Timing of Predictions in Dynamic Cheap Talk: Experts vs. Quacks *

Aleksei Smirnov[†], Egor Starkov[‡]

January 19, 2026

Abstract

This paper studies the dynamics of announcements in prediction markets in the presence of reputation concerns. In our model, an analyst of privately known competence, who cares about his reputation, chooses when to make a prediction regarding the outcome of some future event. We find that the interplay of incentives of the quacks and the real experts produces equilibria in which the earlier reports are more credible and more informative for the receivers. Further, any report hurts the analyst's reputation in the short run, with later reports incurring larger reputation penalties. The reputation of a silent analyst, on the other hand, gradually improves over time.

Keywords: Career concerns, reputation, dynamic games, games of timing, strategic information transmission.

JEL Codes: C73, D82, D83, D84.

*An earlier version of this paper was circulated under the title "Experts, Quacks, and Fortune-Tellers: Dynamic Cheap Talk with Career Concerns". We are grateful to Editor Navin Kartik, the anonymous referees, Dan Bernhardt, Antonio Cabrales, Vincent Crawford, Jeffrey Ely, Yingni Guo, Bård Harstad, Alessandro Ispano, Stephen Morris, Nick Netzer, Wojciech Olszewski, Marek Pycia, Debraj Ray, Armin Schmutzler, Joel Sobel, Bruno Strulovici, seminar participants at Northwestern University, University of Zürich, NHH Bergen and Helsinki GSE, as well as participants of the 2017 OLIGO Workshop, the 9th edition of The Lisbon Meetings in Game Theory and Applications and 2019 Econometric Society Summer School for valuable feedback and helpful comments.

[†]Faculty of Economic Sciences, Higher School of Economics, Pokrovsky Boulevard 11, 109028, Moscow, Russia; e-mail: assmirnov@hse.ru.

[‡]Department of Economics, University of Copenhagen, Øster Farimagsgade 5, 1353 København K, Denmark; e-mail: egor@starkov.email.

1 Introduction

Where there is uncertainty, there are analysts. Build-up to any major public event summons numerous predictions of its outcome from people who claim to be experts in the field. The great recession of 2007-08 has invited plenty of forecasts as it unfolded, as well as extensive judgment of these forecasts in retrospect.[1] Every presidential election cycle in the U.S. turns much of the local and plenty of foreign media into prediction mode for many months.[2] The COVID-19 global pandemic in 2020 has seen a surge of related academic research making prognoses of possible outcomes.[3] It is no surprise, however, that not all predictions are equally informative to the receivers. The quality of a given prediction typically depends on the competence of the analyst making it.

One way for the receivers to learn something about the quality of a prediction (and the competence of its author) is to consider *when* this prediction was made. Specifically, for a given event (macroeconomic outcomes, elections results, company earnings reports, sports outcomes, etc), earlier predictions may be more informative due to the better private information of the early analysts and/or herding effects. Alternatively, the converse may be true instead—waiting for late reports could be worthwhile if the competent analysts take their time to obtain the best possible information, while the less competent have less to learn, and so predict early [Guttman, 2010]. It is not clear at the outset which effect outweighs, and whether earlier or later reports are more informative for the receivers in the end.

Empirically, the former seems to be the case: the literature exploring earnings forecasts has repeatedly shown that the earlier analyst predictions are more informative, suggesting they are made by the more competent experts [Cooper, Day, and Lewis, 2001, Shroff, Venkataraman, and Xin, 2014, Keskek, Tse, and Tucker, 2014]. However, this effect tends to be explained via the indirect/strategic effects of competence, namely, herding behavior: once the first prediction is made, other analysts prefer to herd with this prediction and share the blame if all predictions are wrong, than to predict differently and risk large reputational damages if they end up being wrong alone [Scharfstein and Stein, 1990]. In this paper, we show that herding effects are not necessary to produce this dynamic, and earlier reports can be more informative due to direct effects alone if they are coupled with career concerns.[4] In particular, we show that if analysts differ (not in the quality of prior information, but) in the rate of accumulation of private information and choose the timing of their prediction strategically to maximize their reputation, then earlier reports are more informative than the later ones, even in the absence of any strategic considerations, such as herding effects. This allows us to conclude that a similar informativeness dynamics could arise even when herding behavior is not observed.[5]

[1] `https://www.weforum.org/agenda/2018/11/should-economists-have-foreseen-the-financial-crisis/`

[2] Many major media outlets have large and regularly updated hubs dedicated to the topic. Examples for the 2020 cycle include The Economist (`https://projects.economist.com/us-2020-forecast/president`), Financial Times (`https://ig.ft.com/us-election-2020/`), HuffPost (`https://www.huffpost.com/elections`).

[3] See Dixit [2020] for a contemporary ironic take on the issue.

[4] In doing so, we answer the call of Beyer, Cohen, Lys, and Walther [2010, p.327]: “To understand analysts' incentives, one would ideally start with their overarching objectives such as to maximize their personal utility over multiple periods (potentially including reputational or career concerns).”

[5] Some expert markets feature distinct anti-herding behavior. For example, Drake, Jr, Twedt, and Warren [2023] show that social media financial analysts (SMA, likely of lower competence) do not necessarily herd with institutional analysts (likely of higher competence), with around one sixth of sell-side reports during 2006–17 being preceded—as opposed to followed—by social media analysis. Farrell, Green, Jame, and Markov [2022] further suggest that SMA

Specifically, we present a model of dynamic cheap talk with career concerns. In our model, an analyst, who is privately aware of his competence, makes a choice of whether and *when* to make a prediction about the outcome of some future exogenous event (state of the world). A competent analyst (an expert) accumulates private knowledge about the outcome, while an incompetent analyst (a quack) has no private insight. There is no direct conflict between the analyst and the observer/receiver in our model: the analyst only cares about his reputation, while the observer only cares about the information concerning the outcome. The conflict comes from the reputational concerns within the analysts market, with the quacks trying to blend in with the experts in pursuit of reputation (and benefits that high reputation grants), preventing experts from conveying valuable information to the public.

