Mirror symmetry for invertible singularities

Alexey Basalaev

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Chapter 1

Introduction

The aim of these lecture notes is to give an introduction to mirror symmetry from the point of view of singularity theory. In this text we consider mirror symmetry only from the mathematical point of view. In mathematics mirror symmetry can be formulated in several different languages including homological, algebraic or complex geometry languages. We stick to the language of singularity theory that has connections to all the other languages listed.

From this prospective the best known results today are established for the special class of singularities known as *invertible* singularities. We collect and provide in a systematic way the mirror symmetry results for the invertible singularities in this text. We also discuss more general hypersurface singularities.

The lecture notes are supplied with many examples, open problems and "learn more" sections. In these sections we give the references for the publications that can be useful for the deeper dive into the topic.

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Chapter 2

Polynomials and symmetries

2.1 Polynomials

Consider $f \in \mathbb{C}[x_1, \dots, x_N]$ and its critical set

$$C_f := \left\{ (x_1, \dots, x_N) \in \mathbb{C}^N \mid \frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_N} = 0 \right\}.$$

We will say that f defines an *isolated singularity* at $0 \in \mathbb{C}^N$ if 0 belongs to C_f as an isolated point.

Example 2.1. For any $n \geqslant 2$ the functions below defined isolated singularities

$$f = x_1^{n+1}, \qquad A_n \text{ singularity}, \tag{2.1}$$

$$f = x_1^n + x_1 x_2^2, \qquad D_n \text{ singularity}. \tag{2.2}$$

The corresponding singularities are called A_n and D_n singularity respectively.

Non-Example 2.2. $f = x_1^3 x_2^2$ having $C_f = \{x_1 = 0\} \cup \{x_2 = 0\}$.

Exercise 2.3. Find more examples and non-examples of the polynomials defining isolated singularities.

Note that if $f \in \mathbb{C}[x_1, \dots, x_N]$ defines an isolated singularity, then it does not define an isolated singularity assumed as an element of a bigger ring $\mathbb{C}[x_1, \dots, x_{N+1}]$.

The figures of Figure 2.1 on page 8 are taken from [GPBLS02]. They depict the zero sets of some polynomials defining isolated and non–isolated singularities.

Proposition 2.4. *Let $f \in \mathbb{C}[x_1, \dots, x_k]$ and $g \in \mathbb{C}[x_{k+1}, \dots, x_N]$ be a polynomials defining isolated singularities. Then $f + g \in \mathbb{C}[x_1, \dots, x_N]$ defines an isolated singularity.*

Proof. Skipped. □

The conditions of the proposition above can be reformulated as "f and g depending on the non–intersecting set of variables". In what follows we will write $f \oplus g$ for the polynomial $f + g$ if such condition holds.

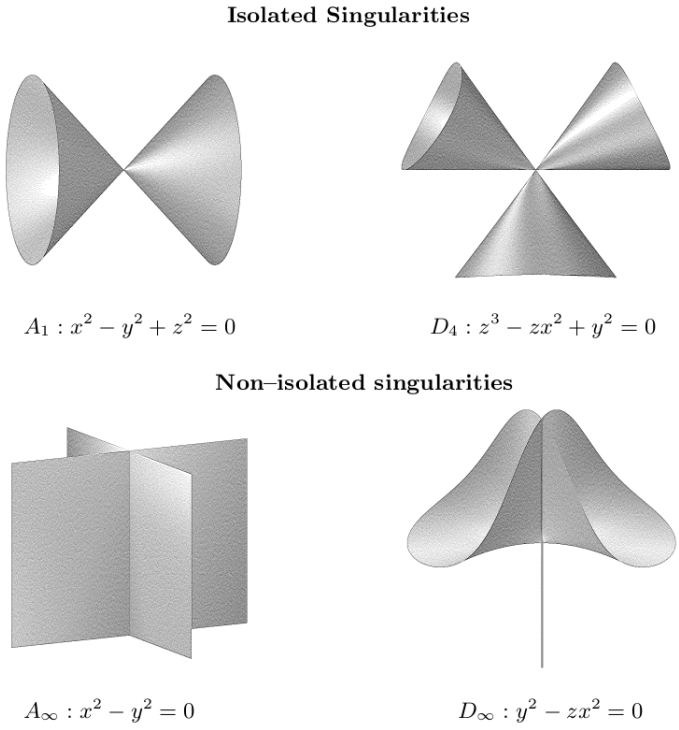

Figure 2.1: Zero–sets of the isolated and non–isolated singularities (taken from [GPBLS02])

Proposition 2.5. *Let $f \in \mathbb{C}[x_1, \dots, x_N]$ define an isolated singularity. Then for every index $j \leqslant N$ one of the following conditions holds*

- *f contains a monomial x_j^a for some a,*
- *f contains a monomial $x_j^a x_k$ for some a and index $k \leqslant N$.*

Proof. Simple. Although the proof can be found in [KS92, Theorem 1] and [S71, Korollar 1.6]. □

Corollary 2.6. *If f defines an isolated singulatiry and has no monomial of the form $x_i x_j$, then the number of the monomials of f is equal or greater than the number of the variables.*

2.1.1 Quasihomogeneity

The polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ is called *quasihomogeneous* if there are positive integers $d_f, d_1, \dots, d_N$, s.t.

$$f(\lambda^{d_1} x_1, \dots, \lambda^{d_N} x_N) = \lambda^{d_f} f(x_1, \dots, x_N), \quad \forall \lambda \in \mathbb{C}. \tag{QH1}$$

understood as the equality in the ring $\mathbb{C}[x_1, \dots, x_N]$. In what follows we will say that f is quasihomogeneous w.r.t. the *weights* $d_f, d_1, \dots, d_N$.

Remark 2.7. The numbers d_k could be understood as the degrees of the variables x_k. Then the set $d_1, \dots, d_N$ provides a grading of the polynomial ring $\mathbb{C}[x_1, \dots, x_N]$. The quasihomogeneity condition on f says that it should be of pure degree d_f w.r.t. this grading.

The equality (QH1) is equivalent to the following condition. Assume

$$f = \sum c_{\alpha_1,\dots,\alpha_N} x_1^{\alpha_1} \cdots x_N^{\alpha_N}$$

for some complex coefficients $c_{\alpha_1,\dots,\alpha_N}$. Then f is quasihomogeneous w.r.t. $d_f, d_1, \dots, d_N$ if and only if for every non–zero coefficient $c_{\alpha_1,\dots,\alpha_N}$ the following condition holds

$$d_f = \alpha_1 d_1 + \cdots + \alpha_N d_N. \tag{QH2}$$

It's clear that the choice of the weights is not unique. Moreover this form allows one to give the third equivalent formulation of the quasihomogeneity property.

The polynomial f is quasihomogeneous if and only if there are positive rational numbers $q_1, \dots, q_N$, s.t. every monomial $x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ of f satisfies

$$1 = \alpha_1 q_1 + \cdots + \alpha_N q_N. \tag{QH3}$$

The set $q_1, \dots, q_N$ as also called the set of *reduced weights*.

Remark 2.8. Still there are polynomials f that are quasihomogeneous w.r.t. to different reduced weights. For example consider $f = x_1 x_2$. It satisfies condition (QH3) above with the reduces weights $(1/2, 1/2)$ and $(1/3, 2/3)$ at the same time.

Example 2.9. The A_n singularity polynomial $f = x_1^{n+1}$ is quasihomogeneous w.r.t. $q_1 = \frac{1}{n+1}$. The D_n singularity polynomial $f = x_1^n + x_1 x_2^2$ is quasihomogeneous w.r.t. $q_1 = \frac{1}{n}$ and $q_2 = \frac{1}{2}(1 - \frac{1}{n})$.

Let f be quasihomogeneous w.r.t. the reduced weights $q_1, \dots, q_N$. These weights introduce the grading on $\mathbb{C}[x_1, \dots, x_N]$. Namely, we get the direct sum decomposition of the polynomial ring assumed as an infinite dimensional vector space

$$\mathbb{C}[x_1, \dots, x_N] = \bigoplus_{\alpha \in \mathbb{Q}} (\mathbb{C}[\mathbf{x}])_\alpha, \tag{2.3}$$

where $(\mathbb{C}[\mathbf{x}])_\alpha$ is a $\mathbb{C}$–span of the monomials of weight α

$$(\mathbb{C}[\mathbf{x}])_\alpha := \mathbb{C}\langle x_1^{\gamma_1} \cdots x_N^{\gamma_N} \mid \sum_{k=1}^{N} q_k \gamma_k = \alpha \rangle. \tag{2.4}$$

The vector spaces $(\mathbb{C}[\mathbf{x}])_\alpha$ are called *graded pieces* of $\mathbb{C}[x_1, \dots, x_N]$.

Note that some of graded pieces $(\mathbb{C}[\mathbf{x}])_\alpha$ might be 0–dimensional vector spaces. The unit of the polynomial ring belongs to $(\mathbb{C}[\mathbf{x}])_0$, and this is the only generator of this graded piece if $q_1, \dots, q_N$ are all positive.

Any $p \in \mathbb{C}[x_1, \dots, x_N]$ is called *homogeneous* if $\exists \alpha \in \mathbb{Q}$, s.t. $p \in (\mathbb{C}[\mathbf{x}])_\alpha$.

Example 2.10. For $f = x_1^{a_1}$ with $q_1 = 1/a_1$ we have $(\mathbb{C}[\mathbf{x}])_\alpha = 0$ if $\alpha \notin \frac{1}{a_1}\mathbb{N}$ and $(\mathbb{C}[\mathbf{x}])_\alpha = \mathbb{C}\langle x_1^k \rangle$ if $\alpha = k/a_1$. The element $1 + x_1$ is not homogeneous.

2.1.2 Quasihomogeneous singularities

The polynomial f defines a *quasihomogeneous singularity* if it is quasihomogeneous and defines an isolated singularity.

Remark 2.11. These two conditions are independent — there are polynomials defining isolated singularities, that are not quasihomogeneous and there are quasihomogeneous polynomials, whose critical sets contain non–isolated points.

Remark 2.12. Quasihomogeneous singularities were investigated extensively by K.Saito in [S71] as a special class and in some parts by V.I. Arnold (cf. [AGV85]).

There was an attempt to classify all quasihomogeneous singularities. It was initiated by Arnold and later pursued by several other groups. Arnold completed the classification for the number of variables $N \leqslant 4$ (cf. [AGV85]) and later Hertling–Kurbel made a thorough treatment in [HK12].

Let $f = \sum c_{\alpha_1,\dots,\alpha_N} x_1^{\alpha_1} \cdots x_N^{\alpha_N}$. Denote

$$\operatorname{supp}(f) := \left\{(\alpha_1,\dots,\alpha_N) \in \mathbb{Z}^N_{\geqslant 0} \mid c_{\alpha_1,\dots,\alpha_N} \neq 0\right\}.$$

Note that $\operatorname{supp}(f)$ only "stores" the exponents of the variables and not the coefficients.

The following theorem generalizes Proposition 2.5.

Proposition 2.13 (Theorem 2.2 in [HK12])**.** *Let f be a quasihomogeneous polynomial. It defines an isolated singularity if and only if any of the following equivalent conditions is satisfied*

(C1) for every non-empty $J \subset \{1,\dots,N\}$ either

$$\prod_{j\in J} x_j^{\alpha_j}$$

is a summand of f for some $\alpha_j \in \mathbb{Z}_{\geqslant 0}$ or there is $K \subset \{1,\dots,N\}\backslash J$, s.t. $|K| = |J|$ and for any $k \in K$, the monomial

$$x_k \prod_{j\in J} x_j^{\alpha_j}$$

is a summand of f for some $\alpha_j \in \mathbb{Z}_{\geqslant 0}$.

(C2) for every non-empty $J \subset \{1,\dots,N\}$ there is $K \subset \{1,\dots,N\}$, s.t. $|K| = |J|$ and for any $k \in K$, the monomial

$$x_k \prod_{j\in J} x_j^{\alpha_j}$$

is contained in f for some $\alpha_j \in \mathbb{Z}_{\geqslant 0}$.

Proof. Need to prove. □

Remark 2.14. The conditions C1 and C2 above can be equivalently rewritten in the combinatorial form operating with a vectors in $\mathbb{Z}^N$ rather than monomials as we choose (cf. [HK12, Lemma 2.1]).

Example 2.15. All quasihomogeneous $N = 2$ isolated singularities are given by the polynomials $f_1 = x_1^{a_1} + x_2^{a_2}$, $f_2 = x_1^{a_1}x_2 + x_2^{a_2}$, $f_1 = x_1^{a_1}x_2 + x_2^{a_2}x_1$ for all positive a_1, a_2.

Example 2.16. All quasihomogeneous $N = 3$ isolated singularities are given by the polynomials $f_1 = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$, $f_2 = x_1^{a_1} + x_2^{a_2}x_3 + x_3^{a_3}$, $f_3 = x_1^{a_1} + x_2^{a_2}x_1 + x_3^{a_3}x_1 + \varepsilon x_2^p x_3^q$, $f_4 = x_1^{a_1} + x_2^{a_2}x_3 + x_3^{a_3}x_1$, $f_5 = x_1^{a_1} + x_2^{a_2}x_1 + x_3^{a_3}x_2$, $f_6 = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}x_1 + \varepsilon x_2^p x_3^q$, $f_7 = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ with some positive a_1, a_2, a_3. The numbers a_i are arbitrary for f_1, f_2, f_4, f_5, f_7, however the polynomials f_3 and f_6 are only quasihomogeneous if $\varepsilon \neq 0$ and some additional combinatorial condition on a_1, a_2, a_3 holds. In particular the least common divisor of (a_2, a_3) should be divisible by $a_1 - 1$ for f_3 to exist.

Example 2.17. Fermat, chain and loop type polynomials

$$\begin{aligned} f &= x_1^{a_1} && \text{Fermat type,} \\ f &= x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m} && \text{Chain type,} \\ f &= x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}x_1 && \text{Loop type,} \end{aligned}$$

are examples of quasihomogeneous singularities for any natural a_j. Note that for all these polynomials the reduced weights $q_1, \ldots, q_m$ are defined in a unique way.

Using the word 'type' we assume the certain structure of the monomials set and do not specify the exponents a_i.

Remark 2.18. The polynomials of Fermat, loop and chain types are also called "of atomic type" in the literature. We refrain from using this addition vocabulary in the current text.

Exercise 2.19. Check that Fermat, loop and chain type polynomials indeed have only one critical point.

2.1.3 Graph of a quasihomogeneous singularity

Given a polynomial $f \in \mathbb{C}[x_1, \ldots, x_N]$ defining a quasihomogeneous singularity construct a map $\kappa : \{1, \ldots, N\} \to \{1, \ldots, N\}$. For any $j \leqslant N$ Proposition 2.5 asserts that f contains either a monomial x_j^a or a monomial $x_j^a x_k$. In the first case set $\kappa(j) := j$ and in the second set $\kappa(j) := k$.

Associate to f the graph Γ_f with N vertices labelled with the numbers $1, \ldots, N$ and the oriented arrows $j \to \kappa(j)$ if $j \neq \kappa(j)$.

Remark 2.20. Such graphs were first considered by Arnold, however with the selfpointing arrows $j \to j$ too. We decide to remove such arrows to reduce complexity.

Exercise 2.21. Check that: 1. the graph of a Fermat type polynomial is a one–vertex graph without edges. 2. the graph of a chain type polynomial is an oriented chain. 3. the graph of a loop type polynomial is an oriented circle (loop).

Exercise 2.22. Check that $\Gamma_{f \oplus g} = \Gamma_f \sqcup \Gamma_g$.

Proposition 2.23. *The graphs occurring as the graphs of the quasihomogeneous singularities are disjoint unions of oriented trees and oriented circles such that the oriented trees have their roots on the oriented circle.*

Proof. This follows from Proposition 2.5. □

Assume we only know the graph Γ_f and not the polynomial f itself. The graph strucuture indicates some monomials that occur as summands of f. Call these monomials *graph monomials*. Additional monomials need to be added if Γ_f is not a disjoint union of the graphs of Fermat, chain or loop type polynomials. However these additional monomials could be rather involved.

Example 2.24. The polynomial $f = x_1^3 + x_1\left(x_2^2 + x_3^2 + x_4^2\right) + \epsilon x_2x_3x_4$ with some non–zero ε defines an isolated singularity. It's also quasihomogeneous with $q_1 = \cdots = q_4 = 1/3$. This example shows that the additional monomials that need to be added to the graph type monomials could be rather complicated.

In [R24] A. Rarovskii found the certain set of monomials that can be added to the graph monomials in order to construct a quasihomogeneous singularity. Note that such a set can never be unique.

2.1.4 Invertible polynomials

The set of all quasihomogeneous singularities contains the following important class. The polynomial f defining an isolated quasihomogeneous singularity having no monomial of the form x_ix_j and as many monomials as the variables is called *invertible polynomial* and is said to define an *invertible singularity*.

Proposition 2.25. *Let f be an invertible polynomial. Then after some rescaling and renumbering of the variables we have $f = f_1 \oplus \cdots \oplus f_n$ for f_k being either of Fermat, chain or loop type.*

Proof. Make use of Proposition 2.5. Assume Γ_f to contain a vertex with two incoming arrows. Then f is of the form

$$\alpha_1 x_i^a x_l^K + \alpha_2 x_j^b x_i + \alpha_3 x_k^c x_i + g(x),$$

where $K \in \{0,1\}$, $\alpha_1\alpha_2\alpha_3 \neq 0$, $b, c \geqslant 2$ and g does not depend on x_k, x_i, x_j. Computing

$$\frac{\partial f}{\partial x_i} = a\alpha_1 x_i^{a-1} x_l^K + \alpha_2 x_j^b + \alpha_3 x_k^c, \tag{2.5}$$

$$\frac{\partial f}{\partial x_j} = b\alpha_2 x_j^{b-1} x_i, \quad \frac{\partial f}{\partial x_k} = a\alpha_3 x_k^{c-1} x_i. \tag{2.6}$$

Setting $x_i = 0$ we see that vanishing $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial x_k} = 0$ is equivalent to $\alpha_2 x_j^b + \alpha_3 x_k^c = 0$ what shows that $x_i = x_j = x_k = 0$ is not an isolated critical point of f. □

Example 2.26. The quasihomogeneous singularities with $N = 2$ are all invertible. In the notation of Example 2.15 we have f_1 is Fermat+Fermat with $n = 2$, f_2 is Chain with $n = 1$ and f_3 is Loop with $n = 1$.

Example 2.27. The quasihomogeneous singularities with $N = 3$ are not all invertible. In the notation of Example 2.16 we have f_1 — Fermat+Fermat+Fermat, f_2 — Fermat+Chain, f_3 — not invertible, f_4 — Fermat+Loop, f_5 — Chain, f_6 — not invertible, f_7 — Loop.

For an invertible polynomial f define the matrix E_f with the entries in $\mathbb{Z}_{\geqslant 0}$. Up to a rescaling of the variables we may assume

$$f = \sum_{i=1}^{N} \prod_{j=1}^{N} x_j^{E_{ij}}$$

for some integers E_{ij}. Treat these integers as the components of the matrix $E_f := \{E_{ij}\}$. In the other words every row of the matrix E_f corresponds to a monomial of f and it's values are exponents of the variables entering the monomial.

Proposition 2.28. *Let f be an invertible polynomial. Then*

(i) The matrix E_f is invertible.

(ii) There is a canonical choice of the weights $(d_f, d_1, \dots, d_N)$.

Proof. Let f be quasihomogeneous with the weight set $(d_f, d_1, \dots, d_N)$. Introduce two $\mathbb{Z}^N$ vectors: $\overline{d} := (d_1, \dots, d_N)^T$ and $\mathbf{1} := (1, \dots, 1)^T$. Then the quasihomogeneity condition is equivalent to the $\mathbb{Z}^N$ vector equality $E_f \cdot \overline{d} = d_f \mathbf{1}$. It follows now from Cramer rule that $\det(E_f) \neq 0$. This completes (i).

The canonical weight set is obtained by taking $d_f := \det(E_f)$. □

Corollary 2.29. *The reduced weight set $q_1, \dots, q_N$ satisfies*

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = E_f^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \Leftrightarrow \quad E_f \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Proof. Consider the RHS vector equality. The k-th row is equivalent to the fact that k-th monomial of f has weight 1 w.r.t. the weight set $(q_1, \dots, q_N)$. □

Corollary 2.30. *For f being of Fermat, loop or chain type the reduced weights are*

$$q_1 = \frac{1}{a_1} \qquad \textit{Fermat type} \tag{2.7}$$

$$q_i = \sum_{j=i}^{N} \frac{(-1)^{j-i}}{a_1 \cdots a_j} \qquad \textit{chain type} \tag{2.8}$$

$$q_m = (-1)^{N-1} \frac{1 - a_m + \sum_{k=2}^{N-1} (-1)^k a_m \prod_{l=2}^{k} a_{m-l+1}}{\prod_{k=1}^{N} a_k - (-1)^N} \qquad \textit{loop type,} \tag{2.9}$$

where one assumes $a_0 := a_N$, $a_{-1} := a_{N-1}$, $a_{-2} := a_{N-2}$ and so on.

Proof. This follow from Cramer's rule. □

Example 2.31. For a loop type with $N = 5$ the weights read:

$$q_1 = \frac{1 - a_5 + a_4 a_5 - a_3 a_4 a_5 + a_2 a_3 a_4 a_5}{D}, \quad q_2 = \frac{1 - a_1 + a_5 a_1 - a_4 a_5 a_1 + a_3 a_4 a_5 a_1}{D},$$
$$q_3 = \frac{1 - a_2 + a_1 a_2 - a_1 a_5 a_2 + a_1 a_4 a_5 a_2}{D}, \quad q_4 = \frac{1 - a_3 + a_2 a_3 - a_1 a_2 a_3 + a_1 a_2 a_5 a_3}{D},$$
$$q_5 = \frac{1 - a_4 + a_3 a_4 - a_2 a_3 a_4 + a_1 a_2 a_3 a_4}{D},$$

for $D = 1 + a_1 \dots a_5$.

2.1.5 Berglund–Hübsch transpose

The space of all invertible polynomials has an involution called *Berglund–Hübsch transposition* of *Berglund–Hübsch duality*. It's defined as follows. Given an invertible polynomial f with the matrix $E_f = \{E_{ij}\}$, s.t.

$$f = \sum_i \prod_j x_j^{E_{ij}},$$

what could be read off as the definition of the matrix E_f, define $\widetilde{f} \in \mathbb{C}[x_1, \dots, x_N]$ by

$$\widetilde{f} := \sum_i \prod_j x_j^{E_{ji}}.$$

Namely, $\widetilde{f}$ is a new polynomial with as many variables as monomials, whose exponent matrix is $(E_f)^T$.

Proposition 2.32. *The function $\widetilde{f}$ defines an isolated singularity. For $f = f_1 \oplus \cdots \oplus f_n$ we have $\widetilde{f} = \widetilde{f}_1 \oplus \cdots \oplus \widetilde{f}_n$*

Proof. Fist of all note that the proposition holds true for f being of Fermat, loop or chain type. For the general type this follows from Proposition 2.25. □

Example 2.33. Fermat type

$$f = x_1^{a_1} \ \Rightarrow \widetilde{f} = x_1^{a_1} = f,$$

Chain type

$$f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m} \ \Rightarrow \widetilde{f} = x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{m-1} x_m^{a_m},$$

Loop type

$$f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_{m-1}^{a_{m-1}} x_m + x_m^{a_m} x_1 \ \Rightarrow \widetilde{f} = x_m x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{m-1} x_m^{a_m}.$$

Exercise 2.34. Check this!

One notes immediately that BH duality preserves the type of an invertible polynomial.

Proposition 2.35. *The graph $\Gamma_{\widetilde{f}}$ is obtained from Γ_f by reversing all arrows.*

Proof. This is obvious for f being of Fermat, chain or loop type. The statement for the general invertible polynomial follows from Proposition 2.32. □

The operation $f \to \widetilde{f}$ looks to be very simple. Note that both f and $\widetilde{f}$ have only graph monomials, hence the type of both is defined by their graphs. It also looks to be rather essential to "just invert" the arrows of the graph and ask what happens. However the operation $f \to \widetilde{f}$ is not essential at all from the point of view of singularity theory. This will be explained later.

2.1.6 Learn more

Why quasihomogeneous?

In fact, quasihomogeneous singularities are those being best understood. Even restricting ourselves to the class of all quasihomogeneous singularities is currently to complicated for mirror symmetry. In particular, there is no clear mirror partner for a generic quasihomogeneous singularity.

Why polynomials?

We took the singularity to be defined by a polynomial. But why? It was shown by Saito [S71, Satz 4.1] that a 'quasihomogeneous' convergent power series defining an isolated singularity can be made into a polynomial by a change of coordinates in the ring of convergent power series $\mathbb{C}\{x_1,\dots,x_N\}$. Here 'quasihomogeneous' means that there are convergent power series $g_k \in \mathbb{C}\{x_1,\dots,x_N\}$, s.t.

$$\sum_{k=1}^{N} g_k \frac{\partial f}{\partial x_k} = f.$$

When f is a polynomial we have $g_k = q_k x_k$ — linear functions given by the reduced weights.

2.2 Symmetries

Given a quasihomogeneous polynomial $f = f(x_1,\dots,x_N)$ consider

$$G_f := \left\{(\alpha_1,\dots,\alpha_N) \in (\mathbb{C}^*)^N \mid f(\alpha_1 x_1,\dots,\alpha_N x_N) = f(x_1,\dots,x_N)\right\}.$$

This is a group w.r.t. component–wise product of $(\mathbb{C}^*)^N$

$$(\alpha'_1,\dots,\alpha'_N)\cdot(\alpha''_1,\dots,\alpha''_N) := (\alpha'_1\alpha''_1,\dots,\alpha'_N\alpha''_N).$$

It's called *maximal group of diagonal symmetries* of f. These are indeed diagonal symmetries because every $\alpha \in G_f$ can be represeted by a diagonal matrix with complex coefficients realizing the linear action $\mathbf{x} = (x_1,\dots,x_N) \mapsto \alpha\cdot\mathbf{x} = (\alpha_1 x_1,\dots,\alpha_N x_N)$.

