \ldots, U_m$ such that $xPy \Leftrightarrow x \in U_i, y \in U_j, i < j$.

The length of the maximum path in $P$ is $m - 1$; therefore, there is a cell of length $m$ in $J_P$.

We note that if $u, u' \in U_i$, where $i \leq m - 1$, then

$$L_P(u) = L_P(u') = \sum_{j > i \text{ or } v \in U_j} v,$$

and if $u \in U_m$, then $L_P(u) = 0$, that is, a nonzero vector of the space $\tilde{U}$. Among the images of the elements of the basis $\tilde{U}$, there are in all $m - 1$ different nonzero vectors and \(\dim \text{Im } L_P \leq m - 1\).

According to Proposition 5, the number of cells of $J_P$ is at least $N - m + 1$. And since among them there is a cell of length $m$ and the sum of the cell lengths is $N$, then only one variant is possible—one cell of the length $m$ and the rest of them of the length 1.

**Remark 1.** The opposite statement is untrue.

As is easy to check, the binary relation in Fig. 2 is the partial order (and even semiorder), but not a weak order.
On the other hand, let us consider the vectors $v_1 = u_1, v_2 = u_2 + u_3, v_3 = u_3, v_4 = u_4 - u_2$ which are linearly independent because the determinant of the matrix of their coefficients in the basis $U = \{u_1, u_2, u_3, u_4\}$ is 1. It follows from $L_P(v_1) = v_2, L_P(v_2) = v_3, L_P(v_3) = 0,$ and $L_P(v_4) = 0$ that it is a Jordan basis. Consequently, $J_P$ has by one cell of lengths 3 and 1. We note that $v_3$ is the eigenvector responsible for the cell of order 3 and $v_2$ and $v_1$ are the attached vectors, respectively, of the first and second orders.

For wider classes of the partial orders, the Jordan forms are more diverse.

**Theorem 2.** Let $J$ be the Young scheme. Then, there exists a semiorder $P$ such that $J_P = J$.

**Proof.** We construct the binary relation $P$ by the Young scheme $J$ whose cells are the elements of $U$. We denote by $u_{i,j}$ the cell in the $i$th row and $j$th column, $P$ is defined as follows:

$$u_{i,j} Pu_{k,l} \iff (i + 1 < k) \text{ or } (i + 1 = k \text{ and } j \leq l).$$

The cell of the Young scheme dominates the cells located at least two rows below or one row below but not to the left. In Fig. 3 the upper cone is shaded by dark gray and the lower one, by light gray.

Let $n$ be the number of the Jordan cells and $x_1, \ldots, x_n$ be their lengths. We assume that $x_1 \geq x_2 \geq \ldots \geq x_n$.

We also introduce the linear order $L$ on $U$: $u_{i,j} Lu_{k,l}$ if $i < k$ or $(i = k \text{ and } j < l)$. At first, the row numbers are compared, and if they are equal, then compared are those of the columns.

We also denote for convenience by $v_i$ the elements of the first row of the Young scheme. There are $n$ such elements, as many as the Jordan cells.

We prove that $P$ is a semiorder.

In virtue of Proposition 3, it suffices to prove that $T$ is the complete preorder, and since $T$ is the preorder by definition, it suffices to prove that $T$ is complete.

Since $L$ is a linear and, consequently, connected order, it suffices to verify that $L \subset T$. In this case, $T$ is connected as well, and since it is reflexive, it is also complete. It remains to prove that

$$xLy \Rightarrow Px \subseteq Py \text{ and } xP \supseteq yP.$$ 

It suffices to check this fact for the neighboring—in terms of $L$—elements which we call $x$ and $y$, $xLy$. In Fig. 4, $Px$ and $yP$ are shaded by light gray, and $Py/Px$ and $xP/yP$, by dark gray.
Let $W = \{w_1, \ldots, w_N\}$ be a Jordan basis and $A$ be the matrix of transition to it from $U = \{u_1, \ldots, u_N\}$, $W = AU$. $u \in U$ will be said to be included in $w_j$ if the coefficient of $u$ is other than zero in the decomposition of $w_j$ with respect to the basis $U$.