We discover that this conflict between the quacks and the experts imposes a lot of structure on equilibrium outcomes. Our first finding is that in equilibrium, the later predictions are less informative than the earlier ones, so the quality of predictions deteriorates over the prediction cycle, consistent with the aforementioned empirical evidence. This has to be the case in order to incentivize the experts to reveal their information early, which requires that there is a higher reputational reward for early reports, which, in turn, requires that the receivers must expect early reports to be of high quality. Otherwise—if later predictions were more informative—these later predictions would be rewarded with higher reputation by the receivers. This would always make the experts prefer to delay their reports, leading to a complete prediction market collapse.

A more surprising finding of our paper is that *all* predictions in such equilibria, although considered informative, are received with solid scepticism by the public. This is in the sense that making any prediction drops the analyst's reputation relative to what he could get by staying quiet. Moreover, the reputation penalties increase with time for as long as the analyst postpones the prediction. This phenomenon is driven by the quack's intertemporal preferences: in order to be indifferent between issuing a report today and issuing one tomorrow (where the latter, by the logic above, leads to a worse reputation), the latter option must be boosted by the high flow payoff from not making a report. Thus, silence is indeed golden in our model—silent analysts see their reputation gradually improving. Those, on the other hand, who choose to make a prediction and take a hit to their reputation, are gambling for the grand prize that is the reputation bonus for predicting the outcome correctly.

A typical path of the analyst's reputation arising from our model is illustrated in Figure 1. In this example the event occurs in period 6 and the analyst starts with reputation b_0 (which is the probability that the observer assigns to the analyst being competent). The analyst makes his report in period 4, and until then his reputation gradually increases. After the report, his reputation drops temporarily until the outcome is revealed, at which point he receives a reputation premium if his prediction turned out correct and is penalized by low reputation otherwise.

Given everything said above, it is not obvious why a quack would ever prefer to make any prediction, i.e., take a risky gamble at the cost of short-run reputation, when staying silent would yield a risk-free high reputation. As we show, equilibria of the form described above only exist if analysts are sufficiently risk-loving or, alternatively, if gains from reputation are sufficiently high at the top—i.e., if the gamble of making a report is appealing enough for the quack. This is the

coverage concentrates on different firms relative to brokerage research coverage, again suggesting differentiation, rather than herding.

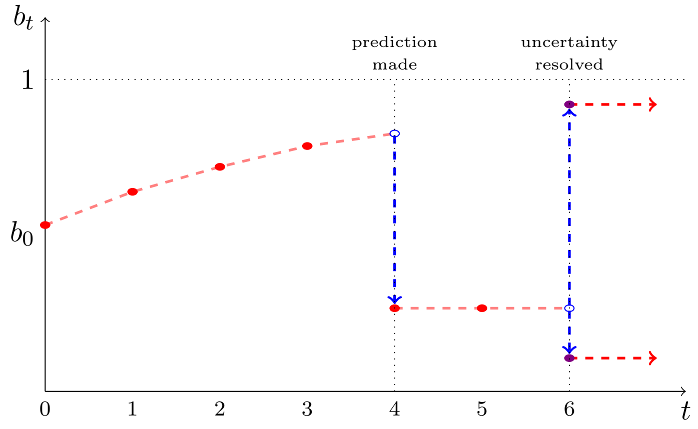

Figure 1: Analyst's example reputation path.

case in, e.g., superstar markets, where the top-regarded analysts obtain a disproportionately higher payoff (in the form of consulting contracts or assets under management). If this is not the case (the top reputation is not worth gambling for), only "static" equilibria can exist, in which all reports are made at some single predetermined date.[6]

The paper is organized as follows. We begin by looking at a stylized example in Section 2 that presents our results in the simplest setting. Section 3 contains a review of the relevant literature. In Section 4 we formulate the general model. The main results are presented in Section 5. Section 6 contains extensions and alternative specifications. Section 7 concludes. All proofs are relegated to the Appendix.

2 Illustrative Example

This section presents an example that showcases our main results in the simplest setting. Suppose there are two periods $t = 1, 2$ and a binary state $\omega \in \{G, B\}$, which is initially not known to anybody and is publicly revealed in the end of period $T = 2$. Assume that players do not discount the future, and that the states are ex ante equally probable, i.e. $\Pr(\omega = G) = \Pr(\omega = B) = \frac{1}{2}$.

There are two players: an analyst and an observer. The analyst is, with equal probabilities, either an *expert*, or a *quack*. The analyst privately knows his type, but the observer does not. The expert has a chance $\lambda_1 \in (0, 1)$ to privately learn the state in period $t = 1$ and a chance $\lambda_2 \in (0, 1)$ to privately learn the state in period $t = 2$ *conditional on not learning the state in period* $t = 1$. Therefore, with a positive probability the expert also remains unaware of the state. The quack never receives any private information about the state. In any of the two periods before the state is publicly revealed, the analyst can send one cheap talk report $m \in \{G, B\}$ to the observer, indicating his prediction about state ω. The report is not verifiable, i.e., the analyst's private information cannot be made observable to the public. At the end of each period, after the report is made (or not), the analyst receives a "reputation payoff" $w(\beta) := w\left(\frac{b}{1-b}\right)$ (it is convenient to use likelihood ratio, rather than probability, as an argument). This payoff is a strictly increasing function of belief b that the observer assigns at the end of period t the analyst being an expert. For

[6]Under special assumptions there also exist degenerate equilibria, in which quack never makes any predictions for the fear of being proved wrong. See Supplementary Appendix (Section B.1) for details.

concreteness, let $w(0) = 0$.

We look for an equilibrium in which the expert is truthful: he reports according to his private information as soon as he obtains it and never makes an unfounded prediction or reports contrary to his information. How would the quack behave in such equilibrium, and how should the market react to either report and to a lack of one?

There are five actions available to the analyst in the game: he can report that the state is G or B at $t = 1, 2$, or stay silent throughout. An honest expert plays each of the five actions with a positive probability. It is immediate then that the quack must do the same in equilibrium—if either action is only taken by the expert and never by the quack, then playing such an action gives the analyst the highest possible reputation from that point onwards and, therefore, the highest possible continuation payoff. This would yield a strictly higher payoff than any alternative path of play available to the quack in the respective period.

