Moment-angle complexes in the flag case (and beyond)

(based on arXiv:2403.18450 and work in progress)

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Plan

For a simplicial complex $\mathcal{K}$, we study the spaces $\mathcal{Z}_{\mathcal{K}}$ and $\mathrm{DJ}(\mathcal{K})$.

1. $\mathcal{K}$ is any complex
   - Definition of $\mathcal{Z}_{\mathcal{K}}$ and $\mathrm{DJ}(\mathcal{K})$
   - Cohomology rings
   - Loop spaces: $\Omega\mathrm{DJ}(\mathcal{K}) \simeq \Omega\mathcal{Z}_{\mathcal{K}} \times \mathbb{T}^m$
   - Loop homology: $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \Bbbk) \subset H_*(\Omega\mathrm{DJ}(\mathcal{K}); \Bbbk) \cong \mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk, \Bbbk)$.
2. $\mathcal{K}$ is a flag complex:
   - Presentation of $H_*(\Omega\mathrm{DJ}(\mathcal{K}); \Bbbk)$ is known
   - Poincaré series of $H_*(\Omega\mathrm{DJ}(\mathcal{K}); \Bbbk)$ is known
   - Homotopy type of $\Omega\mathcal{Z}_{\mathcal{K}}$ is known (L.Stanton + V.)
   - Presentation of $H_*(\Omega\mathcal{Z}_{\mathcal{K}}; \Bbbk)$ can be computed
   - $\mathcal{Z}_{\mathcal{K}}$ is rationally coformal
3. $\mathcal{K}$ is "close to a flag complex" (work in progress):
   - Partial generalisations of results above
   - Colimit decomposition of $H_*(\Omega\mathrm{DJ}(\mathcal{K}); \Bbbk)$
   - "Clique closures" of classes of simplicial complexes

Moment-angle complexes and their partial quotients

Fix an abstract simplicial complex $\mathcal{K}$ on vertex set $[m] = \{1, \dots, m\}$ without ghost vertices, and consider the moment-angle complex

$$\mathcal{Z}_{\mathcal{K}} := \bigcup_{J \in \mathcal{K}} \left(\prod_{i \in J} D^2 \times \prod_{i \in [m] \setminus J} S^1 \right) \subset (D^2)^m.$$

Clearly, $\mathbb{T}^m = (S^1)^m$ acts on $\mathcal{Z}_{\mathcal{K}}$. If a closed subgroup $H \subset \mathbb{T}^m$ acts freely on $\mathcal{Z}_{\mathcal{K}}$, the $\mathbb{T}^m/H$-space $\mathcal{Z}_{\mathcal{K}}/H$ is called a partial quotient of $\mathcal{Z}_{\mathcal{K}}$. Up to an equivariant homotopy equivalence, this class contains all quasitoric manifolds (Davis, Januszkiewicz) and smooth toric varieties (Franz). The Davis–Januszkiewicz space $\mathrm{DJ}(\mathcal{K}) := \bigcup_{J \in \mathcal{K}} (\mathbb{C}P^\infty)^J \subset (\mathbb{C}P^\infty)^m$ is homotopy equivalent to the Borel construction $E\mathbb{T}^m \times_{\mathbb{T}^m} \mathcal{Z}_{\mathcal{K}}$ (Buchstaber, Panov); we obtain the homotopy fibration

$$\mathcal{Z}_{\mathcal{K}} \to \mathrm{DJ}(\mathcal{K}) \hookrightarrow (\mathbb{C}P^\infty)^m.$$

Cohomology rings

$\mathcal{Z}_\mathcal{K} \to \underbrace{\mathrm{DJ}(\mathcal{K})}_{=(\mathbb{C}P^\infty,*)^\mathcal{K}} \hookrightarrow (\mathbb{C}P^\infty)^m$. Fix a principal ideal domain $\Bbbk$.

Proposition

$H^*(\mathrm{DJ}(\mathcal{K});\Bbbk) \cong \Bbbk[\mathcal{K}]$, where $\Bbbk[\mathcal{K}] := \Bbbk[v_1,\dots,v_m]/(\prod_{i\in I} v_i = 0,\ I \notin \mathcal{K})$ is the Stanley–Reisner ring of $\mathcal{K}$. (As graded algebras; $\deg v_i = 2$.)

