Loops on polyhedral products in the flag case

(based on arXiv:2403.18450 and work in progress with Lewis Stanton)

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Outline

Let $\mathcal{K}$ be a flag simplicial complex.

1. Part I: loop homology of moment-angle complexes in the flag case
   - The fibration $(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}} \to (\underline{X},*)^{\mathcal{K}} \to \prod_{i=1}^m X_i$
   - Panov and Ray's results on $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$ and $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\Bbbk)$
   - Dobrinskaya / Cai's results on $H_*(\Omega(\underline{X},*)^{\mathcal{K}};\Bbbk)$ and $H_*(\Omega(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}};\Bbbk)$
   - (F.V.) A homological approach to presentations of $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\Bbbk)$
2. Part II: towards generalisations of the Hilton–Milnor theorem
   - Stanton and Theriault's results on $\Omega\mathcal{Z}_{\mathcal{K}} \in \mathcal{P}$
   - Enumeration of spheres in $\Omega\mathcal{Z}_{\mathcal{K}}$
   - Hilton–Milnor theorem: weak and strong forms
   - (Conjectural) Hilton–Milnor type theorems for flag $\mathcal{K}$
   - Some evidence

Part I: loop homology of moment-angle complexes in the flag case

A fibration of polyhedral products

$\mathcal{K}$ a simplicial complex on the vertex set $[m] = \{1, \dots, m\}$,
$(\underline{X}, \underline{A}) = ((X_1, A_1), \dots, (X_m, A_m))$ a tuple of CW-pairs $\rightsquigarrow$ the polyhedral product

$$(\underline{X}, \underline{A})^{\mathcal{K}} := \bigcup_{I \in \mathcal{K}} \Big(\prod_{i \in I} X_i \times \prod_{j \in [m] \setminus I} A_j \Big) \subset \prod_{i=1}^{m} X_i.$$

We try to compute their algebraic invariants in terms of $\underline{X}$,$\underline{A}$ and $\mathcal{K}$.

Proposition (Buchstaber, Panov / Denham, Suciu / ...)

Let $j : (\underline{X}, *)^{\mathcal{K}} \to \prod_{i=1}^{m} X_i$ be the standard inclusion. Then

(1) There is a fibration $(C\Omega\underline{X}, \Omega\underline{X})^{\mathcal{K}} \to (\underline{X}, *)^{\mathcal{K}} \xrightarrow{j} \prod_{i=1}^{m} X_i$.

(2) Ωj has a homotopy section. □

A special case: $\underbrace{\mathcal{Z}_{\mathcal{K}}}_{=(D^2,S^1)^{\mathcal{K}}} \to \underbrace{\mathrm{DJ}(\mathcal{K})}_{=(\mathbb{C}P^\infty,*)^{\mathcal{K}}} \to (\mathbb{C}P^\infty)^m$ ($\rightsquigarrow$ toric topology).

A fibration of loops on polyhedral products

Since the fibration $\Omega(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}} \to \Omega(\underline{X},*)^{\mathcal{K}} \to \prod_{i=1}^m \Omega X_i$ has a section, we obtain a homotopy equivalence

$$\Omega(\underline{X},*)^{\mathcal{K}} \simeq \Omega(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}} \times \prod_{i=1}^{m} \Omega X_i$$

(not as H-spaces!).

Recall: for a commutative ring $\Bbbk$ with unit,

- $H_*(\Omega X;\Bbbk)$ is an associative $\Bbbk$-algebra;
- $H_*(\Omega X;\Bbbk)$ is a cocommutative $\Bbbk$-Hopf algebra (if free as a $\Bbbk$-module).

In the torsion-free case, we obtain an extension of Hopf algebras

$$H_*(\Omega(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}};\Bbbk) \to H_*(\Omega(\underline{X},*)^{\mathcal{K}};\Bbbk) \to \bigotimes_{i=1}^{m} H_*(\Omega X_i;\Bbbk).$$

For $X_i = \mathbb{C}P^\infty$ (Panov, Ray): the fibration $\Omega\mathcal{Z}_{\mathcal{K}} \to \Omega\mathrm{DJ}(\mathcal{K}) \to \mathbb{T}^m$ of H-spaces; $\Omega\mathrm{DJ}(\mathcal{K}) \simeq \Omega\mathcal{Z}_{\mathcal{K}} \times \mathbb{T}^m$; the extension of Hopf algebras

$$H_*(\Omega\mathcal{Z}_{\mathcal{K}};\Bbbk) \to H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \to \Lambda_{\Bbbk}[u_1,\dots,u_m].$$