Therefore, the quack must be indifferent between all five actions—in particular, between reporting that the state is G at $t = 1$ and $t = 2$. Denote by b_t the belief about the analyst's competence at the end of period t in case no report was made in period t; by $b(m, t)$ the belief after report m was made in period t; and by $b^{\omega}(m, t)$ the belief after report m was made in period t and the state turned out to be ω. We use analogous notation for likelihood ratios $\beta := \frac{b}{1-b}$. The indifference condition between the two reports for the quack is then given by

$$w\left(\beta(G,1)\right) + \left[\frac{1}{2} \cdot w\left(\beta^G(G,1)\right) + \frac{1}{2} \cdot w\left(\beta^B(G,1)\right)\right] =$$
$$= w\left(\beta_1\right) + \left[\frac{1}{2} \cdot w\left(\beta^G(G,2)\right) + \frac{1}{2} \cdot w\left(\beta^B(G,2)\right)\right].$$

Note that the truthful expert is never wrong, since he only makes a report if he knows the state. Therefore, if the analyst made a prediction which turned out to be incorrect, he is definitely a quack: $b^B(G, 1) = b^B(G, 2) = 0$. Therefore, the indifference condition reduces to

$$w\left(\beta(G,1)\right) + \frac{1}{2} \cdot w\left(\beta^G(G,1)\right) = w\left(\beta_1\right) + \frac{1}{2} \cdot w\left(\beta^G(G,2)\right). \tag{1}$$

We assumed that the expert always reveals his information at $t = 1$ if he has it. However, he does have an option to delay his report until the second period if he already knows the state at $t = 1$. To ensure that there is no incentive to delay, the following condition has to hold:

$$w\left(\beta(G,1)\right) + w\left(\beta^G(G,1)\right) \geqslant w\left(\beta_1\right) + w\left(\beta^G(G,2)\right). \tag{2}$$

Note that the expert's expected utility only differs from that of the quack in the probability of guessing the state correctly—the expert knows that his private signal is correct. The two expressions (1) and (2) together produce our main results described below.

Early correct reports are rewarded higher ex post. Subtracting (1) from (2), we immediately obtain that $\beta^G(G, 1) \geqslant \beta^G(G, 2)$ or, equivalently, $b^G(G, 1) \geqslant b^G(G, 2)$. Early correct reports must thus be rewarded with higher reputation to incentivize the expert to reveal his information in a timely manner. Note that this only applies to reputation after the state was revealed.

Reporting harms reputation in the short term. Combining (1) with the observation above, we also infer that $b(G,1) \leqslant b_1$, and we can similarly obtain $b(B,1) \leqslant b_1$. Therefore, any report at $t=1$ must decrease the analyst's reputation relative to not making a report.

Reputation of a silent analyst improves. By belief consistency, we note that $b(G,1), b(B,1)$, and b_1 must average out to b_0. The inequalities we just obtained then imply that $b_1 \geqslant b_0$: if reporting harms reputation then staying silent must improve it. Similarly, one can also show that $b_2 \geqslant b_1$.[7]

Early reports are more precise. The previous observations almost immediately imply that earlier reports contain more information about the state. Indeed, $b^G(G,1) \geqslant b^G(G,2)$ and $b^B(G,1) = b^B(G,2) = 0$, therefore $b(G,1) \geqslant b(G,2)$, since by belief consistency, $b^G(m,t)$ and $b^B(m,t)$ should average out (from the observer's perspective) to $b(m,t)$. Because $b_1 \geqslant b_0$, the latter inequality means that the earlier of the two reports is relatively more likely to be made by the expert—which immediately implies that it is more informative than the latter one. This can be understood in the Blackwell sense: conditional on a report arriving in a given period, report in period $t=1$ is Blackwell-more informative than in period $t=2$.

Reputation function should be convex. So far, we have ignored the fact that quack must be indifferent between any report and staying silent. As we show here, incorporating this indifference condition implies that for the equilibrium described above to exist, the payoff function should be convex.

Let r_t, $t \in \{1,2\}$ denote the respective total report probabilities for the quack in either period. That is, in period t, either report $m \in \{G,B\}$ is sent with probability $r_t/2$, and no report is made with probability $1-r_t$. Using this to calculate all beliefs β, the quack's indifference conditions reduce to

$$w\left(\frac{\lambda_1}{r_1}\right) + \frac{1}{2}w\left(\frac{2\lambda_1}{r_1}\right) = w\left(\frac{1-\lambda_1}{1-r_1}\right) + \frac{1}{2}w\left(\frac{1-\lambda_1}{1-r_1}\cdot\frac{2\lambda_2}{r_2}\right) = w\left(\frac{1-\lambda_1}{1-r_1}\right) + w\left(\frac{1-\lambda_1}{1-r_1}\cdot\frac{1-\lambda_2}{1-r_2}\right).$$

As argued above, given the first indifference, (2) is equivalent to $\beta(G,1) \leqslant \beta_1$, which translates into $r_1 \geqslant \lambda_1$. The first indifference further implies that $r_2 \geqslant \lambda_2$. Therefore,

$$w(1) + \frac{1}{2}w(2) \geqslant w\left(\frac{\lambda_1}{r_1}\right) + \frac{1}{2}w\left(\frac{2\lambda_1}{r_1}\right) = w\left(\frac{1-\lambda_1}{1-r_1}\right) + w\left(\frac{1-\lambda_1}{1-r_1}\cdot\frac{1-\lambda_2}{1-r_2}\right) \geqslant w(1) + w(1).$$

Given $w(0)=0$, if $w(\cdot)$ is *strictly* concave, then $\frac{1}{2}w(2) \geqslant w(1)$ cannot hold, meaning there cannot exist an equilibrium with reports in both periods.