Theorem (Baskakov, Buchstaber, Panov)

$H^*(\mathcal{Z}_\mathcal{K};\Bbbk) \cong \mathrm{Tor}_{\Bbbk[v_1,\dots,v_m]}(\Bbbk[\mathcal{K}],\Bbbk) \cong \bigoplus_{J\subset[m]} \widetilde{H}^*(\mathcal{K}_J;\Bbbk)$ (As graded algebras; products are induced by inclusions $\mathcal{K}_{A\sqcup B} \hookrightarrow \mathcal{K}_A * \mathcal{K}_B$.)

We obtain a natural $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-grading: $\deg v_i = 2e_i \in \mathbb{Z}^m_{\geq 0}$,

$$H^n(\mathcal{Z}_\mathcal{K};\Bbbk) \cong \bigoplus_{\substack{i\geq 0,\ J\subset[m]:\\ -i+2|J|=n}} H^{-i,2J}(\mathcal{Z}_\mathcal{K};\Bbbk),\ H^{-i,2J}(\mathcal{Z}_\mathcal{K};\Bbbk) \cong \widetilde{H}^{|J|-i-1}(\mathcal{K}_J;\Bbbk).$$

Loop space fibrations

Panov, Ray: in the looped fibration $\Omega\mathcal{Z}_\mathcal{K} \xrightarrow{\Omega i} \Omega\mathrm{DJ}(\mathcal{K}) \xrightarrow{\Omega p} \mathbb{T}^m$, the map Ωp has a section. It follows that $\Omega\mathrm{DJ}(\mathcal{K}) \simeq \Omega\mathcal{Z}_\mathcal{K} \times \mathbb{T}^m$ (not an equivalence of H-spaces!).
Recall: for a connected space X and a principal ideal domain $\Bbbk$, the graded $\Bbbk$-module $H_*(\Omega X;\Bbbk)$ is

- an associative algebra (w.r.t. the Pontryagin product);
- a cocommutative Hopf algebra if $H_*(\Omega X;\Bbbk)$ is torsion free.

We obtain:

- $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \simeq H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk) \otimes \Lambda[u_1,\dots,u_m]$ as left $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$-modules (not as algebras!);
- In the torsion-free case, an extension $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk) \to H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \to \Lambda[u_1,\dots,u_m]$ of connected cocommutative Hopf algebras.

(Similarly for $H_*(\Omega(EH \times_H \mathcal{Z}_\mathcal{K});\Bbbk) \to H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \to H_*(\mathbb{T}^m/H;\Bbbk)$).

Loop homology of Davis–Januszkiewicz spaces

Use the (hga-)formality of $\mathrm{DJ}(\mathcal{K})$, Adams' cobar construction and properties of quadratic algebras, one obtains:

Theorem (Panov, Ray'08, Franz'21, V.'23)

1. $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \cong \mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk)$ as graded $\Bbbk$-algebras. We obtain a $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-grading on $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$ (and hence on $H_*(\Omega\mathcal{Z}_{\mathcal{K}};\Bbbk)$):

$$H_n(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \cong \bigoplus_{n=-i+2|\alpha|} H_{-i,2\alpha}, \text{ where } H_{-i,2\alpha} \cong \mathrm{Ext}^i_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk)^{2\alpha}.$$

2. $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk) \cong \mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk)$ as Hopf algebras (in the torsion free case).
3. $H_{-i,2\alpha} = 0$ if $i > |\alpha|$. Diagonal subalgebra $\bigoplus_{i=|\alpha|} H_{-i,2\alpha}$ is isomorphic to $\Bbbk[\mathcal{K}]^! := T(u_1,\dots,u_m)/(u_i^2 = 0;\ [u_i,u_j]=0,\ \{i,j\}\in\mathcal{K})$, $\deg u_i = (-1,2e_i)$.
4. If $\mathcal{K}$ is a flag complex (i.e. $I\in\mathcal{K}$ whenever $\{i,j\}\in\mathcal{K}$ for all $i,j\in I$), then $\mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk,\Bbbk) \cong \Bbbk[\mathcal{K}]^!$.

Stanton's homotopy decomposition

Denote by $\mathcal{P}$ be the class of spaces homotopy equivalent to finite type products of S^1, S^3, S^7 and loops on simply connected spheres.
Simplicial complex is neighbourly if any two vertices are adjacent.