Definition

Maps of Hopf algebras $S \xrightarrow{\iota} A \xrightarrow{\pi} B$ form a Hopf algebra extension if

- ι injective, π surjective, $\pi \circ \iota = \varepsilon$;
- $\mathrm{Ker}(\pi) = I(A) \cdot \iota(S)$;
- $\mathrm{Im}(\iota) = \{x \in A : \sum_\alpha x'_\alpha \otimes \pi(x''_\alpha) = x \otimes 1\}$ for $\Delta(x) = \sum_\alpha x'_\alpha \otimes x''_\alpha$.

Properties:

1. $A \simeq S \otimes B$ as left S-modules and right B-comodules.
2. $0 \to PS \to PA \to PB$ is exact.

Examples:

1. $0 \to L' \to L \to L'' \to 0$ — a s.e.s. of Lie (super)algebras $\rightsquigarrow$ an extension $UL' \to UL \to UL''$.
2. a fibration $F \to E \xrightarrow{p} B$ such that Ωp has a homotopy section $\rightsquigarrow$ an extension $H_*(\Omega F; \Bbbk) \to H_*(\Omega E; \Bbbk) \to H_*(\Omega B; \Bbbk)$ (if torsion free).

So we have $H_*(\Omega(C\Omega\underline{X}, \Omega\underline{X})^{\mathcal{K}}; \Bbbk) \subset H_*(\Omega(\underline{X}, *)^{\mathcal{K}}; \Bbbk)$ with $\bigotimes_i H_*(\Omega X_i; \Bbbk)$ as "quotient".

Loop homology of $(\underline{X}, *)^{\mathcal{K}}$

$\Bbbk[\mathcal{K}] = \Bbbk[v_1, \ldots, v_m]/(\prod_{i \in I} v_i = 0, I \notin \mathcal{K})$ is the Stanley–Reisner ring.
$\mathcal{K}$ is a flag complex if $I \in \mathcal{K}$ whenever $\{i, j\} \in \mathcal{K}$ for all $i \in I$.
Equivalently: $\mathcal{K}$ is a clique complex of a graph; $\Bbbk[\mathcal{K}]$ is quadratic.

Theorem (Panov, Ray / Franz)

1. $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \Bbbk) \cong \mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk, \Bbbk)$ as algebras (as Hopf algebras, if torsion free) for any complex $\mathcal{K}$ and PID $\Bbbk$.
2. $H_*(\Omega \mathrm{DJ}(\mathcal{K}); \Bbbk) \cong T(u_1, \ldots, u_m)/(u_i^2 = 0; u_i u_j + u_j u_i = 0,\ \{i, j\} \in \mathcal{K})$ if $\mathcal{K}$ is flag. $(\deg u_i = 1)$

$H_*(\Omega \mathcal{Z}_{\mathcal{K}})$ is the "commutator subring" of this partial commutative ring.

Theorem (Dobrinskaya / Cai)

If $\mathcal{K}$ is flag and $\Bbbk$ is a field, we have
$H_*(\Omega(\underline{X}, *)^{\mathcal{K}}; \Bbbk) \cong *_{i=1}^m H_*(\Omega X_i; \Bbbk)/\sim$, where $[x_i, x_j] \sim 0$ for $\{i, j\} \in \mathcal{K}$, $x_i \in H_*(\Omega X_i)$, $x_j \in H_*(\Omega X_j)$.

In fact, a $\mathbb{Z}^m_{\geq 0}$-grading on $H_*(\Omega(\underline{X}, *)^{\mathcal{K}})$ can be introduced for any $\mathcal{K}$...

Poincaré series, flag case

From now on, $\mathcal{K}$ is a flag simplicial complex.
Denote $P(V;t) := \sum_n \dim(V_n)\cdot t^n \in \mathbb{Z}[[t]]$ for a graded vector space V.

Theorem (Panov, Ray)

$A = H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$ is Koszul dual to $A^! = \Bbbk[\mathcal{K}]$, hence
$P(A;t) = 1/P(\Bbbk[\mathcal{K}];-t) = (1+t)^{1+\dim\mathcal{K}}/h_{\mathcal{K}}(-t)$;

$$1/P(H_*(\Omega\mathcal{Z}_{\mathcal{K}};\Bbbk);t) = (1+t)^{m-\dim\mathcal{K}-1}h_{\mathcal{K}}(-t).$$

Equivalent formula (F.V.): $1/P(H_*(\Omega\mathcal{Z}_{\mathcal{K}});t) = \sum_{J\subset[m]}(1-\chi(\mathcal{K}_J))\cdot t^{|J|}$.