In what follows we will use the notation

$$\mathbf{e}\left[\alpha\right] := \exp(2\pi\sqrt{-1}\alpha), \quad \alpha \in \mathbb{Q}.$$

Each element $g \in G_f$ has a unique expression of the form

$$g = \operatorname{diag}\left(\mathbf{e}\left[\frac{a_1}{r}\right],\dots,\mathbf{e}\left[\frac{a_N}{r}\right]\right) \quad \text{with } 0 \leqslant a_i < r, \tag{2.10}$$

where r is the order of g. We use the notation $(a_1/r,\dots,a_N/r)$ or $\frac{1}{r}(a_1,\dots,a_N)$ for the element g.

Example 2.36. For $f = x_1^{a_1}$ we have $G_f = \langle g\rangle$ with $g \in \mathbb{C}^*$ acting by $g(x_1) = \mathbf{e}\left[\frac{1}{a_1}\right]\cdot x_1$. Its order is a_1 and in the additive notation we have $g = (1/a_1)$, $G_f \cong \mathbb{Z}/a_1\mathbb{Z}$.

Example 2.37. For $f = x_1^{a_1}x_2 + x_2^{a_2}$ we have $G_f = \langle g_1, g_2\rangle$ with $g_1\cdot(x_1,x_2) = (\mathbf{e}[\frac{1}{a_1}]x_1, x_2)$ and $g_2\cdot(x_1,x_2) = (\mathbf{e}[\frac{1}{a_2}(1-\frac{1}{a_1})]x_1, \mathbf{e}[\frac{1}{a_2}]x_2)$. In the additive notation $g_1 = (1/a_1, 0)$ and $g_2 = ((1-1/a_1)/a_2, 1/a_2)$.

Let $(q_1, \dots, q_N)$ be the reduced weight set of f. Then we have

$$j_f := (\mathbf{e}[q_1], \dots, \mathbf{e}[q_N]) \in G_f.$$

In particular it follows that G_f is not empty whenever f is quasihomogeneous.

Exercise 2.38. Check this!

Denote by $\mathcal{J}$ the group generated by j_f:

$$\mathcal{J} := \langle j_f \rangle \subseteq G_f.$$

It's easy to see that for $f = x_1^{a_1}$ we have $G_f = \mathcal{J}$. However this equality does not hold already for $f = x_1^{2k+1} + x_1 x_2^2$.

Remark 2.39. Every monomial of f adds up a restriction on the symmetry group G_f. Invertible polynomials have the least possible number of the monomials in the class of all polynomials defining isolated singularities. One could expect that the polynomials that are not invertible but still define a quasihomogeneous singularity have $G_f = \mathcal{J}$. However this does not hold. Assume for $\varepsilon \in \mathbb{C}^*$ the polynomial $f = x_1^4 x_2 + x_2^4 x_5 + x_3^4 x_4 + x_4^4 x_5 + x_5^5 + \varepsilon x_2^3 x_3^2$.

Exercise 2.40 (unresolved)**.** Characterize (maybe in terms of Γ_f and additional data) all quasihomogeneous f, s.t. $G_f = \mathcal{J}$.

For each $g \in G$, denote by $\mathrm{Fix}(g)$ *the fixed locus of* g

$$\mathrm{Fix}(g) := \left\{(x_1, \dots, x_N) \in \mathbb{C}^N \mid g \cdot (x_1, \dots, x_N) = (x_1, \dots, x_N)\right\}. \tag{2.11}$$

By $N_g := \dim_{\mathbb{C}} \mathrm{Fix}(g)$ denote its dimension and by $f^g := f|_{\mathrm{Fix}(g)}$ the restriction of f to the fixed locus of g. Note that since G_f acts diagonally on $\mathbb{C}^N$, $\mathrm{Fix}(g)$ is a linear subspace of $\mathbb{C}^N$. It's never empty since it contains $0 = (0, \dots, 0) \in \mathbb{C}^N$.

For each $h \in G_f$, let $I_h := \{i_1, \dots, i_{N_h}\}$ be a subset of $\{1, \dots, N\}$ such that

$$\mathrm{Fix}(h) = \{(x_1, \dots, x_N) \in \mathbb{C}^N \mid x_j = 0, j \notin I_h\}.$$

In the other words, $\mathrm{Fix}(h)$ is indexed by I_h.

In particular, $I_{\mathrm{id}} = \{1, \dots, N\}$ and $N_h = \dim_{\mathbb{C}} \mathrm{Fix}(h)$. Denote by I_h^c the complement of I_h in I_{id} and set $d_h := N - N_h$, the codimension of $\mathrm{Fix}(h)$.

Proposition 2.41. *Let Γ_f be a graph of a quasihomogeneous singularity f and $g \in G_f$. Then if g acts nontrivially on x_k, then it acts nontrivially on all x_i, s.t. there is an oriented path from i-th to the k-th vertex.*

Proof. We first show the statement for the arrows pointing at k. Having an arrow $j \to k$ means that f has a monomial $x_j^{a_j} x_k$ as a summand with a non-zero coefficient. We have $g \cdot x_k \neq x_k$ and therefore the summand can only be preserved under the action of g if $g \cdot x_j \neq x_j$. Having an oriented path $i \to j_1 \to \cdots \to j_n \to k$ we have by using the previous step that $g \cdot x_{j_n} \neq x_{j_n}$ and than $g \cdot x_{j_a} \neq x_{j_a}$ for all a. Hence, for x_i too. □

Exercise 2.42. If f is of Fermat or loop type, then any $g \in G_f$, s.t. $g \neq \mathrm{id}$ satisfies $\mathrm{Fix}(g) = 0$.

Exercise 2.43. If $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}$ is of chain type, then any $g \in G_f$, s.t. $g \neq \mathrm{id}$ satisfies $\mathrm{Fix}(g) = \{(0, \ldots, 0, x_p, \ldots, x_m) \in \mathbb{C}^N \mid x_i \in \mathbb{C}\}$ for some p depending on g.

The polynomial f^g is of chain type again: $f^g = x_p^{a_p}x_{p+1} + \cdots + x_{m-1}^{a_{m-1}}x_m + x_m^{a_m}$.

Denote also

$$G_f^{\mathrm{SL}} := G_f \cap \mathrm{SL}(N, \mathbb{C}) \tag{2.12}$$

$$= \left\{ (\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^*)^N \mid f(\alpha_1 x_1, \ldots, \alpha_N x_N) = f(x_1, \ldots, x_N) \text{ and } \prod_{k=1}^{N} \alpha_k = 1 \right\}. \tag{2.13}$$

This group will be important later on because it preserves the volume form of $\mathbb{C}^N$.

We will see in the next section that the group of diagonal symmetries of an invertible polynomial has an straightforward algebraic and combinatorial description. This is not done at all for those quasihomogeneous singularities, that are not defined by the invertible polynomials.

Exercise 2.44 (unresolved)**.** Describe the group G_f for f not invertible, defining a quasihomogeneous singularity.

2.2.1 Diagonal symmetries of an invertible polynomial

Let $f = f(x_1, \ldots, x_N)$ be an invertible polynomial with an exponents matrix E_f.

We have

$$G_f = \left\{ (\lambda_1, \ldots, \lambda_N) \in (\mathbb{C}^*)^N \;\middle|\; \prod_{j=1}^{N} \lambda_j^{E_{1j}} = \cdots = \prod_{j=1}^{N} \lambda_j^{E_{Nj}} = 1 \right\} \tag{2.14}$$

and hence G_f is a finite group.

An element $g = \frac{1}{r}(\alpha_1, \ldots, \alpha_N)$ belonging to G_f satisfies

$$E_f \cdot \overline{g} \in \mathbb{Z}^N \tag{2.15}$$

for $\overline{g}$ being a vector with the entries α_k/r. This gives yet another characterization of the group G_f

$$G_f \cong \left\{ \overline{g} \in (\mathbb{Q}/\mathbb{Z})^N \mid E_f \cdot \overline{g} \in \mathbb{Z}^N \right\} = E_f^{-1}\mathbb{Z}^N/\mathbb{Z}^N. \tag{2.16}$$

This description allows one to study the group G_f via the Smith normal form of E_f^{-1}. We call this characterization of G_f *additive* because its elements are essentially the additive notation vectors $\overline{g}$.

It follows that every vector $\overline{g}$ giving a G_f–element is a linear combination with integer coefficients of the columns of E_f^{-1} (recall that this matrix if invertible). In particular let $\overline{\rho}_i$ be the i-th column of E_f^{-1}

$$E_f^{-1} = (\overline{\rho}_1 | \ldots | \overline{\rho}_N). \tag{2.17}$$

Proposition 2.45. *The elements ρ_k generate G_f and $j_f = \rho_1 \cdots \rho_N$.*

Proof. We have seen that every G_f–element can be expressed as a linear combination of ρ_k, it remains to show that the elements ρ_k by themselves give the G_f–elements. We have $E_f \cdot \rho_k = E_f(E_f)^{-1}1_k$, where 1_k is vector with 0 everywhere except k–th component that is equal to 1. Hence $E_f \cdot \rho_k \in \mathbb{Z}^N$ and ρ_k give indeed the G_f–elements.

The expression for j_f follows from Corollary 2.29. □

Example 2.46. We have computed G_f for $f = x_1^{a_1}x_2 + x_2^{a_2}$ in Example 2.37. Let's see how it's obtained via the proposition above.

$$E_f = \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix}, \quad E_f^{-1} = \frac{1}{a_1 a_2}\begin{pmatrix} a_2 & -1 \\ 0 & a_1 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} \frac{1}{a_1} \\ 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} -\frac{1}{a_1 a_2} \\ \frac{1}{a_2} \end{pmatrix}.$$

Then we have $\rho_1 \cdot (x_1, x_2) = (\mathbf{e}[\frac{1}{a_1}]x_1, x_2)$ and $\rho_2 \cdot (x_1, x_2) = (\mathbf{e}[-\frac{1}{a_1 a_2}]x_1, \mathbf{e}[\frac{1}{a_2}]x_2)$.

Comparing to Example 2.37 we see that $g_2 = \rho_2$ and $g_1 = \rho_1\rho_2$.

Exercise 2.47. Prove Exercise 2.42 and Exercise 2.43 using the proposition above.

The set of elements $\{\rho_k\}_{k=1}^N$ is usually not free of relations. One notes in the example above that $\rho_1\rho_2^{a_2} = \mathrm{id}$. We will need a good control of all G_f elements later on.

Proposition 2.48. *The columns of E_f generate all relations on $\rho_1, \dots, \rho_N$.*

In particular, for $(E_{1k}, \dots, E_{Nk})$ being a k–th column of E_f we have in G_f

$$\rho_1^{E_{1k}} \cdots \rho_N^{E_{Nk}} = \mathrm{id},$$

and all other relations among $\{\rho_k\}_{k=1}^N$ follow from those written above.

Proof. Let $\prod_{k=1}^N \rho_k^{s_k} = \mathrm{id}$ for some $s_k \in \mathbb{Z}_{\geqslant 0}$. This is equivalent to the existence of some integers $r_1, \dots, r_N$, s.t.

$$E_f^{-1}\begin{pmatrix} s_1 \\ \vdots \\ s_N \end{pmatrix} = \begin{pmatrix} r_1 \\ \vdots \\ r_N \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} s_1 \\ \vdots \\ s_N \end{pmatrix} = E_f \begin{pmatrix} r_1 \\ \vdots \\ r_N \end{pmatrix}. \tag{2.18}$$

Relation above sets up the bijection between the sets of s_i giving a relation of ρ's and the vectors r_i.

Fixing any index k and setting $r_i = 0$ for $i \neq k$ we see that the k–th column of E_f gives a relation on the set of ρ's. Equation above expresses the vector of relations as a linear combination of the columns of E_f with the coefficients $r_1, \dots, r_N$. This completes the statement. □

Example 2.49. For length 2 chain polynomial as in Example 2.46 one notes immediately the relations $\rho_1^{a_1} = \rho_1\rho_2^{a_2} = 1$.

Let $f = x_1^{a_1}x_2 + x_1x_2^{a_2}$ be a length 2 loop polynomial.

$$E_f = \begin{pmatrix} a_1 & 1 \\ 1 & a_2 \end{pmatrix}, \quad E_f^{-1} = \frac{1}{a_1 a_2 - 1}\begin{pmatrix} a_2 & -1 \\ -1 & a_1 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} \frac{a_2}{a_1 a_2 - 1} \\ \frac{-1}{a_1 a_2 - 1} \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} \frac{-1}{a_1 a_2 - 1} \\ \frac{a_1}{a_1 a_2 - 1} \end{pmatrix}.$$

The relations are $\rho_1^{a_1}\rho_2 = \rho_1\rho_2^{a_2} = 1$.

Example 2.50. For an arbitrary loop polynomial we have $\mathrm{id} = \rho_1\rho_2^{a_2} = \cdots = \rho_{N-1}\rho_N^{a_N} = \rho_N\rho_1^{a_1}$. For an arbitrary chain polynomial we have $\mathrm{id} = \rho_1^{a_1} = \rho_1\rho_2^{a_2} = \cdots = \rho_{N-1}\rho_N^{a_N}$.

Proposition 2.51. *For f being of Fermat, loop or chain type we have*

(i) $G_f \cong \mathbb{Z}/a_1\mathbb{Z}$ *for a Fermat type* $f = x_1^{a_1}$,

(ii) $G_f \cong \mathbb{Z}/(a_1\cdots a_N - (-1)^N)\mathbb{Z}$ *for a loop type* $f = x_1^{a_1}x_2 + \cdots + x_N^{a_N}x_1$,

(iii) $G_f \cong \mathbb{Z}/(a_1\cdots a_N)\mathbb{Z}$ *for a chain type* $f = x_1^{a_1}x_2 + \cdots + x_{N-1}^{a_{N-1}}x_N + x_N^{a_N}$.

Proof. (i) is obvious. To show (ii) and (iii) note that $a_1\cdots a_N - (-1)^N = \det E_f$ and $a_1\cdots a_N = \det E_f$ for loop and chain types respectively. It follows from Proposition 2.48 that these numbers are the groups orders.

Denote $D := \det(E_f)$ and consider $g = (\alpha_1, \ldots, \alpha_N)$ with

$$\alpha_1 = (-1)^N\frac{1}{D},\ \alpha_2 = (-1)^{N-1}\frac{a_1}{D},\quad \alpha_k = (-1)^{N+1-k}\frac{a_1\cdots a_{k-1}}{D},\ k > 1.$$

for loop type, and

$$\alpha_k = \frac{(-1)^{N+k}}{D},\quad k \geqslant 1.$$

for chain type.

This is clear that g is of order D. One checks directly that $g \in G_f$. □

Remark 2.52. The additive description Eq.(2.16) of G_f allows one to consider $G \subseteq G_f$ as $G \cong C^{-1}\mathbb{Z}/\mathbb{Z}$ for a $N \times N$ invertible matrix C with the rational coefficients, s.t. $C^{-1} = E_f^{-1}B$ for some $N \times N$ matrix B with the *integer* coefficients.

The propositions above allows us to give a combinatorial description of all G_f–elements for invertible polynomials. Our goal now is to find the set $\mathcal{S}_f$ of all N–tuples $(s_1, \ldots, s_N)$ that every $g \in G_f\backslash\{\mathrm{id}\}$ is written uniquely by

$$g = \prod_{k\in I_g^c}\rho_k^{s_k},$$

and $s_k = 0$ if and only if $k \in I_g$.

Proposition 2.53. *For f being of Fermat, chain or loop type the set $\mathcal{S}_f$ consists of all $s = (s_1, \ldots, s_N)$, s.t.*

- *(Fermat type):* $1 \leqslant s_1 \leqslant a_1 - 1$
- *(loop type):* $1 \leqslant s_k \leqslant a_k$

$$s \neq (a_1, 1, a_3, 1, \ldots, a_{N-1}, 1),\quad s \neq (1, a_2, 1, a_4, 1, \ldots, a_N) \tag{2.19}$$

if N is even.

- *(chain type): s is of the form*

$$(s_1, \dots, s_{p-1}, s_p, 0, \dots, 0), \quad 1 \leqslant p \leqslant N, \tag{2.20}$$

with $1 \leqslant s_p \leqslant a_p - 1$, $1 \leqslant s_k \leqslant a_k$ for $k \leqslant p-1$.

Notation 2.54. We will call condition of Eq. (2.20) the *group chain condition*.

Proof. Fermat type is obvious.

For the loop type assume $g \in G_f$ to be given by $(\beta_1, \dots, \beta_N)$ with the rational $0 \leqslant \beta_k < 1$. We have

$$\begin{pmatrix} s_1 \\ \vdots \\ s_N \end{pmatrix} = E_f \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} = \begin{pmatrix} a_1\beta_1 + \beta_2 \\ \vdots \\ a_N\beta_N + \beta_1 \end{pmatrix}.$$

The upper bounds on β_k give the upper bounds on s_k. To show the lower bound assume $s_k < 1$. Due to the lower bound on β_k, β_{k+1} we have $\beta_k = 0, \beta_{k+1} = 0$. Then x_k and x_{k+1} are fixed by g, hence $g = \mathrm{id}$ (see Exercise 2.42).

For the even N we have in G_f

$$\rho_1 \cdots \rho_N \prod_{k \in 2\mathbb{Z}} \rho_k^{a_k - 1} = \rho_1 \rho_2^{a_2} \cdot \rho_3 \rho_4^{a_4} \cdots \rho_{N-1} \rho_N^{a_N} = \mathrm{id},$$
$$\rho_1 \cdots \rho_N \prod_{k \notin 2\mathbb{Z}} \rho_k^{a_k - 1} = \rho_1^{a_1} \cdot \rho_2 \rho_3^{a_3} \cdots \rho_{N-2} \rho_{N-1}^{a_{N-1}} \cdot \rho_N = \mathrm{id},$$

(recall Example 2.50). These relations in G_f explain Eq. (2.19).

In order to show that these are all relations among the G_f elements obtained from the $\mathcal{S}_f$ introduced one has to count the number of them and compare to the order of the group computed in Proposition 2.51.

For the chain type assume $g \in G_f$ to be given by $(\beta_1, \dots, \beta_N)$ with the rational $0 \leqslant \beta_k < 1$. We have

$$\begin{pmatrix} s_1 \\ \vdots \\ s_N \end{pmatrix} = E_f \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} = \begin{pmatrix} a_1\beta_1 + \beta_2 \\ \vdots \\ a_N\beta_N \end{pmatrix}.$$

The reasoning similar to that given for the loop type polynomial gives the upper bound $s_k \leqslant a_k$.

There is p, s.t. $I_g^c = \{1, \dots, p\}$. Then $s_{p+1} = \cdots = s_N = 0$. If $s_p = a_p$ or $s_p = 0$, then $g \cdot x_p = x_p$ and $p \notin I_g^c$. Let i be the greatest index from $2, \dots, p-1$, s.t. $s_i = 0$. Then assume another presentation of $g = \rho_{i-1}\rho_i^{a_i} \prod_{k=1}^{p} \rho_k^{s_k}$, where we have now a new index $s_i = a_i$. This operation also raises s_{i-1} by one. If it gets equal to $a_{i-1} + 1$, apply $\rho_{i-2}^{-1}\rho_{i-1} = \rho_{i-1}^{a_{i-1}}$. Note that ρ_{i-2}^{-1} can be expressed as $\rho_1^{\alpha_1} \cdots \rho_{i-2}^{\alpha_{i-2}}$.

Applying this procedure step by step we arrive at the presentation $g = \rho_1^{s_1}\rho_2^{s_2} \cdots \rho_p^{s_p}$ with $s_2, \dots, s_p$ as claimed in the proposition. The rest follows from the relation $\rho_1^{a_1} = 1$ that holds in G_f.

For the set $\mathcal{S}_f$ constructed in this way we get again $|\mathcal{S}_f| + 1 = |G_f|$ by Proposition 2.51. This shows that there is no relations between the $\rho_\cdot$ products with the exponents from $\mathcal{S}_f$. □

Note that for f being of chain type we have obvious relations

$$\rho_1^{a_1}\rho_2\rho_3^{a_3} \cdots \rho_{2p}\rho_{2p+1}^{a_{2p+1}} = \mathrm{id}. \tag{2.21}$$

However the set $s = (a_1, 1, a_3, \dots, 1, a_{2p+1}, 0, \dots, 0)$ does not satsify the group chain condition Eq. (2.20) because it's last nonzero entry exceeds $a_{2p+1} - 1$.

Example 2.55. Let $f = x_1^{a_1} x_2 + x_2^{a_2}$. We have

$$G_f = \{\mathrm{id}\} \sqcup \{\rho_1, \rho_1^2, \dots, \rho_1^{a_1-1}\} \bigsqcup_{\substack{1 \leqslant k \leqslant a_1 \\ 1 \leqslant l \leqslant a_2 - 1}} \{\rho_1^k \rho_2^l\}$$

that we divide into the three subsets. The first one acting trivially on both x_1, x_2, the second acting trivially on x_2 and the third having 0 as the fixed locus.

We have

$$\mathcal{S}_f = \bigsqcup_{1 \leqslant s_1 \leqslant a_1 - 1} \{(s_1, 0)\} \bigsqcup_{\substack{1 \leqslant s_1 \leqslant a_1 \\ 1 \leqslant s_2 \leqslant a_2 - 1}} \{(s_1, s_2)\},$$

the first subset corresponding to $p = 1$ and the second to $p = 2$ in the notation of Eq. (2.20).

2.2.2 Dual symmetry group

Given an invertible f let $G \subseteq G_f$. To the pair (f, G) we associate the pair $(\widetilde{f}, \widetilde{G})$, with $\widetilde{f}$ as in Section 2.1.5 and $\widetilde{G} \subseteq G_{\widetilde{f}}$ defined by

$$\widetilde{G} := \mathrm{Hom}(G_f / G, \mathbb{C}^*). \tag{2.22}$$

This is the space of homomorphisms from the finite group to the multiplicative group of the field. Since G_f/G is finite, the homomorphisms above have the image in the unit circle $\{|z| = 1, z \in \mathbb{C}\}$. Defined in this way the group $\widetilde{G}$ gets a natural pointwise multiplication

$$\chi_1 \cdot \chi_2 := \chi_{12}, \quad \text{s.t.} \quad \chi_{12}(a) = \chi_1(a)\chi_2(a), \quad \forall \chi_1, \chi_2 \in \widetilde{G}.$$

This definition is due to Berglund–Henningson who were the first to propose the definition of a dual group back in 1995 [BH95]. Later in 2009 Krawitz gave the definition of another symmetry group $G^T \subset G_{\widetilde{f}}$ as a candidate for a group dual to G. Ebeling and Takahashi proved in 2013 that these two definitions agree [ET13, Proposition 2.11].

Let $\sigma_1, \dots, \sigma_N$ be the rows of the matrix E_f^{-1} and $\rho_1, \dots, \rho_N$ the columns of E_f^{-1}. Due to the definition of $\widetilde{f}$ and Proposition 2.45, the elements σ_k generate $G_{\widetilde{f}}$.

The definition of Krawitz reads

$$G^T := \left\{ \prod_{k=1}^N \sigma_k^{r_k} \mid (r_1, \dots, r_N) E_f^{-1} \begin{pmatrix} s_1 \\ \vdots \\ s_N \end{pmatrix} \in \mathbb{Z} \text{ for all } \prod_{k=1}^N \rho_k^{s_k} \in G \right\}. \tag{2.23}$$

We repeat here the proof of Ebeling and Takahashi extending the details.

Proposition 2.56 (Proposition 2.11 in [ET13])**.** *There is an isomorphism* $\widetilde{G} \cong G^T$.

Proof. Consider the short exact sequence of abelian groups

$$\{1\} \to G \to G_f \to G_f/G \to \{1\}.$$

and its dual short exact sequence

$$\{1\} \to \mathrm{Hom}(G_f/G, \mathbb{C}^*) \to \mathrm{Hom}(G_f, \mathbb{C}^*) \to \mathrm{Hom}(G, \mathbb{C}^*) \to \{1\}.$$

It follows that $\mathrm{Hom}(G_f/G, \mathbb{C}^*)$ is the kernel of the surjective map $\mathrm{Hom}(G_f, \mathbb{C}^*) \to \mathrm{Hom}(G, \mathbb{C}^*)$.

Lemma 2.57. $\mathrm{Hom}(G_f, \mathbb{C}^*) \cong G_{\widetilde{f}}$.

Proof. The group G_f is generated by $\rho_1, \dots, \rho_N$ subject to the relations given by the columns of E_f. It follows that $\mathrm{Hom}(G_f, \mathbb{C}^*)$ consists of the elements χ subject to

$$\chi(\rho_1)^{E_{1k}} \cdots \chi(\rho_N)^{E_{Nk}} = 1 \in \mathbb{C}^*, \quad k = 1, \dots, N$$

for $(E_{1k}, \dots, E_{Nk})$ being the k'th column of E_f. This equality is equivalent to the fact that $(\chi(\rho_1), \dots, \chi(\rho_N))$ assumed as an element of $(\mathbb{C}^*)^N$ preserves the monomial $x_1^{E_{1k}} \cdots x_N^{E_{Nk}}$, what completes the proof. □

Let $h \in G_{\widetilde{f}}$. We show that it belongs of $G^T \subset G_{\widetilde{f}}$ if and only if it lies in the kernel $\mathrm{Hom}(G_f, \mathbb{C}^*) \to \mathrm{Hom}(G, \mathbb{C}^*)$.

Write $h = \prod_{k=1}^N \sigma_k^{r_k}$ for some $r_1, \dots, r_N$. The respective element $\chi \in \mathrm{Hom}(G_f, \mathbb{C}^*)$ under the isomorphism above is given by

$$\chi = (\chi(\rho_1), \dots, \chi(\rho_N)) = \mathbf{e}(\overline{h}).$$

This element belongs to the kernel of $\mathrm{Hom}(G_f, \mathbb{C}^*) \to \mathrm{Hom}(G, \mathbb{C}^*)$ if and only if $\chi(\rho_1)^{s_1} \cdots \chi(\rho_N)^{s_N} = 1$ for all $s_1, \dots, s_N$, s.t. $\prod_{k=1}^N \rho_k^{s_k} \in G$. Formulated for h, this is equivalent to the equiality that defines G^T. □

Assume $G \subseteq G_f$ to be given in the additive notation $G \cong C^{-1}\mathbb{Z}^N/\mathbb{Z}^N$ for some $C^{-1} = E_f^{-1}B$ with integer component matrix B (recall Remark 2.52). The additive description of the dual group $\widetilde{G}$ is given by the following proposition.

