Modular metric spaces, I: Basic concepts

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A B S T R A C T

The notion of a modular is introduced as follows. A (metric) modular on a set $X$ is a function $\rho : (0, \infty) \times X \times X \to [0, \infty]$ satisfying, for all $x, y, z \in X$, the following three properties: $x = y$ if and only if $\rho(\lambda, x, y) = 0$ for all $\lambda > 0$; $\rho(\lambda, x, y) \leq \rho(\lambda, x, z) + \rho(\mu, y, z)$ for all $\lambda, \mu > 0$. We show that, given $X_0 \subset X$, the set $X_0 = \{x \in X : \lim_{\lambda \to \infty} \rho(\lambda, x, x_0) = 0\}$ is a metric space with metric $d_\rho(x, y) = \inf\{\lambda > 0 : \rho(\lambda, x, y) \leq \lambda\}$, called a modular space. The modular $\rho$ is said to be convex if $\rho(\lambda, x, y) \to \rho(\lambda, x, y)$ is also a modular on $X$. In this case $X_\rho$ coincides with the set of all $x \in X$ such that $\rho(\lambda, x, x_0) < \infty$ for some $\lambda = \lambda(x) > 0$ and is metrizable by $d_\rho(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leq 1\}$. Moreover, if $d_\rho(x, y) < 0$ or $\rho(x, y) < 0$, then $d_\rho(x, y)^2 \leq d_\rho(x, y) \leq d_\rho(x, y)$; otherwise, the reverse inequalities hold. We develop the theory of metric spaces, generated by modulars, and extend the results by H. Nakano, J. Musielak, W. Orlicz, Ph. Turpin and others for modulars on linear spaces.

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1. Introduction

The purpose of this paper, which is split into two parts, is to define the notion of a modular on an arbitrary set, develop the theory of metric spaces generated by modulars, called modular metric spaces and, on the basis of it, define new metric spaces of (multivalued) functions of bounded generalized variation of a real variable with values in metric semigroups and abstract convex cones. As an application we present an exhausting description of Lipschitz continuous and some other classes of superposition (or Nemyskii) operators, acting in these modular metric spaces.

In order to motivate our investigations, let us recall the notion of a function of bounded $\varphi$-variation (e.g., [1,2]), where $\varphi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ is a $\varphi$-function, i.e., a continuous nondecreasing unbounded function vanishing at zero only. Let $X$ be the set of all real valued functions $x : I \to \mathbb{R}$ on the closed interval $I = [a, b] \subset \mathbb{R}$ with $a < b$ such that $x(a) = 0$. Clearly, $X$ is a real linear space. The $\varphi$-variation of a function $x \in X$ is the quantity

$$\rho(x) = V_\varphi(x) = \sup \sum_{i=1}^m \varphi(|x(t_i) - x(t_{i-1})|) \in [0, \infty),$$

where the supremum is taken over all partitions $[t_i]_i$ of the interval $I$, i.e., $m \in \mathbb{N}$ and $a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b$. The linear subspace $X_\rho$ of $X$ of functions of bounded generalized $\varphi$-variation is defined by means of the functional $\rho$ on $X$, which gives an example of a modular on $X$, as follows.

According to Orlicz [3], a modular on a real linear space $X$ is a functional $\rho : X \to [0, \infty]$ satisfying the following four conditions: (A.1) $\rho(0) = 0$; (A.2) if $x \in X$ and $\rho(ax) = 0$ for all numbers $a > 0$, then $x = 0$; (A.3) $\rho(-x) = \rho(x)$ for all

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Theorem 3.11

Definition 3.3

2.14

Theorems 2.6, 2.8, 2.10, 2.13 and 2.17. In Section 3 we study the notion of a convex modular and construct the corresponding modular metric spaces, which have applications to the description of certain classes of superposition operators in the second part of the paper. The main results of Section 3 are Theorems 3.6–3.9, 3.11 and 3.14.
2. Metric modulars

Throughout the paper X is a nonempty set, \( \lambda > 0 \) is understood in the sense that \( \lambda \in (0, \infty) \) and, due to the disparity of the arguments, functions \( w : (0, \infty) \times X \times X \to [0, \infty) \) will be written as \( w_\lambda(x, y) = w(\lambda, x, y) \) for all \( \lambda > 0 \) and \( x, y \in X \).

**Definition 2.1.** A function \( w : (0, \infty) \times X \times X \to [0, \infty] \) is said to be a metric modular on X (or simply a modular if no ambiguity arises) if it satisfies the following three axioms:

(i) given \( x, y \in X \), \( w_\lambda(x, y) = 0 \) for all \( \lambda > 0 \) if and only if \( x = y \);
(ii) \( w_\lambda(x, y) = w_\lambda(y, x) \) for all \( \lambda > 0 \) and \( x, y \in X \);
(iii) \( w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(y, z) \) for all \( \lambda, \mu > 0 \) and \( x, y, z \in X \).

If, instead of (i), we have only the condition

(i') \( w_\lambda(x, x) = 0 \) for all \( \lambda > 0 \) and \( x \in X \), then \( w \) is said to be a (metric) pseudomodular on \( X \).

2.2

**Definition 2.1** appeared implicitly in [30, Section 4] and [31, Section 3] and explicitly in [34]. Condition \( w : (0, \infty) \times X \times X \to (-\infty, \infty] \) in it does not lead to a greater generality: in fact, setting \( x = y \) and \( \mu = \lambda > 0 \) in (iii) and taking into account (i'), for all \( y, z \in X \), we find \( 0 = w_{\lambda+\mu}(y, z) \leq 2w_\lambda(y, z) \). If \( w_\lambda(x, y) \) does not depend on \( x, y \in X \), then, by (i'), \( w \equiv 0 \) (although it may formally seem that (iii) in this case is a general subadditivity condition of the function \( \lambda \mapsto w_\lambda \). Now, if \( w_\lambda(x, y) = w(x, y) \) does not depend on \( \lambda > 0 \) and assumes only finite values, then axioms (i)--(iii) mean that \( w \) is a metric on \( X \) (pseudometric if (i) is replaced by (i')).

2.3

The main property of a (pseudo)modular \( w \) on a set \( X \) is the following: given \( x, y \in X \), the function \( 0 < \lambda \mapsto w_\lambda(x, y) \in [0, \infty) \) is nonincreasing on \( (0, \infty) \). In fact, if \( 0 < \mu < \lambda \), then (iii), (i') and (ii) imply

\[
 w_\lambda(x, y) \leq w_{\lambda-\mu}(x, x) + w_\mu(y, x) = w_\mu(y, x) = w_\mu(x, y) = w_{\lambda-\mu}(x, y) \leq \lim_{\mu \to 0} w_\mu(x, y) = \lim_{\mu \to 0} 0 \quad \text{exist in} \quad [0, \infty] \quad \text{and the following two inequalities hold:}
\]

\[
 w_{\lambda+\mu}(x, y) \leq w_\lambda(x, x) \leq w_{\lambda+\mu}(x, y).
\]

(2.1)

**Examples 2.4.** The following indexed objects \( w \) are simple examples of (pseudo)modulars on a set \( X \). Let \( \lambda > 0 \) and \( x, y \in X \). We have:

(a) \( w_\lambda(x, y) = \infty \) if \( x \neq y \), and \( w_\lambda(x, y) = 0 \) if \( x = y \); and if \( (X, d) \) is a (pseudo)metric space with (pseudo)metric \( d \), then we also have:

(b) \( w_\lambda(x, y) = d(x, y)/\phi(\lambda) \), where \( \phi : (0, \infty) \to (0, \infty) \) is a nonincreasing function;

(c) \( w_\lambda(x, y) = \infty \) if \( \lambda \leq d(x, y) \), and \( w_\lambda(x, y) = 0 \) if \( \lambda > d(x, y) \);

(d) \( w_\lambda(x, y) = \infty \) if \( \lambda < d(x, y) \), and \( w_\lambda(x, y) = 0 \) if \( \lambda \geq d(x, y) \).

2.5. The modular set

Let \( w \) be a (pseudo)modular on a set \( X \). The binary relation \( \sim \) on \( X \) defined for \( x, y \in X \) by

\[
 x \sim y \quad \text{if and only if} \quad \lim_{\lambda \to \infty} w_\lambda(x, y) = 0
\]

(2.2)

is, by virtue of axioms (i'), (ii) and (iii), an equivalence relation (e.g., if \( x \sim z \) and \( z \sim y \), then \( w_\lambda(x, y) \leq w_{\lambda/2}(x, z) + w_{\lambda/2}(z, y) \rightarrow 0 \) as \( \lambda \to \infty \), and so, \( x \sim y \)). Denote by \( X / \sim \) the quotient-set of \( X \) with respect to \( \sim \) and by \( X_0^w(x) = \{ y \in X : y \sim x \} \) the equivalence class of the element \( x \in X \) in the quotient-set \( X / \sim \). Note, in particular, that \( x \in X_0^w(x) \) and that the transitivity property of \( \sim \) implies \( y \sim z \) if and only if \( y, z \in X_0^w(x) \) for some \( x \in X \) (e.g., \( x = y \) or \( x = z \)).

It follows from Section 2.3 that the function \( \tilde{d} : (X / \sim) \times (X / \sim) \to [0, \infty] \) given by \( \tilde{d}(X_0^w(x), X_0^w(y)) = \lim_{\lambda \to \infty} w_\lambda(x, y) \), \( x, y \in X \), is well defined (the limit at the right-hand side does not depend on the representatives of the equivalence classes) and satisfies the axioms of a metric, except as Example 2.4(a) shows, that it may take infinite values.

In what follows we are interested in the equivalence classes \( X_0^w(x) \). Note that the quotient-pair \( (X / \sim, \tilde{d}) \) may degenerate in interesting and important cases: e.g., in Example 2.4(c) we have \( X_0^w(x) = X \) for all \( x \in X \) and \( \tilde{d} \equiv 0 \).