Theorem (Li Cai)

$P(H_*(\Omega(\underline{X},*)^{\mathcal{K}});t) = P(H_*(\Omega(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}};t)\cdot\prod_{i=1}^m P(H_*(\Omega X_i);t)$,

$$1/P(H_*(\Omega(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}});t) = \sum_{J\subset[m]}(1-\chi(\mathcal{K}_J))\cdot\prod_{j\in J}P(\widetilde{H}_*(\Omega X_j);t).$$

The last formula is derived from $1/P(A;t) = \sum_{n\geq 0}(-1)^n P(\mathrm{Tor}_n^A(\Bbbk,\Bbbk);t)$.

Generators and relations for $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$: approach

How to study multiplicative presentations of $S = H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$, when $S \to A \to \Lambda$ is a Hopf algebra extension and A, Λ are "well understood"?

1. Fact (C.T.C.Wall / F.V.): multiplicative generators of S correspond to additive generators of $\mathrm{Tor}_1^S(\Bbbk,\Bbbk)$; multiplicative relations correspond to additive generators of $\mathrm{Tor}_2^S(\Bbbk,\Bbbk)$ and relations in $\mathrm{Tor}_1^S(\Bbbk,\Bbbk)$.
2. Moreover (F.V.): generators and relations can be written down, if we know the corresponding cycles in the bar construction $\overline{\mathrm{B}}(S)$.
3. If $(A\otimes M_\bullet, d) \to (\Bbbk, 0)$ is a free resolution of A-modules, then it is a free resolution of S-modules (use $A \simeq S\otimes\Lambda$); hence $\mathrm{Tor}^S(\Bbbk,\Bbbk) \cong H[\Lambda\otimes M_\bullet, \overline{d}]$ can be computed.
4. To pass from $H[\Lambda\otimes M_\bullet,\overline{d}]$ to $H[\overline{\mathrm{B}}(S)]$, we construct a map of free resolutions $(S\otimes\Lambda\otimes M_\bullet, d)\to(\mathrm{B}(S), d_\mathrm{B})$ using the contracting homotopy of the bar resolution.
5. In the case $A = H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$, $A^! = \Bbbk[\mathcal{K}]$ use the Koszul-Priddy(-Fröberg) resolution $(A\otimes(A^!)^*, d)$ as $(A\otimes M_\bullet, d)$.

Generators and relations for $H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{k})$: results

$$H_*(\Omega\mathcal{Z}_\mathcal{K}) \hookrightarrow \underbrace{H_*(\Omega\mathrm{DJ}(\mathcal{K});\mathbb{k})}_{=T(u_1,\dots,u_m)/(\dots)} \to \Lambda[u_1,\dots,u_m]$$

- $\mathrm{Tor}_n^{H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{k})}(\mathbb{k},\mathbb{k}) \cong H_n[\Lambda[m]\otimes(\mathbb{k}[\mathcal{K}])^*,d] \cong \bigoplus_{J\subset[m]} \widetilde{H}_{n-1}(\mathcal{K}_J;\mathbb{k})$.
- Corollary: $H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{k})$ is presented by $\sum_{J\subset[m]}(b_0(\mathcal{K}_J)-1)$ generators modulo $\sum_{J\subset[m]} b_1(\mathcal{K}_J;\mathbb{k})$ relations.
- For non-empty $I=\{i_1<\dots<i_k\}\subset[m]$ and $j\in[m]$, nested commutators $c(I,u_j) := [u_{i_1},[\dots[u_{i_k},u_j]\dots]] \in H_*(\Omega\mathrm{DJ}(\mathcal{K});\mathbb{k})$ belong to $H_*(\Omega\mathcal{Z}_\mathcal{K};\mathbb{k})$.
- A minimal set of generators: the Grbić-Panov-Theriault-Wu elements $\{c(J\setminus j,u_j) : j\in\Theta_\mathcal{K}(J), J\subset[m]\}$, where $\Theta_\mathcal{K}(J)=\{$minimal vertices of connected components of $\mathcal{K}_J$ not containing $\max(J)\}\subset J$.
- Relations: for each cycle $\sum_{\{i<j\}\in\mathcal{K}_J}\alpha_{ij}\{i,j\}\in\mathcal{C}_1(\mathcal{K}_J;\mathbb{k})$ we have a relation $\sum_{\substack{J\setminus\{i,j\}=A\sqcup B:\\ \max(A)>i,\max(B)>j}} \pm\alpha_{ij}\Big[\widehat{c}(A,u_i),\widehat{c}(B,u_j)\Big]=0$, where $\widehat{c}(A,u_i)$ is $c(A,u_i)$ arbitrarily expressed through the GPTW elements.