Proposition 2.58 (see also Section 3 of [ABS11])**.**

$$\left(E_f^{-1}B\mathbb{Z}^N/\mathbb{Z}^N\right)^{\sim} \cong (B^{-1})^T\mathbb{Z}^N/\mathbb{Z}^N.$$

Proof. One computes immediately that $G_f/G \cong B^{-1}\mathbb{Z}^N/\mathbb{Z}^N$. Then $\mathrm{Hom}(G_f/G, \mathbb{C}^*) \cong (B^{-1})^T\mathbb{Z}^N/\mathbb{Z}^N$. □

Corollary 2.59. *There is an isomorphism* $(\widetilde{G})^{\sim} \cong G$ *and* $(G^T)^T \cong G$.

Different formulations of a dual group are all useful and important. In particular, G^T is more useful for computations and combinatorics related to symmetry groups.

We have immediately

Proposition 2.60. *For any invertible polynomial* f *we have*

(i) $(\{\mathrm{id}\})^T \cong G_{\widetilde{f}}$ *and* $G_{\widetilde{f}}^T \cong (\{\mathrm{id}\})$,

(ii) $\mathcal{J}^T \cong G_{\widetilde{f}}^{\mathrm{SL}}$ *and* $(G_{\widetilde{f}}^{\mathrm{SL}})^T \cong \mathcal{J}$.

where {id} *stands for the trivial symmetry group.*

Proof. It's enough to prove only the first statements in both items because the second follows from Corollary 2.59.

To prove (i) we should consider (2.23) with $s_1 = \cdots = s_N = 0$. Then the statement follows from Proposition 2.45 after the transposition.

To show (ii) we should consider $s_1 = s_2 = \cdots = s_N = 1$. It follows from Corollary 2.29 that

$$\mathcal{J}^T := \left\{ \prod_{k=1}^N \sigma_k^{r_k} \mid \sum_{k=1}^N r_k q_k \in \det(E_f)\mathbb{Z} \right\}.$$

To show that $\mathcal{J}^T = G_{\widetilde{f}}^{\mathrm{SL}}$ we should compute det of the diagonal matrix given by $\prod_{k=1}^N \sigma_k^{r_k}$. It reads

$$\begin{aligned} \exp\left(2\pi\sqrt{-1}\sum_{p=1}^N\sum_{k=1}^N r_k (E_f^{-1})_{kp}\right) &= \exp\left(2\pi\sqrt{-1}\cdot(r_1,\ldots,r_N)E_f^{-1}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}\right) \\ &= \exp\left(2\pi\sqrt{-1}\sum_{k=1}^N r_k q_k\right) = 1. \end{aligned}$$

□

Example 2.61. For $f = x_1^{a_1}$ — Fermat type polynomial. Assume also $a_1 = rm$ and consider G to be generated by $g = (1/r)$. Then G^T is generated by $h = (1/m)$.

Proposition 2.62. *The group G_f^{SL} is a proper subgroup of G_f.*

Proof. We have by combining Propositions 2.56 and 2.60.

$$G_f^{\mathrm{SL}} \cong \mathrm{Hom}(G_{\widetilde{f}}/\mathcal{J}, \mathbb{C}^*) \subsetneq \mathrm{Hom}(G_{\widetilde{f}}, \mathbb{C}^*) \cong G_f.$$

□

2.2.3 Learn more

Symmetries have been studied in singularity theory also from the point of view of boundary singularities. In particular, the deformations of the boundary singularities B_n can be understood as $\mathbb{Z}/2\mathbb{Z}$–invariant deformations of the A_{2n+1} singularity. This can be read off in the papers of Slodowy, Kluitman and C.T.C Wall.

The group $\widetilde{G}$ can be given even one more characterization. There is an essential bilinear form

$$b : G_{\widetilde{f}} \times G_f \to \mathbb{Q}/\mathbb{Z}, \quad b([u],[v]) := uE_f^{-1}v$$

for $[u] \in (E_f^{-1})^T\mathbb{Z}^N/\mathbb{Z}^N$ and $[v] \in E_f^{-1}\mathbb{Z}^N/\mathbb{Z}^N$. One can check that the value of b does not depend on the representatives of the classes of u, v. Then one can check that $\widetilde{G}$ is the orthogonal of G w.r.t. b (cf. Section 3.5 of [ABS11]).

Chapter 3

Mirror symmetry as the vector spaces isomorphism

3.1 Jacobian algebra

Let $f \in \mathbb{C}[x_1, \dots, x_N]$ be a quasihomogeneous polynomial. Consider the quotient ring

$$\mathrm{Jac}(f) := \mathbb{C}[x_1, \dots, x_N] \Big/ \left(\tfrac{\partial f}{\partial x_1}, \dots, \tfrac{\partial f}{\partial x_N} \right). \tag{3.1}$$

Call it *Jacobian algebra* of f if it's a finite–dimensional $\mathbb{C}$–vector space. Set $\mu_f := \dim_{\mathbb{C}} \mathrm{Jac}(f)$ and call it the *Milnor number* of f.

Remark 3.1. The algebra $\mathrm{Jac}(f)$ can be also be found in the literature under the names "Milnor algebra" in honour of J. Milnor or "local algebra of a singularity" because it is indeed local.

Theorem 3.2. *f defines an isolated singularity if and only if $\mathrm{Jac}(f)$ is finite–dimensional.*

Proof. Skipped. Look up [AGV85]. □

In what follows we are going to consider $\mathrm{Jac}(f)$ only for isolated singularities. However with an additional convention

$$\text{set } \mathrm{Jac}(f) := \mathbb{C},\ \mu_f := 1 \text{ for the constant function } f = 0. \tag{3.2}$$

Note that $\mathrm{Jac}(f)$, being defined as a quotient ring, has an essential product.

Example 3.3. For $f = x_1^{a_1} + \dots + x_N^{a_N}$ we have $\left(\tfrac{\partial f}{\partial x_1}, \dots, \tfrac{\partial f}{\partial x_N} \right) = (x_1^{a_1 - 1}, \dots, x_N^{a_N - 1})$ and $\mathrm{Jac}(f)$ has the basis $[x_1^{p_1} \cdots x_N^{p_N}]$ with $0 \leqslant p_i \leqslant a_i - 2$. The brackets here represent the classes of the monomials written in the quotient ring.

The product structure is the following

$$[x_1^{p_1} \cdots x_N^{p_N}] \circ [x_1^{q_1} \cdots x_N^{q_N}] = [x_1^{p_1 + q_2} \cdots x_N^{p_N + q_N}]$$

and the RHS expression vanishes unless $p_k + q_k \leqslant a_k - 2$ for all $k = 1, \dots, N$.

Exercise 3.4. Let $f \in \mathbb{C}[x_1, \dots, x_N]$. Then there are the $\mathbb{C}$–algebra isomorphisms

$$\mathrm{Jac}(f) \cong \mathrm{Jac}(f + x_{N+1}^2) \quad \text{and} \quad \mathrm{Jac}(f) \cong \mathrm{Jac}(f + x_{N+1}x_{N+2}).$$

Jacobian algebra is an algebraic invariant of a singularity. In particular, it does not depend on the equivalence class of f.

For f being quasihomogeneous, the role of Jacobian algebra is rather special. In particular, we have $\sum_k q_k x_k \frac{\partial f}{\partial x_k} = f$ what concludes $f \in \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N}\right)$ and $[f] = 0$ in $\mathrm{Jac}(f)$. The converse is also true: K. Saito proved (see [S83, Satz 4.1]) that for $f = f(x_1, \dots, x_N)$ being a convergent power series with an isolated critical point at the origin and $[f] = 0$ in $\mathrm{Jac}(f)$, then there is a change of the variables transforming f to a quasihomogeneous polynomial.

We have the following classical result of Milnor and Orlik.

Proposition 3.5. *Let f be quasihomogeneous with the reduced weights $q_1, \dots, q_N$. Then its Jacobian algebra has a basis given by the classes of monomials and is of dimension*

$$\mu_f = \prod_{k=1}^{N} (\frac{1}{q_k} - 1).$$

Proof. See Theorem 1 in [MO70], and also Corollaries 3 and 4, of II.12 in [AGV85]. □

The weights $q_1, \dots, q_N$ define the $\mathbb{Q}$–grading on $\mathbb{C}[x_1, \dots, x_N]$. Let $\phi_1, \dots, \phi_\mu$ be the classes of monomials, generating $\mathrm{Jac}(f)$ as a $\mathbb{C}$–vector space. Introduce the $\mathbb{Q}$–grading on $\mathrm{Jac}(f)$ by setting

$$\deg([x_1^{\alpha_1} \cdots x_N^{\alpha_N}]) := \alpha_1 q_1 + \cdots + \alpha_N q_N.$$

We say that $\xi \in \mathrm{Jac}(f)$ is of degree κ if it's expressed as a $\mathbb{C}$–linear combination of $\deg = \kappa$ elements $\phi_\bullet$.

Denote by $\mathrm{Jac}(f)_\kappa$ the linear subspace of $\mathrm{Jac}(f)$ spanned by the $\deg = \kappa$ elements.

Proposition 3.6. *The maximal degree of a $\mathrm{Jac}(f)$–element is $K := \sum_{k=1}^{N}(1 - 2q_k)$. For any β, s.t. $0 \leqslant \beta \leqslant K$ there is a vector space isomorphism $\mathrm{Jac}(f)_\beta \cong \mathrm{Jac}(f)_{K-\beta}$.*

Proof. See Corollary 4, of II.12 in [AGV85]. □

Corollary 3.7. *The class of any monomial of degree higher than K is zero in $\mathrm{Jac}(f)$. Moreover we have $\dim \mathrm{Jac}(f)_K = 1$.*

Proof. It follows by construction that $\deg \xi \geqslant 0$ and the equality holds only for the constant ξ. Then for $\mathrm{Jac}(f)_K \cong \mathrm{Jac}(f)_0 \cong \mathbb{C}[1]$ and for $m > K$ we have $\mathrm{Jac}(f)_m \cong \mathrm{Jac}(f)_{K-m} = 0$. □

3.1.1 Jacobian algebras of invertible singularities

One notes immediately from Theorem 3.5 that

$$\mu_f = \begin{cases} a_1 - 1, & f \text{ of Fermat type}, \\ a_1 \cdots a_N + \sum_{k=1}^{N-1} (-1)^k a_{k+1} \cdots a_N + (-1)^N, & f \text{ of chain type}, \\ a_1 \cdots a_N, & f \text{ of loop type}. \end{cases} \tag{3.3}$$

To show this one better uses Corollary 2.29 rather than the computes weights by Corollary 2.30. In particular rewrite

$$\mu_f = \frac{1-q_1}{q_1}\frac{1-q_2}{q_2}\cdots\frac{1-q_N}{q_N}.$$

For the loop type polnomials we have $a_k q_k + q_{k+1} = 1$ and $a_N q_N + q_1 = 1$. This gives $a_k = \frac{1-q_{k+1}}{q_k}$, $a_N = \frac{1-q_1}{q_N}$ and the identity claimed.

Similarly for chain type polynomial we get $\mu_f = (1-q_1)a_1\cdots a_N$. Expanding it now with Corollary 2.30 the identity follows.

Note that for the chain type polynomial we also have

$$\mu_f = \sum_{i=1}^{[N/2]} (a_{2i-1}-1) \prod_{j=i+1}^{N} a_j = \sum_{\substack{k=1\\ k\notin 2\mathbb{Z}}}^{N} (a_k-1) \prod_{j=k+1}^{N} a_j,$$

with $[N/2]$ being the greatest integer less or equal to $N/2$.

Proposition 3.8 (cf. [Kreu94])**.** *Let f be of a Fermat, chain or loop type. Then the Jacobian algebra* $\mathrm{Jac}(f)$ *has a monomial basis given by the classes of all monomials $x_1^{r_1}\cdots x_N^{r_N}$ with $r = (r_1,\dots,r_N)$ s.t.*

- *(Fermat type):* $0 \leqslant r_1 \leqslant a_1 - 2$.
- *(loop type):* $0 \leqslant r_k \leqslant a_k - 1$, *for all* $k = 1,\dots,N$.
- *(chain type):* $0 \leqslant r_k \leqslant a_k - 1$, *for all* $k = 1,\dots,N$; *and* $r = (r_1,\dots,r_N)$ *being of the form*

$$(a_1-1, 0, a_3-1, 0, \dots, a_{2p-1}-1, 0, r_{2p+1}, r_{2p+2}, \dots, r_N), \quad p \geqslant 0, \tag{JCp}$$

with $r_{2p+1} \leqslant a_{2p+1} - 2$.

Notation 3.9. We will call condition of Eq. (JCp) the *Jac chain condition*.

Proof. Fermat type is obvious.

Let f be of a loop type. We show that a class of any monomial $\phi = \prod_{k=1}^N x_k^{s_k}$ with $s_k \geqslant a_k$ is either zero in $\mathrm{Jac}(f)$ or can be expressed via the classes of the monomials claimed in the proposition. There are exactly μ_f such classes by Eq. (3.3), what would complete the proof.

The following relations hold in $\mathrm{Jac}(f)$

$$[x_1^{a_1}] = -a_2[x_2^{a_2-1}x_3], \quad \dots \quad [x_{N-1}^{a_{N-1}}] = -a_N[x_N^{a_N-1}x_1], \quad [x_N^{a_N}] = -[a_1 x_1^{a_1-1}x_2].$$

It also follows from these relations that $[x_1 x_N^{a_N}] = 0$ and $[x_p^{a_p} x_{p+1}] = 0$. Combined together we see that $[x_k^{2a_k}] = 0$. The class $[\phi]$ is non–zero in $\mathrm{Jac}(f)$ only if $0 \leqslant s_k < 2a_k$ and for any s_k, s.t. $s_k \geqslant a_k$ holds $s_{k+1} = 0$, for $s_N \geqslant a_N$ holds $s_1 = 0$.

We procede in two steps.

Step 1. If $s_1 \geqslant a_1$ we have $[\phi] = -a_2[x_1^{s_1-a_1}x_2^{a_2-1}x_3^{s_3+1}x_4^{s_4}\cdots x_N^{s_N}]$. In the similar way we have for some $c_0 \in \mathbb{C}$ that $[\phi] = c_0\prod_{k=1}^N x_k^{s_k}$ with $s_k \leqslant a_k - 1$ for $k = 1,\dots,N-1$.

Step 2. Now for $s_N \geqslant a_N$ and $s_1 = 0$, applying the relation $[x_N^{a_N}] = -[a_1 x_1^{a_1-1}x_2]$ we will get the exponent of x_2 being raised by 1. If $s_2 = a_2 - 1$ repeat Step 1 above.

Let' show that after this we will get either $[\phi]$ being zero or equal to the class of a monomial claimed. To show this we should assume $s_{2k} = a_{2k} - 1$ and $s_{2k+1} = 0$ for all k. If N is even we will have

$$[x_2^{a_2-1}x_4^{a_4-1}\cdots x_{N-2}^{a_{N-2}-1}x_N^{a_N}] = -a_1a_3\cdots a_{N-1}[x_1^{a_1-1}x_3^{a_3-1}\cdots x_{N-1}^{a_{N-1}-1}x_N]$$

as claimed in the proposition. If N is odd we have

$$[x_2^{a_2-1}x_4^{a_4-1}\cdots x_{N-2}^{a_{N-1}-1}x_N^{a_N}] = -a_1a_2\cdots a_N[x_2^{a_2-1}x_4^{a_4-1}\cdots x_{N-2}^{a_{N-1}-1}x_N^{a_N}],$$

from where it follows that the class assumed vanishes.

Let f be of a chain type. We have in $\mathrm{Jac}(f)$

$$[x_1^{a_1-1}x_2] = 0, \quad [x_1^{a_1}] = -a_2[x_2^{a_2-1}x_3], \quad \dots \quad [x_{N-2}^{a_{N-2}}] = -a_{N-1}[x_{N-1}^{a_{N-1}-1}x_N], \quad [x_{N-1}^{a_{N_1}}] = -a_N[x_N^{a_N-1}].$$

The arguments similar to those given above show that the class of any monomial $\phi = \prod_{k=1}^N x_k^{s_k}$ can be expressed via those having $0 \leqslant s_k \leqslant a_k - 1$. However there are relations among the classes of such monomials. In particular, we have $[\phi] = 0$ unless it satisfies Eq (JCp). The number of such monomials coincides with μ_f as computed in Eq (3.3) and we conclude that there is no other relations. □

The condition of Eq. (JCp) was called in [FJJS12] "chain property". In an expanded form it says that the basis of $\mathrm{Jac}(f)$ for the chain type polynomials is given by the classes of the following monomials

$$\{x_1^{r_1}x_2^{r_2}\cdots x_N^{a_N}\}, \quad r_1 \leqslant a_1 - 2,\ r_2 \leqslant a_2 - 1,\ \cdots\ r_N \leqslant a_N - 1, \tag{3.4}$$

$$\{x_1^{a_1-1}x_3^{r_3}x_4^{r_4}\cdots x_N^{a_N}\}, \quad r_3 \leqslant a_3 - 2,\ r_4 \leqslant a_4 - 1,\ \cdots\ r_N \leqslant a_N - 1, \tag{3.5}$$

$$\{x_1^{a_1-1}x_3^{a_3-1}x_5^{r_5}x_6^{r_6}\cdots x_N^{a_N}\}, \quad r_5 \leqslant a_5 - 2,\ r_6 \leqslant a_6 - 1,\ \cdots\ r_N \leqslant a_N - 1, \tag{3.6}$$

and so on. The lines above satisfy Eq. (JCp) with $p = 0, 1, 2$ respectively.

Remark 3.10. Note that group chain condition Eq. (2.20) and also Jac chain condition Eq. (JCp) assume the polynomial to be written as $f = x_1^{a_1}x_2 + x_2^{a_2}x_3 \cdots + x_N^{a_N}$. In particular for $\widetilde{f} = x_1^{a_1} + x_1x_2^{a_2} + \cdots + x_{N-1}x_N^{a_N}$ both conditions will be reversed s.t.

$$s = (0, \dots, 0, s_p, \dots, s_N) \quad \text{and} \quad r = (r_1, \dots, r_{N-(2p+1)}, 0, a_{N-(2p-1)} - 1, \dots, a_N - 1), \tag{3.7}$$

subject to $0 \leqslant r_k \leqslant a_k - 1$ and $r_{N-(2p+1)} \leqslant a_{N-(2p+1)} - 2$, $1 \leqslant s_k \leqslant a_k - 1$.

3.2 State space

We introduce the A- and B-modesl state spaces of the pair (f, G) as the G–invariant subspaces of the bigger spaces $\mathcal{A}_{tot}(f)$ and $\mathcal{B}_{tot}(f)$. To do this we first introduce these bigger spaces and then define the group action on both.

The spaces $\mathcal{A}_{tot}(f)$ and $\mathcal{B}_{tot}(f)$ are isomorphic, however the group action on them is different. Due to this we choose to distinguish between them from the every beginning.

3.2.1 The total spaces

Let f define a quasihomogeneous singularity with a finite diagonal symmetries group G_f. Define

$$\mathcal{A}_{tot}(f) := \bigoplus_{g\in G_f} \mathrm{Jac}(f^g)\omega_g, \quad \text{and} \quad \mathcal{B}_{tot}(f) := \bigoplus_{g\in G_f} \mathrm{Jac}(f^g)\xi_g. \tag{3.8}$$

Here ω_g, ξ_g are the formal letters and $\mathcal{A}_{tot}(f)$, $\mathcal{B}_{tot}(f)$ are the $\mathbb{C}$–vector spaces of dimension $\sum_{g\in G} \dim \mathrm{Jac}(f^g)$. Each direct summand $\mathrm{Jac}(f^g)\xi_g$ will be called *g–th sector*. We will write just $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ when the polynomial is clear from the context.

Right now the only difference between $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ is in the special letter ξ_g or ω_g. This difference will become more profound later.

Remark 3.11. Note that for $g, h \in G$, s.t. $\mathrm{Fix}(g) = \mathrm{Fix}(h)$, we have $f^g = f^h$. Then $\mathrm{Jac}(f^g) = \mathrm{Jac}(f^h)$, but the formal letters $\xi_g \neq \xi_h$ help to distinguish $\mathrm{Jac}(f^g)\xi_g$ and $\mathrm{Jac}(f^h)\xi_h$, s.t. $\mathrm{Jac}(f^g)\xi_g \oplus \mathrm{Jac}(f^h)\xi_h$ is indeed a direct sum of dimension $\dim \mathrm{Jac}(f^g) + \dim \mathrm{Jac}(f^h)$.

Equivalently one can define $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ as follows. Let the polynomials $\phi_k^{(g)}$ be s.t.

$$\mathrm{Jac}(f^g) = \mathbb{C}\langle [\phi_1^{(g)}], \dots, [\phi_{\mu_g}^{(g)}]\rangle.$$

In particular, we may assume $\phi_1^{(g)} = 1$ for any g. Then

$$\mathcal{B}_{tot} := \bigoplus_{g\in G_f} \left\langle [\phi_1^{(g)}]\xi_g, \dots, [\phi_{\mu_g}^{(g)}]\xi_g \right\rangle = \bigoplus_{g\in G_f, k=1,\dots,\mu_g} \mathbb{C}\langle [\phi_k^{(g)}]\xi_g\rangle, \tag{3.9}$$

$$\mathcal{A}_{tot} = \bigoplus_{g\in G_f, k=1,\dots,\mu_g} \mathbb{C}\langle [\phi_k^{(g)}]\omega_g\rangle, \tag{3.10}$$

Warning! The classes $[\phi_k^{(g)}]$ are each taken in it's "own" $\mathrm{Jac}(f^g)$ and not in "one for all g" factor–ring.

Example 3.12. Given $f = x_1^n + x_2^n$ and $G_f = \langle g\rangle$ with $g = (\frac{1}{n}, 0)$. We have $f^h = x_1^n$ if $h \neq \mathrm{id}$.

$$\mathcal{B}_{tot} = \mathbb{C}\left\langle [1]\xi_{\mathrm{id}}, [x_1]\xi_{id}, [x_2]\xi_{id}, \dots, [x_1^{n-2}x_2^{n_2-2}\xi_{\mathrm{id}}]\right\rangle \tag{3.11}$$

$$\oplus\, \mathbb{C}\left\langle [1]\xi_g, [x_1]\xi_g, \dots, [x_1^{n-2}\xi_g]\right\rangle \oplus \mathbb{C}\left\langle [1]\xi_{g^2}, [x_1]\xi_{g^2}, \dots, [x_1^{n-2}\xi_{g^2}]\right\rangle \oplus \dots \tag{3.12}$$

$$\cdots \oplus \mathbb{C}\left\langle [1]\xi_{g^{n-1}}, [x_1]\xi_{g^{n-1}}, \dots, [x_1^{n-2}\xi_{g^{n-1}}]\right\rangle. \tag{3.13}$$

The total dimension of $\mathcal{B}_{tot}$ is $(n-1)(n-1) + (n-1)(n-1) = 2(n-1)^2$.

In what follows we will need a special choice of the basis in $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ in order to establish the mirror symmetry. We need to take into account the "speciality" of both $\mathrm{Jac}(f)$ and G_f outlined in Propositions 2.53 and 3.8.

Loop type polynomial total spaces

Let f be of loop type. We have

$$\mathcal{A}_{tot} = \mathrm{Jac}(f)\omega_{\mathrm{id}} \bigoplus \left(\bigoplus_{g\in G_f\setminus\{\mathrm{id}\}} \mathbb{C}\omega_g \right), \quad \mathcal{B}_{tot} = \mathrm{Jac}(f)\omega_{\mathrm{id}} \bigoplus \left(\bigoplus_{g\in G_f\setminus\{\mathrm{id}\}} \mathbb{C}\xi_g \right).$$

If $N \in 2\mathbb{Z}$ denote

$$\mathrm{Jac}^{(1)}(f) := \mathbb{C}\left\langle [\prod_{k\in 2\mathbb{Z}} x_k^{a_k-1}], [\prod_{k\notin 2\mathbb{Z}} x_k^{a_k-1}] \right\rangle, \qquad \mathrm{Jac}^{(0)}(f) := \mathrm{Jac}(f)\backslash \mathrm{Jac}^{(1)}(f),$$

where $[\cdot]$ stands for the class of a monomial in $\mathrm{Jac}(f)$. If $N \notin 2\mathbb{Z}$ we will use

$$\mathrm{Jac}^{(1)}(f) := 0, \qquad \mathrm{Jac}^{(0)}(f) := \mathrm{Jac}(f),$$

to unify the formulae.

Denote

$$\mathcal{A}_{tot}^{(1)} := \mathrm{Jac}^{(1)}(f)\omega_{\mathrm{id}} \subset \mathcal{A}_{tot}, \qquad \mathcal{B}_{tot}^{(1)} := \mathrm{Jac}^{(1)}(f)\xi_{\mathrm{id}} \subset \mathcal{B}_{tot}. \tag{3.14}$$

$$\mathcal{A}_{tot}^{(0)} := \mathcal{A}_{tot}\backslash\mathcal{A}_{tot}^{(1)}, \qquad \mathcal{B}_{tot}^{(0)} := \mathcal{B}_{tot}\backslash\mathcal{B}_{tot}^{(1)}. \tag{3.15}$$

Chain type polynomial total spaces

Let f be of chain type. Denote

$$\mathrm{Jac}^{(p)}(f) := \mathbb{C}\left\langle \left[\prod_{i=1}^{N} x_i^{r_i}\right] \mid (r_1,\dots,r_N) \text{ satisfies (JCp)} \right\rangle.$$

By Proposition 3.8 we have

$$\mathrm{Jac}(f) = \bigoplus_{p=0}^{\lfloor N/2\rfloor} \mathrm{Jac}^{(p)}(f).$$

For any $g \in G_f$ we have f^g of chain type or zero. We have

$$\mathcal{A}_{tot} = \bigoplus_{g\in G_f} \bigoplus_{p=0}^{\lfloor N_g/2\rfloor} \mathrm{Jac}^{(p)}(f^g)\omega_g, \qquad \mathcal{B}_{tot} = \bigoplus_{g\in G_f} \bigoplus_{p=0}^{\lfloor N_g/2\rfloor} \mathrm{Jac}^{(p)}(f^g)\xi_g.$$

Denote

$$\mathcal{A}_{tot,g}^{(p)} := \mathrm{Jac}^{(p)}(f^g)\omega_g \subset \mathcal{A}_{tot}, \qquad \mathcal{B}_{tot,g}^{(p)} := \mathrm{Jac}^{(p)}(f^g)\xi_g \subset \mathcal{B}_{tot}.$$

3.2.2 A- and B-model group actions

Note that G_f acts in a natural way on $\mathrm{Jac}(f^g)$ for every g by

$$h\cdot[x_1^{\alpha_1}\cdots x_N^{\alpha_N}] := [h\cdot(x_1^{\alpha_1}\cdots x_N^{\alpha_N})] := \prod_{k=1}^{N} h_k^{\alpha_k}\cdot[(x_1^{\alpha_1}\cdots x_N^{\alpha_N})].$$

for any $h = (h_1,\dots,h_N) \in G_f$.