Since $\det A \neq 0$, the decomposition of the determinant

$$
\det A = \sum_{\sigma \in S_N} (-1)^\sigma a_1\sigma(1) \cdots a_N\sigma(N)
$$

has nonzero addends. Let us consider one of them defined by the permutation $\sigma$. Its distinction from zero implies that for $1 \leq i \leq N$, $a_i\sigma(i) \neq 0$, that is, for each element of $U$ there exists its element $W$ into which it is included. In particular, this is true for each of the elements of the first row of the Young scheme $v_i$.

Having numerated the elements of the basis $W$, one may assume that $v_i$ in included in the decomposition $w_i$ for all $i$, $1 \leq i \leq n$.

If any of $v_1, \ldots, v_n$ is included in the vector of the Jordan basis, then it is not contained in $\text{Im} L_P$ and, according to Proposition 7, it is senior. Therefore, at least $w_1, \ldots, w_n$ are the senior vectors of $W$, that is, the number of the Jordan cells $J_P$ is at least $n$.

We prove that the length of the Jordan cell corresponding to $w_i$ is not smaller than $x_i$. We take the minimum $j$ such that $v_j$ is included in $w_i$. Since $j \leq i$, we get $x_i \geq x_j$. Therefore, it suffices to check that the length of the Jordan cell corresponding to $w_i$ is not smaller than $x_j$.

We prove by induction that for $k < x_j$

$$
L_P^k(w_i) \sim u_{k+1,j} + u,
$$

where $u$ lies within the linear hull of the elements of $U$ dominated by $u_{k+1,j}$ relative to $L$, $\sim$ standing for proportionality.

For $k = 0$, it is possible to state that for $k < j$ $u_{k,1} = v_k$ are not included in $w_i$.

Let $L_P^{k-1}(w_i) \sim u_{k,j} + u$. Then, $L_P^j(w_i) \sim L_P(u_{k,j} + u) = L_P(u_{k,j}) + P(u)$. Yet $u_{k,j}$ is the minimum of the elements $v \in U$ such that $L_P(v)$ contains $u_{k+1,j}$. Therefore, $L_P(u)$ does not contain $u_{k+1,j}$, whereas $L_P(v_{k,j})$ contains them, and, therefore, their sum contains them as well.

As follows from the aforementioned, $L_P^{j-1}(w_i) \neq 0$, which implies that the Jordan cell with the senior vector $w_i$ has length at least $x_j$.

Therefore, $J_P$ contains at least $n$ cells among which there $n$ cells such that the length of the $i$th one is not smaller than $x_i$. Yet, since $\sum x_i = N$ and the sum of the lengths of the Jordan cells is $N$, there are precisely $n$ cells of lengths $x_1, \ldots, x_n$ in $J_P$, which is what we set out to prove.

Example. We illustrate the proof of Theorem 2 for the case of two Jordan cells of lengths 2 and 3.

Let $U$ consist of the elements $u_{1,1}, u_{1,2}, u_{2,1}, u_{2,2}, u_{3,1}$. The adjacency matrix $P$ is as follows:

$$
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

The Jordan basis consists of the vectors $v_1 = u_{1,1}$, $v_2 = L_P(v_1) = u_{2,1} + u_{2,2} + u_{3,1}$, $v_3 = L_P(v_2) = u_{3,1}$, $v_4 = u_{1,2}$, and $v_5 = L_P(v_4) = u_{2,2} + u_{3,1}$, and the operator’s JNF has two Jordan cells of lengths 3 and 2.
3. CONCLUSIONS

The following results were obtained:
— the partial order is linear if and only if the JNF of the adjacency matrix of its corresponding
graph has only one Jordan cell;
— the JNF of the adjacency matrix of the graph corresponding to the weak order has at most
one Jordan cell of length greater than 1, the opposite statement being untrue;
— there are no constraints on the JNF of the adjacency matrix of the graph corresponding to
the semiorder or a wider class of acyclic binary relations, that is, for any JNF with zero eigenvalues
there exists a semiorder such that the matrix of its corresponding graph has such JNF.

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