Theorem (Stanton'24)

If $\Omega\mathcal{Z}_{\mathcal{K}_A} \in \mathcal{P}$ for any neighbourly $\mathcal{K}_A$, then $\Omega\mathcal{Z}_{\mathcal{K}} \in \mathcal{P}$.
In particular: if $\mathcal{K}$ is flag, then $\Omega\mathcal{Z}_{\mathcal{K}} \in \mathcal{P}$.

Remark: $\mathcal{K}_A$ is neighbourly iff $A \in \mathcal{K}^f$, where $\mathcal{K}^f$ is the flagification of $\mathcal{K}$.

Lemma (Stanton; see [Vylegzhanin'24])

If $\Omega\mathcal{Z}_{\mathcal{K}} \in \mathcal{P}$, then $\Omega\mathcal{Z}_{\mathcal{K}} \simeq \prod_{n\geq 3}(\Omega S^n)^{\times D_n}$ for some $D_n \geq 0$.

Strategy: find D_n by computing the Poincaré series of $\mathrm{Ext}_{\mathbb{k}[\mathcal{K}]}(\mathbb{k},\mathbb{k})$. By the results above, it should be equal to $(1+t)^m / \prod_{n\geq 3}(1-t^{n-1})^{D_n}$.

$\pi_*(\mathcal{Z}_\mathcal{K})$, flag complexes

In the flag case, $\mathrm{Ext}_{\Bbbk[\mathcal{K}]}(\Bbbk, \Bbbk) \cong \Bbbk[\mathcal{K}]^!$ is a Koszul algebra (i.e. Koszul-Priddy complex $(\Bbbk[\mathcal{K}]^! \otimes (\Bbbk[\mathcal{K}])^*, d)$ is acyclic). Hence, Poincaré series of $\Bbbk[\mathcal{K}]^!$ and $\Bbbk[\mathcal{K}]$ are related by the Fröberg formula.

Theorem (V.'24)

Let $\mathcal{K}$ be a flag simplicial complex on vertex set $[m]$. Then there is a homotopy equivalence $\Omega\mathcal{Z}_\mathcal{K} \simeq \prod_{n\geq 3}(\Omega S^n)^{\times D_n}$, where the numbers D_n satisfy

$$\prod_n (1-t^{n-1})^{D_n} = \sum_{J\subset[m]} (1-\chi(\mathcal{K}_J))\cdot t^{|J|}. \tag{0.1}$$

In particular, $\pi_N(\mathcal{Z}_\mathcal{K}) \simeq \bigoplus_{n\geq 3} \pi_N(S^n)^{\oplus D_n}$.

This allows to describe homotopy groups of smooth toric varieties and quasitoric manifolds corresponding to flag complexes!
In fact, the RHS of (0.1) equals $(1+t)^{m-\dim(\mathcal{K})}\cdot \sum_{i\geq 0} h_i(\mathcal{K})\cdot(-t)^i$.

$\pi_*(\mathcal{Z}_\mathcal{K})$, flag complexes with holes

We say that $\mathcal{K}$ is a flag complex with holes if $\mathcal{K} = \mathcal{L} \setminus M$, where $\mathcal{L}$ is a flag complex and $M \subset \mathcal{L}$ are some maximal faces of $\mathcal{L}$ (of dim ≥ 2). By Stanton's results, $\Omega\mathcal{Z}_\mathcal{K} \simeq \prod_{n\geq 3}(\Omega S^n)^{\times D_n}$ if $\mathcal{K}$ is flag with holes.

Theorem (V., work in progress)

Let $\mathcal{K} = \mathcal{L} \setminus M$ be a flag complex with holes. Then

$$H_*(\Omega \mathrm{DJ}(\mathcal{K}); \Bbbk) \cong T(u_1, \dots, u_m; w_I, I \in M)/\mathcal{I},$$

$$\mathcal{I} = \Big(u_i^2 = 0;\ [u_i, u_j] = 0,\ \{i,j\} \in \mathcal{K};\ [u_i, w_I] = 0,\ i \in I \in M\Big).$$

In fact, RHS is a Koszul algebra, so we can compute D_n:

$$\prod_{n\geq 3}(1 - t^{n-1})^{D_n} = \sum_{J\subset[m]} (1 - \chi(\mathcal{L}_J)) \cdot t^{|J|} - \sum_{I\in M} t^{2|I|-2}(1+t)^{m-|I|}.$$

Presentations of connected algebras ($\mathbb{k}$ is a field)

Let $\mathbb{k}$ be a commutative ring with unit. Graded associative $\mathbb{k}$-algebra $A = \bigoplus_{n \geq 0} A_n$ with unit is connected if $A_0 = \mathbb{k} \cdot 1$. A presentation of A is any isomorphism $A \simeq T(a_1, \ldots, a_N)/(r_1 = \cdots = r_M = 0)$, where a_i, r_j are homogeneous elements of positive degree.