An example: 5-cycle

For $\mathcal{K}$ a 5-cycle, the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is presented by 10 generators and a single relation, first computed by Veryovkin (2016, computer bruteforce). The GPTW generators correspond to $\widetilde{H}_0(\bullet\bullet)\simeq\Bbbk$ and $\widetilde{H}_0(\bullet\ \bullet\!\!-\!\!\bullet)\simeq\Bbbk$:

$$[u_3,u_1],\ [u_4,u_1],\ [u_4,u_2],\ [u_5,u_2],\ [u_5,u_3],\ [u_1,[u_5,u_3]],$$

$$[u_2,[u_4,u_1]],\ [u_3,[u_4,u_1]],\ [u_3,[u_5,u_2]],\ [u_4,[u_5,u_2]].$$

The single relation corresponds to $\widetilde{H}_1(\mathcal{K})\simeq\Bbbk$:

$$-\Big[[u_3,u_1],[u_4,[u_5,u_2]]\Big]+\Big[[u_4,u_1],[u_3,[u_5,u_2]]\Big]+\Big[[u_3,[u_4,u_1]],[u_5,u_2]\Big]$$

$$+\Big[[u_4,u_2],[u_1,[u_5,u_2]]\Big]+\Big[\underline{[u_1,[u_4,u_2]]},[u_5,u_3]\Big]=0.$$

The underlined element is not a GPTW generator; however, $-[u_1,[u_4,u_2]]=[u_2,[u_4,u_1]]$ in $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$, and $[u_2,[u_4,u_1]]$ is a generator.

Possible generalisations

In the similar way, one can compute presentations of

- $H_*(\Omega(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}})$ for flag $\mathcal{K}$. (Tor module and multiplicative generators already described by Li Cai.)
- Loop homology of partial quotients $\mathcal{Z}_{\mathcal{K}}/H$ of moment-angle complexes, including quasitoric manifolds and smooth toric varieties, for flag $\mathcal{K}$. (F.V., master thesis: we compute a presentation of $H_*(\Omega(\mathcal{Z}_{\mathcal{K}}/H))$ if we understand the groups $H_*(\mathcal{Z}_{\mathcal{K}}/H)$.)
- $H_*(\Omega\mathcal{Z}_{\mathcal{K}})$ for non-flag $\mathcal{K}$ whenever $H_*(\Omega\mathrm{DJ}(\mathcal{K}))$ is known.
- $H_*(\Omega F)$, if $F\to E\to B$ has a section after looping and the algebras $H_*(\Omega E)$, $H_*(\Omega B)$ are well understood. (Other examples?)

Part II: towards generalisations of the Hilton–Milnor theorem

Loop space decompositions: work of Stanton and Theriault

Denote $\mathcal{P} := \{$finite type products of S^1,S^3,S^7 and loops on spheres$\}$. We have:

- $\mathcal{P}$ is closed under retracts (Stanton'23);
- If $\mathcal{Z}_\mathcal{K}$ is a wedge of spheres, then $\Omega\mathcal{Z}_\mathcal{K} \in \mathcal{P}$ (by Hilton–Milnor);
- If $\Omega\mathcal{Z}_\mathcal{K} \in \mathcal{P}$, then $\Omega\mathcal{Z}_\mathcal{K} \simeq \prod_{n\geq 3}(\Omega S^n)^{\times D_n}$ (Stanton, see F.V.'24);
- $\underline{\text{If } \mathcal{K} \text{ is flag, then } \Omega\mathcal{Z}_\mathcal{K} \in \mathcal{P}}$ (Stanton'23);
- If $\Omega\mathcal{Z}_{\mathcal{K}_I} \in \mathcal{P}$ for all 1-neighbourly $\mathcal{K}_I$, then $\Omega\mathcal{Z}_\mathcal{K} \in \mathcal{P}$ (Stanton'24);
- If $\Omega\mathcal{Z}_\mathcal{K}, \Omega\mathcal{Z}_{\mathcal{L}_i} \in \mathcal{P}$, then $\Omega\mathcal{Z}_{\mathcal{K}(\mathcal{L}_1,\dots,\mathcal{L}_m)} \in \mathcal{P}$ (Eldridge'24);
- If $\mathcal{K}$ is a triangulation of S^3 or a closed oriented surface, then $\Omega\mathcal{Z}_\mathcal{K} \in \mathcal{P}$ (Stanton,Theriault'24).