Remark 3.13. Writing all variables $x_1,\dots,x_N$ in the product above we assume $\alpha_k = 0$ for all $k \notin I_g$.

Extend this action to the G_f–action on $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$. Set h to act on ξ_g and ω_g by

$$h: \quad \xi_g \mapsto h^*(\xi_g) := \prod_{k\in I_g^c} h_k^{-1}\cdot\xi_g \tag{3.16}$$

or

$$h: \quad \xi_g \mapsto h^*(\omega_g) := \prod_{k\in I_g} h_k \cdot \omega_g. \tag{3.17}$$

Finally we have

$$h^*\left([\prod_{k=1}^N x_k^{\alpha_k}]\xi_g\right) := \left([h\cdot\prod_{k=1}^N x_k^{\alpha_k}]h^*(\xi_g)\right) \tag{3.18}$$

$$h^*\left([\prod_{k=1}^N x_k^{\alpha_k}]\omega_g\right) := \left([h\cdot\prod_{k=1}^N x_k^{\alpha_k}]h^*(\omega_g)\right) \tag{3.19}$$

Because $I^c_{\rm id} = \varnothing$, we have $h^*(\xi_{\rm id}) = \xi_{\rm id}$ and $h^*(\omega_{\rm id}) = \det(h)\omega_{\rm id}$. Similarly $h^*(\xi_J) = \det(h)\xi_J$ and $h^*(\omega_J) = \omega_J$.

Exercise 3.14. Show that this is indeed a group action. Namely, $(h_1h_2)^* = h_1^*\cdot h_2^*$.

Remark 3.15. The action of $h\in G_f$ on a vector from $\mathcal{A}_{tot}$, $\mathcal{B}_{tot}$ is just by a scalar multiplication.

Finally define the *A–model state space* $\mathcal{A}(f,G)$ and the *B–model state space* $\mathcal{B}(f,G)$ by

$$\mathcal{A}(f,G) := (\mathcal{A}_{tot})^G \quad \text{and} \quad \mathcal{B}(f,G) := (\mathcal{B}_{tot})^G. \tag{3.20}$$

Namely

$$\mathcal{A}(f,G) = \mathbb{C}\langle [\phi]\omega_g \in \mathcal{A}_{tot} \ \text{ s.t. } h\cdot([\phi]\omega_g) = [\phi]\omega_g \ \forall h\in G\rangle$$

— a linear span of the $\mathcal{A}_{tot}$ vectors that are invariant w.r.t. the action of all elements of G,

$$\mathcal{B}(f,G) = \mathbb{C}\langle [\phi]\xi_g \in \mathcal{B}_{tot} \ \text{ s.t. } h\cdot([\phi]\xi_g) = [\phi]\xi_g \ \forall h\in G\rangle$$

— a linear span of the $\mathcal{B}_{tot}$ vectors that are invariant w.r.t. the action of all elements of G.

Example 3.16. Let $f = x_1^{a_1}$ — the Fermat type polynomial. Assume $a_1 = rm$ and consider G to be generated by $g = (1/r)$. We have

$$\mathcal{A}_{tot} = \mathbb{C}\langle [1]\omega_{\rm id}, [x_1]\omega_{\rm id}, \dots, [x_1^{rm-2}]\omega_{\rm id}\rangle \oplus \mathbb{C}\langle [1]\omega_g, \dots, [1]\omega_{g^{r-1}}\rangle,$$
$$\mathcal{B}_{tot} = \mathbb{C}\langle [1]\xi_{\rm id}, [x_1]\xi_{\rm id}, \dots, [x_1^{rm-2}]\xi_{\rm id}\rangle \oplus \mathbb{C}\langle [1]\xi_g, \dots, [1]\xi_{g^{r-1}}\rangle.$$

Because $I^c_g = I^c_{g^k} = \{1\}$, we have

$$(g^k)^*(\omega_{g^l}) = \omega_{g^l} \text{ and } (g^k)^*(\xi_{g^l}) = \exp\left(-2\pi i\cdot\frac{k}{r}\right)\xi_{g^l}.$$

However $(g^k)^*([x_1^l]) = \exp\left(2\pi i\cdot\frac{kl}{r}\right)[x_1^l]$ and the G–invariant monomials are x_1^{rn} with $n\in\mathbb{Z}$. This gives

$$\mathcal{A}(f,G) = \mathbb{C}\langle [x_1^{r-1}]\omega_{\rm id}, \dots, [x_1^{(r-1)m}]\omega_{\rm id}, [1]\omega_g, \dots, [1]\omega_{g^{r-1}}\rangle,$$
$$\mathcal{B}(f,G) = \mathbb{C}\langle [1]\xi_{\rm id}, [x_1^r]\xi_{\rm id}, \dots, [x_1^{r(m-1)}]\xi_{\rm id}\rangle.$$

Obviously, as a vector space $\mathrm{Jac}(f,G) \cong \mathrm{Jac}(x_1^{r+1})$.

Proposition 3.17. *Let $f_1 \in \mathbb{C}[x_1, \dots, x_k]$ and $f_2 \in \mathbb{C}[x_{k+1}, x_N]$ define quasihomogeneous singularities. Then*

$$\mathcal{B}_{tot}(f_1 \oplus f_2) \cong (\mathcal{B}_{tot}(f_1)) \otimes (\mathcal{B}_{tot}(f_2)).$$

In particular, if $\mathcal{B}_{tot}(f_1) = \mathbb{C}\langle[\phi_k^{(g)}]\xi_g\rangle$ with the basis indexed by $(k, g) \in I_1$ and $\mathcal{B}_{tot}(f_2) = \mathbb{C}\langle[\psi_l^{(h)}]\xi_h\rangle$ with the basis indexed by $(l, h) \in I_2$, then

$$\mathcal{B}_{tot}(f_1 \oplus f_2) = \bigoplus \mathbb{C}\langle[\phi_k^{(g)}\psi_l^{(h)}]\xi_{gh}\rangle$$

where the summation if taken over $I_1 \times I_2$.

The same holds for $\mathcal{A}_{tot}$.

Remark 3.18. Note that the classes $[\cdot]$ are taken in the three different quotient rings.

Proof. TBA fill in the proof □

3.2.3 Bigrading

Let $\mathbb{C}[x_1, \dots, x_N]$ be graded by $q_1, \dots, q_N$. For a homogeneous polynomial p let $\deg(p)$ be the degree of p w.r.t. this grading. Extend this grading to both $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ as follows.

For $g = \frac{1}{r}(\alpha_1, \dots, \alpha_N)$ with $0 \leqslant \alpha_k < r$, the *age* of g is defined as the rational number

$$\operatorname{age}(g) := \frac{1}{r}\sum_{k=1}^{N} \alpha_k. \tag{3.21}$$

Exercise 3.19. If $g \in G_f^{\mathrm{SL}}$ then $\operatorname{age}(g) \in \mathbb{Z}$.

Exercise 3.20. For any $g \in G_f$ we have

$$\operatorname{age}(g) + \operatorname{age}(g^{-1}) = N - N_g = d_g. \tag{3.22}$$

For any homogeneous $p \in \mathbb{C}[x_1, \dots, x_N]$ assume the element $[p]\xi_g \in \mathcal{A}_{tot}$ and $[p]\omega_g \in \mathcal{B}_{tot}$. Define for both $[p]\xi_g$ and $[p]\omega_g$ its *left charge* q_l and *right charge* q_r to be

$$(q_l, q_r) = \left(\deg p - \sum_{k \in I_g^c} q_k + \operatorname{age}(g), \deg p - \sum_{k \in I_g^c} q_k + \operatorname{age}(g^{-1})\right). \tag{3.23}$$

This definition endows $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ with the structure of a $\mathbb{Q}$-bigraded vector space. It follows immediately that

$$q_\bullet(\xi_u) + q_\bullet(\xi_v) = q_\bullet(\xi_{uv})$$

for $u, v \in G$, s.t. $I_u^c \cap I_v^c = \varnothing$.

This bigrading restricts to $\mathcal{A}(f, G)$ and $\mathcal{B}(f, G)$. In particular, we see that $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ are isomorphic as the graded vector spaces.

Exercise 3.21. For any $[p]\xi_g$ we have $J^*([p]\xi_g) = \mathbf{e}[\deg(p) - \sum_{k \in I_g^c} q_k] \cdot [p]\xi_g$.

Calabi–Yau condition

Assume $N \geqslant 3$ and the weight set of f to satisfy $\sum_{k=1}^{N} q_k = 1$. The latter equality is called *Calabi–Yau* condition [1] and we will say that f *satisfies CY condition*.

Under this condition we have $J \subseteq G_f^{\mathrm{SL}}$ and for any $g, h \in G \subseteq G_f^{\mathrm{SL}}$ we have

$$h^*(\xi_g) = \prod_{k \in I_g^c} h_k^{-1} \cdot \xi_g = \frac{\prod_{k \in I_g} h_k}{\prod_{k=1}^{N} h_k} \cdot \xi_g = \prod_{k \in I_g} h_k \cdot \xi_g.$$

It follows that $[p]\xi_g \in \mathcal{A}_{tot}$ is G–invariant if and only if $[p]\omega_g \in B_{tot}$ is G–invariant and there is a vector spaces isomorphism $\mathcal{A}(f,G) \tilde{\to} \mathcal{B}(f,G)$. It follows directly from the definition that this isomorphism respects the bigrading.

Another important property is the following

Proposition 3.22. *For f satisfying CY condition and G s.t. $J \subseteq G \subseteq G_f^{\mathrm{SL}}$ both left and right charges q_l and q_r of any $X \in \mathcal{A}(f,G)$ and any $Y \in \mathcal{B}(f,G)$ are integer.*

Proof. Note that $q_r([p]\xi_g) = q_l([p]\xi_g) - (N - N_g) + 2\mathrm{age}(g)$. By Exercise 3.19 the right charge $q_r([p]\xi_g)$ is integral if and only if $q_l([p]\xi_g)$ is integral because $g \in G_f^{\mathrm{SL}}$. Let $[p]\xi_g$ be nonzero in $\mathcal{A}(f,G)$, then it's invariand under the action of J and $\mathbf{e}[\deg(p) - \sum_{k \in I_g^c} q_k] = 1$ by Exercise 3.21. This gives integrality of q_l. The case of $[p]\omega_g$ is treated similarly. □

The following two propositions state that the graded pieces of $\mathcal{A}(f,G)$ and $\mathcal{B}(f,G)$ are organized into a diamond when CY condition holds.

Proposition 3.23. *For f satisfying CY condition and G s.t.* $\mathrm{age}(g)$ *is integer for any $g \in G$ let V be either $\mathcal{A}(f,G)$ or $\mathcal{B}(f,G)$ and $V^{a,b}$ stand for the bidegree (a,b)–subspace. We have*

(i) $V^{a,b} = 0$ *for* $a < 0$ *or* $b < 0$,

(ii) $V^{0,0} \cong \mathbb{C}$, *generated by* $[1]\xi_{\mathrm{id}}$ *or* $[1]\omega_{\mathrm{id}}$,

(iii) $V^{a,b} = 0$ *for* $a > N-2$ *or* $b > N-2$,

(iv) $V^{N-2,N-2} \cong \mathbb{C}$, *generated by* $[\mathrm{hess}(f)]\xi_{\mathrm{id}}$ *or* $[\mathrm{hess}(f)]\omega_{\mathrm{id}}$,

Proof. Assume $X = [p]\xi_g$ or $X = [p]\omega_g$ for p being a polynomial in the variables, fixed by g.

(i) Let $g = \mathrm{id}$. Then $q_l(X) = q_r(X) = \deg p + 1 \geqslant 1$. For $g \neq \mathrm{id}$ we have $\mathrm{age}(g) \in \mathbb{N}_{\geqslant 1}$ by Exercise 3.19. Rewriting

$$q_l(X) = \deg p + \sum_{k \in I_g} q_k - 1 + \mathrm{age}(g)$$

we see that $q_l(X) \geqslant 0$ because the weights $q_\bullet$ are all strictly positive. Similarly for $q_r(X)$ by the same argument applied to $\mathrm{age}(g^{-1})$.

(ii) For $g = \mathrm{id}$ we have that $q_l(X) = q_r(X) = 0$ if and only if $\deg p = 0$. By Proposition 3.6 we have $[p] = [1]$ in $\mathrm{Jac}(f)$.

[1] see "Learn more" section for the explanation of the terminology

For $g \neq \mathrm{id}$ in order to satisfy $q_l(X) = 0$ we need $\deg p + \sum_{k\in I_g} q_k = 0$. Both summands are nonnegative so that the equality hold only if $\deg p = 0$ and $|I_g| = 0$. Then $N_g = 0$. If $q_r(X) = 0$ we have $q_l(X) - q_r(X) = 2\mathrm{age}(g) - N + N_g = 0$ what gives $\mathrm{age}(g) = N/2 = \mathrm{age}(g^{-1})$. We conclude

$$q_l(X) = -1 + \frac{N}{2} > 0 \quad \text{for } N \geqslant 3.$$

(iii) If $g = \mathrm{id}$, the statement follows from Proposition 3.6. For $g \neq \mathrm{id}$ apply the same proposition again to estimate $\deg p$ in $\mathrm{Jac}(f^g)$. It gives

$$q_l(X) \leqslant N_g - 2\sum_{k\in I_g} q_k + \sum_{k\in I_g^c} q_k + \mathrm{age}(g) = N_g - \sum_{k\in I_g} q_k - 1 + \mathrm{age}(g).$$

At the same time we have $N_g + \mathrm{age}(g) = N - \mathrm{age}(g^{-1}) \leqslant N - 1$ because $\mathrm{age}(g^{-1}) \in \mathbb{N}_{\geqslant 1}$. Combining this with the inequality above we get

$$q_l(X) \leqslant N - 2 - \sum_{k\in I_g} q_k \leqslant N - 2.$$

One gets in the similar way that $q_r(X) \leqslant N - 2$.

(iv) If $g = \mathrm{id}$, by Proposition 3.6 we see that $q_l([\mathrm{hess}(f)]\xi_{\mathrm{id}}) = q_r([\mathrm{hess}(f)]\xi_{\mathrm{id}}) = q_l([\mathrm{hess}(f)]\omega_{\mathrm{id}}) = q_r([\mathrm{hess}(f)]\omega_{\mathrm{id}}) = N - 2$.

If $g \neq \mathrm{id}$, we have $N_g < N$ and $\deg p \leqslant N_g - 2\sum_{k\in I_g} q_k$. If $q_l(X) = q_r(X) = N - 2$ we have $N = N_g + 2\mathrm{age}(g)$ and

$$\deg p = N_g + \mathrm{age}(g) - 1 - \sum_{k\in I_g} q_k > N_g - 2\sum_{k\in I_g} q_k,$$

what concludes the proof. □

The condition $J \subset G \subset G_f^{\mathrm{SL}}$ can be relaxed in the proposition above. In particular, the statement of this proposition also holds for the subspaces of $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$, spanned by all the g–sectors with integral $\mathrm{age}(g)$.

Proposition 3.24. *For f satisfying CY condition let V be either $\mathcal{A}(f,G)$ or $\mathcal{B}(f,G)$ and $V^{a,b}$ stand for bidegree (a,b)–subspace. Then we have the $\mathbb{C}$–vector spaces isomorphisms*

$$V^{a,b} \cong V^{b,a} \quad \text{and} \quad V^{a,b} \cong V^{N-2-a,N-2-b}.$$

Proof. We construct the two isomorphism explicitly.

Let $\psi : \mathcal{A}_{tot} \to \mathcal{A}_{tot}$ be given by $[p]\xi_g \mapsto [p]\xi_{g^{-1}}$. This is an isomorphism because $f^g = f^h$ and $\mathrm{Jac}(f^g) = \mathrm{Jac}(f^h)$ for $gh = \mathrm{id}$. Under this isomorphism we have directly by the definition that $q_l(\psi(X)) = q_r(X)$ and $q_r(\psi(X)) = q_l(X)$. This gives the first isomorphism.

Let $\phi_g : \mathrm{Jac}(f^g) \to \mathrm{Jac}(f^g)$ be the isomorphism of Proposition 3.6, (ii). Denote by $\phi : \mathcal{A}_{tot} \to \mathcal{A}_{tot}$ the linear map acting by $[p]\xi_g \mapsto \phi_g([p])\xi_{g^{-1}}$. This is an isomorphism again. We have

$$\begin{aligned} q_l\left(\phi_g([p])\xi_{g^{-1}}\right) &= N_g - 2\sum_{k\in I_g} q_k - \deg p - \sum_{k\in I_g^c} q_k + \mathrm{age}(g^{-1}) \\ &= N - 2 + \sum_{k\in I_g^c} q_k - \deg p - \mathrm{age}(g) = N - 2 - q_l([p]\xi_g). \end{aligned}$$

Similarly $q_r\left(\phi_g([p])\xi_{g^{-1}}\right) = N-2-q_r([p]\xi_g)$, what shows that this isomorphism is exactly the one claimed. □

For a fixed pair (f,G) set

$$h^{a,b}_{\mathcal{A}(f,G)} := \dim_{\mathbb{C}}\{X \in \mathcal{A}(f,G) \mid (q_l(X), q_r(X)) = (a,b)\}, \tag{3.24}$$

$$h^{a,b}_{\mathcal{B}(f,G)} := \dim_{\mathbb{C}}\{X \in \mathcal{B}(f,G) \mid (q_l(X), q_r(X)) = (a,b)\}. \tag{3.25}$$

It follows from Proposition 3.23 that for f satisfying CY condition and G s.t. $J \subseteq G \subseteq G_f^{\mathrm{SL}}$, both the numbers $h^{a,b}_{\mathcal{A}(f,G)}$ and $h^{a,b}_{\mathcal{B}(f,G)}$ form a diamond.

$$\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\
 & \iddots & & & \vdots & & & \ddots & \\
h^{n,0} & & \cdots & & \curvearrowleft & & \cdots & & h^{0,n} \quad \updownarrow \text{ Hodge} \\
 & & & & \text{Serre} & & & & \\
 & \ddots & & & \vdots & & & \iddots & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & &
\end{array}$$

where $h^{a,b}$ is either $h^{a,b}_{\mathcal{A}(f,G)}$ or $h^{a,b}_{\mathcal{B}(f,G)}$ and we denote $n := N-2$.

By Proposition 3.24 this diamond is subject to two symmetries that we call Serre and Hodge symmetries. Plotting the numbers as above this is equivalent to say that this diamond is symmetric under the reflections by both the horizontal line going from $h^{n,0}$ to $h^{0,n}$ and by the vertical line going from $h^{0,0}$ to $h^{n,n}$.

Example 3.25. Consider $f = x_1^2x_2 + x_2^2 + x_2x_3^6 + x_4^6 + x_1x_3^9$ and $G = G_f^{\mathrm{SL}}$. Then $G = \langle J \rangle$ with

$$\overline{J} = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{12}, \frac{1}{6}\right).$$

The basis of $\mathcal{A}(f,G)$ is given by the elements

$$\xi_{J^3},\ \xi_{J^5},\ \xi_{J^7},\ \xi_{J^9},\ [x_1]\xi_{J^4},\ [x_1]\xi_{J^8},\ [x_4^2]\xi_{J^6}, \tag{3.26}$$

$$[x_3^4x_4^4]\xi_{\mathrm{id}},\ [x_1x_3x_4^4]\xi_{\mathrm{id}},\ [x_1x_3^3x_4^3]\xi_{\mathrm{id}},\ [x_2x_4^3]\xi_{\mathrm{id}},\ [x_1^2x_4^3]\xi_{\mathrm{id}},\ [x_1x_3^5x_4^2]\xi_{\mathrm{id}},\ [x_2x_3^2x_4^2]\xi_{\mathrm{id}}, \tag{3.27}$$

$$[x_1^2x_3^2x_4^2]\xi_{\mathrm{id}},\ [x_2x_3^4x_4]\xi_{\mathrm{id}},\ [x_1x_2x_3x_4]\xi_{\mathrm{id}},\ [x_1^3x_3x_4]\xi_{\mathrm{id}},\ [x_1^2]\xi_{\mathrm{id}},\ [x_2^2]\xi_{\mathrm{id}}. \tag{3.28}$$

all having the bigrading $(1,1)$, and the elements

$$\xi_J, \quad \xi_{J^{11}}, \quad [1]\xi_{\mathrm{id}}, \quad [x_1x_2^2x_3x_4^4]\xi_{\mathrm{id}},$$

having the bigrading $(0,2)$, $(2,0)$, $(0,0)$ and $(2,2)$ respectively. Note that the bigrading of these elements is fully determined by the propositions above.

One gets the following diamond

$$1$$

$$1 \qquad 20 \qquad 1$$

$$1$$

3.3 Vector spaces mirror symmetry conjecture

Conjecture 3.26 (Vector spaces mirror symmetry conjecture)**.** *For any $f \in \mathbb{C}[x_1, \dots, x_N]$ defining a quasihomogeneous singularity and $G \subseteq G_f^{\mathrm{SL}}$ there is some $\widetilde{f} \in \mathbb{C}[x_1, \dots, x_N]$ defining a quasihomogeneous singularity and some $\widetilde{G} \subseteq G_{\widetilde{f}}$ s.t. there is a vector space isomorphism*

$$\tau : \mathcal{B}(f, G) \to \mathcal{A}(\widetilde{f}, \widetilde{G}),$$

s.t.

$$q_l(\tau(X)) = q_l(X), \quad q_r(\tau(X)) + q_r(X) = \sum_{k=1}^{N} (1 - 2q_k), \quad \forall X \in \mathcal{B}(f, G). \tag{3.29}$$

Such isomorphism τ is called the mirror map.

It's important to stress that the starting data of the conjecture above if the pair (f, G), while the mirror pair $(\widetilde{f}, \widetilde{G})$ is the *right* pair that should be found.

The theorem above appeared first in [Kra09] however without the full proof.

Theorem 3.27. *Vector spaces mirror symmetry conjecture holds for the invertible polynomials f with the pair $(\widetilde{f}, \widetilde{G})$ given by the Berglund–Hübsch–Henningson construction.*

Remark 3.28. The reason for that is for example that we don't know up to now anything about mirror symmetry for the quasihomogeneous singularities that are not defined by an invertible polynomial.

One of the problems on extending mirror symmetry outside of the class of invertible polynomials is that the set of quasihomogeneous singularities is very big and much more complicated than those defined by invertible polynomials (cf. previous sections). At the same time it's not enough to prove the conjecture for say four or five cases. In order to give good evidence that the conjecture holds for a larger class of singularities, one has to show it at least for some infinite class.

Exercise 3.29 (far from being resolved)**.** Solve vector spaces mirror symmetry conjecture for the quasihomogeneous singularities that are not defined by an invertible polynomial.

We prove the theorem in two steps. Step 1: construct the isomorphism $\tau : \mathcal{B}_{tot}(f) \to \mathcal{A}_{tot}(\widetilde{f})$, Step 2: show that it restricts to the G and $\widetilde{G}$ invariants on the B and A sides, respectively. Step 1 is accomplished in Section 3.3.3 and Step 2 in Section 3.3.4.

By Proposition 3.17 it's enough to define the map τ for the polynomials f of Fermat, loop and chain type.

Remark 3.30. The mirror map τ was first proposed by Kreuzer (see Sections 4 and 5 in [Kreu94]) for $G = \{\mathrm{id}\}$ and later used by Krawitz (cf. [Kra09, Section 3.2]) for arbitrary $G \subseteq G_f$. However, Krawitz did not prove the bijectivity of the mirror map and his map τ for chain and loop type polynomials with the even number of variables is not bijective.

3.3.1 Mirror symmetry under the Calabi–Yau condition

Note that the conjectural mirror map τ is not a bidegree preserving isomorphism. However, when CY condition $\sum_k q_k = 1$ holds, relations Eq. (3.29) get the special meaning. In particular, in terms of the Hodge diamonds of $\mathcal{B}(f,G)$ and $\mathcal{A}(\widetilde{f},\widetilde{G})$ (recall Section 3.2.3), Eq. (3.29) is equivalent to

Hodge diamond of $\mathcal{B}(f,G)$ coincides with the Hodge diamond of $\mathcal{A}(\widetilde{f},\widetilde{G})$ reflected by $q_r = \dfrac{N-2}{2}$.

If $N-2=3$

3.3.2 Mirror map for a Fermat type polynomial

Consider $f = x_1^{a_1}$ — the Fermat type polynomial. We have $f = \widetilde{f}$. The map

$$\tau : [x_1^k]\xi_{\mathrm{id}} \mapsto [1]\omega_{\rho^{k+1}}, \quad [1]\xi_{\rho^{k+1}} \mapsto [x_1^k]\omega_{\mathrm{id}}, \qquad 0 \leqslant k \leqslant a_1 - 2 \tag{3.30}$$

sets up the isomorphism $\mathcal{B}_{tot}(f) \to \mathcal{A}_{tot}(\widetilde{f})$. It's easy to check that this isomorphism satisfies Eq (3.29).

Assume $a_1 = rm$ and consider G to be generated by $g = (1/r)$. The dual group $\widetilde{G}$ was computed in Example 2.61 to be generated by $h = (1/m)$. By Example 3.16

$$\mathcal{B}(f,G) = \mathbb{C}\langle [1]\xi_{\mathrm{id}}, [x_1^r]\xi_{\mathrm{id}}, \ldots, [x_1^{r(m-1)}]\xi_{\mathrm{id}}\rangle.$$
$$\mathcal{A}(\widetilde{f},\widetilde{G}) = \mathbb{C}\langle [x_1^{m-1}]\omega_{\mathrm{id}}, \ldots, [x_1^{(m-1)r}]\omega_{\mathrm{id}}, [1]\omega_g, \ldots, [1]\omega_{g^{m-1}}\rangle.$$

The B–model state space is not isomorphic to the A–model states space whatever r and m are. However $G \nsubseteq G_f^{\mathrm{SL}}$.