Theorem (C.T.C.Wall'60)

Let $\mathbb{k}$ be a field and A be an associative graded $\mathbb{k}$-algebra. Let $n \geq 0$.

1. $\forall$ presentation of A has $\geq \dim_{\mathbb{k}} \mathrm{Tor}_1^A(\mathbb{k}, \mathbb{k})_n$ generators and $\geq \dim_{\mathbb{k}} \mathrm{Tor}_2^A(\mathbb{k}, \mathbb{k})_n$ relations of degree n.
2. $\exists$ a presentation of A with $\dim_{\mathbb{k}} \mathrm{Tor}_1^A(\mathbb{k}, \mathbb{k})_n$ generators and $\dim_{\mathbb{k}} \mathrm{Tor}_2^A(\mathbb{k}, \mathbb{k})_n$ relations of degree n.

Presentations of connected algebras ($\Bbbk$ is a PID)

Let $\Bbbk$ be a commutative ring with unit. Graded associative $\Bbbk$-algebra $A = \bigoplus_{n \geq 0} A_n$ with unit is connected if $A_0 = \Bbbk \cdot 1$. A presentation of A is any isomorphism $A \simeq T(a_1, \ldots, a_N)/(r_1 = \cdots = r_M = 0)$, where a_i, r_j are homogeneous elements of positive degree.

Theorem (V.)

Let $\Bbbk$ be a PID and A be an associative graded $\Bbbk$-algebra. Let $n \geq 0$.

1. $\forall$ presentation of A has $\geq \operatorname{gen} \operatorname{Tor}_1^A(\Bbbk, \Bbbk)_n$ generators and $\geq \left(\operatorname{gen} \operatorname{Tor}_2^A(\Bbbk, \Bbbk)_n + \operatorname{rel} \operatorname{Tor}_1^A(\Bbbk, \Bbbk)_n\right)$ relations of degree n.
2. $\exists$ a presentation of A with $\operatorname{gen} \operatorname{Tor}_1^A(\Bbbk, \Bbbk)_n$ generators and $\left(\operatorname{gen} \operatorname{Tor}_2^A(\Bbbk, \Bbbk)_n + \operatorname{rel} \operatorname{Tor}_1^A(\Bbbk, \Bbbk)_n\right)$ relations of degree n.

Here $M \simeq \Bbbk^{\operatorname{gen} M}/\Bbbk^{\operatorname{rel} M}$, where $\operatorname{gen} M$ and $\operatorname{rel} M$ are the smallest possible. For example, $\operatorname{gen}(\mathbb{Z}/6 \oplus \mathbb{Z}) = 2$, $\operatorname{rel}(\mathbb{Z}/6 \oplus \mathbb{Z}) = 1$ if $\Bbbk = \mathbb{Z}$.

Presentations and the bar construction

The $\Bbbk$-module $\mathrm{Tor}^A(\Bbbk, \Bbbk)$ is isomorphic to homology of the bar construction $(\overline{\mathrm{B}}(A), d)$, where $\overline{\mathrm{B}}(A)_n = (A_{>0})^{\otimes n}$.

Theorem (V.)

Choose the elements

- $a_1, \ldots, a_N \in A_{>0} \simeq \overline{\mathrm{B}}_1(A)$ whose images generate $\mathrm{Tor}_1^A(\Bbbk, \Bbbk)$;
- $\rho_r = \sum_\beta K_{r,\beta} \otimes L_{r,\beta} \in A_{>0} \otimes A_{>0} \simeq \overline{\mathrm{B}}_2(A)$ so that their boundaries $d_{\overline{\mathrm{B}}}(\rho_r) = \sum_{i=1}^N \lambda_{ri} \cdot a_i \in \overline{\mathrm{B}}_1(A)$ produce a sufficient set of additive relations $\sum_i \lambda_{ri} \cdot [a_i] = 0$ between $[a_1], \ldots, [a_N] \in \mathrm{Tor}_1^A(\Bbbk, \Bbbk)$;
- $\sum_\alpha P_{j,\alpha} \otimes Q_{j,\alpha} \in A_{>0} \otimes A_{>0} \simeq \overline{\mathrm{B}}_2(A)$ whose images generate $\mathrm{Tor}_2^A(\Bbbk, \Bbbk)$ as a $\Bbbk$-module.