Conversely, if $w(\cdot)$ is convex, then both the indifference condition for the quack and the incentive constraint for the expert can be satisfied simultaneously. Indeed, if $w(\cdot)$ is convex and $r_1 = \lambda_1$ (and $r_2 = \lambda_2$), the value of a report is higher than the value of staying silent. Then increase r_1 and r_2

[7] By belief consistency, $b(G,2), b(B,2)$, and b_2 must average out to b_1, and we know that $b(m,2) \leqslant b(m,1) \leqslant b_1$ for $m = G, B$. Therefore, $b_2 \geqslant b_1$.

simultaneously so as to preserve the indifference between reports in either period. The payoff from reporting would decrease, while the payoff from staying silent would grow. Clearly, when r_2 reaches 1, the value of no report is strictly higher than any report. Therefore, there exist r_1, r_2 such that the quack is indifferent between all possible options. By construction, this solution also satisfies the incentive constraint for the expert because $\beta(G,1) \leqslant \beta_1$. Therefore, with convex $w(\cdot)$, the described equilibrium exists.

The construction above implies that the quack faces a choice between improving his reputation by staying silent ($\beta_2 \geqslant \beta_1 \geqslant \beta_0$) and making a report in either period, which entails a reputational gamble with a lower mean ($\beta(m,1) \leqslant \beta_0$). In order for the latter to be an attractive strategy, the high payoff from a correct prediction must significantly outweigh a possible loss after an incorrect prediction. In other words, there must exist a disproportionate premium for becoming a highly-regarded analyst in order for dynamic equilibria to exist.[8]

The remainder of the paper expands on the analysis of this example in a general framework and shows that the insights demonstrated above are quite general.

3 Relation to the Literature

Our paper mainly contributes to two strands of literature: communication with *career concerns* and the *timing of communication*. Possibly the most related to ours is the contemporary paper by Shahanaghi [2025] which similarly lies in the intersection of both literatures. Shahanaghi [2025] explores a setting, in which a media firm chooses the timing of its report in order to maximize the readers' opinion about its competence. Some similar predictions are obtained, namely that later reports yield lower reputation and that incorrect reports are punished heavily. That paper, however, assumes the firm only cares about its terminal reputation, after the state is revealed. In contrast, our sender cares about both flow reputation and terminal reputation, and it is the interaction between the two that drives our predictions.

Another highly related paper is by Boleslavsky and Taylor [2024], who consider a timing game, in which an agent decides when to propose his project to the principal. The agent wants the project to be accepted, while the principal wants the project to be finished/ready by the time it is proposed. They assume an honest agent proposes as soon as the project is ready (which happens at a random time), and an opportunistic agent chooses whether to do the same or to offer a fake/unfinished project—and if so, when to do it. Their model yields some predictions that are similar to ours, e.g., that reputation of a non-proposing agent increases over time, but differs substantially in payoff structure and in terms of focus (they mainly explore measures that can help the principal screen the projects).

Optimal timing of communication is also explored by Guttman [2010], Acharya, DeMarzo, and Kremer [2011], Guttman, Kremer, and Skrzypacz [2014], Gratton, Holden, and Kolotilin [2017], and Aghamolla and An [2021] in the context of dynamic disclosure of verifiable information. In contrast to these papers, we deal with soft information, which cannot be credibly disclosed. Grenadier, Malenko, and Malenko [2016] study a setting in which the informed expert uses the timing of his

[8] If this does not hold, then only "static" equilibria exist, in which all analysts coordinate on issuing their reports in the same period.

(non-verifiable) report to manipulate the timing of the observer's decision. Frug [2018] adds strategic information acquisition by the expert and shows that when learning takes time, communication can be more informative than if the sender was informed from the start. A separate literature [see, e.g., Aumann and Hart, 2003] explores dynamic communication of static information and shows it can improve on one-shot communication. Finally, there is a large literature on dynamic cheap talk exploring repeated interactions, see Margaria and Smolin [2018], Pavesi and Scotti [2022] for recent examples. All of the aforementioned communication models assume direct conflict of interest between the sender(s) and the receiver(s). Our model of career concerns is different in this regard, since all barriers to truthful communication stem instead from the conflict within the senders' market, namely between competent and incompetent analysts.[9]

The importance of career concerns for informative communication was first argued by Holmström [1999]. One of Holmström's original examples illustrates that an analyst may be reluctant to truthfully reveal his private information for fear of making a mistake and appearing incompetent, preferring instead to herd with public information or reports of other analysts. This idea was picked up and greatly extended upon by the literature that followed: see Scharfstein and Stein [1990], Trueman [1994], Ely and Välimäki [2003], Ottaviani and Sørensen [2006a], Dasgupta and Prat [2008], Andina-Díaz and García-Martínez [2020]. Other papers have argued that some cohort of analysts—or even all of them in some settings—may, conversely, resort to extreme reports, overstating their private signals in order to separate themselves from "herders", see Prendergast and Stole [1996], Graham [1999], Hong, Kubik, and Solomon [2000], Lamont [2002], Ottaviani and Sørensen [2006b], Mariano [2012], Kang and Kim [2022]. Either way, it is generally agreed that analysts' career concerns make information transmission noisy. Dewatripont, Jewitt, and Tirole [1999], Prat [2005], Ottaviani and Sørensen [2006c], and Vong [2025] give various general characterizations of communication outcomes in the presence of career concerns and their dependence on the information structure of the game.

Of the papers mentioned above, only a few look at the dynamics of announcements. In the model of Prendergast and Stole [1996], the analyst obtains his private information gradually over time, and his competence determines the speed of learning. They establish that the analysts overreact to early pieces of information in order to establish their reputation for competency early on, while as time progresses, they become too reluctant to change their decisions and thus underreact to late information. Predictions of a model by Graham [1999] can be interpreted in a similar way.[10] Hong, Kubik, and Solomon [2000] and Lamont [2002] find a completely opposite pattern in the data: as the analysts become older and more established, they usually make more extreme predictions. Li [2007] shows theoretically that when an analyst acquires multiple pieces of information over time, changing one's prediction can act as a signal of competence. However, timing of the prediction or a decision is never a choice variable for the analyst in these papers. Our paper fills this gap by examining how an analyst can manipulate his reputation by strategically choosing the timing of his prediction. It is also worth noting that much of the career concerns literature assumes analysts are uninformed about their type, whereas we focus instead on the case where the analyst is privately

[9]Effects similar to career concerns models can be obtained in communication settings with sender-receiver conflict where the sender's deceit can be detected with positive probability. For examples of such models see Dziuda and Salas [2019] and Drugov and Troya-Martinez [2019].