Let us fix an element \( x_0 \in X \) arbitrarily and set \( X_0^w = X_0^w(x_0) \). The set \( X_0^w \) is called a modular set. We note that condition (2.2) in the definition of \( X_0^w \) is an analogue of the condition in the definition of \( X_0 \) from (1.2), which is known in the literature as condition (B.1) or B1 (cf. [19,4,1,5] and the end of Section 2.15).
Theorem 2.6. If $w$ is a metric (pseudo)modular on $X$, then the modular set $X_w$ is a (pseudo)metric space with (pseudo)metric given by

$$d^w_m(x, y) = \inf\{\lambda > 0 : w_{\lambda}(x, y) \leq \lambda\}, \quad x, y \in X_w.$$ 

Proof. Given $x, y \in X_w$, the value $d^w_m(x, y) \in \mathbb{R}^+$ is well defined: in fact, since $x \sim y$, then, by virtue of (2.2), there exists $\lambda_0 > 0$ such that $w_{\lambda}(x, y) \leq 1$ for all $\lambda \geq \lambda_0$, and so, setting $\lambda_1 = \max(1, \lambda_0)$, we get $w_{\lambda_1}(x, y) \leq 1 \leq \lambda_1$, which together with the definition of $d^w_m(x, y)$ gives $d^w_m(x, y) \leq \lambda_1 < \infty$.

Given $x \in X_w$, (i) implies $w_{\lambda}(x, x) = 0$ for all $\lambda > 0$, and so, $d^w_m(x, x) = 0$. Let $w$ satisfy (i), $x, y \in X_w$ and $d^w_m(x, y) = 0$. Then $w_{\lambda}(x, y)$ does not exceed $\mu$ for all $\mu > 0$. Hence for any $\lambda > 0$ and $0 < \mu < \lambda$, in view of Section 2.3, we have $w_{\lambda}(x, y) \leq w_{\lambda}(x, x) \leq \mu \rightarrow 0$ as $\mu \rightarrow +0$. It follows that $w_{\lambda}(x, y) = 0$ for all $\lambda > 0$, and so, axiom (i) implies $x = y$.

Due to axiom (ii), the equality $d^w_m(x, y) = d^w_m(x, x)$, $x, y \in X_w$, is clear.

Let us show that $d^w_m(x, y) \leq d^w_m(x, z) + d^w_m(y, z)$ for all $x, y, z \in X_w$. In fact, by the definition of $d^w_m$, for any $\lambda > d^w_m(x, z)$ and $\mu > d^w_m(y, z)$ we find $w_{\lambda}(z, x) \leq \lambda$ and $w_{\mu}(y, z) \leq \mu$, and so, axiom (iii) implies

$$w_{\lambda + \mu}(x, y) \leq w_{\lambda}(x, z) + w_{\mu}(y, z) \leq \lambda + \mu.$$ 

It follows from the definition of $d^w_m$ that $d^w_m(x, y) \leq \lambda + \mu$, and it remains to pass to the limits as $\lambda \rightarrow d^w_m(x, z)$ and $\mu \rightarrow d^w_m(y, z)$. □

The metric $d^w_m$ is a counterpart of the $F$-norm $|\cdot|_\mu$ from (1.3), and Theorem 2.6 generalizes the results from [5, Section 1.82] and [11, Section 1.21] for $F$-norms generated by classical modulaters on linear spaces.

Examples 2.7. For modulaters $w = w^d, w^b, w^e, w^d$ from Example 2.4 we have, respectively:

(a) $X_w = \{x_0\}$ and $d^w_m(x, y) = 0$.

(b) If the function $\psi$ is bounded from above, then $X_w = \{x_0\}$ and $d^w_m(x, y) = 0$. On the other hand, if $\psi(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$, then $X_w = X$ and $d^w_m(x, y) = \psi^{-1}(d(x, y))$, where $\psi(\lambda) = \lambda \psi(\lambda), \lambda > 0$, is a strictly increasing function on $(0, \infty)$ such that $\psi(\lambda) \rightarrow 0$ as $\lambda \rightarrow +0$ and $\psi(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$ and $\psi^{-1}$ is the inverse function for $\psi$. In particular, if $\psi(\lambda) = \lambda^p$ with $p = \text{const} > 0$, then we get: $d^w_m(x, y) = (d(x, y))^{1/(p+1)}$.

(c), (d) $X_w = X$ and $d^w_m(x, y) = d(x, y)$.

Theorem 2.8. Let $w$ be a (pseudo)modular on a set $X$. Put

$$d^w_w(x, y) = \inf_{\lambda>0}(\lambda + w_{\lambda}(x, y)), \quad x, y \in X_w.$$ 

Then $d^w_w$ is a (pseudo)metric on $X_w$ with $d^w_w \leq d^w_w \leq 2d^w_m$ on $X_w \times X_w$.

Proof. Since, for $x, y \in X_w$, the value $w_{\lambda}(x, y)$ is finite due to (2.2) for $\lambda > 0$ large enough, then the set $(\lambda + w_{\lambda}(x, y) : \lambda > 0) \subseteq \mathbb{R}^+$ is nonempty and bounded from below, and so, $d^w_w(x, y) \in \mathbb{R}^+$.

If $x \in X_w$, then, by (i), $\lambda + w_{\lambda}(x, x) = \lambda$ for all $\lambda > 0$, and so, $d^w_w(x, x) = 0$. Now let $w$ satisfy (i), $x, y \in X_w$ and $d^w_w(x, y) = 0$. The equality $x = y$ will follow from (i) if we show that $w_{\lambda}(x, y) = 0$ for all $\lambda > 0$. On the contrary, suppose that $w_{\lambda}(x, y) > 0$ for some $\lambda > 0$. Then for $\lambda \geq \lambda_0$ we find $\lambda + w_{\lambda}(x, y) \geq \lambda_0$, and if $0 < \lambda < \lambda_0$, then it follows from Section 2.3 that

$$0 < w_{\lambda_0}(x, y) \leq w_{\lambda_0}(x, y) \leq \lambda + w_{\lambda_0}(x, y).$$ 

Thus, $\lambda + w_{\lambda}(x, y) \geq \lambda_1 = \min(\lambda_0, w_{\lambda_0}(x, y))$ for all $\lambda > 0$, and so, by the definition of $d^w_w$, $d^w_w(x, y) \geq \lambda_1 > 0$, which contradicts the assumption.

Axiom (ii) implies the symmetry property of $d^w_w$.

Let us establish the triangle inequality: $d^w_w(x, y) \leq d^w_w(x, z) + d^w_w(y, z)$. By the definition of $d^w_w$, for any $\epsilon > 0$ we find $\lambda = \lambda(\epsilon) > 0$ and $\mu = \mu(\epsilon) > 0$ such that

$$\lambda + w_{\lambda}(x, z) \leq d^w_w(x, z) + \epsilon \quad \text{and} \quad \mu + w_{\mu}(y, z) \leq d^w_w(y, z) + \epsilon,$$

whence, applying axiom (iii),

$$d^w_w(x, y) \leq (\lambda + \mu) + w_{\lambda+\mu}(x, y) \leq \lambda + \mu + w_{\lambda}(x, z) + w_{\mu}(y, z) \leq d^w_w(x, z) + \epsilon + d^w_w(y, z) + \epsilon,$$

and it remains to take into account the arbitrariness of $\epsilon > 0$.

Let us prove that metrics $d^w_w$ and $d^w_m$ are equivalent on $X_w$. In order to obtain the left-hand side inequality, suppose that $\lambda > 0$ is arbitrary. If $w_{\lambda}(x, y) \leq \lambda$, then the definition of $d^w_w$ implies $d^w_m(x, y) \leq \lambda$. Now if $w_{\lambda}(x, y) > \lambda$, then $d^w_m(x, y) \leq w_{\lambda}(x, y)$: in fact, setting $\mu = w_{\lambda}(x, y)$ we find $\mu > \lambda$, and so, it follows from Section 2.3 that $w_{\mu}(x, y) \leq w_{\lambda}(x, y) = \mu$, which
whence \( d_w^p(x,y) \leq \mu = w_\lambda(x,y) \). Therefore, for any \( \lambda > 0 \) we have \( d_w^p(x,y) \leq \max\{\lambda, w_\lambda(x,y)\} \leq \lambda + w_\lambda(x,y) \), and so, taking the infimum over all \( \lambda > 0 \), we arrive at the inequality \( d_w^p(x,y) \leq d_w^1(x,y) \).

To obtain the right-hand side inequality, we note that, given \( \lambda > 0 \) such that \( d_w^p(x,y) < \lambda \), by the definition of \( d_w^p \), we get \( w_\lambda(x,y) \leq \lambda \), and so, \( d_w^1(x,y) \leq \lambda + w_\lambda(x,y) \leq 2\lambda \). Passing to the limit as \( \lambda \to d_w^p(x,y) \), we get \( d_w^1(x,y) \leq 2d_w^p(x,y) \). \( \square \)

We remark that the metric \( d_w^1 \) and Theorem 2.8 are more general variants of the \( F \)-norm and the result from [8] (see also [19, Theorem 1.1(b)]).

**Examples 2.9.** (a) Let us consider the (pseudo)modular \( w = w^b \) from Example 2.4(b) with \( \varphi(\lambda) = \lambda^p \) and \( p = \text{const} > 0 \). In order to calculate \( d_w^p(x,y) = \inf_{\lambda>0} f(\lambda) \), where \( f(\lambda) = \lambda + (d(x,y)/\lambda)^p \), we note that the derivative \( f'(\lambda) = 1 - (pd(x,y)/\lambda^{p+1}) \) vanishes only at the point \( \lambda_0 = (pd(x,y))^{1/(p+1)} \) and assumes negative values at the left of this point and positive values at the right, and so, \( \lambda_0 \) is the point where \( f \) attains the global minimum on \((0, \infty)\). It follows that

\[
d_w^p(x,y) = f(\lambda_0) = (p + 1)p^{-p/(p+1)}(d(x,y))^{1/(p+1)}, \quad x, y \in X.
\]

Note that this expression is coherent with Theorem 2.8 if we take into account Example 2.7(b) and the inequalities \( 1 \leq (p + 1)p^{-p/(p+1)} \leq 2 \) for all \( p > 0 \). For classical modulars an example of this type was elaborated in [19, p. 5]. In particular, if \( p = 1 \), the expressions for \( d_w^p(x,y) \) and \( d_w^1(x,y) \) are of the form, respectively:

\[
d_w^1(x,y) = \sqrt{d(x,y)} \quad \text{and} \quad d_w^1(x,y) = 2\sqrt{d(x,y)}, \quad x, y \in X.
\]

(b) The results in Example 2.7(b) and 2.9(a) with \( \varphi(\lambda) = \lambda^p \) are valid in a somewhat more general case when the (pseudo)modular \( w \) on the set \( X \) is \( p \)-homogeneous with \( p > 0 \), i.e., satisfies the condition

\[
w_\lambda(x,y) = w_1(x,y)/\lambda^p, \quad \lambda > 0, \quad x, y \in X.
\]

If this is the case, \( d(x,y) \) should be replaced by \( w_1(x,y) \) in the expressions for \( d_w^p(x,y) \) from Example 2.4(b) and for \( d_w^1(x,y) \) from Example 2.9(a).