Then we have a presentation

$$A \simeq T(a_1, \ldots, a_N)/(\textstyle\sum_\beta \pm K_{r,\beta} \cdot L_{r,\beta} = \sum_{i=1}^N \lambda_{ri} a_i,\ \sum_\alpha \pm P_{j,\alpha} \cdot Q_{j,\alpha} = 0),$$

where $K_{r,\beta}, L_{r,\beta}, P_{j,\alpha}, Q_{j,\alpha}$ are considered as words on $a_1, \ldots, a_N$.

Our approach to loop homology

Let $\mathcal{K}$ be a flag complex. We have: $S = H_*(\Omega\mathcal{Z}_\mathcal{K}; \Bbbk)$ is a subalgebra in the known algebra $A = H_*(\Omega \mathrm{DJ}(\mathcal{K}); \Bbbk)$, and $A \simeq S \otimes_\Bbbk \Lambda[m]$ as a S-module. A presentation of S is computed as follows.

1. We have the Fröberg resolution $(A \otimes \Bbbk\langle\mathcal{K}\rangle, d)$ of the left A-module $\Bbbk$. Consider it as a free resolution $(S \otimes \Lambda[m] \otimes \Bbbk\langle\mathcal{K}\rangle, \widehat{d})$ of the left S-module $\Bbbk$.
2. Compute $\mathrm{Tor}^S(\Bbbk, \Bbbk)$ as homology of $(\Lambda[m] \otimes \Bbbk\langle\mathcal{K}\rangle, \overline{d})$.
3. Construct a homology isomorphism $\overline{\varphi} : (\Lambda[m] \otimes \Bbbk\langle\mathcal{K}\rangle, \overline{d}) \to (\overline{\mathrm{B}}(S), d)$.
4. Obtain elements in $\overline{\mathrm{B}}(S)$ corresponding to additive generators and relations in $\mathrm{Tor}^S(\Bbbk, \Bbbk)$.
5. Use the previous slide to give a presentation of S.

Theorem (V.'22)

On step 2 we obtain $\mathrm{Tor}_q^{H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)}(\Bbbk, \Bbbk) \cong \bigoplus_{J \subset [m]} \widetilde{H}_{q-1}(\mathcal{K}_J; \Bbbk)$.

Results: generators for $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$, flag case

Recall that the algebra $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$ is generated by elements $u_1,\dots,u_m$ of degree 1. For $I=\{i_1<\dots<i_k\}\subset[m]$, $x\in H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$ denote

$$c(I,x):=[u_{i_1},[u_{i_2},\dots[u_{i_k},x]\dots]]\in H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk).$$

One can show that $c(I,u_j)\in H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)\subset H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$ if $I\neq\varnothing$.

For every $J\subset[m]$ let $\Theta(J)\subset J$ contain exactly one vertex from every path component of $\mathcal{K}_J$ not containing $\max(J)$ (for example, the minimal vertices of path components). We have $|\Theta(J)|=b_0(\mathcal{K}_J)-1$.

Theorem (Grbić, Panov, Theriault, Wu'16 / V.)

$H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is multiplicatively generated by the GPTW generators $\{c(J\setminus j,u_j):\ J\subset[m],j\in\Theta(J)\}$. It is a minimal set of generators.

Results: relations in $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$, flag case

Each $c(I,u_j)$ can be expressed through the GPTW generators by an explicit recursive algorithm. Denote any such expression as $\widehat{c}(I,u_j)$.