[10]Bernhardt, Wan, and Xiao [2016] observe inertia in financial analysts' predictions, but their explanation of this phenomenon does not rely on career concerns.

informed about his competence.

Keskek, Tse, and Tucker [2014] provide evidence from the field that competent experts tend to make their reports earlier—so earlier reports are more informative and are perceived more favorably—and explain this through preemption mechanisms. We show that competition is *not* necessary for this phenomenon to arise. Bernhardt, Campello, and Kutsoati [2006] also explore competitive prediction markets and discover strong anti-herding dynamics in the data.

4 The Model

4.1 Primitives

Time is discrete and finite: $t \in \{0\} \cup \mathcal{T}$ where $\mathcal{T} := \{1, \ldots, T\}$ for some $T > 1$.

Players. There are two players: an observer (she) and an analyst (he). Both players live for T periods and do not discount the future. The analyst has a binary type $\gamma \in \{E, Q\}$: he can be competent or incompetent or, as we call them, an expert (E) or a quack (Q) respectively. The type is privately known by the analyst, but is not known by the observer. The observer's prior belief that the analyst is competent is $b_0 \in (0, 1)$.

Information. A binary state of the world ω can be either *good* or *bad*: $\omega \in \{G, B\}$. The state is initially not known to anyone; the common prior belief is $p_0 := \Pr(\omega = G) \in \left[\frac{1}{2}, 1\right)$. At the end of period T the state is publicly revealed. The expert can privately observe ω at some random time $t^* \sim F(t)$.[11] We assume that $F(t)$ has full support on $\mathcal{T}$ and that $F(T) < 1$, i.e., there is a positive probability that the signal arrives at any time t, and it is also possible that it does not arrive by time T. We let $\eta_t^\gamma \in \{\varnothing, G, B\}$ denote the private information that the analyst of type γ has about the state at time t, so $\eta_t^E = \omega$ for $t \geqslant t^*$, and $\eta_t^\gamma = \varnothing$ otherwise.

Actions. In any period $t \in \mathcal{T}$, the analyst can send a report $m \in \{G, B\}$ to the observer, indicating his prediction about state ω, or stay silent, which we interpret as an empty message, $m = \varnothing$. The report is not verifiable, i.e., the analyst's private information is not ever observable and/or contractible. Additionally, we assume that the analyst can send at most one (non-empty) report throughout the game.[12]

The observer has no actions in the model. In every period t she updates her beliefs p_t about the state of the world and b_t about the analyst's competence given the analyst's report or lack thereof.[13] It will prove convenient to represent these beliefs as likelihood ratios rather than probabilities, so let $\rho_t := \frac{p_t}{1-p_t}$ and $\beta_t := \frac{b_t}{1-b_t}$.

Payoffs. At the end of every period $t < T$, the analyst receives a "flow reputation payoff" $w(\beta_t)$ which depends on the observer's belief about the analyst's competence held at the end of period t. After the state is revealed in period T, the analyst receives a "terminal reputation payoff" $w^c(\beta_T)$, representing the analyst's continuation value from the reputation he has accumulated. We assume that both $w(\cdot)$ and $w^c(\cdot)$ are strictly increasing and normalize $w(0) = w^c(0) = 0$. Payoffs can be

[11] In Section 6 we show that all results continue to hold in case of imperfect signals, given additional conditions.

[12] This constraint should not be seen as restrictive since the analyst receives at most one private signal by time T and there are no public signals.

[13] In the discussion surrounding the model, we assume that she is interested in information about state. To fix ideas, one may think that the observer chooses a binary action from $\{G, B\}$, receives a fixed reward if and only if her action matches the state, and potentially faces some cost of delay—but we do not model this decision explicitly.

interpreted as either competitive wages from the observer, or coming from some external source (e.g., a highly regarded analyst can bargain higher wage from employers in the labor market or get more followers on social media).

4.2 Timing

In period $t = 0$, the state of the world ω, the analyst's type γ, and the expert's signal arrival time t^* are drawn. Type γ (but not time t^*) is privately observed by the analyst. After that, in every period $t \in \{1, ..., T-1\}$, the stage game proceeds as follows:

1. If $t = t^*$, the expert $\gamma = E$ receives a private signal and updates his belief about the state;
2. The analyst decides whether to send report $m \in \{G, B\}$ to the observer or to stay silent, $m = \varnothing$;
3. The observer updates her beliefs, ρ_t about the state and β_t about the analyst's competence, given the analyst's report or lack of thereof;
4. The analyst receives payoff $w(\beta_t)$;

In period T, steps 1 and 2 take place as above, but instead of steps 3 and 4 the following happens:

3. State ω is publicly revealed;
4. All players update their beliefs accordingly;
5. The analyst receives terminal lump-sum payoff $w^c(\beta_T)$.

4.3 The Analyst's Problem

A *public history* h_t consists of the variables that are publicly observable at the beginning of period t, which is current period t and the analyst's past report (m, s) if report m has been made in period $s \leqslant t$. The expert's private history also includes his private information η_t^E.[14] The quack never receives any private signals, so $\eta_t^Q = \varnothing$ for all t. The quack's private histories are then equivalent to public histories, and hereinafter we will treat them as such.

Since the analyst is restricted to sending at most one report, only histories at which no report had been made involve a non-trivial choice of action. Therefore, we introduce the analyst's *behavioral strategy* as $r_\eta^\gamma(m, t)$, which denotes the probability of analyst γ making report m at time t conditional on having private information η and having not made any report prior to t. Further, let $r^\gamma(m, t) := \mathbb{E}_\eta r_\eta^\gamma(m, t)$ denote the hazard rate of report (m, t) by an analyst of type γ as perceived by the observer (who does not observe the analyst's private information η).