Another example of a \( p \)-homogeneous modular on a metric space \((X, d)\) can be given as \( w_\lambda(x,y) = (d(x,y)/\lambda)^p = w_1(x,y)/\lambda^p \).

(c) Given a (pseudo)metric space \((X, d)\) and a convex function \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) vanishing at zero only (it follows that \( \varphi \) is strictly increasing, continuous and admits the continuous inverse function \( \varphi^{-1} \)), we set:

\[
w_\varphi(x,y) = \lambda \varphi(d(x,y)/\lambda), \quad \lambda > 0, \quad x, y \in X.
\]

Then \( w \) is a metric (pseudo)modular on \( X \) (note that modulars of this type are not allowed in the classical theory of modulars on linear spaces). In order to see this, it suffices to verify axiom (iii):

\[
w_{\lambda+\mu}(x,y) \leq (\lambda + \mu)\varphi\left(\frac{\lambda}{\lambda+\mu}, \frac{d(x,z)}{\lambda} + \frac{\mu}{\lambda+\mu}, \frac{d(y,z)}{\mu}\right)
\]

\[
\leq (\lambda + \mu)\left[\varphi\left(\frac{\lambda}{\lambda+\mu}, \frac{d(x,z)}{\lambda}\right) + \varphi\left(\frac{\mu}{\lambda+\mu}, \frac{d(y,z)}{\mu}\right)\right]
\]

\[
= w_\lambda(x,z) + w_\mu(y,z).
\]

If, in addition, the function \( \varphi \) satisfies the condition \( \varphi(\lambda) = o(\lambda) \) as \( \lambda \to +0 \), that is, \( \varphi'(0) = \lim_{\lambda \to +0} \varphi(\lambda)/\lambda = 0 \), then, given \( x \in X \), we get:

\[
\lim_{\lambda \to +0} w_\lambda(x,0) = \lim_{\lambda \to +0} \varphi\left(\frac{d(x,0)}{\lambda}\right) = \lim_{\mu \to +0} \frac{d(x,0)}{\mu} \varphi(\mu) = 0,
\]

and so, \( X_\lambda = X \). Then \( d_w^p(x,y) \) is of the form:

\[
d_w^p(x,y) = \inf\{\lambda > 0 : \varphi(d(x,y)/\lambda) \leq 1\} = d(x,y)/\varphi^{-1}(1).
\]

Let \( \varphi(\lambda) = \lambda^p \) with \( p > 1 \). Then \( d_w^p(x,y) = d(x,y) \). In order to calculate the value \( d_w^1(x,y) \), we turn to Example 2.9(b) and note that

\[
w_\lambda(x,y) = \lambda^p(d(x,y)/\lambda^p) = (d(x,y))^p/\lambda^{p-1} = w_1(x,y)/\lambda^{p-1}.
\]

It follows that \( d_w^1(x,y) = p(p - 1)^{1-p/p}d(x,y) \) for all \( x, y \in X \).

(d) Setting \( \varphi(\lambda) = e^\lambda, \lambda > 0, \) and \( w = w^b \) in Example 2.4(b), we find that \( d_w^p(x,y) = d(x,y) \) if \( d(x,y) \leq 1 \), and

\[
d_w^1(x,y) = 1 + \log(d(x,y)) \quad \text{if} \quad d(x,y) > 1.
\]

Now, let us exhibit specific relations between \( \lambda > 0, w_\lambda(x,y) \) and \( d_w^p(x,y) \).

**Theorem 2.10.** Given a (pseudo)modular \( w \) on \( X \), \( x, y \in X_w \) and \( \lambda > 0 \), we have:

(a) if \( d_w^p(x,y) < \lambda \), then \( w_\lambda(x,y) \leq d_w^p(x,y) < \lambda \);

(b) if \( w_\lambda(x,y) = \lambda \), then \( d_w^p(x,y) = \lambda \);
(c) if $\lambda = d_{w}^{\circ}(x, y) > 0$, then $w_{x} \leq \lambda \leq w_{x}^{\circ}(x, y)$.

If the function $\mu \mapsto w_{\mu}(x, y)$ is continuous from the right on $(0, \infty)$, then along with (a)–(c) we have:

(d) $d_{w}^{\circ}(x, y) \leq \lambda$ if and only if $w_{x}(x, y) \leq \lambda$.

If the function $\mu \mapsto w_{\mu}(x, y)$ is continuous from the left on $(0, \infty)$, then along with (a)–(c) we have:

(e) $d_{w}^{\circ}(x, y) < \lambda$ if and only if $w_{x}(x, y) < \lambda$.

If the function $\mu \mapsto w_{\mu}(x, y)$ is continuous on $(0, \infty)$, then along with (a)–(e) we have:

(f) $d_{w}^{\circ}(x, y) = \lambda$ if and only if $w_{x}(x, y) = \lambda$.

**Proof.** (a) For any $\mu > 0$ such that $d_{w}^{\circ}(x, y) < \mu < \lambda$, by the definition of $d_{w}^{\circ}$ and Section 2.3, we have $w_{\mu}(x, y) \leq \mu$ and $w_{x}(x, y) \leq w_{\mu}(x, y)$, whence $w_{x}(x, y) \leq \mu$, and it remains to pass to the limit as $\mu \to d_{w}^{\circ}(x, y)$.

(b) By the definition, $d_{w}^{\circ}(x, y) \leq \lambda$, and item (a) implies $d_{w}^{\circ}(x, y) = \lambda$.

(c) For any $\mu > \lambda = d_{w}^{\circ}(x, y)$, the definition of $d_{w}^{\circ}$ implies $w_{\mu}(x, y) \leq \mu$, and so, $w_{x}^{\lambda, \circ}(x, y) = w_{\mu, \lambda} \to w_{\mu}(x, y) \leq d_{w}^{\circ}(x, y)$.

(d) The implication $\Leftarrow$ follows from the definition of $d_{w}^{\circ}$. Let us prove the reverse implication. If $d_{w}^{\circ}(x, y) < \lambda$, then, by virtue of item (a), $w_{x}(x, y) < \lambda$, and if $d_{w}^{\circ}(x, y) = \lambda$, then $w_{x}(x, y) = w_{x}^{\lambda, \circ}(x, y) \leq \lambda$, which is a consequence of the continuity from the right of the function $\mu \mapsto w_{\mu}(x, y)$ and item (c).

(e) By virtue of (a), it suffices to prove the implication $\Leftarrow$. The definition of $d_{w}^{\circ}$ gives $d_{w}^{\circ}(x, y) \leq \lambda$, but if, on the contrary, $\lambda = d_{w}^{\circ}(x, y)$, then, by item (c), we would have $w_{x}(x, y) = w_{x}^{\lambda, \circ}(x, y) \geq \lambda$, which contradicts the assumption.

(f) $\Leftarrow$ follows from (b). For the reverse inequalities $w_{x}(x, y) \leq \lambda \leq w_{x}(x, y)$ follow from (c). $\Box$

**Example 2.11.** The limit $w_{x}^{\lambda, \circ}(x, y)$ in Theorem 2.10(c) cannot be replaced by $w_{x}(x, y)$, in general. To see this, consider the (pseudo)modular $w = w^{\circ}$ from Example 2.4(c) (see also 2.7(c)). It is clear that the function $\mu \mapsto w_{\mu}(x, y)$ is continuous from the left on $(0, \infty)$, but not from the right (at the point $\lambda = d(x, y)$). If $x \neq y$, then for $\lambda = d_{w}^{\circ}(x, y) = d(x, y) > 0$ we find

$$w_{x}^{\lambda, \circ}(x, y) = 0 < \lambda = d(x, y) < \infty = w_{x}^{\lambda, \circ}(x, y),$$

whereas $w_{x}(x, y) = \infty > \lambda$. This example shows also that item (d) may fail without the assumption of the right continuity of the function $\mu \mapsto w_{\mu}(x, y)$.

A similar situation holds with respect to the left limit in Theorem 2.10(c), (e); an appropriate example is 2.4(d).

2.12

In addition to Theorem 2.10 let us recall two implications, which are always valid: if $w_{x}(x, y) \leq \lambda$, then $d_{w}^{\circ}(x, y) \leq \lambda$, and if $w_{x}(x, y) > \lambda$, then $d_{w}^{\circ}(x, y) \leq w_{x}(x, y)$; they are established in the proof of Theorem 2.8.

In the next theorem we show that the convergence in metric $d_{w}^{\circ}$ and modular convergence of sequences from $X_{w}$ are equivalent.

**Theorem 2.13.** Let $w$ be a (pseudo)modular on a set $X$. Given a sequence $(x_{n})_{n=1}^{\infty} \subset X_{w}$ and $x \in X_{w}$, we have: $d_{w}^{\circ}(x_{n}, x) \to 0$ as $n \to \infty$ if and only if $w_{x}^{\circ}(x_{n}, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$. A similar assertion holds for Cauchy sequences.

**Proof.** Sufﬁciency. Given arbitrary $\varepsilon > 0$, the assumption implies that $w_{x}^{\circ}(x_{n}, x) \to 0$ as $n \to \infty$, and so, there is a number $n_{0}(\varepsilon)$ such that $w_{x}^{\circ}(x_{n}, x) \leq \varepsilon$ for all $n \geq n_{0}(\varepsilon)$, whence $d_{w}^{\circ}(x_{n}, x) \leq \varepsilon$ for all $n \geq n_{0}(\varepsilon)$.