Theorem (V.'24)

For every $J\subset[m]$ choose a set of simplicial 1-cycles $\sum_{\{i<j\}\in\mathcal{K}_J}\lambda_{ij}^{\alpha}[\{i,j\}]$ in $C_1(\mathcal{K}_J;\Bbbk)$, whose images generate the $\Bbbk$-module $H_1(\mathcal{K}_J;\Bbbk)$. Then $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is presented by GPTW generators modulo the relations

$$\sum_{\{i<j\}\in\mathcal{K}_J}\lambda_{ij}^{(\alpha)}\sum_{\substack{J\setminus\{i,j\}=A\sqcup B:\\ \max(A)>i,\\ \max(B)>j.}}(-1)^{\cdots}\left[\widehat{c}(A,u_i),\widehat{c}(B,u_j)\right]=0.$$

In particular, there is a presentation by $\sum_{J\subset[m]}(b_0(\mathcal{K}_J)-1)$ generators and $\sum_{J\subset[m]}\operatorname{gen}H_1(\mathcal{K}_J;\Bbbk)$ relations. It is minimal among $\mathbb{Z}\times\mathbb{Z}^m_{\geq 0}$-homogeneous presentations, if $\Bbbk$ is a PID.

An example: 5-cycle

For $\mathcal{K}$ a 5-cycle, the algebra $H_*(\Omega\mathcal{Z}_\mathcal{K};\Bbbk)$ is presented by 10 generators and a single relation, first computed by Veryovkin (2016, computer bruteforce). The GPTW generators:

$$[u_3,u_1],\ [u_4,u_1],\ [u_4,u_2],\ [u_5,u_2],\ [u_5,u_3],\ [u_1,[u_5,u_3]],$$

$$[u_2,[u_4,u_1]],\ [u_3,[u_4,u_1]],\ [u_3,[u_5,u_2]],\ [u_4,[u_5,u_2]].$$

Our formula gives the relation:

$$-\Big[[u_3,u_1],[u_4,[u_5,u_2]]\Big]+\Big[[u_4,u_1],[u_3,[u_5,u_2]]\Big]+\Big[[u_3,[u_4,u_1]],[u_5,u_2]\Big]$$

$$+\Big[[u_4,u_2],[u_1,[u_5,u_2]]\Big]+\Big[\underline{[u_1,[u_4,u_2]]},[u_5,u_3]\Big]=0.$$

The underlined element is not a GPTW generator; however, $-[u_1,[u_4,u_2]]=[u_2,[u_4,u_1]]$ in $H_*(\Omega\mathrm{DJ}(\mathcal{K});\Bbbk)$, and $[u_2,[u_4,u_1]]$ is a generator.

$\mathbb{Q}$-coformality in the flag case

Simply connected space X is $\Bbbk$-coformal if $C_*(\Omega X;\Bbbk) \sim H_*(\Omega X;\Bbbk)$ as dga algebras over $\Bbbk$. It is known that $\mathrm{DJ}(\mathcal{K})$ is coformal if and only if $\mathcal{K}$ is flag.

Proposition (V.'24)

If $\mathcal{K}$ is flag, then $\mathcal{Z}_{\mathcal{K}}$ (and all $EH \times_H \mathcal{Z}_{\mathcal{K}}$) are $\mathbb{Q}$-coformal.

This follows from the following theorem.

Theorem (Huang'23)

Let $F \to E \to B$ be a fibration of nilpotent spaces of finite type, such that E is $\mathbb{Q}$-coformal and $\pi_*(F) \otimes \mathbb{Q} \to \pi_*(E) \otimes \mathbb{Q}$ is injective. Then F is $\mathbb{Q}$-coformal.

$\Bbbk$-coformality in the flag case?

X a simply connected space such that $H_*(\Omega X; \Bbbk)$ is $\Bbbk$-free $\rightsquigarrow$
Milnor-Moore spectral sequence $E^2_{p,q} = \mathrm{Tor}_p^{H_*(\Omega X;\Bbbk)}(\Bbbk, \Bbbk)_q \Rightarrow H_{p+q}(X; \Bbbk)$.
We have $E^2 = E^\infty$ if X is $\Bbbk$-coformal.

Theorem (V.'22)

If $\mathcal{K}$ is flag, then $E^2 = E^\infty$ for $\mathcal{Z}_\mathcal{K}$ for any $\Bbbk$.

Conjecture

If $\mathcal{K}$ is flag, then $\mathcal{Z}_\mathcal{K}$ (and all $EH \times_H \mathcal{Z}_\mathcal{K}$) are coformal over any $\Bbbk$.

It would follow from the following generalisation of Huang's result.

Conjecture

Let $F \to E \xrightarrow{p} B$ be a fibration of simply connected spaces of finite type, such that E is $\Bbbk$-coformal and Ωp has a homotopy section. Then F is $\Bbbk$-coformal.

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Thank you for your attention!

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