Assume $G = \{\mathrm{id}\}$. We have $\widetilde{G} = G_f \cong \mathbb{Z}/a_1\mathbb{Z}$. We get

$$\mathcal{B}(f,\{\mathrm{id}\}) = \mathbb{C}\langle [1]\xi_{\mathrm{id}}, [x_1]\xi_{\mathrm{id}}, \ldots, [x_1^{a_1-2}]\xi_{\mathrm{id}}\rangle.$$
$$\mathcal{A}(f,G_f) = \mathbb{C}\langle [1]\omega_\rho, [1]\omega_{\rho^2}, \ldots, [1]\omega_{\rho^{a_1-1}}\rangle$$

and the map τ above sets up the mirror isomorphism $\mathcal{B}(f,\{\mathrm{id}\}) \cong \mathcal{A}(f,G_f)$.

Exercise 3.31. Consider $f = x_1^{a_1} + \ldots, x_N^{a_N}$ with $a_k = r_k m_k$. Establish the mirror isomorphism $\mathcal{B}_{f,\{\mathrm{id}\}} \to \mathcal{A}_{\widetilde{f},G_{\widetilde{f}}}$.

3.3.3 Mirror map for loop and chain type polynomials

Let f be of loop type. We define τ separately on $\mathcal{B}_{tot}^{(0)}$ and $\mathcal{B}_{tot}^{(1)}$ (recall Section 3.2.1). Set $\tau : \mathcal{B}_{tot}^{(0)}(f) \to \mathcal{A}_{tot}^{(0)}(\widetilde{f})$ to be given as follows. For any fixed $g \in G\backslash\{\mathrm{id}\}$ let $s_1, \ldots, s_N$ be the unique numbers such that $g = \prod_{k\in I_g^c} \rho_k^{s_k+1}$ (recall Proposition 2.53). Set

$$\tau : [\prod_{k\in I_g} x_k^{r_k}]\xi_g \mapsto [\prod_{k\in I_g^c} x_k^{s_k}]\omega_h, \tag{3.31}$$

with $h := \prod_{k\in I_g}^{N} \sigma_k^{r_k+1}$.

For $N \in 2\mathbb{Z}$ extend it to $\mathcal{B}^{(1)}_{tot}(f) \to \mathcal{A}^{(1)}_{tot}(\widetilde{f})$ by

$$\tau([\prod_{k\notin 2\mathbb{Z}} x_k^{a_k}]\xi_{\mathrm{id}}) := [\prod_{k\notin 2\mathbb{Z}} x_k^{a_k}]\omega_{\mathrm{id}}, \quad \tau([\prod_{k\notin 2\mathbb{Z}} x_k^{a_k}]\xi_{\mathrm{id}}) := [\prod_{k\notin 2\mathbb{Z}} x_k^{a_k}]\omega_{\mathrm{id}}. \tag{3.32}$$

Let f be of chain type. Introduce $\tau : \mathcal{B}^{(p)}_{tot,g}(f) \to \mathcal{A}^{(p)}_{tot,h}(\widetilde{f})$ (recall Section 3.2.1) as follows. For any fixed $g \in G\backslash\{\mathrm{id}\}$ let $s_1, \dots, s_N$ be the unique numbers s.t. $g = \prod_{k\in I_g^c} \rho_k^{s_k+1}$. Set

$$\tau : [\prod_{i=1}^{p} x_{2i-1}^{a_{2i-1}-1} \prod_{k\in I_g} x_k^{r_k}]\xi_g \mapsto [\prod_{k\in I_g^c} x_k^{s_k} \prod_{i=1}^{p} x_{N-(2i-1)}^{a_{N-(2i-1)}-1}]\omega_h, \tag{3.33}$$

with $h := \prod_{k\in I_g}^{N} \sigma_k^{r_k+1}$.

Extend this definition to any linear combination of $\mathcal{B}_{tot}$–elements v_m as above by

$$\tau\left(\sum_m \alpha_m v_m\right) := \sum_m \alpha_m \tau\left(v_m\right), \tag{3.34}$$

for any $\alpha_m \in \mathbb{C}$.

Proposition 3.32. *Assume f to be of loop or chain type.*

(i) *The map τ above is well–defined,*

(ii) *It established the vector spaces isomorphism $\mathcal{B}_{tot}(f) \cong \mathcal{A}_{tot}(\widetilde{f})$,*

(iii) *This isomorphism preserves the grading q_l.*

Proof. The proof is given separately for loop and chain types. We use Proposition 3.8 and Proposition 2.53 to fix the bases of the vector spaces assumed.

Loop type. Fix $g \in G_f\backslash\{\mathrm{id}\}$. Then $f^g = 0$. By Proposition 2.53, all elements $X = [1]\xi_g \in \mathcal{B}_{tot}$ are indexed by the vectors s

$$s = (s_1, \dots, s_N), \quad \text{s.t.} \quad 1 \leqslant s_k \leqslant a_k - 1, \quad g = \prod_{k=1}^{N} \rho_k^{s_k}. \tag{3.35}$$

By Proposition 3.8 all elements $Y = [\phi]\xi_{\mathrm{id}} \in B_{tot}$ are indexed by

$$r = (r_1, \dots, r_N), \quad \text{s.t.} \quad 0 \leqslant r_k \leqslant a_k - 2, \quad \phi = \prod_{k=1}^{N} x_k^{r_k}. \tag{3.36}$$

Combine the two vectors s, r assuming $s = (0, \dots, 0)$ if $g = \mathrm{id}$ and $r = (0, \dots, 0)$ if $g \neq \mathrm{id}$. We conclude that $\mathcal{B}_{tot}$ has the basis given by the elements indexed by a pair of vectors (r, s).

The same indexing statement holds for $\mathcal{A}_{tot}$ in a straightforward way.

Set

$$\tau(s) := (s_1 - 1, \dots, s_N - 1), \quad \text{and} \quad \tau(r) := (r_1 + 1, \dots, r_N + 1). \tag{3.37}$$

Then τ maps a $\mathcal{B}_{tot}$ basis element (r,s) to the $\mathcal{A}_{tot}$ basis element $(\tau(s),\tau(r))$. This map is a bijection by Proposition 2.53 and Proposition 3.8 if $N \notin 2\mathbb{Z}$.

If $N \in 2\mathbb{Z}$ consider

$$Y_1 := \prod_{k\notin 2\mathbb{Z}} x_k^{a_k-1}\xi_{\mathrm{id}}, \quad Y_2 := \prod_{k\in 2\mathbb{Z}} x_k^{a_k-1}\xi_{\mathrm{id}}, \tag{3.38}$$

$$Y_1' := \prod_{k\notin 2\mathbb{Z}} x_k^{a_k-1}\omega_{\mathrm{id}}, \quad Y_2' := \prod_{k\in 2\mathbb{Z}} x_k^{a_k-1}\omega_{\mathrm{id}}. \tag{3.39}$$

Due to the relations $\prod_{k\notin 2\mathbb{Z}} \sigma_k^{a_k}\sigma_{k+1} = \mathrm{id}$ and $\prod_{k\in 2\mathbb{Z}} \sigma_k^{a_k}\sigma_{k-1} = \mathrm{id}$ the map τ acting as above is not well–defined for Y_1 and Y_2. We have

$$\tau(Y_1) = Y_1', \quad \tau(Y_2) = Y_2'. \tag{3.40}$$

Bigrading. Compute the left and right charges for X and $\tau(X)$ as above. Denote by $q_1,\dots,q_N$ the reduced weight set of f and by $\widetilde{q}_1,\dots,\widetilde{q}_N$ the reduced weight set of $\widetilde{f}$. We have

$$q_l(X) = \mathrm{age}(g) - \sum_{k=1}^N q_k = (1,\dots,1)E_f^{-1}\begin{pmatrix} r_1 \\ \vdots \\ r_N \end{pmatrix} - (1,\dots,1)E_f^{-1}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \tag{3.41}$$

$$= (1,\dots,1)E_f^{-1}\begin{pmatrix} r_1-1 \\ \vdots \\ r_N-1 \end{pmatrix} = (r_1-1,\dots,r_N-1)\left(E_f^{-1}\right)^T\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \tag{3.42}$$

$$= \sum_{k=1}^N \widetilde{q}_k(r_k-1) = q_l(\tau(X)). \tag{3.43}$$

Similarly

$$q_r(X) = N - \mathrm{age}(g) - \sum_{k=1}^N q_k = N - 2\sum_{k=1}^N q_k - \left(\mathrm{age}(g) - \sum_{k=1}^N q_k\right) = \sum_{k=1}^N (1-2q_k) - q_l(X). \tag{3.44}$$

$$q_r(\tau(X)) = q_l(\tau(X)), \tag{3.45}$$

and we have $q_r(X) + q_r(\tau(X)) = \sum_{k=1}^N (1-2q_k)$ by the computations of q_l above.

$$q_l(Y_1) = q_r(Y_1) = \sum_{k=1}^{N/2}(a_{2k-1}-1)q_{2k-1} = \sum_{k=1}^{N/2}(a_{2k-1}q_{2k-1}+q_{2k}) - \sum_{k=1}^N q_k \tag{3.46}$$

$$= \frac{N}{2} - \sum_{k=1}^N q_k = \frac{N}{2} - \sum_{k=1}^N \widetilde{q}_k = q_l(Y_1') = q_r(Y_1'). \tag{3.47}$$

Chain type. Fix $g \in G_f\backslash\{\mathrm{id}\}$. Then f^g is zero or of a chain type again. By Proposition 2.53 applied to

f and Proposition 3.8 applied to f^g, an element $X = [\phi]\xi_g \in \mathcal{B}^{(p)}_{tot,g}$ can be indexed by a pair of vectors s and r

$$s = (s_1, \dots, s_b, 0, \dots, 0), \quad r = (0, \dots, 0, a_{b+1} - 1, 0, \dots, a_{b+2p-1} - 1, 0, r_{b+2p+1}, \dots, r_N), \tag{3.48}$$

$$\text{s.t.} \quad g = \prod_{k=1}^{b} \rho_k^{s_k}, \quad \phi = \prod_{k=1}^{p} x_{b+2k-1}^{a_{b+2k-1}-1} \prod_{k=b+2p+1}^{N} x_k^{r_k}, \tag{3.49}$$

where s is subject to the group chain condition in G_f (see Eq.(2.20)) and r subject to Jac chain condition in $\mathrm{Jac}(f^g)$ (see Eq. (JCp)). The number b is s.t. $I_g = \{b, \dots, N\}$. We also have $N - b \geqslant 2p$.

For a untwisted sector element $X = [\phi]\xi_{\mathrm{id}}$ we have $b = 0$ and $s = (0, \dots, 0)$.

We have

$$\tau(X) = \left[\prod_{k=1}^{b} x_k^{s_k - 1} \cdot \prod_{k=1}^{p} x_{b+2k}^{a_{b+2k}-1}\right] \omega_h, \quad h = \prod_{k=b+1}^{N} \sigma_k^{r_k+1} = \prod_{k=b+2p+1}^{N} \sigma_k^{r_k+1} \tag{3.50}$$

An element $Y = [\psi]\omega_h \in \mathcal{A}^{(p)}_{tot,h}$ can be indexed by (recall Remark 3.10)

$$\widetilde{s} = (0, \dots, \widetilde{s}_{i+1}, \dots, \widetilde{s}_N), \quad \widetilde{r} = (\widetilde{r}_1, \dots, \widetilde{r}_{i-2p-2}, 0, a_{i-2p} - 1, \dots, 0, a_i - 1, 0, \dots, 0), \tag{3.51}$$

$$\text{s.t.} \quad h = \prod_{k=i+1}^{N} \sigma_k^{\widetilde{s}_k}, \quad \phi = \prod_{k=1}^{i} x_k^{\widetilde{r}_k}, \tag{3.52}$$

and $i \geqslant 2p$. The relations between i and p, b and p guarantee that the equation $i + 1 = b + 2p + 1$ has a solution for given b and p. According to Propositions 2.53 and 3.8 the ranges of $s_\bullet$ and $\widetilde{r}_\bullet$ are the same as well as the ranges of $r_\bullet$ and $\widetilde{s}_\bullet$. This gives the bijectivity of τ for chain type polynomials.

Bigrading. Compute the left and right charges for X and $\tau(X)$ as above. We have

$$q_l(X) = \sum_{k=1}^{p} (a_{b+2k-1} - 1) q_{b+2k-1} + \sum_{k=b+2p+1}^{N} r_k q_k + \mathrm{age}(g) - \sum_{k=1}^{b} q_k \tag{3.53}$$

$$= p + \sum_{k=b+2p+1}^{N} (r_k + 1) q_k + \mathrm{age}(g) - \sum_{k=1}^{N} q_k \tag{3.54}$$

The same computation applied to $\tau(X)$ gives

$$q_l(\tau(X)) = p + \sum_{k=1}^{b} s_k \widetilde{q}_k + \mathrm{age}(h) - \sum_{k=1}^{N} \widetilde{q}_k. \tag{3.55}$$

Similarly to the loop type case above we have

$$\mathrm{age}(g) = \sum_{k=1}^{b} s_k \widetilde{q}_k, \quad \sum_{k=b+2p+1}^{N} (r_k + 1) q_k = \mathrm{age}(h), \quad \sum_{k=1}^{N} q_k = \sum_{k=1}^{N} \widetilde{q}_k,$$

what gives $q_l(X) = q_l(\tau(X))$. □

We have seen in the proof of the proposition above that for a loop type polynomial with the even number of variables, the spaces $\mathcal{A}^{(1)}_{tot}(f)$ and $\mathcal{B}^{(1)}_{tot}(f)$ are G–invariant for any group assumed in mirror symmetry conjecture.

Example 3.33. Let $f = x_1^{a_1}x_2 + x_2^{a_2}$ be a 2–variable chain. We have $\widetilde{f} = x_1^{a_1} + x_1x_2^{a_2}$ and

$$\mathcal{B}_{tot}(f) = \bigoplus_{\substack{0\leqslant k_1\leqslant a_1-2\\ 0\leqslant k_2\leqslant a_2-2}} \mathbb{C}\langle[x_1^{k_1}x_2^{k_2}]\rangle \bigoplus_{0\leqslant k_1\leqslant a_1-2} \mathbb{C}\langle[x_1^{k_1}x_2^{a_2-1}]\rangle \oplus \mathbb{C}\langle[x_1^{a_1-1}]\rangle \tag{3.56}$$

$$\bigoplus_{\substack{1\leqslant l_1\leqslant a_1-1\\ 0\leqslant p_2\leqslant a_2-2}} \mathbb{C}\langle[x_2^{p_2}]\rho_1^{l_1}\rangle \bigoplus_{\substack{1\leqslant b_1\leqslant a_1-1\\ 1\leqslant b_2\leqslant a_2-1}} \mathbb{C}\langle\rho_1^{b_1}\rho_2^{b_2}\rangle \bigoplus_{1\leqslant l_2\leqslant a_2-1} \mathbb{C}\langle\rho_1^{a_1}\rho_2^{l_2}\rangle, \tag{3.57}$$

$$\mathcal{A}_{tot}(\widetilde{f}) = \bigoplus_{\substack{1\leqslant b_1\leqslant a_1-1\\ 1\leqslant b_2\leqslant a_2-1}} \mathbb{C}\langle\sigma_1^{b_1}\sigma_2^{b_2}\rangle \bigoplus_{1\leqslant l_1\leqslant a_1-1} \mathbb{C}\langle\sigma_1^{l_1}\sigma_2^{a_2}\rangle \bigoplus_{\substack{1\leqslant l_2\leqslant a_2-1\\ 0\leqslant p_1\leqslant a_1-2}} \mathbb{C}\langle[x_1^{p_1}]\sigma_2^{l_2}\rangle \tag{3.58}$$

$$\bigoplus_{\substack{0\leqslant k_1\leqslant a_1-2\\ 0\leqslant k_2\leqslant a_2-2}} \mathbb{C}\langle[x_1^{k_1}x_2^{k_2}]\rangle \bigoplus_{0\leqslant k_2\leqslant a_2-2} \mathbb{C}\langle[x_1^{a_1-1}x_2^{k_2}]\rangle \bigoplus \mathbb{C}\langle[x_2^{a_2-1}]\rangle. \tag{3.59}$$

The map τ gives

$$\tau([x_1^{k_1}x_2^{k_2}]) = [\sigma_1^{k_1+1}\sigma_2^{k_2+1}]. \quad \tau([x_1^{k_1}x_2^{a_2-1}]) = [\sigma_1^{k_1+1}\sigma_2^{a_2}], \quad \tau([x_1^{a_1-1}]) = [x_2^{a_2-1}]. \tag{3.60}$$

$$\tau([x_2^{p_2}]\rho_1^{l_1}) = [x_1^{l_1-1}\sigma_2^{p_2+1}], \quad \tau(\rho_1^{b_1}\rho_2^{b_2}) = [x_1^{b_1-1}x_2^{b_2-1}]\omega_{\mathrm{id}}, \quad \tau([\rho_1^{a_1}\rho_2^{l_2}) = [x_1^{a_1-1}x_2^{l_2-1}]\omega_{\mathrm{id}}. \tag{3.61}$$

Exercise 3.34 (unresolved)**.** For $N \in 2\mathbb{Z}$ and f of loop type classify all symmetry groups $G \subset G_f$, s.t. the monomial $\prod_{k\in 2\mathbb{Z}} x_k^{a_k-1}$ is G–invariant.

3.3.4 Proof of Theorem 3.27

In order to prove Theorem 3.27 we should investigate the group action of both A– and B–side. In particular, it is enough to show that

(A) if $[\prod_{k\in I_g} x_k^{r_k}]\xi_g$ is G–invariant, then $h := \prod_{k\in I_g}^N \sigma^{r_k+1} \in \widetilde{G}$,

(B) $X \in \mathcal{B}_{tot}$ is G–invariant if and only if $\tau(X)$ is $\widetilde{G}$–invariant.

On the B–side assume for some $g = \prod_{k\in I_g^c} \rho_k^{s_k+1} \in G$ the $\mathcal{B}_{tot}$–element $[\prod_{k\in I_g} x_k^{r_k}]\xi_g$ to be G–invariant. Then for any $o = (o_1, \dots, o_N) \in G$ we have

$$\frac{\prod_{k\in I_g} o_k^{r_k}}{\prod_{k\in I_g^c} o_k} = \frac{\prod_{k\in I_g} o_k^{r_k+1}}{\det(o)} = \prod_{k\in I_g} o_k^{r_k+1} \in \mathbb{Z},$$

where we used that $G \subseteq G_f^{\mathrm{SL}}$. Then $[\prod_{k\in I_g} x_k^{r_k}]\xi_g$ is invariant w.r.t. any $o \in G$ if and only if the monomial $\prod_{k\in I_g} x_k^{r_k+1}$ is invariant w.r.t. the action of any $o \in G$. Consider the following lemma.

Lemma 3.35. *Let $p'_1, \dots, p'_N$ and $p''_1, \dots, p''_N$ be some nonnegative integers. Then $\prod_{k=1}^N \rho_k^{p''_k}$ preserves the monomial $\prod_{k=1}^N x_k^{p'_k}$ if and only if $\prod_{k=1}^N \sigma_k^{p'_k}$ preserves the monomial $\prod_{k=1}^N x_k^{p''_k}$.*

Proof. Both statements are equivalent to

$$(p'_1, \dots, p'_N)E_f^{-1}\begin{pmatrix} p''_1 \\ \vdots \\ p''_N \end{pmatrix} \in \mathbb{Z}. \tag{3.62}$$

□

The monomial $[\prod_{k\in I_g} x_k^{r_k}]\xi_g$ is invariant w.r.t. any $o \in G$ if and only if

$$(r_1+1,\dots,r_N+1)E_f^{-1}\begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} \in \mathbb{Z}, \quad \forall b_1,\dots,b_N, \text{ s.t. } \prod_{k=1}^{N} \rho_k^{b_k} \in G. \tag{3.63}$$

The latter condition holds if and only if $\prod_{k\in I_g} \sigma_k^{r_k+1} \in \widetilde{G}$. This completes (A).

To show (B) assume $f = f_1 \oplus \cdots \oplus f_m$ with each f_i being of Fermat, loop or chain type. Let $X \in \mathcal{B}_{tot}(f)$ be given by $X = X_1 \otimes \cdots \otimes X_m$, $X_i \in \mathcal{B}_{tot}(f_i)$. Then $\tau(X) = \tau(X_1) \otimes \cdots \otimes \tau(X_m)$. For any $w \in \widetilde{G}$ we have $w = w_1 \cdots w_m$ with $w_i \in \widetilde{G}_{w_i}$ and

$$w^*(\tau(X)) = \lambda_1 \cdots \lambda_m \tau(X), \qquad \text{where} \qquad w_k^*(\tau(X_k)) = \lambda_k \tau(X_k).$$

Assume first that for all of X_i the map τ is given by Eq.(3.31). Namely, either f_i is of Fermat type or $X_i \in B_{tot}^{(0)}$ for chain or loop type f_i. Then for $X = [\prod_{k\in I_g} x_k^{r_k}]\xi_g$ with $g = \prod_{k\in I_g^c} \rho_k^{s_k}$ we have $h = \prod_{k\in I_g} \sigma_k^{r_k+1}$, $I_h = I_g^c$ and

$$\tau(X) = \left[\prod_{k\in I_h} x_k^{s_k-1}\right]\omega_h.$$

Repeating the arguments above, $\tau(X)$ is invariant w.r.t. $w \in \widetilde{G}$ if and only if $g \in \widetilde{\widetilde{G}} = G$. In particular, we get that the number λ_i above is obtained by

$$w_i(\prod_{b\in I_{h_i}} x_b^{s_b}) = \lambda_k \prod_{b\in I_{h_i}} x_b^{s_b}.$$

Consider now the special cases: $X_i \in \mathcal{B}_{tot}^{(1)}$ for a loop type polynomial or $X_i \in \mathcal{B}_{tot}^{(p\geqslant 1)}$ for a chain type polynomial. In the first case it's easy to see that $w_i^*(\tau(X_i)) = X_i$. In the second case assume X_i to be given as in Eq. (3.50). Then the number λ_i satisfies

$$w_i(\prod_{k=1}^{b} x_k^{s_k}) = \lambda_i \prod_{k=1}^{b} x_k^{s_k}.$$

This reduces the treatment of the special cases to that of the general cases above.

Remark 3.36. The first big achievement in vector spaces mirror symmetry was given in [Kra09] without any doubts. However the loc.cit. has several huge mistakes. In particular, the mirror map itself was not totally fixed — it was not clarified how to fix the numbers r_k that depend heavily on the choice of $\mathcal{R}$; the analogue of Proposition 3.32 (being called "unprojected mirror theorem") was not proved and even appeared to be wrong — this can be checked even by a simple dimension count of the identity sector and group order of the twisted sector; the final step concerning the group action (Section 3.3.4 above) was just claimed to be obvious.

Important result of [Kra09] is the combinatorial presentation of $\widetilde{G}$. Namely, the group G^T. This was important for the definition of the mirror map. However one can not call the dual group $\widetilde{G}$ to be an invention of Krawitz because it's been known since Henningson.

We are sure that the author could have done all these steps carefully, however we insist that [Kra09] could only be considered as a starting point for the serious investigation.

Because of all this we also don't see any reason to say "Berglund-Hübsch-Krawitz mirror symmetry".

3.4 Learn more

3.4.1 Pairing

The algebra $\mathrm{Jac}(f)$ can be endowed with the $\mathbb{C}$–bilinear pairing η called *residue pairing*. There are several ways to introduce it (see [GH94, Chapter 5], [AGV85, Section 5.11]). The easiest one makes use of the *Hessian* of f defined as the following determinant:

$$\mathrm{hess}(f) := \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,N}. \tag{3.64}$$

Its class is non-zero in $\mathrm{Jac}(f)$ and therefore one can take $[\mathrm{hess}(f)]$ as one of the $\mathbb{C}$–basis elements of the Jacobian algebra. The value $\eta([u],[v])$ is than taken as the $[\mathrm{hess}(f)]$ coefficient of $[u][v]$ written in this basis. This pairing makes $\mathrm{Jac}(f)$ a Frobenius algebra: $\eta([u][v],[w]) = \eta([u],[v][w])$ for any $[u],[v],[w] \in \mathrm{Jac}(f)$.

Similarly, one can introduce the pairing on $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ that descends to $\mathcal{A}(f,G)$ and $\mathcal{B}(f,G)$ respectively. To do this note that $\mathrm{Fix}(g) = \mathrm{Fix}(g^{-1})$ for any $g \in G$. Therefore the g-th sectors of $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ are isomorphic to the g^{-1}-th sectors of $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ respectively and one can make use of the residue pairings of f^g. Recall that for $f^g = 0$ we had $\mathrm{Jac}(f^g) = \mathbb{C}$. It has the unique $\mathbb{C}$–bilinear pairing that we denote by η too.

Define for $[u]\xi_g, [v]\xi_h \in \mathcal{A}_{tot}$

$$\eta_{f,G}([u]\xi_g,[v]\xi_h) := \begin{cases} 0, & \text{if } gh \neq \mathrm{id}, \\ \eta([u],[v]), & \text{if } gh = \mathrm{id}, \end{cases}$$

where η on the second line is computed in $\mathrm{Jac}(f^g) \cong \mathrm{Jac}(f^h)$. This bilinear form is immediately non–degenerate. The same definition also works for $\mathcal{B}_{tot}$.

One can show that the mirror map $\mathcal{A}(f,G) \xrightarrow{\tau} \mathcal{B}(\widetilde{f},\widetilde{G})$ preserves the pairing on both sides in the sense that $\eta_{\widetilde{f},\widetilde{G}}(\tau(X),\tau(Y)) = \eta_{f,G}(X,Y)$ for all $X,Y \in \mathcal{A}(f,G)$.

3.4.2 Milnor lattice

to be filled in

3.4.3 Calabi–Yau A–model

Given $f \in \mathbb{C}[x_1,\dots,x_N]$, defining a quasihomogeneous singularity with the reduced weights $q_k = d_k/d_0$, s.t. $\gcd(d_0,d_1,\dots,d_N) = 1$, and $G \subseteq G_f$ consider the variety

$$X_{f,G} := \{(x_1,\dots,x_N) \in \mathbb{P}(d_1,\dots,d_N) \mid f(x_1,\dots,x_N) = 0\}/(G/J).$$

The quasihomogeneity condition on f implies that the zero–set $f = 0$ is well–defined in $\mathbb{P}(d_1, \dots, d_N)$. If f satisfies the Calabi–Yau condition $\sum_k q_k = 1$, then $X_{f,G}$ is Calabi–Yau in the sense that its canonical class vanishes. If G/J is non–trivial, $X_{f,G}$ is an orbifold.