Necessity. Let us ﬁx $\lambda > 0$ arbitrarily. Then, for each $\varepsilon > 0$, we have: either (a) $0 < \varepsilon < \lambda$, or (b) $\varepsilon \geq \lambda$. In case (a), by the assumption, there is a number $n_{0}(\varepsilon)$ such that $d_{w}^{\circ}(x_{n}, x) < \varepsilon$ for all $n \geq n_{0}(\varepsilon)$, and so, by Theorem 2.10(a), we get $w_{x}(x_{n}, x) < \varepsilon$ for all $n \geq n_{0}(\varepsilon)$. Since $\varepsilon < \lambda$, then, in view of Section 2.3, we find $w_{x}(x_{n}, x) \leq w_{x}^{\circ}(x_{n}, x) < \varepsilon$ for all $n \geq n_{0}(\varepsilon)$.

In case (b) we set $n_{1}(\varepsilon) = n_{0}(\varepsilon)/2$. From Section 2.3 and the just established fact (when $\varepsilon = \lambda/2 < \lambda$), we get:

$$w_{x}(x_{n}, x) \leq w_{x}^{\circ}(x_{n}, x) < \lambda/2 \leq \varepsilon/2 < \varepsilon$$

for all $n \geq n_{1}(\varepsilon)$.

Hence, $w_{x}(x_{n}, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$. $\Box$

In order to present the example in Section 2.15, we need the following notion.

2.14. Metric semigroup

([30, Section 4], [31, Section 3]). The triple $(M, d, +)$ is said to be a metric semigroup if $(M, d)$ is a metric space with metric $d$, $(M, +)$ is an Abelian semigroup with respect to the addition operation + and the metric $d$ is translation invariant: $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in M$. The element $0 \in M$ such that $x + 0 = 0 + x = x$ for all $x \in M$ is called the zero. The metric semigroup $(M, d, +)$ is said to be complete, if the metric space $(M, d)$ is complete.
Given a metric semigroup \((M, d, +)\) and any \(x, y, \bar{x}, \bar{y} \in M\), we have:
\[
d(x, y) \leq d(x + \bar{x}, y + \bar{y}) + d(\bar{x}, \bar{y}),
\]
\[
d(x + \bar{x}, y + \bar{y}) \leq d(x, y) + d(\bar{x}, \bar{y}).
\]
If sequences \(\{x_n\}, \{y_n\}, \{\bar{x}_n\}\) and \(\{\bar{y}_n\}\) of elements from \(M\) converge to \(x, y, \bar{x}\) and \(\bar{y}\) as \(n \to \infty\), respectively, then, by virtue of (2.4),
\[
d(x_n + \bar{x}_n, y_n + \bar{y}_n) \to d(x + \bar{x}, y + \bar{y}) \quad \text{as} \quad n \to \infty,
\]
and, in particular, the addition operation \((x, y) \mapsto x + y\) in \(M\) is a continuous mapping from \(M \times M\) into \(M\).

2.15. The modular generating functions of finite \(\varphi\)-variation

Let \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) be a \(\varphi\)-function (cf. Introduction), \((M, d, +)\) be a metric semigroup, \(I = [a, b]\) a closed interval in \(\mathbb{R}\) and \(X = M^I\) the set of all functions mapping \(I\) into \(M\). Given \(\lambda > 0\) and \(x, y \in X\), we define a function \(w : (0, \infty) \times X \times X \to [0, \infty]\) by the rule:
\[
w_\lambda(x, y) = \sup \sum_{i=1}^m \varphi\left(\frac{1}{\lambda} d(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1}))\right),
\]
where the supremum is taken over all partitions \(\{t_i\}_{i=0}^m\) of the interval \(I\).

Let us show that \(w\) is a pseudomodular on \(X\), having verified only axiom (iii). We make use of the following simple observation: if \(\alpha, \beta \geq 0\), 
\(\alpha + \beta \leq 1\) and \(A, B \geq 0\), then
\[
\varphi(\alpha A + \beta B) \leq \max\{\varphi(A), \varphi(B)\} \leq \varphi(A) + \varphi(B).
\]
If \(\lambda, \mu > 0\), \(x, y, z \in X\), \(m \in \mathbb{N}\), \(a = t_0 < t_1 < \cdots < t_{m-1} < t_m = b\) and \(i \in \{1, \ldots, m\}\), then, by virtue of (2.3) and the translation invariance of \(d\), we have:
\[
C_i \equiv d(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1})) \\
\leq d(x(t_i) + y(t_{i-1}) + z(t_{i-1}), y(t_i) + x(t_{i-1}) + z(t_i) + y(t_{i-1}))) + d(y(t_i) + z(t_{i-1}), z(t_i) + y(t_{i-1})) \\
= d(x(t_i) + z(t_{i-1}), z(t_i) + x(t_{i-1})) + d(y(t_i) + z(t_{i-1}), z(t_i) + y(t_{i-1})) \\
= A_i + B_i,
\]
and so, the monotonicity of \(\varphi\) and the observation above imply
\[
\varphi\left(\frac{C_i}{\lambda + \mu}\right) \leq \varphi\left(\frac{\lambda}{\lambda + \mu} A_i + \frac{\mu}{\lambda + \mu} B_i\right) \leq \varphi\left(\frac{A_i}{\lambda}\right) + \varphi\left(\frac{B_i}{\mu}\right).
\]
Summing over \(i = 1, \ldots, m\) and taking the supremum over all partitions of the interval \(I\), we obtain the inequality in axiom (iii).

Let us establish that, given \(x, y \in X\), the function \(\lambda \mapsto w_\lambda(x, y)\) is continuous from the right on \((0, \infty)\). Let \(\lambda > 0\). By virtue of (2.1), it suffices to show that \(w_\lambda(x, y) \leq w_{\lambda+0}(x, y)\). For each partition \(\{t_i\}_{i=0}^m\) of the interval \(I\) and any \(\mu > \lambda\) we have:
\[
\sum_{i=1}^m \varphi\left(\frac{1}{\mu} d(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1}))\right) \leq w_\mu(x, y),
\]
and so, as \(\mu \to \lambda + 0\), the continuity of \(\varphi\) implies
\[
\sum_{i=1}^m \varphi\left(\frac{1}{\lambda} d(x(t_i) + y(t_{i-1}), y(t_i) + x(t_{i-1}))\right) \leq w_{\lambda+0}(x, y).
\]
It remains to take the supremum over all partitions \(\{t_i\}_{i=0}^m\) of the interval \(I\).

For a constant function \(x_0(t) = x_0\) for all \(t \in I\), where \(x_0 \in M\), the definition (2.6) of the pseudomodular \(w\) gives (note that the expression on the right does not depend on \(x_0\)):
\[
w_\lambda(x, x_0) = \sup \sum_{i=1}^m \varphi\left(\frac{1}{\lambda} d(x(t_i), x(t_{i-1}))\right), \quad x \in M^I.
\]
If \(x \in X_w = X_w^0(x_0)\), then \(x\) is said to be a function of bounded generalized \(\varphi\)-variation in the sense of Wiener and Young (if \(\varphi\) is convex, more details can be found in [31, Section 3]). To the best of the author’s knowledge, if the function \(\varphi\) is nonconvex, the question concerning the complete description of the space \(X_w\) remains open even in the case when \(M = \mathbb{R}\) (cf. (1.1) and (1.2)); in particular, it is not known whether the modular space \(X_w\) coincides with the larger set \(X_{\varphi}^0\) defined in Section 3.1 below. For a certain discussion and a partial solution of this problem when \(M = \mathbb{R}\) see [16,35].
2.16. The right inverse (pseudo)modular

Let \( w \) be a (pseudo)modular on a set \( X \). The right inverse for \( w \) is the function \( w^+ : (0, \infty) \times X \times X \to [0, \infty] \) defined, for all \( \mu > 0 \) and \( x, y \in X \), by the rule:

\[
w^+_\mu(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \leq \mu\} \quad (\inf \varnothing = \infty).
\]

The properties of \( w^+ \) are gathered in the following theorem.

**Theorem 2.17.** Let \( w \) be a (pseudo)modular on \( X \). Then \( w^+ \) is also a (pseudo)modular on \( X \) such that the following (in)equalities in \([0, \infty)\) hold:

(a) \( w^+\mu_\mu(x, y) = w^+\mu(x, y) \) for all \( \mu > 0 \) and \( x, y \in X \);

(b) \( w^+\mu_\lambda(x, y) \leq w^+\lambda(x, y) \) for all \( \lambda > 0 \) and \( x, y \in X \), where \( w^+ = (w^+)^+ \).

**Proof.** Let us verify only axioms (i) and (ii) for \( w^+ \). (i) Let \( x, y \in X \) and \( w^+\mu(x, y) = 0 \) for all \( \mu > 0 \). Then \( \lambda > w^+\mu(x, y) \) for all \( \lambda > 0 \), and so, the definition of \( w^+ \) implies \( w_\lambda(x, y) \leq \mu \) for all \( \mu > 0 \), whence \( w_\lambda(x, y) = 0 \) for all \( \lambda > 0 \), and so, by (i) for \( w \), we get \( x = y \). (ii) Let us show that \( w^+\lambda_\mu(x, y) \leq w^+\lambda(x, z) + w^+\mu(y, z) \). If \( w^+\lambda(x, z) = \infty \) or \( w^+\mu(y, z) = \infty \), then the inequality is obvious. So, suppose that \( w^+\lambda(x, z) < \infty \) and \( w^+\mu(y, z) < \infty \). Then, given \( \xi > w^+\lambda(x, z) \) and \( \eta > w^+\mu(y, z) \), the definition of \( w^+ \) implies \( w_\xi(x, z) \leq \lambda \) and \( w_\eta(y, z) \leq \mu \), and so,

\[
w^+\xi+y(x, z) \leq w^+\xi(x, z) + w^+\eta(y, z) \leq \lambda + \mu.
\]

It follows that \( \xi + \eta \in \{y > 0 : w_\xi(x, y) \leq \lambda + \mu\} \), and so, \( w^+\lambda_\mu(x, y) \leq \xi + \eta \), and it remains to pass to the limits as \( \xi \to w^+\lambda(x, z) \) and \( \eta \to w^+\mu(y, z) \).

(A) By virtue of (2.1), we have \( w^+\mu_\mu(x, y) \leq w^+\mu(x, y) \). If \( w^+\mu(x, y) = \infty \), then, by the last inequality, \( w^+\mu(x, y) = \infty \).