Everything above works for $(C\underline{A}, \underline{A})^\mathcal{K}$ assuming that ΣA_i are wedges of spheres. However, the results are not quantitative. To compute the numbers D_n, we use loop homology.

Homotopy type of $\Omega\mathcal{Z}_\mathcal{K}$, flag case

Proposition (F.V.'24)

For a flag simplicial complex $\mathcal{K}$ without ghost vertices, we have a homotopy equivalence $\Omega\mathcal{Z}_\mathcal{K} \simeq \prod_{n\geq 3}(\Omega S^n)^{\times D_n}$, where the numbers D_n are determined by the formula

$$\boxed{\prod_{n\geq 3}(1-t^{n-1})^{D_n} = \sum_{J\subset[m]}(1-\chi(\mathcal{K}_J))t^{|J|}} \in \mathbb{Z}[[t]].$$

(This is a combination of Panov,Ray and Stanton's results: compute $P(H_*(\Omega-);t)$ for LHS and RHS.)

Corollary

Let X be a smooth (quasi)toric manifold such that the underlying simplicial complex $\mathcal{K}$ is flag. Then $\pi_N(X) = \bigoplus_{n=3}^N \pi_N(S^n)^{\oplus D_n}$ for each $N\geq 3$.

Comparison with the Hilton–Milnor theorem

Hilton's theorem: from the weakest to the strongest from

Let $W = S^{d_1+1} \vee \cdots \vee S^{d_m+1}$, $d_i \geq 1$. Then

(1) ΩW is a finite type product of loops on spheres;

(2) $\Omega W \simeq \prod_{n\geq 2} (\Omega S^n)^{\times D_n}$, where $\prod_{n\geq 2}(1-t^n)^{D_n} = 1 - \sum_{i=1}^m t^{d_i}$;

(3) $\Omega W \simeq \prod_{b\in B(L)} \Omega S^{1+|b|}$, where $L = FL(x_1,\dots,x_m)$ is the free Lie algebra, $\deg x_i = d_i$, and $B(L)$ is the set of basic commutators (explicit Lie monomials, an additive basis of L);

(4) For each $b \in B(L)$ consider the corresponding Whitehead bracket $w_b : S^{1+|b|} \to W$. Then the map $\prod_b \Omega w_b : \prod_{b\in B(L)} \Omega S^{1+|b|} \to \Omega W$ is a weak homotopy equivalence.

Assuming (1), other parts require

(2) knowledge of $P(H_*(\Omega W); t)$ (i.e. Bott–Samelson theorem)

(3) knowledge of $P(UL; t)$ (+ PBW theorem)

(4) some commutator calculus

Comparison with the Hilton–Milnor theorem

Hilton–Milnor's theorem: from the weakest to the strongest from

Let $W = \Sigma(X_1 \vee \cdots \vee X_m)$. Then

(1) ΩW is a finite type product of $\Omega\Sigma(\underline{X}^{\wedge\alpha})$'s, where $\underline{X}^{\wedge\alpha} = \bigwedge_{i=1}^m X_i^{\wedge\alpha_i}$ for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m_{\geq 0}$;

(2) $\Omega W \simeq \prod_{\alpha\in\mathbb{Z}^m_{\geq 0}} \Omega\Sigma(\underline{X}^{\wedge\alpha})^{\times D_\alpha}$, where $D_\alpha \geq 0$ are determined by $\prod_{\alpha\in\mathbb{Z}^m_{\geq 0}}(1-t^\alpha)^{D_\alpha} = 1 - \sum_{i=1}^m t_i$ in $\mathbb{Z}[[t_1, \ldots, t_m]]$, $t^\alpha := \prod_{i=1}^m t_i^{\alpha_i}$;