The orbifold cohomology or Chen–Ruan cohomology $H^*_{orb}(X_{f,G})$ provide *another* A–model of (f, G).

3.4.4 Hodge diamond

The numbers $h^{a,b}$ (both $h^{a,b}_{\mathcal{A}}$ and $h^{a,b}_{\mathcal{B}}$) may be organized into the so-called E–function

$$E(f,G)(t,\bar{t}) := \sum_{p,q\in\mathbb{Q}} (-1)^{p+q-N} h^{p,q} \cdot t^{p-\frac{N}{2}} \bar{t}^{q-\frac{N}{2}},$$

that is a function of two independent formal variables t and $\bar{t}$. Serre and Hodge symmetries are equivalent to the following equalities

$$E(f,G)(t,\bar{t}) = E(f,G)(\bar{t},t), \quad E(f,G)(t,\bar{t}) = E(f,G)(t^{-1},\bar{t}^{-1}).$$

These equalities together with the explicit formula for $E(f,G)$ were proved in [**?**] by Ebeling and Takahashi. They have also proved that the mean of $h^{a,b}$ is $N/2$. Namely

$$\sum_{p,q\in\mathbb{Q}} (-1)^{p+q-N}(q - \frac{N}{2}) h^{p,q} = 0$$

see Corollary 9 of loc.cit..

More on the Hodge diamond of a Landau–Ginzburg orbifold (for not necessarily invertible singularity) can be found in [BI24].

3.4.5 On the bigrading

The bigrading on $\mathcal{A}_{tot}$ and $\mathcal{B}_{tot}$ was first introduced by Intriligator and Vafa in [IV90, Eq.3.2]. However, they reference [V89] at this point. The grading operators q_l and q_r are equal to the grading operators of Intriligator–Vafa shifted by $\frac{1}{2}\sum_{k=1}^{N}(1 - q_k)$.

The bigrading of Intriligator–Vafa was later used by Krawitz in [Kra09]. However he has introduced 8 more grading operators on p.13 of loc.cit., so that it was not really clear which exactly should be used. We also think, some of these operators were not introduced at all (like deg^A_+ and deg^A_-) and the other defined in the specific situation only (deg^B_- in Eq.16 of loc.cit.). The grading operators that are indeed used could be read off in the proofs given in Section 3.3. It took us some time to check that the grading preserving statement of Krawitz (Section 2.4 of loc.cit.) is equivalent to ours (see Eq. (3.29)).

Chapter 4

Nonabelian symmetry groups

Let $f = f_1 \oplus \cdots \oplus f_p$ be an in invertible polynomial with each f_k of Fermat, loop or chain type. Together with the symmetry groups $G \subseteq G_f$ acting diagonally on the variables of f, one may assume the symmetry groups that may interchanges different variables. The main examples are

$$f = x_1^4 + x_2^4 + x_2, \quad \sigma : (x_1, x_2, x_3) \mapsto (x_2, x_1, x_3) \tag{4.1}$$

when the polynomials in the $\oplus$–decomposition get interchanged or

$$f = x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1, \quad \sigma : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1) \tag{4.2}$$

with the group acting on the variables of just one loop type polynomial. Both directions are not yet settled even on the level of the vector space isomorphisms.

Speaking about the nonabelian symmetry groups one assumes the action of $G \subseteq S \ltimes G_f$, where $S \subseteq S_N$ and $f(u \cdot \mathbf{x}) = f(\mathbf{x})$ for any $u \in G$. Let's expand these words.

Any $u \in S \ltimes G_f$ can be written as a pair $u = \sigma \cdot g$ for some $\sigma \in S$ and $g \in G_f$. Every such u acts on $\mathbb{C}^N$ by

$$u \cdot \mathbf{x} = \sigma(g \cdot \mathbf{x}).$$

Namely, it first rescales $\mathbf{x}$ via the action of g and then permutes the coordinates of the rescaled vector obtained. Note that for $u_1 = \sigma_1 \cdot g_1 \in S \ltimes G_f$ and $u_2 = \sigma_2 \cdot g_2 \in S \ltimes G_f$ we have

$$u_1 u_2 = \sigma_1 \sigma_2 \cdot \left(\sigma_2^{-1}(g_1) g_2\right),$$

where $\sigma_2^{-1}(g_1) g_2 \in G_f$ is obtained as a product of two G_f elements, the first $\sigma_2^{-1}(g_1)$ being obtained from g_1 via the action of the permutation σ_2^{-1} permuting the components of g_1.

Mirror pairs

Mirror symmetry for nonabelian groups is still an open topic and there are just five papers from two groups of people on this subject ([EGZ18, EGZ20, BI21, BI22], an example is computed in [PWW20]). Currently, the candidates for the mirror pairs were proposed by Takahashi to be

$$(f, S \ltimes G) \text{ and } (\widetilde{f}, S \ltimes \widetilde{G}),$$

where $\widetilde{f}$ and $\widetilde{G}$ are the BHH duals for $G \subseteq G_f$.

It's not clear how far mirror symmetry should go. One should not also consider to be a good practice a big goal to find a mirror symmetry even though there is none.

In particular, the leading researchers in this field Ebeling, Gusein–Zade and Takahashi, conjecture that additional condition should hold in order for mirror symmetry to take place. They call it PC for "parity condition".

$$\text{for any } T \subseteq S \text{ holds } \dim \left(\mathbb{C}^N\right)^T \equiv N \mod 2. \tag{PC}$$

In particular, S can only satisfy PC if $S \subseteq A_N$. In [BI22] the authors gave more geometric reformulation of PC.

Remark 4.1. In [BI21] the authors have found several examples of the mirror pairs, for which the vector spaces mirror symmetry conjecture holds, but PC is not satisfied. The mirror maps in that cases became in some sense "wild". This is still an open question if such mirror pairs are the fluctuations of a small dimension or not.

Remark 4.2. Note that one should still take $S = \{\mathrm{id}\} \subset S_N$ and get a diagonal symmetry group as discussed in the previous chapters. Therefore working with the nonabelian symmetry groups one should provide the mirror maps that are compatible with all the results obtained before.

The best result up to now concerning vector space mirror symmetry is Theorem 4.15 below. And it only assumes the symmetries of $f = x_1^n + \cdots + x_N^n$ for $n = N$ and prime N with $G = \mathrm{SL}_f$. It will be shown that no mirror symmetry should be expected for $n \neq N$.

Exercise 4.3 (unresolved)**.** Set up the vector spaces mirror symmetry for $f = x_1^N + \cdots + x_N^N$ for non-prime N or G different from SL_f or $\mathcal{J}$.

From now on to the end of the chapter assume

$$f = x_1^n + \cdots + x_N^n,$$

with $n, N \in \mathbb{Z}_{\geqslant 2}$. In this chapter we also call the $\oplus$ sum of several Fermat type polynomials to be of Fermat type too

In order to work with the nonabelian symmetry groups one has to supply some reformulation of the notation and objects of diagonal symmetry groups.

4.1 Preliminaries and notation

For any $u \in S_N \ltimes G_f$ let $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ be the eigenvalues of the linear transformation $\mathbf{x} \mapsto u\mathbf{x}$. We may assume $\lambda_k = \exp(2\pi\sqrt{-1}a_k)$ for some $a_k \in \mathbb{Q} \cap [0, 1)$. Denote:

$$\mathrm{age}(u) := \sum_{k=1}^{N} a_k.$$

Then for the inverse element u^{-1} we have

$$\operatorname{age}(u) + \operatorname{age}(u^{-1}) = N - N_u = d_u.$$

Assuming u as an element of $\mathrm{GL}_N(\mathbb{C})$ set $\mathrm{Fix}(u)$ to be the eigenvalue 1 subspace of $\mathbb{C}^N$ and I_u^c be the set of all indices k, s.t. $u \cdot x_k \neq x_k$. Decompose further

$$\mathrm{Fix}(u) = \mathrm{Fix}^T(u) \oplus \mathrm{Fix}^E(u)$$

into a *tautological fixed locus* $\mathrm{Fix}^T(u) := \{x \in \mathbb{C}^N \mid x_j = 0, j \in I_u^c\}$ and a *eigen fixed locus* $\mathrm{Fix}^E(u)$.

Example 4.4. Consider a special cycle $\sigma g \in S_N \ltimes G_f$ with $\sigma = (i_1, \dots, i_k)$. Let g act trivially on x_j with $j \notin \{i_1, \dots, i_k\}$.

For $\zeta_k = \exp(2\pi\sqrt{-1}/k)$ consider the change of the variables

$$\widetilde{x}_{i_b} := \frac{1}{k} \sum_{a=1}^{k} \zeta_k^{(1-b)(a-1)} \widetilde{g}_{i_a} x_{i_a}, \quad 1 \leqslant b \leqslant k,$$

with $\widetilde{g}_{i_1} := 1$ and $\widetilde{g}_{i_a} := g_{i_1} \cdot \dots \cdot g_{i_{a-1}}$ for $1 \leqslant a \leqslant k$. Then we have

$$\sigma g(\widetilde{x}_{i_b}) = \zeta_k^{b-1} \widetilde{x}_{i_b}.$$

In particular, the variable $\widetilde{x}_{i_1}$ is preserved by the action of σg. We have:

$$x_{i_a} = \frac{1}{\widetilde{g}_{i_a}} \sum_{b=1}^{k} \zeta_k^{(b-1)(a-1)} \widetilde{x}_{i_b}, \quad 1 \leqslant a \leqslant k,$$

and also

$$f^{\sigma g} = k \cdot \widetilde{x}_{i_1}^n + \sum_{\substack{l \notin \{i_1, \dots, i_k\} \\ 1 \leqslant l \leqslant N}} x_l^n.$$

We have

$$\mathrm{Fix}^T((\sigma, g)) = \{x \in \mathbb{C}^N \mid x_{i_1} = \dots = x_{i_k} = 0\}, \tag{4.3}$$

$$\mathrm{Fix}^E((\sigma, g)) = \{x \in \mathbb{C}^N \mid x_j = 0,\ j \notin \{i_1, \dots, i_k\} \text{ and } x_{i_a} = t \cdot \widetilde{g}_{i_a}^{-1},\ t \in \mathbb{C}\}. \tag{4.4}$$

For every x_{i_a}, considered as a function of $\widetilde{x}_\bullet$ we have

$$x_{i_1}|_{\mathrm{Fix}(\sigma g)} = \widetilde{g}_{i_2} x_{i_2}|_{\mathrm{Fix}(\sigma g)} = \dots = \widetilde{g}_{i_k} x_{i_k}|_{\mathrm{Fix}(\sigma g)} = \widetilde{x}_{i_1}|_{\mathrm{Fix}(\sigma g)}, \tag{4.5}$$

$$\widetilde{x}_{i_b}|_{\mathrm{Fix}(\sigma g)} = 0 \text{ for } b > 1. \tag{4.6}$$

From this we conclude $\lfloor \left(\widetilde{g}_{i_p} x_{i_p}\right)^q \rfloor = \lfloor \widetilde{x}_{i_1}^q \rfloor$ in $\mathrm{Jac}(f^{\sigma g})$.

In what follows denote

$$\widetilde{x}_{\sigma g} := \widetilde{x}_{i_1} = \frac{1}{k} \sum_{a=1}^{k} \widetilde{g}_{i_a} x_{i_a}.$$

Proposition 4.5. *Restriction of f to $\mathrm{Fix}(u)$, $f^u := f\,|_{\mathrm{Fix}(u)}$ is a Fermat type polynomial again or a constant zero–function.*

Proof. We have $f^u = \left(f\mid_{\mathrm{Fix}^E(u)}\right)\mid_{\mathrm{Fix}^T(u)}$. The polynomial $f\mid_{\mathrm{Fix}^E(u)}$ is of Fermat type by the example above and restricting to Fix^E one only sets some variables to be equal to zero, what does not spoil the property of being of Fermat type. □

Let $\sigma = \prod_{a=1}^p \sigma_a$ be the decomposition into the non–intersecting cycles. Denote by $|\sigma_a|$ the length of the cycle σ_a. We will also allow σ_a to be of length 1, so that we always have $\sum_{a=1}^p |\sigma_a| = N$. There exists the unique set $g_1, \dots, g_p$ of G_f–elements, s.t. g_a acts non–trivially only on $I^c_{\sigma_a}$ and $\sigma \cdot g = \prod_{a=1}^p \sigma_a g_a$. We call the product $\sigma \cdot g = \prod_{a=1}^p \sigma_a g_a$ *generalized cycle decomposition* of u.

Said differently, the $\mathrm{GL}_N(\mathbb{C})$–element u can be represented by a block–diagonal matrix with the blocks given by the $\mathrm{GL}_{|\sigma_a|}(\mathbb{C})$–elements $\sigma_a g_a$.

A generalized cycle $\sigma_a g_a$ is said to be *special* if $\det(g_a) = 1$ and *non–special* otherwise. It is clear that $\mathrm{Fix}(\sigma_a g_a) \cap \mathbb{C}^{I^c_{\sigma_a g_a}} = 0$ for a non–special cycle, where by $\mathbb{C}^{I^c_u}$ we mean the subspace of $\mathbb{C}^N$ spanned by standard basis vectors with indices in I^c_u. For a special cycle we have $\dim \mathrm{Jac}(f^{\sigma_a g_a}\mid_{\mathbb{C}^{I^c_{\sigma_a g_a}}}) = n-1$.

Lemma 4.6. *We have*

(1) *If $u \in G_f$ is a non–special cycle or $u \in G^d$. Then* $\mathrm{Fix}^E(u) = 0$.

(2) *If $u \in G_f$ satisfies $u = \prod_{i=1}^k (\sigma_i, g_i)$ for some non-intersecting cycles (σ_i, g_i). Then*

$$\mathrm{Fix}^T(u) = \cap_{i=1}^k \mathrm{Fix}^T((\sigma_i, g_i)) \quad \textit{and} \quad \mathrm{Fix}^E(u) = \cup_{i=1}^k \mathrm{Fix}^E((\sigma_i, g_i)).$$

Proof. 1) Let σ be of length k. Assumed as a GL_N–element, u is given by a matrix that is the product of the permutation matrix (fixed by σ) and the diagonal matrix, fixed by g. The characteristic polynomial of u equals $\lambda^k - \det(g)$. It follows that σg has the eigenvalue 1 if and only if g is special.

2) Decomposition into the non-intersecting cycles provides the decomposition of $\mathbb{C}^N$ together with the operator σg into the direct sums with the summand $\{x \in \mathbb{C}^N \mid x_j = 0, j \in I^c_{\sigma g}\}$ and the summands on which one of the cycles of the decomposition act nontrivially. □

Denote by $\lfloor \phi(\mathbf{x}) \rfloor$ the class of the polynomial $\phi(\mathbf{x})$ in $\mathrm{Jac}(f^{\sigma_a g_a})$. Let $\widetilde{x}_{i_a}$ be the $\sigma_a g_a$-invariant linear combination of $x_\bullet$ with indexes in $I^c_{\sigma_a g_a}$, s.t. $\mathrm{Jac}(f^{\sigma_a g_a}|_{I^c_{\sigma_a g_a}})$ has the basis $\lfloor \widetilde{x}^k_\bullet \rfloor$, $k = 0, \dots, n-2$. Set

$$\mathcal{A}'_{\sigma_a g_a} := \langle \lfloor 1 \rfloor, \lfloor \widetilde{x}_{i_a} \rfloor, \dots, \lfloor \widetilde{x}^{n-2}_{i_a} \rfloor \rangle \xi_{\omega_a g_a},$$
$$\mathcal{B}'_{\sigma_a g_a} := \langle \lfloor 1 \rfloor, \lfloor \widetilde{x}_{i_a} \rfloor, \dots, \lfloor \widetilde{x}^{n-2}_{i_a} \rfloor \rangle \xi_{\sigma_a g_a},$$

where we denote by $\xi_{\sigma_a g_a}$ the formal letter associated to $\sigma_a g_a$. The elements of $\mathcal{B}'_{\sigma_a g_a}$ will be denoted by $\lfloor \phi(\mathbf{x}) \rfloor \xi_{\sigma_a g_a}$.

In particular, for $g_a = \mathrm{id}$ we have $\widetilde{x}_{i_a} = \sum_i x_i$ where the summation is taken over $i \in I^c_{\sigma_a}$. We adopt the notation above for the non–special cycles too, assuming $\lfloor \widetilde{x}^0_{i_a} \rfloor \xi_{\sigma_a g_a} = \lfloor 1 \rfloor \xi_{\sigma_a g_a}$. For $u \in G$ with generalized cycle decomposition $u = \prod_{a=1}^p \sigma_a g_a$ we have

$$\mathcal{A}'_{f,u} = \bigotimes_{a=1}^p \mathcal{A}'_{\sigma_a g_a} \quad \mathcal{B}'_{f,u} = \bigotimes_{a=1}^p \mathcal{B}'_{\sigma_a g_a}.$$

Fix $\zeta_n := \exp(2\pi\sqrt{-1})/n)$ and $t_k \in G_f$ with $k = 1, \dots, N$ by

$$t_k : (x_1, \dots, x_N) \to (x_1, \dots, \zeta_n x_k, \dots, x_N).$$

Then the Fermat type polynomial maximal diagonal symmetries group G_f is generated by $t_1, \dots, t_N$. Denote also

$$\mathrm{SL}_f := \{g \in G_f \mid \det(g) = 1\}, \quad J := t_1 \cdots t_N, \quad \mathcal{J} = \langle J \rangle.$$

The groups $S \ltimes \mathrm{SL}_f$ and $S \ltimes \mathcal{J}$ with $S \subseteq S_N$ will be in particular important in this paper.

4.1.1 The phase space

Consider the spaces

$$\mathcal{A}_{tot} := \bigoplus_{u \in S \ltimes G_f} \mathcal{A}'_{f,u}, \quad \mathcal{B}_{tot} := \bigoplus_{u \in S \ltimes G_f} \mathcal{B}'_{f,u}.$$

We call its direct summands $\mathcal{A}'_{f,u}$ and $\mathcal{B}'_{f,u}$ the *u-th sectors*. The subspaces $\mathcal{A}'_{f,u}$ and $\mathcal{B}'_{f,u}$ are called *narrow sectors* if $\mathrm{Fix}(u) = 0$ and *broad sectors* otherwise.

For any $G \subseteq S_N \ltimes G_f$ we denote by $\mathcal{A}_{f,G}$ and $\mathcal{B}_{f,G}$ the *phase space* of G, being the subspace of $\mathcal{A}_{f,G}$ and $\mathcal{B}_{tot}$ respectively, defined as follows.

Let $\mathcal{C}^G$ stand for the set of representatives of the conjugacy classes of G and $Z(u)$ for the centralizer of u. Denote

$$\mathcal{A}_{f,G} := \bigoplus_{u \in \mathcal{C}^G} \left(\mathcal{A}'_{f,u}\right)^{Z(u)}, \quad \mathcal{B}_{f,G} := \bigoplus_{u \in \mathcal{C}^G} \left(\mathcal{B}'_{f,u}\right)^{Z(u)}. \tag{4.7}$$

The action of $v \in Z(u)$ on $\mathcal{A}'_{f,u}$ and $\mathcal{B}'_{f,u}$ is computed as follows.

Let λ_k^u, λ_k^v be the eigenvalues of u and v computed in their common eigenvectors basis. For $Y = \lfloor \phi(\mathbf{x}) \rfloor \omega_u \in \mathcal{A}'_{f,u}$ and $v \in Z(u)$ we have

$$v^*(Y) = \prod_{\substack{k=1,\dots,N \\ \lambda_k^u = 1}} \lambda_k^v \cdot \lfloor \phi(v \cdot \mathbf{x}) \rfloor \omega_u. \tag{4.8}$$

For $X = \lfloor \phi(\mathbf{x}) \rfloor \xi_u \in \mathcal{B}'_{f,u}$ and $v \in Z(u)$ we have

$$v^*(X) = \prod_{\substack{k=1,\dots,N \\ \lambda_k^u \neq 1}} \frac{1}{\lambda_k^v} \cdot \lfloor \phi(v \cdot \mathbf{x}) \rfloor \xi_u. \tag{4.9}$$

Remark 4.7. The G–action for the nonabelian groups is more complicated. For $v \in G$ we have $v^* : \mathcal{B}'_{f,u} \to \mathcal{B}'_{f,vuv^{-1}}$ for any $u \in G$. This is the reason why we have the sum over the conjugacy classes in Eq. (4.7).

For any $G_1, G_2 \subseteq S_N \ltimes G_f$ we have the natural inclusion $i_1 : \mathcal{B}'_{f,G_1} \to \mathcal{B}_{tot}$ and the projections $\pi_2 : \mathcal{B}_{tot} \to \mathcal{B}'_{f,G_2}$. In what follows we will consider the maps $\psi : \mathcal{B}'_{f,G_1} \to \mathcal{B}'_{f,G_2}$ via the maps $\widetilde{\psi} : \mathcal{B}_{tot} \to \mathcal{B}_{tot}$ by $\psi := \pi_2 \circ \widetilde{\psi} \circ i_1$.

With respect to a generalized cycle decomposition $u = \prod_a \sigma_a g_a$ we have the relation $\xi_u = \prod_{a=1}^p \xi_{\sigma_a g_a}$ between the generators of the different vector spaces $\mathcal{B}_{f,G_1}$, $\mathcal{B}_{f,G_2}$. This extends to the product of arbitrary $X_1 = \lfloor \phi_1 \rfloor \xi_u$ and $X_2 = \lfloor \phi_2 \rfloor \xi_v$ assumed as $\mathcal{B}_{tot}$–elements by $X_1 X_2 := \lfloor \phi_1 \phi_2 \rfloor \xi_{uv}$ when $I_u^c \cap I_v^c = \varnothing$[1].

[1] this should not be confused with the cup-product on Hochschild cohomology as in [**?**]

4.2 Mirror map

In this section we define the mirror map $\tau : \mathcal{B}_{tot} \to \mathcal{A}_{tot}$. It will be used later in Theorem 4.15 to set up a mirror isomorphism. First consider the following examples.

4.2.1 Examples

Example 1 shows that there is no mirror map if $N \neq n$. Example 2 considers the case when there is a mirror map, interchanging broad and narrow sectors. Example 3 deals with the symmetry group that does not satisfy PC condition of Ebeling–Gusein–Zade. Even though, the mirror map exists. Example 4 depicts the situation when mirror map exists but does not always interchange broad and narrow sectors as it is for diagonal symmetry groups.

Examples 2 and 3 are the particular cases of Theorem 4.15 and Example 4 is the particular case of Theorem 4.17.

In all the examples beneath we consider the groups $G = S \ltimes \mathrm{SL}_f$ and $\widetilde{G} = S \ltimes \mathcal{J}$ with different $S \subseteq S_N$. There is a decomposition

$$\mathcal{B}_{f,S\ltimes \mathrm{SL}_f} = \mathcal{B}_{\mathrm{SL},d} \oplus \mathcal{B}_{\mathrm{SL},s}, \quad \mathcal{A}_{f,S\ltimes\langle J\rangle} = \mathcal{A}_{\langle J\rangle,d} \oplus \mathcal{A}_{\langle J\rangle,s}$$

for $\mathcal{B}_{\mathrm{SL},d}$ and $\mathcal{A}_{\mathcal{J},d}$ being the direct sums of all u–th sectors of $\mathcal{B}_{f,S\ltimes\mathrm{SL}_f}$ and $\mathcal{A}_{f,S\ltimes\mathcal{J}}$, s.t. $u \in \mathrm{id}\cdot\mathrm{SL}_f$ and $u \in \mathrm{id}\cdot\mathcal{J}$ respectively.

Example 1: $N = 3$ **and** $n = 4$**,** $S = S_3$

We have

$$\mathcal{B}_{S_3\ltimes\mathrm{SL}_f,s} = \mathbb{C}\left\langle \lfloor (x_2 - x_1)^2 \rfloor \xi_{(1,2,3)t_1^2t_2^2t_3^2}, \lfloor (x_1+x_2+x_3)^2\rfloor \xi_{(1,2,3)} \right\rangle, \tag{4.10}$$

$$\mathcal{A}_{S_3\ltimes\mathcal{J},s} = \mathbb{C}\left\langle \lfloor (x_1+x_2+x_3)^2\rfloor \omega_{(1,2,3)} \right\rangle \tag{4.11}$$

showing that there is no mirror map between $\mathcal{B}_{f,S_3\ltimes\mathrm{SL}_f}$ and $\mathcal{A}_{f,S_3\ltimes\mathcal{J}}$ when $N \neq n$.

Example 2: $N = n = 5$ **and** $S = \langle (1,2)(3,4)\rangle \subset S_5$

Denote

$$\phi_{a,b,c}(\mathbf{x}) := (x_1+x_2)^a (x_3+x_4)^b x_5^c, \quad a,b,c \geqslant 0. \tag{4.12}$$

We have

$$\mathcal{A}_{\mathcal{J},s} = \bigoplus_{k=1}^{4} \mathbb{C}\langle \omega_{(1,2)(3,4)J^k}\rangle \bigoplus_{\substack{a+b+c=2\\ \text{or } a+b+c=7}} \mathbb{C}\langle \lfloor \phi_{a,b,c}\rfloor \omega_{(1,2)(3,4)}\rangle, \tag{4.13}$$

$$\mathcal{B}_{\mathrm{SL},s} = \bigoplus_{\substack{a+b=2\\ \text{or } a+b=7}} \mathbb{C}\langle \lfloor \phi_{a,a,b}\rfloor \xi_{(1,2)(3,4)}\rangle \bigoplus_{\substack{a,b=1\\ a+b\neq 0}}^{4} \mathbb{C}\langle \xi_{(1,2)(3,4)t_1^at_3^bt_5^{5-a-b}}\rangle. \tag{4.14}$$

The vector space isomorphism $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f} \to \mathcal{A}_{f,S\ltimes\mathcal{J}}$ is

$$\lfloor H^k \rfloor \xi_{\mathrm{id}} \mapsto \omega_{J^{k-1}}, \tag{4.15}$$

$$\xi_{(1,2)(3,4)t_1^a t_4^b t_5^c} \mapsto \lfloor \phi_{a-1,b-1,c-1} \rfloor \omega_{(1,2)(3,4)} \tag{4.16}$$

$$\lfloor \phi_{a,a,c} \rfloor \xi_{(1,2)(3,4)} \mapsto \omega_{(1,2)(3,4)J^{c-1}}. \tag{4.17}$$

This map interchanges bidegree $(1,2)$ classes with bidegree $(1,1)$ classes and bidegree $(2,1)$ classes with bidegree $(2,2)$ classes.