Now, let \( w^+\mu_\lambda(x, y) < \infty \) and \( \lambda > 0 \) be any number such that \( w^+\mu_\mu(x, y) < \lambda \). Then there exists \( \varepsilon_0 > 0 \) such that \( w^+\mu_\lambda(x, y) < \lambda \) for all \( 0 < \varepsilon < \varepsilon_0 \). From the definition of \( w^+ \) we find \( w_\lambda(x, y) \leq \mu + \varepsilon \) for all \( \varepsilon \in (0, \varepsilon_0) \), whence we get \( w^+\lambda(x, y) \leq \mu \). It follows from Section 2.16 that \( w^+\mu(x, y) \leq \lambda \). Passing to the limit as \( \lambda \to w^+\mu_\lambda(x, y) \) we conclude that \( w^+\lambda(x, y) \leq w^+\mu_\lambda(x, y) \).

(b) First, given \( \lambda > 0 \) and \( x, y \in X \), we obtain the inequality in (b). If \( w_\lambda(x, y) = \infty \), the inequality is clear. Now if \( w_\lambda(x, y) < \infty \), then, for each \( \mu > w_\lambda(x, y) \), the definition of \( w^+ \) implies \( w^+\mu(x, y) \leq \lambda \). Applying the definition of \( w^+ \) we get \( w^+\mu_\lambda(x, y) \leq \mu \) for all \( \mu > w^+\mu_\lambda(x, y) \), and so, \( w^+\mu_\lambda(x, y) \leq w^+\lambda_\mu(x, y) \).

Now we prove the equality in (b). In the last paragraph we have shown that \( w^+\lambda_\mu(x, y) \leq w^+\mu_\mu(x, y) \). Since, by item (a), \( w^+\lambda_\mu(x, y) = w^+\lambda(x, y) \), then \( w^+\lambda(x, y) \leq w^+\lambda_\mu(x, y) \).

If \( w^+\lambda(x, y) = \infty \), then the last inequality in the previous paragraph implies \( w^+\lambda_\mu(x, y) = \infty \). So, assume that \( w^+\lambda(x, y) < \infty \). Then, for any \( \mu > w^+\lambda(x, y) \), the definition of \( w^+ \) gives \( w^+\mu(x, y) \leq \lambda \). If \( w^+\mu(x, y) < \lambda \), then \( w^+\mu(x, y) \leq \mu \), and so, \( w^+\mu_\lambda(x, y) \leq w^+\mu_\mu(x, y) \) \( \leq \lambda \). Now if \( w^+\mu(x, y) = \lambda \), then for each \( \xi > \lambda = w^+\mu(x, y) \), the definition of \( w^+ \) implies \( w_\xi(x, y) \leq \mu \), whence \( w^+\mu_\lambda(x, y) = \lim_{\varepsilon \to \lambda}(w^+\varepsilon(x, y) \leq \mu) \). Thus, we have shown that \( w^+\mu_\lambda(x, y) \leq \mu \) for all \( \mu > w^+\mu(x, y) \), and so, \( w^+\lambda_\mu(x, y) \leq w^+\lambda(x, y) \).

**Remark 2.18.** Let \( w \) be a (pseudo)modular on a set \( X \).

Note that \( w^{++} = w^+ \); in fact, by Theorem 2.17(a), (b), we have \( w^+\mu(x, y) = w^+\mu_\mu(x, y) = w^+\mu(x, y) \). Moreover, if the function \( \lambda \mapsto w_\lambda(x, y) \) is continuous from the right on \((0, \infty)\) for all \( x, y \in X \), then \( w^+ = w \).

The term the ‘right inverse (pseudo)modular’ for \( w^+ \) can be motivated as follows. For the sake of convenience we temporarily employ also the notation \( w^+(\xi; x, y) = w^+(x, y) \). If \( \lambda > 0 \) and \( w^+(x, y) \in \infty \), then the inequalities \( w^+(x, y) \leq \lambda \) and \( w^+_\mu(x, y) \leq w^+(x, y) \) are equivalent (cf. the inequality in Theorem 2.17(b)). On the other hand, if \( \mu > 0 \) and \( w^+(x, y) \in \infty \), then we have the inequalities: \( w^+(x, y) + \mu \leq \mu \) and \( \mu \leq w^+(x, y) \). Therefore, if the function \( \lambda \mapsto w_\lambda(x, y) \) is continuous on \((0, \infty)\), then \( w^+(x, y) = \mu \), \( \mu > 0 \).

**Examples 2.19.** (a) If \( w \) is the (pseudo)modular \( w^b \) from Example 2.4(b) with \( \varphi(\lambda) = \lambda^p \) and \( p > 0 \), then \( w^+\mu(x, y) = (d(x, y)/\mu)^{1/p} \). In particular, if \( p = 1 \), then \( w^+ = w \).

(b) Applying the notation from Example 2.4(c), (d), we have: if \( x, y \in X, \mu > 0 \) and \( \lambda > 0 \), then \( w^+_\mu(x, y) = d(x, y) \), \( w^+_\mu(x, y) = w^+_\mu(x, y) \), and so, the inequality in Theorem 2.17(b) is sharp. Note also that \( X_{\mu^+} = X \), whereas \( X_{\mu^+} = \{x_0\} \).

3. Convex metric modulars

3.1. The set \( X^*_w \)

Given a metric (pseudo)modular \( w \) on a set \( X \), along with the modular set \( X_w \) from Section 2.5 we also put

\[
X^*_w = X^*_w(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0 \text{ such that } w_\lambda(x, x_0) < \infty\} \quad (x_0 \in X).
\]
It is clear from (2.2) that $X_w \subset X_w^*$, and this inclusion is, in general, strict: it is seen from Example 2.7(b) with bounded function $\varphi$ that $X_w = \{x_0\}$ and $X_w^* = X$. Now let us expose a more subtle example of this type.

**Example 3.2.** Let $(M, d)$ be a metric space and $X = M^\mathbb{N}$ be the set of all sequences $x : \mathbb{N} \to M$. Define a function $w : (0, \infty) \times X \times X \to [0, \infty]$ by

$$w_x(y) = \sup_{n \in \mathbb{N}} \frac{d(x(n), y(n))}{\lambda}, \quad \lambda > 0, \quad x, y \in X.$$

Then $w$ is a metric modular on $X$, for which the function $\lambda \mapsto w_x(y)$ is continuous from the right on $(0, \infty)$ for all $x, y \in X$. Fixing some elements $x_0, x \in M, x \neq x_0$, let $x_0, x \in X$ be the corresponding constant functions: $x_0(n) = x_0$ and $x(n) = x$ for all $n \in \mathbb{N}$. It follows that $x \in X_w^*(x_0) \setminus X_w^*(x_0) - X$: in fact, if $\lambda > d(x, x_0)$, then

$$w_x(x_0) = \sup_{n \in \mathbb{N}} d(x(n), x_0)/\lambda^{1/n} = \lim_{n \to \infty} d(x(n), x_0)/\lambda^{1/n} = 1.$$

**Definition 3.3.** A function $w : (0, \infty) \times X \times X \to [0, \infty]$ is said to be a convex (metric) modular on a set $X$ if it satisfies the axioms (i) and (ii) from Definition 2.1 as well as the following axiom

$$(iv) \quad \frac{w_{x+y}}{\lambda + \mu} \leq \frac{\lambda}{\lambda + \mu} w_x(x, z) + \frac{\mu}{\lambda + \mu} w_y(y, z) \quad \forall \lambda, \mu > 0, \quad x, y, z \in X.$$

If, instead of (i), we have only condition (i') from Definition 2.1, then $w$ is called a convex (metric) pseudomodular on $X$.

**Examples 3.4.** (a) The (pseudo)modulars from Example 2.4 are convex, except item (b); this is true in item (b) as well only if $\varphi(\lambda) = \lambda$ (and not $\varphi(\lambda) \equiv 1$). In Section 2.15 the pseudomodular $w$ is convex if the function $\varphi$ is convex, and in Example 3.2 the modular $w$ is not convex (cf. (3.2)).

(b) Given a (pseudo)metric space $(X, d)$ and a convex function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, we set $w_x(y) = \varphi(d(x, y)/\lambda)$ for all $\lambda > 0$ and $x, y \in X$. Since the function $\varphi$ coincides with the (pseudo)modular from Example 2.9(c), the (pseudo)modular $w$ is convex.

(c) Let $(M, d)$ be a metric space, $\varphi_i : \mathbb{R}^+ \to \mathbb{R}^+$ be convex functions for all $i \in \mathbb{N}$ and $X = M^\mathbb{N}$. Setting

$$w_x(y) = \sum_{i=1}^{\infty} \varphi_i\left(\frac{d(x_i, y_i)}{\lambda}\right), \quad \lambda > 0, \quad x = \{x_i\}_{i=1}^{\infty}, \quad y = \{y_i\}_{i=1}^{\infty} \in M^\mathbb{N},$$

we find that $w$ is a convex modular on $X$.

3.5

**The main property of a convex (pseudo)modular** $w$ on a set $X$ can be expressed as follows: given $x, y \in X$, the functions $\lambda \mapsto w_\lambda(x, y)$ and $\lambda \mapsto \hat{w}_\lambda(x, y) = \lambda w_\lambda(x, y)$ are nonincreasing on $(0, \infty)$:

$$\text{if } 0 < \mu \leq \lambda, \text{ then } w_\lambda(x, y) \leq \frac{\mu}{\lambda} w_\mu(x, y) \leq w_\mu(x, y).$$

Moreover, condition (iv) is equivalent to the following condition:

$$\text{(iv')} \quad \frac{w_{x+y}}{\lambda + \mu} \leq \frac{\lambda}{\lambda + \mu} w_x(x, z) + \frac{\mu}{\lambda + \mu} w_y(y, z) \quad \forall \lambda, \mu > 0, \quad \lambda + \mu \leq v, \quad x, y, z \in X.$$
The next theorem shows that a metric $d_w^*$ on $X_w^*$ can be introduced by the same rule independently of whether the last inclusion in (3.3) is strict or not.

**Theorem 3.6.** Given a convex (pseudo)modular $w$ on a set $X$, put:

\[ d_w^*(x, y) = \inf \{ \lambda > 0 : w_\lambda(x, y) \leq 1 \}, \quad x, y \in X_w^*. \]

Then $(X_w^*, d_w^*)$ is a (pseudo)metric space.