The analyst's problem is hence as follows: at every private history (h_t, η_t^γ) such that no report has yet been made, the analyst of type $\gamma \in \{E, Q\}$ chooses a contemporaneous reporting strategy

[14]In principle, the informed expert's private history also contains t^*, the arrival time of the private signal. It is, however, straightforward that t^* is not payoff-relevant to any player, and neither is it observable by anyone except the expert, so it can be seen as nothing more than the expert's private randomization device.

$r_\eta^\gamma(m,t)$ that solves

$$V_{t,\eta}^\gamma := \max_{r_\eta^\gamma} \mathbb{E}\left[\sum_{s=t}^{T-1} w\left(\beta(h_s)\right) + w^c\left(\beta(h_T)\right) \middle| h_t, \eta_t^\gamma\right] \tag{3}$$

subject to the evolution of $\beta(h_s)$. The expectation is taken over all possible future histories. As the quack never receives the private signal, we suppress the subscript η when talking about $V_{t,\eta}^Q$ and refer to it simply as V_t^Q.

4.4 Equilibrium Definition

We are looking for Weak Perfect Bayesian Equilibria of the game, which consist of a strategy profile $\{r_\eta^\gamma(m,t)\}$ and a belief profile $(\beta(h_t), \rho(h_t))$ such that:

1. strategies r_η^γ solve (3) given the observer's updating rule for $\beta(h_t)$,
2. all players update their beliefs via Bayes' rule on path,

Cheap talk games are infamous for equilibrium multiplicity problems, and the dynamic setting exacerbates the problem. Therefore, we adopt some equilibrium refinements in order to focus our analysis. Before we introduce the refinements, let us first define the equilibrium *support* $\mathcal{S}$, which is the set of times at which reports are made in equilibrium:[15]

$$\mathcal{S} := \{t \in \mathcal{T} \mid r^\gamma(m,t) > 0 \text{ for some } \gamma \in \{E,Q\}, m \in \{G,B\}\}. \tag{4}$$

We refer to the first point of the support as t_1, the second point as t_2, etc, and the final point as $\bar{t}$.

The following refinements are implied whenever we talk about an "equilibrium", unless stated otherwise:

(OP) Off-path Pessimism: off the equilibrium path the beliefs are $\rho = \rho_0$ and $\beta = 0$, with the exception that an extreme belief $\beta = +\infty$ $(b = 1)$ is not updated,

(TE) Truthful Expert: $r_G^E(G,t) = r_B^E(B,t) = r_\varnothing^E(\varnothing,t) = 1$ for all $t \in \mathcal{S}$.

Off-path pessimism (OP) is a simplifying restrictiong on the off-path beliefs. Note that (OP) is *almost* without loss in the sense that it makes it easier to sustain any given strategy profile as equilibrium, because it makes deviations extremely unappealing for the analyst. In particular, if there is some PBE with some off-equilibrium path beliefs, then the same profile of strategies and on-path beliefs would still constitute a PBE when paired with the off-path beliefs prescribed by (OP). The only loss of generality comes from the exception, which activates only at specific histories.[16]

[15] More formally, support $\mathcal{S}$ is a subset of public histories h_t for which $r^\gamma(m,t) > 0$ for some γ and m. Since a public history in our model consists of current time and report record, and the analyst can make at most one report, it is unambiguous to define the support simply as a set of times.

[16] The exception is known as a "never dissuaded once convinced" assumption in the signaling and bargaining literature, introduced by Osborne and Rubinstein [1990] (see Starkov, 2023 for a broad discussion of this assumption in dynamic games). As the name suggests, it requires that once the observer makes up her mind (converges to a degenerate belief) about the analyst's type, she does not update her belief further. This can be supported by a perturbation of the model, in which either the expert, or the receiver observe the state with noise. Section B.1 discusses alternatives to (OP) in case of no noise.

Truthful expert (TE) is a non-trivial restriction, which refines away some equilibria. It requires that the expert always reports according to the private signal whenever an opportunity presents, and abstains from reporting otherwise. We emphasize that (TE) is an equilibrium refinement, rather than an assumption—which means that this strategy must be optimal for the expert in equilibrium.

5 Equilibrium Analysis

The main question that is answered in this section is as follows: assuming that in equilibrium, reports are only made at some set of periods $S \subseteq \mathcal{T}$, how do the analyst's strategies look and how does the informativeness of the reports change across different periods? It turns out that all equilibria have quite a lot of common structure, described in Section 5.2. Section 5.4 then provides a characterization of when the respective equilibria exist. Section 5.5 compares equilibria with different supports. Proofs of all statements presented in this chapter can be found in the Appendix.

5.1 Belief Updating

The two important characteristics of any public history h_t are the observer's beliefs $\beta(h_t)$ and $\rho(h_t)$ about the analyst's type and the state of the world, respectively.

Given a time-t history $h_t = (t, (m, \tau))$ where report $m \in \{G, B\}$ has been made at time $\tau \leqslant t$, we define the following shorthand notation:

$$\beta(m, \tau) := \beta(t, (m, \tau)) \qquad \rho(m, \tau) := \rho(t, (m, \tau))$$
$$\beta_t := \beta(t, \varnothing) \qquad \rho_t := \rho(t, \varnothing)$$

Here, $\beta(m, \tau)$ is the observer's belief at time t about the analyst's type given report m made at time t, and β_t is the belief held at time t in the absence of any reports. Note that $\beta(m, \tau)$ is well defined, because once a report has been made, no further information can be conveyed from the analyst to the observer, hence the belief is frozen in place. The same applies to her belief about state, ρ, and we will use the same notation for the respective probabilities b and p whenever applicable. Finally, we let $\beta^{\omega}(m, \tau)$ denote the belief about the analyst's type given a terminal history $h_T = (T, (m, \tau))$ and the revealed state ω.

Moving to beliefs about the state, condition (TE) implies that any information about ω is immediately (at first opportunity) conveyed via a report. Since the expert's learning technology is symmetric across states, this means that the observer's belief ρ_t remains constant in the absence of a report: $\rho_t = \rho_0$ for all t. Conversely, reports are informative: $\rho(G, t) > \rho_0 > \rho(B, t)$. We can then treat the arrival of a report in period t as a binary Blackwell experiment $\phi_t(m|\omega)$ that prescribes a distribution over reports $m \in \{G, B\}$ conditional on each state $\omega \in \{G, B\}$ and conditional on some report being issued in period t. We can then compare these experiments using the standard Blackwell order.