The group considered can be diagonalized, however this requires also the change of the polynomial f so that we will not have the relation $\widetilde{f} = f$ any more.

Example 3: $N = n = 5$**,** $S = \langle (1,2,3), (1,2) \rangle \subset S_5$

Denote

$$\phi_{a,b,c}(\mathbf{x}) := (x_1 + x_2 + x_3)^a x_4^b x_5^c, \quad a, b, c \geqslant 0. \tag{4.18}$$

We have

$$\mathcal{A}_{\mathcal{J},s} = \bigoplus_{k=1}^{4} \mathbb{C}\langle \omega_{(1,2,3)J^k} \rangle \bigoplus_{\substack{a+b+c=2 \\ \text{or } a+b+c=7}} \mathbb{C}\langle \lfloor \phi_{a,b,c} \rfloor \omega_{(1,2,3)} \rangle, \tag{4.19}$$

$$\mathcal{B}_{\mathrm{SL},s} = \bigoplus_{\substack{a+b=2 \\ \text{or } a+b=7}} \mathbb{C}\langle \lfloor \phi_{a,b,b} \rfloor \xi_{(1,2,3)} \rangle \bigoplus_{\substack{a,b=1 \\ a+b\neq 0}}^{4} \mathbb{C}\langle \xi_{(1,2,3)t_1^{5-a-b} t_4^a t_5^b} \rangle. \tag{4.20}$$

The vector space isomorphism $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f} \to \mathcal{A}_{f,S\ltimes\mathcal{J}}$ is

$$\lfloor H^k \rfloor \xi_{\mathrm{id}} \mapsto \omega_{J^{k-1}}, \tag{4.21}$$

$$\xi_{(1,2,3)t_1^a t_4^b t_5^c} \mapsto \lfloor \phi_{a-1,b-1,c-1} \rfloor \omega_{(1,2,3)} \tag{4.22}$$

$$\lfloor \phi_{a,b,b} \rfloor \xi_{(1,2,3)} \mapsto \omega_{(1,2,3)J^{b-1}}. \tag{4.23}$$

This map interchanges bidegree $(1,2)$ classes with bidegree $(1,1)$ classes and bidegree $(2,1)$ classes with bidegree $(2,2)$ classes.

Example 4: $N = n = 5$**,** $S = S_5$

Denote

$$\phi_{a,b,c}(\mathbf{x}) := (x_1 + x_2 + x_3)^a (x_4^b x_5^c + x_4^c x_5^b),$$
$$\psi_{a,b}(\mathbf{x}) := (x_1 + x_2 + x_3)^a (x_4 x_5)^b.$$

We have $\mathcal{A}_{\mathcal{J},s} = \mathcal{A}^{(1)}_{\mathcal{J},s} \oplus \mathcal{A}^{(2)}_{\mathcal{J},s}$, $\mathcal{B}_{\mathrm{SL},s} = \mathcal{B}^{(1)}_{\mathrm{SL},s} \oplus \mathcal{B}^{(2)}_{\mathrm{SL},s}$ with

$$\mathcal{A}^{(1)}_{\mathcal{J},s} = \mathbb{C}\langle \lfloor \phi_{0,0,2} \rfloor \omega_{(1,2,3)}, \lfloor \phi_{1,0,1} \rfloor \omega_{(1,2,3)}, \lfloor \phi_{2,2,3} \rfloor \omega_{(1,2,3)}, \lfloor \phi_{3,1,3} \rfloor \omega_{(1,2,3)} \rangle, \tag{4.24}$$

$$\mathcal{A}^{(2)}_{\mathcal{J},s} = \mathbb{C}\langle \lfloor \psi_{0,1} \rfloor \omega_{(1,2,3)}, \lfloor \psi_{1,3} \rfloor \omega_{(1,2,3)}, \lfloor \psi_{2,0} \rfloor \omega_{(1,2,3)}, \lfloor \psi_{3,2} \rfloor \omega_{(1,2,3)} \rangle, \tag{4.25}$$

$$\mathcal{B}^{(1)}_{\mathrm{SL},s} = \mathbb{C}\langle \xi_{(1,2,3)t_1 t_4 t_5^3}, \xi_{(1,2,3)t_1^2 t_4 t_5^2}, \xi_{(1,2,3)t_1^3 t_4^3 t_5^4}, \xi_{(1,2,3)t_1^4 t_4^2 t_5^4} \rangle \tag{4.26}$$

$$\mathcal{B}^{(2)}_{\mathrm{SL},s} = \mathbb{C}\langle \lfloor \psi_{0,1} \rfloor \xi_{(1,2,3)}, \lfloor \psi_{1,3} \rfloor \xi_{(1,2,3)}, \lfloor \psi_{2,0} \rfloor \xi_{(1,2,3)}, \lfloor \psi_{3,2} \rfloor \xi_{(1,2,3)} \rangle. \tag{4.27}$$

We see that $\mathcal{A}_{\mathcal{J},d}$ contains only narrow sectors, $\mathcal{A}_{\mathcal{J},s}$ only broad sectors. $\mathcal{B}_{\mathrm{SL},d}$ contains only broad sectors while $\mathcal{B}_{\mathrm{SL},s}$ is a direct sum of both broad and narrow sectors.

It's easy to guess the vector space isomorphism $\mathcal{B}_{f,S_5\ltimes \mathrm{SL}_f} \to \mathcal{A}_{f,S_5\ltimes\mathcal{J}}$ in this case.

$$\lfloor H^k \rfloor \xi_{\mathrm{id}} \mapsto \omega_{J^{k-1}}, \tag{4.28}$$

$$\xi_{(1,2,3)t_1^a t_4^b t_5^c} \mapsto \lfloor \phi_{a-1,b-1,c-1} \rfloor \omega_{(1,2,3)} \tag{4.29}$$

$$\lfloor \psi_{a,b} \rfloor \xi_{(1,2,3)} \mapsto \lfloor \psi_{a,b} \rfloor \omega_{(1,2,3)} \tag{4.30}$$

The first two lines of it interchange degree $(1,2)$ and $(2,1)$ classes with the degree $(1,1)$ and $(2,2)$ classes respectively. However the last line sets up the isomorphism $\mathcal{A}^{(2)}_{\mathcal{J},s} \to \mathcal{B}^{(2)}_{\mathrm{SL},s}$ mapping the degree $(1,1)$ and $(2,2)$ classes to the degree $(1,1)$ and $(2,2)$ classes again.

4.2.2 Definition of the mirror map

For any $u \in S \ltimes G_f$ let $u = \prod_{a=1}^p \sigma_a g_a$ be its generalized cycle decomposition. Arbitrary element $X \in \mathcal{B}'_{f,u}$ reads

$$X = \prod_{a=1}^p \lfloor \widetilde{x}_{i_a}^{r_a} \rfloor \xi_{\sigma_a g_a}.$$

with $r_a = 0$ if $\sigma_a g_a$ is non–special and $0 \leqslant r_a \leqslant N-2$ if $\sigma_a g_a$ is special.

Set

$$\tau\left(X\right) := \prod_{a=1}^p \tau\left(\lfloor \widetilde{x}_{i_a}^{r_a} \rfloor \xi_{\sigma_a g_a}\right)$$

for the product on the right hand side understood in the sense of Section 4.1. The map τ is defined on the generalized cycles as follows.

Case 1: $u = \sigma_1 \cdot g_1$ is a non–special cycle

Assume $g_1 = \prod_p t_p^{d_{1,p}}$ with p running over $I^c_{g_1}$. Denote $d_1 := \sum_p d_{1,p} \mod N$ with $1 \leqslant d_1 \leqslant N$. Set

$$\tau(\xi_{\sigma_1 \cdot g_1}) := \lfloor \widetilde{x}_{i_1}^{d_1-1} \rfloor \cdot \omega_{\sigma_1}. \tag{4.31}$$

Case 2: $u = \sigma_1 \cdot g_1$ is a special cycle

For any $0 \leqslant r_1 \leqslant N-2$ set

$$\tau(\lfloor \widetilde{x}_{i_1}^{r_1} \rfloor \xi_{\sigma_1 \cdot g_1}) := \omega_{\sigma_1 \cdot h} \tag{4.32}$$

for $h := (\prod_a t_a)^{k_1}$ with a running over $I^c_{\sigma_1}$ and $k_1 \in 1, \ldots, N-1$, the unique integer, s.t. $r_1 + 1 \equiv k_1 |\sigma_1|$ modulo N.

4.2.3 Mirror map of Krawitz

The mirror map τ generalizes the mirror map of Krawitz (see). Namely $\tau_{Kr} : \mathcal{B}_{f,\mathrm{SL}_f} \to \mathcal{A}_{f,\mathcal{J}}$ by the following rule

$$\prod_{a=1}^p x_p^{r_p} \xi_{\mathrm{id}} \mapsto \omega_{\prod_{a=1}^p t_p^{r_p+1}}, \quad \xi_{\prod_{a=1}^p t_p^{d_p}} \mapsto \prod_{a=1}^p x_p^{d_p-1} \omega_{\mathrm{id}}$$

Let's see that our mirror map coincides with the mirror maps of Krawitz on $\mathcal{B}'_{f,u}$, s.t. $u = \mathrm{id} \cdot g$. The generalized cycle decomposition of such elements is given by $u = \prod_{a=1}^{N}(a)t_a^{d_a}$ for $g = \prod t_a^{d_a}$. We have $x_a = \widetilde{x}_a$ if $d_a \equiv 0$.

When $g = \mathrm{id}$ we have

$$\tau(\prod_{a=1}^{p}\lfloor x_a^{r_a}\rfloor\xi_{\mathrm{id}}) = \prod_{a=1}^{p}\tau(\lfloor \widetilde{x}_a^{r_a}\rfloor\xi_{\mathrm{id}}) = \prod_{a=1}^{p}\left(\lfloor 1\rfloor\omega_{t_a^{r_a+1}}\right) = \lfloor 1\rfloor\omega_{\prod_{a=1}^{p} t_a^{r_a+1}}. \tag{4.33}$$

For $d_a \neq 0$ for all a we have

$$\tau(\lfloor 1\rfloor\xi_{\prod_{a=1}^{p} t_a^{d_a}}) = \prod_{a=1}^{p}\tau(\lfloor 1\rfloor\omega_{t_a^{d_a}}) = \prod_{a=1}^{p}\lfloor \widetilde{x}_a^{d_a-1}\rfloor\omega_{\mathrm{id}}. \tag{4.34}$$

4.3 The vector space structure

In this section consider prime $N = n$ and arbitrary $S \subseteq S_N$. The aim of this section is to describe the structure of the vector spaces $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$ and $\mathcal{B}_{S\ltimes\mathcal{J}}$.

According to the definition, we should consider the sets of representatives of the conjugacy classes $\mathcal{C}^{S\ltimes \mathrm{SL}_f}$ and $\mathcal{C}^{S\ltimes\mathcal{J}}$. Let $\mathcal{C}^S$ be some set of representatives of the conjugacy classes of S. We have

$$\mathcal{C}^{S\ltimes\mathcal{J}} = \{\sigma \cdot J^k \mid \sigma \in \mathcal{C}^S, k = 0, \dots, N-1\}.$$

The map $S \ltimes \mathrm{SL}_f \to S$ is compatible with the conjugation action. We chose representatives in $\mathcal{C}^{S\ltimes \mathrm{SL}_f}$ in a way compatible with the choice of $\mathcal{C}^S$. The set $\mathcal{C}^{S\ltimes \mathrm{SL}_f}$ is described in Proposition 4.10 below.

Concerning the bases of $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$, $\mathcal{B}_{f,S\ltimes\mathcal{J}}$ we show the following proposition.

Proposition 4.8.

(A). The basis of $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$ can be chosen to consists of narrow vectors $\prod_a \xi_{\sigma_a t_{i_a}^{d_a}}$ and broad vectors $\prod_a \lfloor \widetilde{x}_{i_a}^{r_a}\rfloor\xi_{\sigma_a}$,

(B). The basis of $\mathcal{A}_{f,S\ltimes\mathcal{J}}$ can be chosen to consists of narrow vectors $\prod_a \omega_{\sigma_a J^q}$ and broad vectors $\prod_a \lfloor \widetilde{x}_{i_a}^{r_a}\rfloor\omega_{\sigma_a}$,

with $0 \leqslant r_a \leqslant N-2$, $1 \leqslant d_a \leqslant N-1$ and $1 \leqslant q \leqslant N-1$ in both cases.

Note that this proposition does not guarantee that all vectors of the form given do appear in the bases of $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$ and $\mathcal{A}_{f,S\ltimes\mathcal{J}}$. The proof is given in the following sections beneath.

4.3.1 The group $G = S \ltimes \mathrm{SL}_f$

Proposition 4.9. *For any $g \in \mathrm{SL}_f$ and length N cycle σ we have $\left(\mathcal{B}'_{f,\sigma\cdot g}\right)^{Z(\sigma\cdot g)} = 0$.*

Proof. The element $\sigma \cdot g$ is special with 1–dimensional fixed locus. We have $\mathcal{B}'_{f,\sigma\cdot g} \cong \mathbb{C}\langle\lfloor \widetilde{x}_1^p\rfloor\xi_{\sigma\cdot g}\rangle$ for $p = 0, \dots, N-2$. We also have $J \in G$ and $J \in Z(\sigma \cdot g)$. The action of J gives

$$J^*\left(\lfloor \widetilde{x}_1^p\rfloor\xi_{\sigma\cdot g}\right) = \zeta_N^{p+1} \cdot \lfloor \widetilde{x}_1^p\rfloor\xi_{\sigma\cdot g},$$

because $\det(J) = 1$. The vector assumed is invariant under the action J if and only if $p + 1 \equiv 0 \mod N$, what never holds for the range of p given.

□

In the obvious way one gets that for a length $N-1$ cycle σ, an element $\sigma \cdot g$ is conjugate to $\sigma \cdot t_{i_1}^{d_1} t_{i_2}^{d_2}$ in the group G for $d_1 + d_2 \equiv 0 \mod N$ and $i_1 \in I_\sigma^c$, $i_2 \notin I_\sigma^c$. This can be generalized to the following statement.

Proposition 4.10. *Let $u = \sigma \cdot g \in G$ be an element with the cycle decomposition $\sigma = \prod_a \sigma_a$. Let i_a be the first index of $I_{\sigma_a}^c$. Then for some $d_a \in \mathbb{Z}$, u is conjugate to $\prod_a \sigma_a t_{i_a}^{d_a}$ in the group G.*

Proof. It is sufficient to consider the case, when $\sigma = (i_1, i_2, \ldots, i_p)$ is a single cycle and $g = t_{i_1}^{d_{i_1}} t_{i_2}^{d_{i_2}} \ldots t_{i_p}^{d_{i_p}}$. It is also sufficient to consider the bigger group $S \ltimes G_f^d$, since it is generated by its center and G. Now we can conjugate by $h = t_{i_2}^{d_{i_2}+d_{i_3}+\ldots+d_{i_p}} t_{i_3}^{d_{i_3}+\ldots+d_{i_p}} \ldots t_{i_p}^{d_{i_p}}$ to get the desired result. □

The following corollary makes a clear distinction between broad and narrow sectors for the group $S \ltimes \mathrm{SL}_f$.

Corollary 4.11. *Let $X \in \mathcal{B}_{S \ltimes \mathrm{SL}_f}$ be a non–zero element with $X = \lfloor \phi(\mathbf{x}) \rfloor \xi_u$, then $\lfloor \phi(\mathbf{x}) \rfloor$ is non–constant only if u is conjugate to $\prod_a \sigma_a$.*

In particular for every such X, the conjugacy class of u contains exactly one $v = wuw^{-1}$, s.t. $\lfloor \phi(w \cdot \mathbf{x}) \rfloor \xi_v \in \mathcal{B}_S$.

Proof. Put $u = \sigma g$ with $\sigma \in S, g \in G^d$. If $\lfloor \phi(\mathbf{x}) \rfloor$ is non constant, then at least one of the cycles of σ is special. Let it be σ_1. Assume further, that there is a nonspecial cycle σ_2 in the cycle decomposition of σ.

Let d be such that $|\sigma_1| + d|\sigma_2| \equiv 0 \mod N$. Consider the element $h = (\prod_i t_i)(\prod_j t_j)^d \in \mathrm{SL}_f$ with i running over $I_{\sigma_1}^c$ and j running over $I_{\sigma_2}^c$. Then h commutes with u. We have $h^* \lfloor \tilde{x}_{i_1}^r \rfloor \xi_u = \zeta_N^{r+1} \xi_u$. Since, h fixes other variables and r varies between 0 and $n-2$, there are no invariant elements in the sector. □

This completes proof of Proposition 4.8, case (A).

Remark 4.12. The conjugacy classes of S_N are indexed by the partitions of N. In particular, any length N–cycle of S_N is conjugate to $\sigma = (1, 2, \ldots, N)$. We conclude that the representatives of the conjugacy classes of $S_N \ltimes \mathrm{SL}_f$ are given by two partitions of $n = N$ of the same length. The first one being given by the cycle type of σ and the second by the set of exponents $d_\bullet$.

4.3.2 The group $\widetilde{G} = S \ltimes \langle J \rangle$

Let $u = \sigma \cdot J^q$ and $\sigma = \prod_{a=1}^p \sigma_a$ be the cycle decomposition. If $q \equiv 0$, all σ_a are special. If $q \not\equiv 0$, for $n = N$ being a prime number, either $p = 1$ and $\sigma = \sigma_1$ is special again, or all σ_a are non–special.

For the group assumed $J \in Z(u)$. Consider the action of this element on $\mathcal{A}'_{f,u}$.

Assume $X = \lfloor\phi(\mathbf{x})\rfloor\omega_{\sigma\cdot J^0}$ and $Y = \omega_{\sigma_1\cdot J^q}$ with some $q \not\equiv 0$. Up to a constant multiple we have for some $0 \leqslant r_a \leqslant N-2$

$$X = \prod_{a=1}^{p}\lfloor\widetilde{x}_{i_a}^{r_a}\rfloor\omega_{\sigma_a} \quad \text{giving} \quad J^*(X) = \zeta_N^{\sum_a (r_a+1)}\cdot X, \tag{4.35}$$

$$Y = \lfloor\widetilde{x}_{i_1}^{r_1}\rfloor\omega_{\sigma_1 J^q} \quad \text{giving} \quad J^*(Y) = \zeta_N^{r_1+1}\cdot Y \tag{4.36}$$

Therefore X is only J–invariant if $\sum_a (r_a+1) \in N\mathbb{Z}$ and Y is not ever J–invariant.

Similarly we have that $\omega_{\sigma\cdot J^q}$ is J–invariant for any $q \not\equiv 0$.

This gives the following proposition beneath and completes proof of Proposition 4.8, case (B).

Proposition 4.13. *Consider $X := \lfloor\phi(\mathbf{x})\rfloor\omega_{\sigma\cdot g} \in \mathcal{A}'_{S\ltimes\langle J\rangle}$. Assume X to be $\langle J\rangle$–invariant. Then either $\lfloor\phi(\mathbf{x})\rfloor$ is a constant or $g = \mathrm{id}$.*

4.3.3 Symmetric group action

For any $u = \sigma\cdot g \in G \subseteq S_N \ltimes G_f$ we have $u \in Z(u)$ and for any $X = \lfloor\phi(\mathbf{x})\rfloor\xi_u$ we have $\lfloor\phi(u\cdot\mathbf{x})\rfloor = \lfloor\phi(\mathbf{x})\rfloor$. However the action of u on the generator ξ_u is not necessarily trivial. It was proved in Corollary 39 of [**?**], or can be deduced from Eq. (4.9), that we have

$$u^*(\xi_u) = (-1)^{\mathrm{sgn}(\sigma)}\xi_u.$$

Consider the $\langle u\rangle$–invariant subspace of $\mathcal{B}'_{f,u}$.

$$\left(\mathcal{B}'_{f,u}\right)^{\langle u\rangle} = 0 \;\text{ if }\; \sigma \text{ is odd and } \;\left(\mathcal{B}'_{f,u}\right)^{\langle u\rangle} = \mathcal{B}'_{f,u} \text{ otherwise.} \tag{4.37}$$

Due to this reason no sector of $u = \sigma\cdot g$ with σ odd appears in $\mathcal{B}_{f,G}$. However such elements u, if they exist in the group, can still affect the conjugacy classes decomposition and also the centralizers of the other group elements, contributing non–trivially to $\mathcal{B}_{f,G}$.

In the following proposition we consider group–action approach to the parity condition of Ebeling–Gusein-Zade.

Proposition 4.14. *Let $u, v \in A_N$ be even commuting permutations. For $T := \langle u, v\rangle$ we have.*

- *PC holds for T if and only if $u^*(\xi_v) = \xi_v$ and $v^*(\xi_u) = \xi_u$,*
- *if PC doesn't hold for T we have $u^*(\xi_v) = -\xi_v$ or $v^*(\xi_u) = -\xi_u$.*

Proof. Let u and v share a common eigenbasis $\widetilde{x}_1, \ldots, \widetilde{x}_N$. Let $u(\widetilde{x}_k) = \lambda'_k\widetilde{x}_k$ and $v(\widetilde{x}_k) = \lambda''_k\widetilde{x}_k$. Assume also $\lambda'_1 = \cdots = \lambda'_p = 1$ and $\lambda''_1 = \cdots = \lambda''_q = 1$. We have

$$v^*(\xi_u) = \frac{1}{\lambda'_{p+1}\cdots\lambda'_N}\,\xi_u = \frac{1}{\lambda'_{p+1}\cdots\lambda'_N}\,\xi_u \quad q \leqslant p \tag{4.38}$$

$$v^*(\xi_u) = \frac{1}{\lambda'_{q+1}\cdots\lambda'_N}\,\xi_u = \frac{\lambda'_{p+1}\cdots\lambda'_q}{\lambda'_{p+1}\cdots\lambda'_N}\,\xi_u \quad q > p. \tag{4.39}$$

It follows that for both cases

$$v^*(\xi_u) = \frac{\lambda'_1\cdots\lambda'_q}{\det(v)}\xi_u = \frac{\det(v_{|\mathrm{Fix}(u)})}{\det(v)}\xi_u, \tag{4.40}$$

where $v_{|\mathrm{Fix}(u)}$ stands for the restriction of v to the fixed locus of u. Assume $q > p$. The action of v on $\mathrm{Fix}(u)$ is given by a permutation of the set $\{1, \dots, q\}$. Therefore both determinants assumed are either $+1$ or -1. We have

$$\det(v) = (-1)^{N - \dim \mathrm{Fix}(v)}, \quad \det(v_{|\mathrm{Fix}(u)}) = (-1)^{\dim \mathrm{Fix}(u) - \dim \mathrm{Fix}(v)}.$$

This completes the proof. □

4.4 Mirror isomorphism

We have seen in Example 1 of Section 4.2.1 that the condition $N = n$ is necessary for mirror symmetry to hold. Assume it to hold true. Assume further N to be prime. Fix some $S \subseteq S_N$.

Let $G \subset S \ltimes G_f$. We call u–th sector of $\mathcal{B}_{f,G}$ *stable* if

$$\sigma^*(\xi_u) = \xi_u \quad \text{for any} \quad \sigma \in Z(u) \cap S \subset G \tag{4.41}$$

and the u–th sector of $\mathcal{A}_{f,G}$ *stable* if

$$\sigma^*(\omega_u) = \omega_u \quad \text{for any} \quad \sigma \in Z(u) \cap S \subset G. \tag{4.42}$$

Denote by $\mathcal{A}_{f,G}^{stable} \subseteq \mathcal{A}_{f,G}$ and $\mathcal{B}_{f,G}^{stable} \subseteq \mathcal{B}_{f,G}$ the direct sum of all stable sectors $\mathcal{A}_{f,u}$ and $\mathcal{B}_{f,u}$ respectively. It's important to note that our stability property considers the generator of the u–th sector rather than an arbitrary element $\lfloor \phi(\mathbf{x}) \rfloor \xi_u$ of it.

In particular according to Proposition 4.14, we have $\mathcal{B}_{f,G}^{stable} = \mathcal{B}_{f,G}$ if S satisfies (PC) of Ebeling–Gusein-Zade. The converse is not true. In particular these vector spaces coincide in Examples 2 and 3 of Section 4.2.1. In the case of Example 4 we have $\mathcal{B}_{f,G}^{stable} = \mathcal{B}_{G,d} \oplus \mathcal{B}_{G,s}^{(1)} \subsetneq \mathcal{B}_{f,G}$.

Theorem 4.15. *The map τ establishes an isomorphism $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}^{stable} \to \mathcal{A}_{f,S\ltimes \mathcal{J}}^{stable}$. Under this isomorphism we have $q_l(X) = q_l(\tau(X))$ and $q_r(X) = N - 2 - q_r(\tau(X))$ for any homogeneous $X \in \mathcal{B}_{f,S\ltimes \mathrm{SL}_f}^{stable}$.*

Proof. See Sections 4.4.1 and 4.4.2 for the first claim and Section 4.4.3 for the second claim. □

To show Theorem 4.15 we need to consider the action of the centralizers of the elements $u = \sigma g \in S \ltimes \mathrm{SL}_f$ and $v = \sigma J^k \in S \ltimes \mathcal{J}$ with the same $\sigma \in S$.

4.4.1 Narrow sectors

Let $u \in S \ltimes \mathrm{SL}_f$ be s.t. u–th sector is stable and narrow. Assume u to be decomposed into generalized cycles as $u = \prod_{a=1}^{p} \sigma_a t_{i_a}^{d_a}$. Let $Y := \xi_u \in \mathcal{B}'_{f,S\ltimes \mathrm{SL}_f}$. Then $X := \tau(Y) = \prod_a \lfloor \widetilde{x}_{i_a}^{d_a - 1} \rfloor \omega_{\sigma_a} \in \mathcal{A}'_{f,S\ltimes \mathcal{J}}$.

As u-th sector is narrow, any $v \in Z(u) \subset S \ltimes \mathrm{SL}_f$ gives $v^*(Y) = (\det(v))^{-1} Y = Y$ and we know that Y is non–zero in $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$. We show that X is non–zero in $\mathcal{A}_{f,S\ltimes \mathcal{J}}$ too.