**Proof.** The value $d_w^*(x, y) \in \mathbb{R}^+$ is well defined for all $x, y \in X_w^*$. In fact, there are numbers $\lambda = \lambda(x) > 0$ and $\mu = \mu(y) > 0$ such that $w_\lambda(x, x_0) < \infty$ and $w_\lambda(y, x_0) < \infty$, and so, applying the inequality (iv) with $z = x_0$ we get $w_\lambda(x, y) \leq 1$. It follows that there exists $v_0 > 0$ such that $w_\lambda(x, y) \leq 1$ for all $\lambda \geq v_0$, whence the definition of $d_w^*$ implies $d_w^*(x, y) \leq v_0 < \infty$. The remaining part of the proof coincides essentially with the proof of Theorem 2.6, and so, we verify only two axioms of metric $d_w^*$. If $x, y \in X_w^*$ and $d_w^*(x, y) = 0$, then $w_\lambda(x, y) \leq 1$ for all $\lambda > 0$. It follows from (3.1) that, given $\lambda > 0$ and $0 < \mu < \lambda$, $w_\lambda(x, y) \leq (\mu/\lambda)w_\mu(x, y) \leq \mu/\lambda \to 0$ as $\lambda \to 0$, and so, $w_\lambda(x, y) = 0$ for all $\lambda > 0$, whence $x = y$. Now, let $x, y, z \in X_w^*$. If $\lambda > d_w^*(x, z)$ and $\mu > d_w^*(y, z)$, the definition of $d_w^*$ gives $w_\lambda(x, z) \leq 1$ and $w_\mu(y, z) \leq 1$, and so, by axiom (iv),

\[ w_{\lambda + \mu}(x, y) \leq \frac{\lambda}{\lambda + \mu} w_\lambda(x, z) + \frac{\mu}{\lambda + \mu} w_\mu(y, z) \leq \frac{\lambda}{\lambda + \mu} + \frac{\mu}{\lambda + \mu} = 1. \]

Thus, $d_w^*(x, y) \leq \lambda + \mu$, and the arbitrariness of $\lambda$ and $\mu$ as above implies $d_w^*(x, y) \leq d_w^*(x, z) + d_w^*(y, z)$. □

The metric $d_w^*$ is a counterpart of the norm (1.4), and Theorem 3.6 is a generalization of the results from [10,4,6,3] (see also [19, Theorem 1.2] and [21, Theorem 1.5]).

In the next theorem we present a generalization of the norm, which is known in the classical theory of modulars as the Anemiy norm (cf. [17], [19, Theorem 1.2 and Remark 2], [21, p. 165], [7, p. 218]).

**Theorem 3.7.** Let $w$ be a convex (pseudo)modular on $X$. Set

\[ d_w^A(x, y) = \inf_{\lambda > 0} (\lambda + \lambda w_\lambda(x, y)), \quad x, y \in X_w^*. \]

Then $d_w^A$ is a (pseudo)metric on $X_w^*$ and $d_w^A(x, y) \leq d_w^A(x, y) \leq 2d_w^*(x, y)$ for all $x, y \in X_w^*$.

**Proof.** Since $0 < \lambda + \lambda w_\lambda(x, y) < \infty$ for $\lambda$ large enough (see the beginning of the proof of Theorem 3.6), the set $\{ \lambda w_\lambda(x, y) : \lambda > 0 \}$ is nonempty and bounded from below, and so, $d_w^A$ is well defined: $0 \leq d_w^A(x, y) < \infty$ for all $x, y \in X_w^*$.

Let us verify only the nondegeneracy of $d_w^A$ and the triangle inequality for $d_w^A$. Suppose that $w$ satisfies (i), $x, y \in X_w^*$ and $d_w^*(x, y) = 0$, and show that $w_\lambda(x, y) = 0$ for all $\lambda > 0$. If this is not so, then $w_{\lambda_0}(x, y) > 0$ for some $\lambda_0 > 0$. We have: if $\lambda \geq \lambda_0$, then $\lambda + \lambda w_\lambda(x, y) \geq \lambda_0$, and if $0 < \lambda < \lambda_0$, then, by virtue of (3.1), we find $w_{\lambda_0}(x, y) \leq (\lambda/\lambda_0)w_\lambda(x, y)$, and so, $\lambda_0 w_{\lambda_0}(x, y) \leq \lambda w_\lambda(x, y) \leq \lambda + \lambda_0 w_\lambda(x, y)$. It follows that, for all $\lambda > 0$, we have $\lambda + \lambda w_\lambda(x, y) \geq \min(\lambda_0, \lambda_0 w_{\lambda_0}(x, y)) \equiv \lambda_1 > 0$ implying $d_w^A(x, y) \geq \lambda_1 > 0$, which contradicts the assumption.

Now we show that $d_w^A(x, y) \leq d_w^A(x, z) + d_w^A(y, z)$, $x, y, z \in X_w^*$. By the definition of $d_w^A$, given $\varepsilon > 0$, there exist $\lambda = \lambda(\varepsilon) > 0$ and $\mu = \mu(\varepsilon) > 0$ such that

\[ \lambda + \lambda w_\lambda(x, z) \leq d_w^A(x, z) + \varepsilon \quad \text{and} \quad \mu + \mu w_\mu(y, z) \leq d_w^A(y, z) + \varepsilon, \]

and so, the convexity of $w$ implies

\[ d_w^A(x, y) \leq (\lambda + \mu) + (\lambda + \lambda)w_{\lambda + \mu}(x, y) \]

\[ \leq \lambda + \mu + (\lambda + \mu) \left[ \frac{\lambda}{\lambda + \mu} w_\lambda(x, z) + \frac{\mu}{\lambda + \mu} w_\mu(y, z) \right] \]

\[ = \lambda + \lambda w_\lambda(x, z) + \mu + \mu w_\mu(y, z) \]

\[ \leq d_w^A(x, z) + \varepsilon + d_w^A(y, z) + \varepsilon, \]

and it remains to pass to the limit as $\varepsilon \to +0$.

Let us prove the equivalence of metrics $d_w^A$ and $d_w^*$. Given $\lambda > 0$, we have: if $w_\lambda(x, y) \leq 1$, then, by the definition of $d_w^*$, $d_w^*(x, y) \leq \lambda$, and if $w_\lambda(x, y) > 1$, then $d_w^*(x, y) \leq \lambda w_\lambda(x, y)$; in fact, setting $\mu = \lambda w_\lambda(x, y)$, we find $\mu > \lambda$, and so, (3.1) implies $w_\mu(x, y) \leq (\lambda/\mu)w_\lambda(x, y) = \lambda w_\lambda(x, y)/\mu = 1$, whence $d_w^*(x, y) \leq \mu = \lambda w_\lambda(x, y)$. It follows that $d_w^*(x, y) \leq \max(\lambda, \lambda w_\lambda(x, y)) \leq \lambda + \lambda w_\lambda(x, y)$ for all $\lambda > 0$, and the inequality at the left-hand side follows. In order to establish the right-hand side inequality, note that the definition of $d_w^*$ implies $w_\lambda(x, y) \leq 1$ for all $\lambda > d_w^*(x, y)$, and so, $d_w^*(x, y) \leq \lambda + \lambda w_\lambda(x, y) \leq 2\lambda$, and it remains to pass to the limit as $\lambda \to d_w^*(x, y)$. □
For the sake of applications in part II we reformulate Theorem 2.10 somehow: in the next theorem we establish certain relations between a number $\lambda > 0$, convex modular $w_\lambda(x, y)$ and metric $d^*_w(x, y)$.

**Theorem 3.8.** Let $w$ be a convex metric (pseudo)modular on a set $X$, $\lambda > 0$ and $x, y \in X^*_w$. We have:

(a) if $d^*_w(x, y) < \lambda$, then $w_\lambda(x, y) \leq d^*_w(x, y)/\lambda < 1$;
(b) if $w_\lambda(x, y) = 1$, then $d^*_w(x, y) = \lambda$;
(c) if $\lambda = d^*_w(x, y) > 0$, then $w_{\lambda+0}(x, y) \leq 1 \leq w_{\lambda-0}(x, y)$.

If the function $\mu \mapsto w_\mu(x, y)$ is continuous from the right on $(0, \infty)$, then along with (a)–(c) we have:

(d) $d^*_w(x, y) \leq \lambda$ if and only if $w_\lambda(x, y) \leq 1$.

If the function $\mu \mapsto w_\mu(x, y)$ is continuous from the left on $(0, \infty)$, then along with (a)–(c) we have:

(e) $d^*_w(x, y) < \lambda$ if and only if $w_\lambda(x, y) < 1$.

If the function $\mu \mapsto w_\mu(x, y)$ is continuous on $(0, \infty)$, then along with (a)–(e) we have:

(f) $d^*_w(x, y) = \lambda$ if and only if $w_\lambda(x, y) = 1$.

**Proof.** (a) If $\mu$ is arbitrary such that $d^*_w(x, y) < \mu < \lambda$, then the definition of $d^*_w$ and (3.1) imply $w_\mu(x, y) \leq 1$ and $w_\lambda(x, y) \leq (\mu/\lambda)w_\lambda(x, y) \leq \mu/\lambda$.

(b) The definition $d^*_w$ and item (a) imply, respectively, $d^*_w(x, y) \leq \lambda$ and $d^*_w(x, y) = \lambda$.

(c) Since $w_\mu(x, y) \leq 1$ for all $\mu > \lambda = d^*_w(x, y)$, we have $w_{\lambda+0}(x, y) \leq 1$, and since $\mu \mapsto w_\mu(x, y)$ is continuous from the left on $(0, \infty)$, then along with (a)–(c) we have:

Items (d)–(f) are established in the same manner as the corresponding items in the proof of Theorem 2.10 taking into account the convexity of the (pseudo)modular.

Now we establish a specific equivalence of metrics $d^*_w$ and $d^*_w$ for a convex modular $w$, which is similar to the one encountered in the Introduction. In particular, it will imply an analogue of Theorem 2.13 for metric $d^*_w$ on the modular set $X^*_w = X^*_w$.