(3) $\boxed{\Omega W \simeq \prod_{b\in B(L)} \Omega\Sigma\underline{X}^{\wedge \deg(b)},}$ where $L = FL(x_1, \ldots, x_m)$ is the free Lie algebra, $\deg x_i = e_i \in \mathbb{Z}^m_{\geq 0}$, and $B(L)$ is the set of basic commutators (explicit Lie monomials, an additive basis of L);

(4) For each $b \in B(L)$ consider the corresponding Whitehead bracket $w_b : \Sigma\underline{X}^{\wedge \deg(b)} \to W$. Then the map $\prod_b \Omega w_b : \prod_{b\in B(L)} \Omega\Sigma\underline{X}^{\wedge \deg(b)} \to \Omega W$ is a weak homotopy equivalence.

Partially commutative Lie algebras

Let $\Bbbk$ be a commutative ring with unit. Consider the partially commutative Lie algebra (in the ungraded sense!)

$$L_{\mathcal{K}} := FL_{\Bbbk}(x_1,\dots,x_m)/([x_i,x_j]=0,\ \forall\{i,j\}\in\mathcal{K}),\ \ \deg x_i := e_i \in \mathbb{Z}^m_{\geq 0}.$$

$L_{\mathcal{K}}$ is always torsion free; an additive basis is known (Poroshenko'10). Consider the Lie subalgebra $N_{\mathcal{K}} \subset [L_{\mathcal{K}}, L_{\mathcal{K}}]$ generated by Lie monomials without repeating indices. Equivalently, $N_{\mathcal{K}} := \langle GPTW\rangle_{Lie} \subset L_{\mathcal{K}}$, where

$$GPTW := \{c(J\setminus j, x_j) : j\in\Theta_{\mathcal{K}}(J), J\subset[m]\} \subset [L_{\mathcal{K}}, L_{\mathcal{K}}].$$

Theorem* (F.V.)

Suppose that $\Bbbk = \mathbb{Z}/2$. Then $U(N_{\mathcal{K}}) \cong H_*(\Omega\mathcal{Z}_{\mathcal{K}};\Bbbk)$ as algebras.

Conjecture

The additive structure of $N_{\mathcal{K}}$ does not depend on $\Bbbk$.

Hilton–Milnor type theorems for flag $\mathcal{K}$

Below, $N_{\mathcal{K}} := \langle GPTW\rangle_{Lie} \subset L_{\mathcal{K}}$ is considered for $\Bbbk = \mathbb{Z}/2$. Denote $A_1 \sim_P A_2$ if $P(H_*(A_1;\Bbbk);t) = P(H_*(A_2;\Bbbk);t)$ for any field $\Bbbk$.

Theorem* (F.V., Lewis Stanton)

For a flag simplicial complex $\mathcal{K}$, we have

- $\Omega(\Sigma\underline{Y},*)^{\mathcal{K}} \sim_P \prod_{b\in B(L_{\mathcal{K}})} \Omega\Sigma\underline{Y}^{\wedge\deg(b)}$;
- $\Omega(C\Omega\Sigma\underline{Y},\Omega\Sigma\underline{Y})^{\mathcal{K}} \sim_P \prod_{b\in B([L_{\mathcal{K}},L_{\mathcal{K}}])} \Omega\Sigma\underline{Y}^{\wedge\deg(b)}$;
- $\Omega(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}} \sim_P \prod_{b\in B(N_{\mathcal{K}})} \Omega\Sigma(\Omega\underline{X})^{\wedge\deg(b)}$;
- $\Omega(\underline{X},*)^{\mathcal{K}} \sim_P \prod_{i=1}^m \Omega X_i \times \prod_{b\in B(N_{\mathcal{K}})} \Omega\Sigma(\Omega\underline{X})^{\wedge\deg(b)}$.

Theorem* (F.V., L.Stanton)

$\Omega(\Sigma\underline{Y},*)^{\mathcal{K}}$ is a finite type product of $\Omega\Sigma\underline{Y}^{\wedge\alpha}$'s.
$\Omega(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}}$ is a finite type product of $\Omega\Sigma(\Omega\underline{X})^{\wedge\alpha}$'s.

Hilton–Milnor type theorems for flag $\mathcal{K}$

Below, $N_{\mathcal{K}} := \langle GPTW \rangle_{Lie} \subset L_{\mathcal{K}}$ is considered for $\mathbb{k} = \mathbb{Z}/2$. Denote $A_1 \sim_P A_2$ if $P(H_*(A_1;\mathbb{k});t) = P(H_*(A_2;\mathbb{k});t)$ for any field $\mathbb{k}$.