Definition. *Given an equilibrium with support $\mathcal{S}$, we say that reports are more informative about state at $t' \in \mathcal{S}$ rather than at $t'' \in \mathcal{S}$ if the Blackwell experiment $\phi_{t'}$ corresponding to reports (m, t'), $m \in \{G, B\}$ is Blackwell-more informative than experiment $\phi_{t''}$ corresponding to (m, t''), $m \in \{G, B\}$.*

The exact expression for $\phi_t(m|\omega)$ is presented in the Appendix (see (11)). With binary state and binary messages, experiment ϕ' is more informative than experiment ϕ'' if and only if ϕ' produces a wider distribution of posterior beliefs, which is equivalent to likelihood ratios satisfying $\frac{\phi'(G|G)}{\phi'(G|B)} \geqslant \frac{\phi''(G|G)}{\phi''(G|B)}$ and $\frac{\phi'(B|B)}{\phi'(B|G)} \geqslant \frac{\phi''(B|B)}{\phi''(B|G)}$.

5.2 Characterization of Dynamic Equilibria

This section explores the properties of an arbitrary dynamic equilibrium (with support $\mathcal{S}$ such that $|\mathcal{S}| \geqslant 2$), assuming it exists. We start by stating our formal result and proceed by explaining the equilibrium logic that produces it. All monotonicity statements in the proposition below are understood in the weak sense.

Proposition 1. *In any equilibrium with $|\mathcal{S}| \geqslant 2$, the following are true for both $m \in \{G, B\}$:*

1. *reports are more informative about state at $t' \in \mathcal{S}$ rather than at $t'' \in \mathcal{S}$ whenever $t' < t''$;*
2. *the reputation of a silent analyst improves over time: β_t is increasing in t on $\mathcal{S}$;*
3. *making any report decreases reputation as compared to no report: $\beta(m,t) \leqslant \beta_t$ for any $t \in \mathcal{S}$.*

To start on the path towards understanding the result above, notice first that neither report can perfectly reveal the analyst's type in equilibrium. This is readily seen by contradiction: if some report (m,t) is only ever made by an expert, then this report gives the highest reputation: $b(m,t) = 1$ and $b^\omega(m,t) = 1$ for any realized state ω. Reporting (m,t) is then strictly preferred by the quack to any other alternative. Conversely, if some report (m,t) is only ever made by a quack, then this report gives the lowest reputation: $b(m,t) = b^\omega(m,t) = 0$ for any ω. The quack then has no incentive to send this report, and he strictly prefers to mimic an expert. This argument is formalized by Lemma 6 in the Appendix.

Lemma 6 together with condition (OP) clearly imply that the quack should be indifferent between all reports made in equilibrium, as well as staying silent. This indifference condition is a significant driver in our model, together with (TE) when it is interpreted as an incentive constraint. In particular, consider two reports (G,t') and (G,t'') predicting the state is $\omega = G$ and made in periods t' and t'', respectively. A quack must be indifferent between these reports, as argued above. An expert who has received signal $\eta = G$ at t' must be willing to make the corresponding report immediately rather than delay—i.e., weakly prefer (G,t') over (G,t''). The expert and the quack receive the same flow payoffs from either report. The only difference between their payoffs—the difference that must generate different incentives—comes from the probabilities of guessing the state correctly and receiving high terminal reputation, $\beta^G(G,t)$, instead of low, $\beta^B(G,t)$. Given (TE), an incorrect report reveals a quack perfectly: $\beta^B(G,t) = 0$. Therefore, given the quack's indifference, the informed expert picks a report that maximizes $\beta^G(G,t)$ or, equivalently, $\beta(G,t)$. Then delaying a report is unprofitable if and only if $\beta^G(G,t') \geqslant \beta^G(G,t'')$ and $\beta(G,t') \geqslant \beta(G,t'')$. This argument applies to $\eta = B$ as well and to any pair of reporting periods, hence we conclude that both $\beta(m,t)$ and $\beta^m(m,t)$ must be weakly decreasing in t for either $m \in \{G, B\}$, meaning that later reports yield lower reputation.

This dynamics then immediately implies that any report must be associated with a hit to the analyst's reputation, as compared to the counterfactual reputation the analyst could have had by

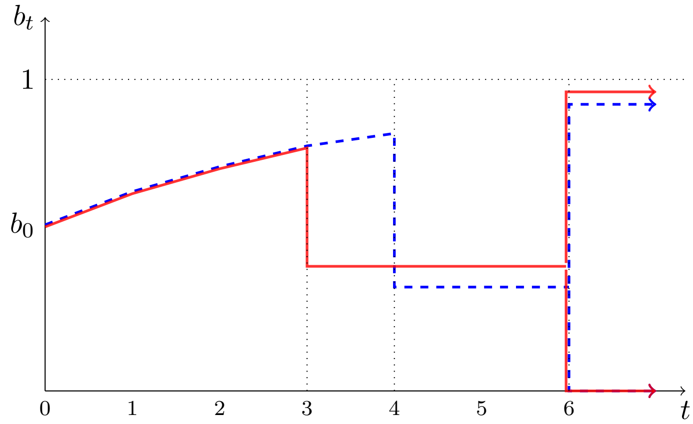

Figure 2: Report penalties increase over time.

staying silent. Indeed, consider the first period in the support, t_1, and suppose, by contradiction, that $\beta(m, t_1) > \beta_{t_1}$, meaning that staying silent at t_1 yields a higher reputation than reporting m. Then report (m, t_1) yields a strictly higher payoff to the quack than (m, t_2)—the former grants a higher payoff at t_1, a higher payoff between t_2 and T, and a higher continuation payoff after T. This contradicts the quack's indifference, hence it must be that $\beta(m, t_1) \leqslant \beta_{t_1}$ By the martingale property of beliefs, $b(m, t_2)$ and b_{t_2} should average out to b_{t_1}, so the inequality above together with $\beta(m, t_1) \geqslant \beta(m, t_2)$ implies

$$\beta(m, t_2) \leqslant \beta(m, t_1) \leqslant \beta_{t_1} \leqslant \beta_{t_2}. \tag{5}$$

This argument unravels to all $t \in \mathcal{S}$, granting the second and third statements of Proposition 1. This logic is exemplified in Figure 2, where the red solid line shows the reputation path of an analyst who makes a report at $t = 3$, and the blue dashed line shows the counterfactual reputation path if the same report is made at $t = 4$.