**Theorem 3.9.** Let $w$ be a convex (pseudo)modular on a set $X$. Then, given $x, y \in X^*_w$, for the (pseudo)metrics $d^*_w$ and $d^*_w$ we have:

(a) conditions $d^*_w(x, y) < 1$ and $d^*_w(x, y) < 1$ are equivalent, and if at least one of them holds, then $d^*_w(x, y) \leq d^*_w(x, y) \leq \sqrt{d^*_w(x, y)}$;

(b) conditions $d^*_w(x, y) \geq 1$ and $d^*_w(x, y) \geq 1$ are equivalent, and if at least one of them holds, then $\sqrt{d^*_w(x, y)} \leq d^*_w(x, y) \leq d^*_w(x, y)$.

**Proof.** 1. First, we show that $d^*_w(x, y) < 1$ implies $d^*_w(x, y) \leq d^*_w(x, y)$. In fact, for any $\lambda$ such that $d^*_w(x, y) < \lambda < 1$ the definition of $d^*_w$ gives $w_\lambda(x, y) \leq \lambda < 1$, whence, by virtue of the definition of $d^*_w$, $d^*_w(x, y) \leq \lambda$, and it remains to pass to the limit as $\lambda \to d^*_w(x, y)$.

2. Now we show that if $d^*_w(x, y) < 1$, then $d^*_w(x, y) \leq \sqrt{d^*_w(x, y)}$. In fact, in this case $d^*_w(x, y) \leq \sqrt{d^*_w(x, y)} < 1$, and so, by Theorem 3.8(a), for any $\lambda$ such that $\sqrt{d^*_w(x, y)} < \lambda < 1$ we find $w_\lambda(x, y) \leq d^*_w(x, y)/\lambda < \lambda$, whence the definition of $d^*_w$ implies $d^*_w(x, y) \leq \lambda$, and it suffices to let $\lambda$ go to $\sqrt{d^*_w(x, y)}$.

Thus, in steps 1 and 2 we have established the desired inequalities and showed that the inequalities $d^*_w(x, y) < 1$ and $d^*_w(x, y) < 1$ are equivalent, and so, as a consequence, the inequalities $d^*_w(x, y) \geq 1$ and $d^*_w(x, y) \geq 1$ are equivalent as well.

3. The inequality $d^*_w(x, y) \geq 1$ implies $d^*_w(x, y) \leq d^*_w(x, y)$ by the definition of $d^*_w$, $w_\lambda(x, y) \leq 1$ for all $\lambda > d^*_w(x, y)$, but $\lambda > 1$, and so, $w_\lambda(x, y) < \lambda$. It follows from the definition of $d^*_w$ that $d^*_w(x, y) \leq \lambda$.

4. Finally, we show that if $d^*_w(x, y) \geq 1$, then $\sqrt{d^*_w(x, y)} \leq d^*_w(x, y)$. In fact, given $\lambda > d^*_w(x, y)$, we find $w_\lambda(x, y) \leq \lambda$, but $\lambda > 1$, and so, $\lambda^2 > \lambda > 1$. By the convexity of $w$ and (3.1), we get

$$w_{\lambda^2}(x, y) \leq (\lambda/\lambda^2)w_{\lambda^2}(x, y) \leq (\lambda/\lambda^2) \cdot \lambda = 1,$$

whence $d^*_w(x, y) \leq \lambda^2$, and it remains to let $\lambda \to d^*_w(x, y)$.

**Examples 3.10.** (a) If the modular $w$ is not convex, Theorem 3.9 may be false. The idea of the following example is borrowed from the classical theory of modulars (cf. [19, p. 8]). Let $(X, d)$ be a metric space. We set

$$w_\lambda(x, y) = \frac{d(x, y)}{\lambda + d(x, y)}, \quad \lambda > 0, \quad x, y \in X.$$

Then $w$ is a nonconvex metric modular on $X$ and $d^*_w(x, y) \equiv 0$, whereas

$$d^*_w(x, y) = \frac{1}{2}(\sqrt{d(x, y)^2 + 4d(x, y)} - d(x, y)), \quad x, y \in X.$$
Note that in the proof of Theorem 3.9 the implications in steps 1 and 3 do not rely on the convexity of \( w \):
\[
d^\alpha_w(x, y) < 1 \Rightarrow d^\gamma_w(x, y) \leq d^\Delta_w(x, y), \quad \text{and} \quad d^\mu_w(x, y) \geq 1 \Rightarrow d^\nu_w(x, y) \leq d^\xi_w(x, y).
\]

The example above corresponds to the former implication.

(b) Consider the \( p \)-homogeneous convex (pseudo)modular \( w_j(x, y) = (d(x, y)/\lambda)^p \) from Example 3.4(b), where \( \lambda > 0 \), \( x, y \in X \) and \( p = \text{const} \geq 1 \). Then \( X^*_w = X_w = X \) and, by virtue of Example 2.9(b), (c), we have:
\[
d^\alpha_w(x, y) = (d(x, y))^{p/(p+1)}, \quad d^\gamma_w(x, y) = (p + 1)^{p/(p+1)}(d(x, y))^{p/(p+1)},
\]
\[
d^\Delta_w(x, y) = \begin{cases} d(x, y) & \text{if } p = 1, \\ p(p - 1)^{(1-p)/p}d(x, y) & \text{if } p > 1. \end{cases}
\]
The last two formulas hold for \( 0 < p < 1 \) as well.

(c) The inequalities in Theorem 3.9 are the best possible: in fact, if \( w \) is the (pseudo)modular from the previous example, then for \( p = 1 \) we have \( d^\alpha_w(x, y) = \sqrt{d^\gamma_w(x, y)} \), and if \( p > 1 \), then
\[
d^\alpha_w(x, y) = (d^\gamma_w(x, y))^{p/(p+1)} \to d^\alpha_w(x, y) \quad \text{as } p \to \infty.
\]

Now we expose the coherence between our theory and the classical theory of modulars and modular linear spaces (cf. [34, Theorem 3]). The following theorem is a straightforward consequence of the corresponding definitions and axioms.

**Theorem 3.11.** Let \( X \) be a real linear space.

(a) Given a functional \( \rho : X \to [0, \infty] \), we set
\[
w_j(x, y) = \rho \left( \frac{x - y}{\lambda} \right), \quad \lambda > 0, \; x, y \in X. \tag{3.4}
\]

Then we have: \( \rho \) is a modular (convex modular) on \( X \) in the sense of classical axioms (A.1)–(A.4) from the Introduction if and only if \( w \) is a metric modular (convex metric modular, respectively) on \( X \).

(b) On the other hand, let the function \( w : (0, \infty) \times X \times X \to [0, \infty] \) satisfy the following two conditions:

(i) \( w_\lambda(\mu x, 0) = w_\mu(\lambda x, 0) \) for all \( \lambda, \mu > 0 \) and \( x \in X \);

(ii) \( w_\lambda(x + z, y + z) = w_\lambda(x, y) \) for all \( x, y, z \in X \).

Given \( x \in X \), we set \( \rho(x) = w_1(x, 0) \). Then we have: \( w \) is a metric modular (convex metric modular) on \( X \) if and only if \( \rho \) is a classical modular (convex modular, respectively) on \( X \). Moreover, the equality (3.4) holds, the set \( X_\rho = X^*_\rho(0) \) is a linear subspace of \( X \) and the functional \( |x|_\rho = d^\rho_w(x, 0) \), \( x \in X_\rho \), is an F-norm on \( X_\rho \) (and if \( w \) is convex, then \( X^*_\rho \equiv X^*_w(0) = X^*_\rho \) is a linear subspace of \( X \) and the functional \( \|x\|_\rho = d^\rho_w(x, 0) \), \( x \in X^*_w \), is a norm on \( X^*_w \), respectively). Similar assertions hold if we replace the word ‘modular’ by ‘pseudomodular’.

In the second part of the paper we will need a certain observation (cf. Theorem 3.14), whose construction is encountered several times. It is based on the notions from Section 2.14 and the following Section 3.12.

3.12. Abstract convex cone

([30, Section 4], [31, Section 3], [36, Section 2], [37]). The quadruple \( (M, d, +, \cdot) \) is said to be an abstract convex cone if the triple \( (M, d, +) \) is a metric semigroup with zero \( 0 \in M \) and the operation \( \cdot : \mathbb{R}^+ \times M \to M \) of multiplication of elements from \( M \) by nonnegative numbers, given by \( (\alpha, x) \mapsto \alpha x \), for all \( \alpha, \beta \in \mathbb{R}^+ \) and \( x, y \in M \) has the following properties:
\[
d(\alpha x, \alpha y) = \alpha d(x, y) \quad \text{and} \quad (\alpha + \beta)x = \alpha x + \beta x, \quad \alpha(\beta x) = (\alpha \beta)x, \quad 1 \cdot x = x \tag{3.5}
\]
(cf. examples in Example 3.13). If the metric space \( (M, d) \) is complete, then the corresponding abstract convex cone is called complete.

The following equality holds in an abstract convex cone \( (M, d, +, \cdot) \):
\[
d(\alpha x + \beta y, \alpha y + \beta x) = |\alpha - \beta|d(x, y), \quad \alpha, \beta \in \mathbb{R}^+, x, y \in M. \tag{3.6}
\]
It follows that \( d(\alpha x, \beta y) \leq |\alpha - \beta|d(x, y) + |\alpha - \beta|d(y, 0) \), and so, the operation of multiplication of numbers from \( \mathbb{R}^+ \) by elements from \( M \) is a continuous mapping from \( \mathbb{R}^+ \times M \) into \( M \).