Theorem* (F.V., Lewis Stanton)

For a flag simplicial complex $\mathcal{K}$, we have

- $\Omega(\Sigma\underline{Y},*)^{\mathcal{K}} \sim_P \prod_{b\in B(L_{\mathcal{K}})} \Omega\Sigma\underline{Y}^{\wedge \deg(b)}$;
- $\Omega(C\Omega\Sigma\underline{Y},\Omega\Sigma\underline{Y})^{\mathcal{K}} \sim_P \prod_{b\in B([L_{\mathcal{K}},L_{\mathcal{K}}])} \Omega\Sigma\underline{Y}^{\wedge \deg(b)}$;
- $\Omega(C\Omega\underline{X},\Omega\underline{X})^{\mathcal{K}} \sim_P \prod_{b\in B(N_{\mathcal{K}})} \Omega\Sigma(\Omega\underline{X})^{\wedge \deg(b)}$;
- $\Omega(\underline{X},*)^{\mathcal{K}} \sim_P \prod_{i=1}^m \Omega X_i \times \prod_{b\in B(N_{\mathcal{K}})} \Omega\Sigma(\Omega\underline{X})^{\wedge \deg(b)}$.

We plan to use Gröbner–Shirshov theory for $L_{\mathcal{K}}$ (and its superLie analogue) to prove the Hilton–Milnor type theorems:

Conjecture/Claim/Work in progress (F.V., L.Stanton)

Under torsion-free assumptions, the maps $LHS \leftarrow RHS$ given by looped Whitehead brackets are weak homotopy equivalences.

Hilton–Milnor type theorems "for Golod $\mathcal{K}$"

Suppose that the BBCG decomposition for $(C\underline{A},\underline{A})^{\mathcal{K}}$ desuspends, $(C\underline{A},\underline{A})^{\mathcal{K}} \simeq \bigvee_{J\subset[m]} \Sigma|\mathcal{K}_J| \wedge \underline{A}^{\wedge J}$.

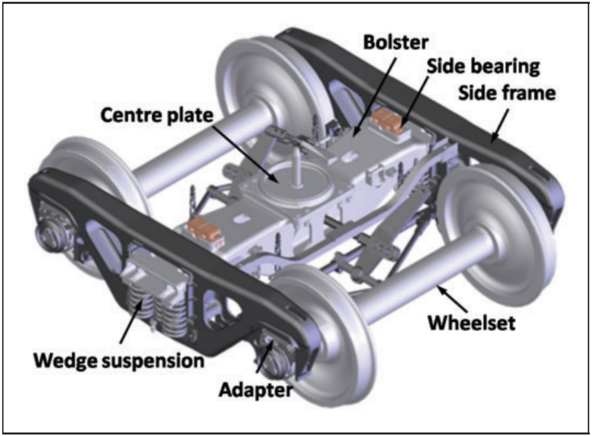
1. $\Omega(C\underline{A},\underline{A})^{\mathcal{K}} \simeq$ product of $\Omega\Sigma|\mathcal{K}_{J_1}| \wedge \dots \wedge |\mathcal{K}_{J_s}| \wedge \underline{A}^{\wedge\alpha}$'s (use the Hilton–Milnor theorem)
2. $\Omega(\underline{X},*)^{\mathcal{K}} \simeq \prod_{i=1}^m \Omega X_i \times$ product of $\Omega\Sigma|\mathcal{K}_{J_1}| \wedge \dots \wedge |\mathcal{K}_{J_s}| \wedge (\Omega\underline{X})^{\wedge\alpha}$'s
3. $\Omega(\Sigma\underline{Y},*)^{\mathcal{K}} \simeq$ product of $\Omega\Sigma|\mathcal{K}_{J_1}| \wedge \dots \wedge |\mathcal{K}_{J_s}| \wedge \underline{Y}^{\wedge\alpha}$'s (use the James splitting)

Iriye, Kishimoto: for $\mathcal{K}$ totally fillable, the factors correspond to Whitehead products. (First result in this direction: Porter'64, $\mathcal{K} = \mathrm{sk}_i\,\Delta_{[m]}$)

Problem

- Enumerate the factors using bases in Lie algebras
- Find a common generalisation with the HM theorems for flag $\mathcal{K}$

Thank you for your attention!



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