The centralizer $Z(\prod_a \sigma_a) \subset S \ltimes \mathcal{J}$ is generated by the element J and all permutations $\sigma' \in S$, commuting with $\prod_a \sigma_a$. We have that $J^*(X) = X$ if and only if $\sum_a d_a \equiv 0$ modulo N, what is equivalent to the condition $\prod_a t_{i_a}^{d_a} \in \mathrm{SL}_f$. The action of σ' is trivial on X because the u–th sector is stable.

4.4.2 Broad sectors

Let $u \in S \ltimes \mathrm{SL}_f$ be s.t. u–th sector is stable and broad. Assume u to be decomposed into generalized cycles as $u = \prod_{a=1}^{p} \sigma_a$. Take $X := \prod_a \lfloor \widetilde{x}_{i_a}^{d_a - 1} \rfloor \xi_{\sigma_a} \in \mathcal{B}'_{f, S \ltimes \mathrm{SL}_f}$.

Proposition 4.16. *Let $X = \prod_{a=1}^{p} \lfloor \widetilde{x}_{i_a}^{r_a} \rfloor \xi_{\sigma_a}$ be non–zero in $\mathcal{B}_{f, S \ltimes \mathrm{SL}_f}$. Let $k_1, \ldots, k_p$ be as in Section 4.2.2. Then $k_a = k_b$ for all $1 \leqslant a, b \leqslant p$.*

Proof. Take $a = 1$ and $b = 2$. Let $g_1 := (\prod_{a \in I^c_{\sigma_1}} t_a)^{\alpha}$ and $g_2 := (\prod_{a \in I^c_{\sigma_2}} t_a)^{\beta}$. Then for $\alpha |\sigma_1| + \beta |\sigma_2| \equiv 0$ modulo N we have $v := g_1 g_2 \in Z(\prod_a \sigma_a) \subset S \ltimes \mathrm{SL}_f$.

We have $v^*(X) = \lambda_1^{r_1 + 1} \lambda_2^{r_2 + 1} \cdot X$ with $\lambda_1 = \zeta_N^{\alpha}$, $\lambda_2 = \zeta_N^{\beta}$. We conclude that X is invariant w.r.t. v if and only if $(r_1 + 1)\alpha + (r_2 + 1)\beta \equiv 0$. For N being prime this holds for α and β assumed if and only if $k_1 \equiv k_2$. □

It follows by the proposition above that $Y := \tau(X) = \omega_{\sigma_a J^k} \in \mathcal{B}'_{f, S \ltimes \mathcal{J}}$ is well–defined. We have $J^*(Y) = \det(J)^{-1} Y = Y$ and it remains to consider the action of symmetric part elements on both X and Y. Every S–element $\sigma' \in Z(\prod_a \sigma_a J^k) \subset S \ltimes \mathcal{J}$ gives $(\sigma')^* Y = \det(\sigma')^{-1} Y = Y$. At the same time we have $\sigma' \in Z(u)$ and $(\sigma')^* X = X$ because the sector is stable.

4.4.3 Bidegree

Let $X = \xi_u \in \mathcal{B}_{tot}$ with narrow $u = \prod_{a=1}^{p} \sigma_a g_a$. Set $Y := \tau(X) \in \mathcal{A}_{tot}$. Let $d_1, \ldots, d_p$ be the exponents associated to $g_1, \ldots, g_p$ as in Section 4.2.2. We have

$$q_l(Y) = q_r(Y) = \sum_{a=1}^{p} \frac{d_a - 1}{N} + \frac{N - p}{2} - \frac{N - p}{N} = \sum_{a=1}^{p} \frac{d_a}{N} - 1 + \frac{N - p}{2}, \tag{4.43}$$

$$q_l(X) = \frac{N - p}{2} + \sum_{a=1}^{p} \frac{d_a}{N} - 1, \quad q_r(X) = \frac{N + p}{2} - \sum_{a=1}^{p} \frac{d_a}{N} - 1. \tag{4.44}$$

This gives the desired statement for the map τ, sending narrow sector $X \in \mathcal{B}'_{f, S \ltimes \mathrm{SL}_f}$ to $Y = \tau(X) \in \mathcal{A}'_{f, S \ltimes \mathcal{J}}$.

Because τ is involutive, this also approves the case when τ maps a broad sector element of $\mathcal{B}'_{f, S \ltimes \mathrm{SL}_f}$ to a narrow element of $\mathcal{A}'_{f, S \ltimes \mathcal{J}}$. However we find it useful to provide the full prove in this case too.

Let $X = \prod_{a=1}^{p} \lfloor x_{i_a}^{r_a} \rfloor \xi_{\sigma_a}$ and $Y := \tau(X) = \prod_{a=1}^{p} \omega_{\sigma_a J^k}$. According to the definition of the mirror map we have for some $l_a \geqslant 0$ that $r_a + 1 = k \cdot |\sigma_a| - l_a N$. Moreover we can assume $l_a < |\sigma_a|$ due to our assumptions on k and r_a.

We have

$$q_l(X) = q_r(X) = \frac{\sum_{a=1}^{p} r_a}{N} + \frac{N - p}{2} - \frac{N - p}{N} = \frac{\sum_{a=1}^{p} (r_a + 1)}{N} - 1 + \frac{N - p}{2}.$$

The bigrading of Y has more complicated expression. The eigenvalues of $\prod_{a=1}^{p} \sigma_a J^k$ are

$$\lambda_{a,q} := \exp(2\pi \sqrt{-1} \left(\frac{q}{|\sigma_a|} + \frac{k}{N} \right))$$

for $a = 1, \dots, p$ and $q = 1, \dots, |\sigma_a|$. To compute $\mathrm{age}(\prod_a \sigma_a J^k)$ note that that the expression in the brackets is not always smaller than 1. We have

$$\lambda_{a,q} := \exp(\frac{2\pi\sqrt{-1}}{|\sigma_a|}\left(q + l_a + \frac{r_a+1}{N}\right)).$$

Because $r_a + 1 \leqslant N - 1$, we conclude that there are exactly l_a values of q, s.t. the expression inside brackets is greater of equal to $|\sigma_a|$. This gives

$$\mathrm{age}(\prod_a \sigma_a J^k) = \sum_{a=1}^{p}\left(\sum_{q=1}^{|\sigma_a|}\frac{q}{|\sigma_a|} + |\sigma_a|\frac{k}{N} - l_a\right) = \sum_{a=1}^{p}\left(\frac{|\sigma_a|-1}{2} + \frac{k|\sigma_a|}{N} - l_a\right). \tag{4.45}$$

Summing up we have $q_l(Y) = \frac{N-p}{2} + k - \sum_{a=1}^{p} l_a - 1 = q_l(X)$. It's easy to see that $N - 2 = q_l(Y) + q_r(Y)$, what finishes the proof.

4.5 Mirror map for Fermat quintic

The purpose of this section is to prove the following theorem.

Theorem 4.17. *For $N = 5$ there is a vector space isomorphism $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f} \to \mathcal{A}_{f,S\ltimes\mathcal{J}}$.*

Corollary 4.18. *Denote $h^{p,q}_{\mathrm{SL}} := h^{p,q}(f, S \ltimes \mathrm{SL})$ and $h^{p,q}_{\mathcal{J}} := h^{p,q}(f, S \ltimes \mathcal{J})$. We have*

$$h^{1,1}_{\mathrm{SL}} + h^{2,1}_{\mathrm{SL}} = h^{1,1}_{\mathcal{J}} + h^{2,1}_{\mathcal{J}}.$$

Proof. In both spaces the subspaces of $(0,0)$, $(3,3)$, $(3,0)$ and $(0,3)$ classes are all 1–dimensional. Both $\mathcal{B}_{f,S\ltimes\mathrm{SL}}$ and $\mathcal{A}_{f,S\ltimes\mathcal{J}}$ have a pairing, respecting the bidegree (cf. Theorem 32 of [**?**]). The statement follows now from the equality of the dimensions of $\mathcal{B}_{f,S\ltimes\mathrm{SL}}$ and $\mathcal{A}_{f,S\ltimes\mathcal{J}}$. □

The proof of the theorem above occupies the rest of this section.

To show the theorem we first need to redefine the mirror map.

4.5.1 Definition

As in general case set $\widehat{\tau}(X) := \prod_{a=1}^{p}\widehat{\tau}\left(\lfloor\widetilde{x}_{i_a}^{r_a}\rfloor\xi_{\sigma_a g_a}\right)$ for $X = \prod_{a=1}^{p}\lfloor\widetilde{x}_{i_a}^{r_a}\rfloor\xi_{\sigma_a g_a}$. We (re)define $\widehat{\tau}$ on the generalized cycles.

Case 1: $u = \sigma_1 \cdot g_1$ is a non–special cycle

Assume also $g = \prod_p t_p^{d_{1,p}}$ with p running over I_g^c. Denote $d_1 := \sum_p d_{1,p} \mod N$ with $1 \leqslant d_1 \leqslant N-1$.

$$\widehat{\tau}(\xi_{\sigma\cdot g}) := \widehat{\tau}_1(\xi_{\sigma\cdot g}) + \widehat{\tau}_2(\xi_{\sigma\cdot g}), \tag{4.46}$$

$$\text{for}\quad \widehat{\tau}_1(\xi_{\sigma\cdot g}) := \lfloor\widetilde{x}_1^{d_1-1}\rfloor\cdot\omega_\sigma,\quad \widehat{\tau}_2(\xi_{\sigma\cdot g}) := \omega_{\sigma\cdot h}, \tag{4.47}$$

for $h = (\prod_a t_a)^{k_1}$ and $k_1 \in 1, \dots, N-1$, the unique integer, s.t. $d_1 \equiv k_1|\sigma|$ modulo N.

Case 2: $u = \sigma \cdot g$ is a special cycle

We have necessarily $g = \mathrm{id}$ or $\sigma = \mathrm{id}$. For any $0 \leqslant r_1 \leqslant N-2$ set

$$\widehat{\tau}(\lfloor \widetilde{x}_{i_1}^{r_1} \rfloor \xi_{\sigma\cdot g}) := -\widehat{\tau}_3(\lfloor \widetilde{x}_{i_1}^{r_1} \rfloor \xi_{\sigma\cdot g}) + \widehat{\tau}_4(\lfloor \widetilde{x}_{i_1}^{r_1} \rfloor \xi_{\sigma\cdot g}), \tag{4.48}$$

$$\text{for} \quad \widehat{\tau}_3(\lfloor \widetilde{x}_{i_1}^{r_1} \rfloor \xi_{\sigma\cdot g}) := \lfloor \widetilde{x}_{i_1}^{r_1} \rfloor \omega_{\sigma\cdot g}, \quad \widehat{\tau}_4(\lfloor \widetilde{x}_{i_1}^{r_1} \rfloor \xi_{\sigma\cdot g}) := \omega_{\sigma\cdot h}, \tag{4.49}$$

with $h := (\prod_a t_a)^{k_1}$ and $k_1 \in 1, \ldots, N-1$, the unique integer, s.t. $r_1 + 1 \equiv k_1|\sigma|$ modulo N.

4.5.2 The properties of $\widehat{\tau}$

The map $\widehat{\tau}$ mixes up the special and non–special generators $\xi_{\sigma_a g_a}$. However we have in $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$ and $\mathcal{B}_{f,S\ltimes \mathcal{J}}$ only broad or narrow elements. It's easy to check that the map $\widehat{\tau}$ still generalizes the mirror map of Krawitz.

Proposition 4.19. *For G being SL_f or $\mathcal{J}$, let $X, Y \in \mathcal{B}'_{S\ltimes G}$ be narrow and broad basis elements respectively. We have in $\mathcal{A}'_{f,S\ltimes G}$*

$$\widehat{\tau}(X) = \widehat{\tau}_1(X) + \widehat{\tau}_2(X), \quad \widehat{\tau}(Y) = -\widehat{\tau}_3(Y) + \widehat{\tau}_4(Y). \tag{4.50}$$

Proof. This follows immediately from Proposition 4.8. □

The following proposition considers the elements for which both $\widehat{\tau}_1 \neq 0$, $\widehat{\tau}_2 \neq 0$ or both $\widehat{\tau}_3 \neq 0$,$\widehat{\tau}_4 \neq 0$.

Proposition 4.20. *Let $Y := \xi_v$ be a non–zero narrow element of $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$. We have*

(A) *both $\widehat{\tau}_1(Y) \neq 0$ and $\widehat{\tau}_2(Y) \neq 0$ in $\mathcal{A}_{f,S\ltimes\langle J\rangle}$ if and only if $\exists X$ narrow non-zero in $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$, s.t. $\widehat{\tau}_1(Y) = \widehat{\tau}_3(X)$ and $\widehat{\tau}_2(Y) = \widehat{\tau}_4(X)$.*

(B) *define the following $\mathbb{C}$–linear map $K : \mathcal{B}_{f,S\ltimes \mathrm{SL}_f} \to \mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$. Let it act by*

$$K(X) := Y \text{ and } K(Y) := -X$$

on X and Y as in (A) above and act by identity otherwise. Then the map

$$\widetilde{\tau} := \frac{1}{2}\left(\widehat{\tau} \circ K + \widehat{\tau}\right)$$

maps the basis elements of $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$ to the basis elements of $\mathcal{A}_{f,S\ltimes\langle J\rangle}$.

Proof.

(A). Assume $\widehat{\tau}_1(Y) \neq 0$, $\widehat{\tau}_2(Y) \neq 0$ and $v = \sigma \cdot g$. Let $\sigma = \prod_{a=1}^{p} \sigma_a$ be the cycle decomposition.

Set $X := \widehat{\tau}_1(Y)$. By the construction $X \in \mathcal{A}'_{f,u}$ with $u = \sigma$. For d_a as in Section 4.2.2 we have $X = \prod_{a=1}^{p} \lfloor x_{i_a}^{d_a - 1} \rfloor \omega_u$. Let $k_a|\sigma_a| = d_a \mod N$. Because $\widehat{\tau}_2(Y) \neq 0$, we have that $k_1 = \cdots = k_p := \kappa$.

To show that X is non–zero in $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$, consider the centralizer $Z(u)$ in $S \ltimes \langle J\rangle$. It's generated by the permutations $\sigma' \in S$, commuting with σ and also the diagonal element J. In $S \ltimes \mathrm{SL}_f$ the centralizer $Z(v)$ is generated by the same set of $\sigma' \in S$ as above and also the diagonal elements $h \in \mathrm{SL}_f$ commuting

with σ. Denote $J_a := \prod_i t_i$ with $i \in I^c_{\sigma_a}$. Then for some $l_1, \ldots, l_p$ we have $h = \prod_{a=1}^p J_a^{l_a}$. By Eq. (4.9) we have

$$h^*(X) = \prod_{a=1}^{p} \zeta_N^{l_a d_a} \cdot X.$$

Using the κ above, modulo N we have $\sum_{a=1}^p l_a d_a \equiv \kappa\left(\sum_{a=1}^p l_a |\sigma_a|\right) \equiv 0$ because $h \in \mathrm{SL}_f$.

The action of σ' considered is the same on X both considered in $\mathcal{B}'_{f,S\ltimes \mathrm{SL}_f}$ and $\mathcal{A}'_{f,S\ltimes\langle J\rangle}$. It is trivial because non–special $\widehat{\tau}_2(Y)$ also has σ' in it's centralizer and is non–zero.

For $\widehat{\tau}_3(X) \neq 0$, $\widehat{\tau}_4(X) \neq 0$. Set $Y := \widehat{\tau}_4(X)$. This case is treated completely similarly.

(B). The statement is obvious for X and Y, s.t. $\widehat{\tau}_1(X)$, $\widehat{\tau}_2(X)$ and $\widehat{\tau}_3(Y)$, $\widehat{\tau}_4(Y)$ are not simultaneously non–zero. If this doesn't hold, by case (A) above we have $\widetilde{\tau}(Y) = \widehat{\tau}_1(Y)$ and $\widetilde{\tau}(X) = \widehat{\tau}_4(X)$. □

Proposition 4.21. *The map $\widetilde{\tau}$ is an isomorphism for $N = 5$.*

Proof. Let's follow the steps of Section 4.4. Let X, Y be non–zero broad and narrow elements of $\mathcal{B}_{f,S\ltimes \mathrm{SL}_f}$ respectively. By using Proposition 4.20 (B) it's enough to show that for non–zero X, Y we have $\widehat{\tau}_1(Y)$, $\widehat{\tau}_2(Y)$ are not simultaneously zero and $\widehat{\tau}_3(X)$, $\widehat{\tau}_4(X)$ are not simultaneously zero. Moreover it's enough to consider the action of the symmetric part elements $\sigma' \in S$ on both sides.

Assume $\widehat{\tau}_1(Y) = 0$ in $\mathcal{A}_{f,S\ltimes\mathcal{J}}$. Then there is $\sigma' \in Z(Y)$, s.t. $\sigma' \in Z(\widehat{\tau}_1(Y))$ and $(\sigma')^*(\widehat{\tau}_1(Y)) = -\widehat{\tau}_1(Y)$. In this case we should have $\sigma = (i,j)(k,l)$ for some pairwise distinct $1 \leqslant i,j,k,l \leqslant 5$. Without loss of generality assume $\{i,j,k,l\} = \{1,2,3,4\}$. Then we should have $u = \sigma g = (1,2)(3,4) t_1^d t_3^d t_5^{-2d}$. Then we have $\widehat{\tau}_2(Y) = \xi_{\sigma J^{-2d}}$ that is non–zero in $\mathcal{A}_{f,S\ltimes\mathcal{J}}$. The case of $\widehat{\tau}_4(X) = 0$ in $\mathcal{A}_{f,S\ltimes\mathcal{J}}$ is treated completely similarly. □

4.5.3 Examples

We follow the notation of Section 4.2.1. In particular, we make use of the spaces $\mathcal{B}_{\mathrm{SL},s}$ and $\mathcal{B}_{\mathcal{J},s}$.

Case 1: Klein 4–group

Consider $S = \langle \sigma_1, \sigma_2 \rangle \subset S_5$ with $\sigma_1 := (1,2)(3,4)$, $\sigma_2 := (1,3)(2,4)$. Define also $\sigma_3 := \sigma_1\sigma_2$. The spaces $\mathcal{B}_{\mathrm{SL},s}$ and $\mathcal{A}_{\mathcal{J},s}$ are 24–dimensional. Let

$$\phi^{(1)}_{a,b,c} := \left((x_1+x_2)^a(x_3+x_4)^b - (x_1+x_2)^b(x_3+x_4)^a\right) x_5^c, \tag{4.51}$$

$$\phi^{(2)}_{a,b,c} := \left((x_1+x_3)^a(x_2+x_4)^b - (x_1+x_3)^b(x_2+x_4)^a\right) x_5^c, \tag{4.52}$$

$$\phi^{(3)}_{a,b,c} := \left((x_1+x_4)^a(x_2+x_3)^b - (x_1+x_4)^b(x_2+x_3)^a\right) x_5^c. \tag{4.53}$$

We have

$$\mathcal{B}_{\mathrm{SL},s} = \bigoplus_{\substack{a,b=1,\ldots,4\\ a\leqslant b}} \mathbb{C}\langle \xi_{\sigma_1 t_1^a t_3^b t_5^{5-a-b}} \rangle \bigoplus_{\substack{a,b=1,\ldots,4\\ a\leqslant b}} \mathbb{C}\langle \xi_{\sigma_2 t_1^a t_2^b t_5^{5-a-b}} \rangle \bigoplus_{\substack{a,b=1,\ldots,4\\ a\leqslant b}} \mathbb{C}\langle \xi_{\sigma_3 t_1^a t_4^b t_5^{5-a-b}} \rangle. \tag{4.54}$$

$$\mathcal{A}_{\mathcal{J},s} = \bigoplus_{k=1}^{3} \bigoplus_{a=1}^{4} \mathbb{C}\langle \omega_{\sigma_k} J^a \rangle \bigoplus_{k=1}^{3} \mathbb{C}\langle \phi^{(k)}_{1,0,1}\omega_{\sigma_k}, \phi^{(k)}_{2,0,0}\omega_{\sigma_k}, \phi^{(k)}_{3,1,3}\omega_{\sigma_k}, \phi^{(k)}_{3,2,2}\omega_{\sigma_k} \rangle. \tag{4.55}$$

The point that the polynomials $\phi^{(k)}_{a,b,c}$ are skew-symmetric w.r.t. action of σ_l with $l \neq k$ reflects the fact that $\sigma_l^*(\xi_{\sigma_k}) = -\xi_{\sigma_k}$.

Because $\mathcal{B}_{\mathrm{SL},s}$ only contains broad sectors, we should only consider the maps $\widetilde{\tau}_1$ and $\widetilde{\tau}_2$. One computes

$$\widehat{\tau}_1(\xi_{\sigma_1 t_1^a t_3^b t_5^c}) = \begin{cases} \phi^{(1)}_{a-1,b-1,c-1} & \text{if } a \neq b, \\ 0 & \text{if } a = b. \end{cases} \quad \widehat{\tau}_2(\xi_{\sigma_1 t_1^a t_3^b t_5^c}) = \begin{cases} 0 & \text{if } a \neq b, \\ \omega_{\sigma_1 J^{c+1}} & \text{if } a = b. \end{cases} \tag{4.56}$$

The last equality looks to be mysterious because it only depends on c. However, the mystery is resolved by the fact that we have coincidence of the indices a, b in this case and a, b, c are all connected by the degree condition.

In this example the map K appeared to be identity and $\widetilde{\tau} = \widehat{\tau}$.

Case 2: $S \cong S_3 \times S_2$

Consider $S = \langle (1,2,3), (1,2), (4,5) \rangle \subset S_5$. The spaces $\mathcal{B}_{\mathrm{SL},s}$ and $\mathcal{A}_{\mathcal{J},s}$ are 8–dimensional. Let

$$\phi_{a,b,c} := (x_1 + x_2 + x_3)^a (x_4^b x_5^c + x_4^c x_5^b). \tag{4.57}$$

We have

$$\mathcal{B}_{\mathrm{SL},s} = \mathbb{C}\langle \xi_{(1,2,3)t_1 t_4 t_5^3}, \xi_{(1,2,3)t_1^2 t_4 t_5^2}, \xi_{(1,2,3)t_1^3 t_4^3 t_5^4}, \xi_{(1,2,3)t_1^4 t_4^2 t_5^4} \rangle \tag{4.58}$$

$$\bigoplus \mathbb{C}\langle \lfloor \phi_{0,1,1} \rfloor \xi_{(1,2,3)}, \lfloor \phi_{2,0,0} \rfloor \xi_{(1,2,3)}, \lfloor \phi_{3,2,2} \rfloor \xi_{(1,2,3)}, \lfloor \phi_{1,3,3} \rfloor \xi_{(1,2,3)} \rangle. \tag{4.59}$$

$$\mathcal{A}_{\mathcal{J},s} = \bigoplus \mathbb{C}\langle \lfloor \phi_{0,0,2} \rfloor \omega_{(1,2,3)}, \lfloor \phi_{1,0,1} \rfloor \omega_{(1,2,3)}, \lfloor \phi_{2,2,3} \rfloor \omega_{(1,2,3)}, \lfloor \phi_{3,1,3} \rfloor \xi_{(1,2,3)} \rangle \tag{4.60}$$

$$\bigoplus \mathbb{C}\langle \lfloor \phi_{0,1,1} \rfloor \omega_{(1,2,3)}, \lfloor \phi_{2,0,0} \rfloor \omega_{(1,2,3)}, \lfloor \phi_{3,2,2} \rfloor \omega_{(1,2,3)}, \lfloor \phi_{1,3,3} \rfloor \omega_{(1,2,3)} \rangle. \tag{4.61}$$

The mirror map gives

$$\widehat{\tau}_3(\lfloor \phi_{a,b,c} \rfloor \xi_{(1,2,3)}) = \lfloor \phi_{a,b,c} \rfloor \omega_{(1,2,3)}, \quad \widehat{\tau}_4(\lfloor \phi_{a,b,c} \rfloor \xi_{(1,2,3)}) = 0, \quad b \neq c, \tag{4.62}$$

$$\widehat{\tau}_1(\xi_{(1,2,3)t_1^a t_4^b t_5^c}) = \lfloor \phi_{a-1,b-1,c-1} \rfloor \omega_{(1,2,3)}, \quad \widehat{\tau}_2(\xi_{(1,2,3)t_1^a t_4^b t_5^c}) = 0. \tag{4.63}$$

In this example the map K appeared to be identity and $\widetilde{\tau} = \widehat{\tau}$.

Case 3: $S \cong S_3$ **revisited**

To illustrate how the map K works, consider $S := \langle (1,2,3), (1,2) \rangle$ as in Example 3 of Section 4.2.1. We have

$$\widehat{\tau}(\lfloor H^k \rfloor \xi_{\mathrm{id}}) = \omega_{J^{k-1}}, \tag{4.64}$$

$$\widehat{\tau}(\xi_{(1,2,3)t_1^a t_4^b t_5^c}) = \lfloor \phi_{a-1,b-1,c-1} \rfloor \omega_{(1,2,3)}, \quad \text{if } b \neq c, \tag{4.65}$$

$$\widehat{\tau}(\xi_{(1,2,3)t_1^a t_4^b t_5^b}) = \lfloor \phi_{a-1,b-1,b-1} \rfloor \omega_{(1,2,3)} + \omega_{(1,2,3)J^b}, \tag{4.66}$$

$$\widehat{\tau}(\lfloor \phi_{a,b,b} \rfloor \xi_{(1,2,3)}) = -\lfloor \phi_{a,b,b} \rfloor \omega_{(1,2,3)} + \omega_{(1,2,3)J^{b-1}}. \tag{4.67}$$

Then $K(\xi_{(1,2,3)t_1^a t_4^b t_5^b}) = -\lfloor \phi_{a-1,b-1,b-1} \rfloor \xi_{(1,2,3)}$, $K(\lfloor \phi_{a,b,b} \rfloor \xi_{(1,2,3)}) = \xi_{(1,2,3)t_1^a t_4^b t_5^b}$ and $K = \mathrm{id}$ on all the other basis vectors.

4.6 Learn more

This is exactly the bigrading introduced in [**?**, IV90].

Chapter 5

Notations

Throughout the text we keep the notation.

- μ – the Milnor number,
- $\mathbf{x} := (x_1, \ldots, x_N)$ — a point in $\mathbb{C}^N$,
- N — number of the complex variables x_i, $i = 1, \ldots, N$.
- $\mathbf{e}\left[\alpha\right] := \exp(2\pi\sqrt{-1}\alpha)$ for any $\alpha \in \mathbb{Q}$.

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