**Examples 3.13.** A simple example of an abstract convex cone is a normed linear space \( (Z, | \cdot |) \) with the induced metric \( d(y, z) = |y - z| \), \( y, z \in Z \), and operations of addition \( + \) and multiplication \( \cdot \) of elements from \( Z \) by nonnegative numbers. If \( K \subseteq Z \) is a convex cone (i.e., \( y + z, \alpha y \in K \) for all \( y, z \in K \) and \( \alpha \geq 0 \)), then \( (K, d, +, \cdot) \) is an abstract convex cone, which is complete if \( Z \) is a Banach space and \( K \) is closed in \( Z \).
Let \((Z, |·|)\) be a real normed linear space. Denote by \(\text{cb}(Z)\) the family of all nonempty closed bounded convex subsets of \(Z\) and, given \(P, Q \in \text{cb}(Z)\), set: \(P + Q = \{ p + q : p \in P, q \in Q \}\) (Minkowski's sum), \(\alpha P = \{ \alpha p : p \in P \}\) for \(\alpha \in \mathbb{R}^+\) and \(P + Q = \text{cl}(P + Q)\), where \(\text{cl}(R)\) designates the closure in \(Z\) of the set \(R \subseteq Z\). The operations in \(\text{cb}(Z)\) have the following properties \([38,39]\): \(\frac{P + Q = \text{cl}(P) + \text{cl}(Q)}{\alpha P = \alpha \text{cl}(P)}\), \(\alpha \leq \beta \Rightarrow P + \beta P, \alpha \beta P = (\alpha \beta)P \), \(1 \cdot P = P\) for all \(\alpha, \beta \in [0, \infty)\). The Abelian semigroup \(\text{cb}(Z)\) is endowed with the Hausdorff metric \(d_H\), generated by the norm \(|·|\) in \(Z\): if \(P, Q \in \text{cb}(Z)\), then we set
\[
d_H(P, Q) = \max \left\{ \sup_{p \in P} \inf_{q \in Q} |p - q|, \sup_{q \in Q} \inf_{p \in P} |p - q| \right\} = \inf \left\{ \alpha > 0 : P \subseteq Q + \alpha S \text{ and } Q \subseteq P + \alpha S \right\},
\]
where \(S = \{ z \in Z : |z| \leq 1 \}\). The properties of \(d_H\) are as follows \([40, \text{Lemma 2.2}], [32, \text{Lemma 3}]\): if \(P, Q, R \in \text{cb}(Z)\) and \(\alpha \leq 0\), then \(d_H(\alpha P, Q) = \alpha d_H(P, Q)\) and
\[
d_H(P + R, Q + R) = d_H(P + R, Q + R) = d_H(P, Q).
\]
Consequently, \(\text{cb}(Z), d_H, +, \cdot\) is an abstract convex cone, which is complete if \(Z\) is a Banach space (this follows from the properties of the Hausdorff metric \(d_H\), cf. [41, Theorems II-9 and II-14]). Note that, by the above (Sections 2.14 and 3.12), the operations of \(+, \cdot\) addition in \(\text{cb}(Z)\) and multiplication by numbers from \(\mathbb{R}^+\) are continuous. More examples of metric semigroups and abstract convex cones, necessary for our purposes, will be presented in the second part of the paper.

**Theorem 3.14.** (a) Let \((X, +)\) be an Abelian semigroup with zero \(0\) and \(w\) be a metric (pseudo)modular on \(X\), which is translation invariant:
\[
w_{\lambda}(x + z, y + z) = w_{\lambda}(x, y) \quad \text{for all } \lambda > 0 \text{ and } x, y, z \in X.
\]

Set \(X_w = X_w^0\), then the triple \((X_w, d_w^0, +)\) is a (pseudo)metric semigroup. A similar assertion holds for a convex (pseudo)modular \(w\) on \(X\) if we replace \(X_w\) by \(X_w^\ast\) (3.9) and \(d_w^0\) by \(d_w^\ast\).

(b) In addition to conditions in (a) suppose that an operation of multiplication by numbers from \(\mathbb{R}^+\) is defined in \(X\) satisfying \((3.5)\) and \(w\) is a convex (pseudo)modular on \(X\), which is homogeneous in the sense:
\[
w_{\lambda}(\mu x, \mu y) = w_{\lambda/\mu}(x, y) \quad \text{for all } \lambda, \mu > 0 \text{ and } x, y \in X.
\]

Set \(X_w^\ast = X_w^0\). Then the quadruple \((X_w^\ast, d_w^\ast, +, \cdot)\) is an abstract convex (pseudo)cone.

**Proof.** (a) If \(x, y \in X_w\), then, by virtue of Definition 2.1(iii) and (3.7), we have, as \(\lambda \to \infty\):
\[
w_{\lambda}(x + y, 0) \leq w_{\lambda/2}(x + y, y) + w_{\lambda/2}(0, y) = w_{\lambda/2}(x, 0) + w_{\lambda/2}(y, 0) \to 0,
\]
whence \(x + y \in X_w\), and so, \((X_w, +)\) is a semigroup. The translation invariance of \(d_w^0\) follows from the corresponding property of the (pseudo)modular \(w\). If \(w\) is a modular on \(X\), then, as was shown in Theorem 2.6, \(d_w^0\) is a metric on \(X_w\), and so, \((X_w, d_w^0, +)\) is a metric semigroup in the sense of Section 2.14.

Now if we suppose that \(w\) is convex, then \(x, y \in X_w^\ast\), implies \(x + y \in X_w^\ast\); in fact, \(w_{\lambda}(x, 0)\) and \(w_{\mu}(y, 0)\) are finite for some positive numbers \(\lambda = \lambda(x)\) and \(\mu = \mu(y)\), and so,
\[
w_{\lambda + \mu}(x + y, 0) = \frac{\lambda}{\lambda + \mu} w_{\lambda}(x + y, y) + \frac{\mu}{\lambda + \mu} w_{\mu}(0, y) = \frac{\lambda}{\lambda + \mu} w_{\lambda}(x, 0) + \frac{\mu}{\lambda + \mu} w_{\mu}(y, 0) < \infty.
\]

(b) By item (a), the triple \((X_w^\ast, d_w^\ast, +)\) is a (pseudo)metric semigroup. Let us show that \(\alpha x \in X_w^\ast\) for all \(\alpha \in \mathbb{R}^+\) and \(x \in X_w^\ast\). From (3.7) and properties (3.5), given \(\lambda > 0\), we find
\[
w_{\lambda}(0, x) = w_{\lambda/2}(0 + x, 0 + x) = w_{\lambda/2}(0 + 1)x, x) = w_{\lambda}(x, x) = 0,
\]
and so, \(0 \cdot x \in X_w^\ast\) (moreover, if \(w\) is a convex modular on \(X\), then (3.9) implies \(0 \cdot x = 0\)). Let \(\alpha > 0\). Applying (3.7) and (3.8), we get:
\[
w_{\lambda}(\alpha \cdot 0, 0) = w_{\lambda}(\alpha \cdot 0 + \alpha \cdot 0, \alpha \cdot 0) = w_{\lambda}(\alpha(1 + 1) \cdot 0, \alpha \cdot 0)
\]
\[
= w_{\lambda/\alpha}(1 \cdot 0 + 1 \cdot 0, \alpha \cdot 0) = w_{\lambda/\alpha}(0, 0) = 0, \lambda > 0,
\]
and so, \(\alpha \cdot 0 \in X_w^\ast\) (moreover, if \(w\) is a convex modular on \(X\), then (3.10) implies \(\alpha \cdot 0 = 0\)). Now, if \(\alpha > 0\) and \(x \in X_w^\ast\), so that \(w_{\mu}(x, 0) < \infty\) for some \(\mu > 0\), then the definition of a convex (pseudo)modular implies
\[
\frac{1}{2} w_{\alpha x}(\alpha x, \alpha x) + \frac{1}{2} w_{\mu}(\alpha x, 0) = \frac{1}{2} w_{\mu}(\alpha x, 0) \leq w_{\mu}(x, 0) < \infty,
\]
and so, $\alpha x \in X_w^*$ (if $w$ is a convex modular, then $\alpha \cdot 0 = 0$ and

$$w_{\alpha 0}(\alpha x, 0) = w_{\alpha 0}(\alpha, \alpha \cdot 0) = w_{\alpha 0}(\alpha, 0) = w_{\alpha 0}(x, 0) < \infty.$$  

In addition to the translation invariance, the (pseudo)metric $d_w^\alpha$ on $X_w^*$ is homogeneous, i.e., if $\alpha > 0$ and $x, y \in X_w^*$, then

$$d_w^\alpha(\alpha x, \alpha y) = \inf (\lambda > 0 : w_{\alpha}(\alpha x, \alpha y) \leq 1) = \inf (\lambda > 0 : w_{\lambda \alpha}(x, y) \leq 1)$$

$$= \inf (\mu > 0 : w_{\mu}(x, y) \leq 1) = \alpha d_w^\alpha(x, y).$$

And in the case when $\alpha = 0$, for all $\lambda > 0$ we have:

$$w_{\lambda 0}(\lambda x, 0) = \frac{1}{2} w_{\lambda 0}(\lambda x, 0) + \frac{1}{2} w_{\lambda 0}(\lambda y, 0) = 0,$$

and so, $d_w^\alpha(0, 0) = 0 = d_w^\alpha(x, y).$

Finally, note that if $w$ is a convex pseudomodular on $X$, then the quadruple $(X_w^*, d_w^\alpha, +, \cdot)$ satisfies all the conditions of Section 3.12, but $d_w^\alpha$ is only a pseudometric, and so, this quadruple was called an abstract convex pseudocone in the formulation of Theorem 3.14. Now, if $w$ is a convex modular on $X$, then $d_w^\alpha$ is a metric on $X_w^*$ and the above quadruple is an abstract convex cone in the sense of Section 3.12.  □

Remark 3.15. The inclusion $\alpha x \in X_w^*$ for all $\alpha \in \mathbb{R}^+$ and $x \in X_w^*$ in item (b) of Theorem 3.14 holds also if the (pseudo)modular $w$ is not necessarily convex. However, for the (pseudo)metric $d_w^\alpha$ the equality $d_w^\alpha(\alpha x, \alpha y) = \alpha d_w^\alpha(x, y)$ for all $\alpha \geq 0$ and $x, y \in X_w^*$ may fail even if $w$ is convex. To see this, let $(X, d, +, \cdot)$ be an abstract convex cone (e.g., $X = \mathbb{R}$) and a convex modular $w$ on $X$ is of the form $w_d(x, y) = d(x, y)/\lambda$. Then, given $\alpha \geq 0$ and $x, y \in X_w^* = X$, we have:

$$d_w^\alpha(\alpha x, \alpha y) = \sqrt{\alpha} d_w^\alpha(x, y) = \sqrt{\alpha} \cdot \sqrt{d_w^\alpha(x, y)} = \sqrt{\alpha} d_w^\alpha(x, y).$$

Thus, Theorem 3.14 fails for the pair $(X_w^*, d_w^\alpha)$.  

References