The first statement of Proposition 1 formally follows from the reputation dynamics (5) obtained above, as well as the updating rules for belief about the state. Intuitively, the informativeness monotonicity can be explained as follows. As t increases, β_t increases while $\beta(m, t)$ decreases. Therefore, in equilibrium, later reports cause a larger drop in reputation. Given (TE), it means that later reports are relatively more likely to be made by a quack, which immediately implies that later reports must also be less informative about the state.

It is worth noting that while the inequalities in the statement of Proposition 1 are weak, they can only hold with equality in non-generic cases. The logic presented above implies that any strict inequalities unravel into further strict inequalities and vice versa, equalities translate into equalities. For example, assuming $\beta(m, t_1) = \beta_{t_1}$ for some $m \in \{G, B\}$ would eventually require (by the martingale property) that $\beta(m, t) = \beta_t = \beta_0$ for all $t \in \mathcal{S}$ and both m, and consequently, $\beta^G(G, t) = \frac{\beta_0}{p_0}$ and $\beta^B(B, t) = \frac{\beta_0}{1-p_0}$. Therefore, to sustain the quack's indifference, the terminal payoff function $w^c(\cdot)$ has to be such that $w^c(\beta_0) = p_0 \cdot w^c\left(\frac{\beta_0}{p_0}\right)$ and $w^c(\beta_0) = (1 - p_0) \cdot w^c\left(\frac{\beta_0}{1-p_0}\right)$ both hold, which is not generally true unless it is linear.

Finally, the third statement, despite being inherently static, also requires $|\mathcal{S}| \geqslant 2$. If $|\mathcal{S}| = 1$, then the statement no longer holds: one may construct an equilibrium with $\mathcal{S} = \{t\}$ such that $\beta(m, t) > \beta_t$ for both $m \in \{G, B\}$. In this equilibrium, either report is more likely to be made by an expert than a quack, and so generates a reputation premium. Therefore, "static" equilibria with $|\mathcal{S}| = 1$ can potentially be more informative than "dynamic" equilibria, and allowing for reports to be made at more than one point in time may actually be harmful to the informativeness of these reports. We discuss this issue in more details in Section 5.5.

5.3 Static Equilibria

We now discuss the properties of static equilibria—those with a singleton support $\mathcal{S}$. In such equilibria, all reports must be made in a single period. In equilibria satisfying (TE), if an expert has received a private signal by time $\bar{t}$, he reveals it truthfully, and stays silent otherwise. A quack then has to mix between making a report at $\bar{t}$ and staying silent.

Properties of dynamic equilibria above relied on dynamic incentive constraints—namely, on the preferences of the informed expert regarding making a given report in different periods. However, these constraints do not apply in static equilibria. Instead, we only need to create the right balance of incentives between making a report and staying silent, as illustrated by the following example.

Example. Suppose $w(\beta) = w^c(\beta) = \frac{\beta}{1+\beta} = b$, $\mathcal{S} = \{t\}$, and $p_0 = \frac{1}{2}$. Then the quack's indifference condition amounts to

$$\theta \cdot b(G,t) + (1-\theta) \cdot p_0 \cdot b^G(G,t) = b_t,$$

where $\theta := \frac{T-t}{T-t+1} \in (0,1)$. Note, however, that the martingale property of beliefs implies that $b(G,t) = p(G,t) \cdot b^G(G,t)$, where $p(G,t)$ is the posterior belief after report (G,t) was made, and $p(G,t) > p_0$ because reports are informative. Therefore, for the indifference condition to hold, it must be that $b(G,t) > b_t$, i.e., a report must increase the analyst's reputation.[17] ■

In the example above, we obtain a conclusion that is in direct opposition to part 3 of Proposition 1. If the payoff from reputation is linear in belief b, then the quack is reluctant to report because this exposes him to scrutiny. Silence offers a sure payoff, while reporting gives rise to the possibility of guessing the state wrong and revealing own incompetence. In this case, quacks need an extra stimulus to send reports relative to staying silent—a reputational lottery associated with making a report must have a higher mean than reputation from silence.

We obtain this result for a static equilibrium for a given payoff function, but the same incentive constraints apply in a dynamic equilibrium as well (at least in the final period in the support). It follows that in a dynamic equilibrium with this payoff function, a report must seemingly both decrease and increase reputation—a contradiction? Indeed, we argue in the following section that dynamic equilibria need not always exist due to such contradictory requirements embedded in different incentive constraints.

5.4 Equilibrium Existence

So far we have discussed properties of equilibria without proving that any equilibria actually exist. An uninformative equilibrium with $\mathcal{S} = \varnothing$ always exists, but the existence of other equilibria is not a trivial concern.

[17]We thank an anonymous referee for presenting this example to us.

The following Proposition outlines some necessary and sufficient conditions for existence of informative equilibria, which allow to understand some driving forces behind their existence and non-existence.

Proposition 2. *Suppose $w(\cdot)$ and $w^c(\cdot)$ are continuous. Then*

1. *If $\lim_{\beta\to+\infty} w^c(\beta) = +\infty$, for any t, there exists a unique equilibrium with support $\mathcal{S} = \{t\}$;*[18]
2. *If $w^c(\beta)$ is linear then for any $\mathcal{S} \subseteq \mathcal{T}$, there exists a unique equilibrium with support $\mathcal{S}$;*
3. *If $w^c(\beta)$ is strictly convex, there exists $\bar{p}_0 > \frac{1}{2}$ such that for any $p_0 \in [\frac{1}{2}, \bar{p}_0]$ and any $\mathcal{S} \subseteq \mathcal{T}$, there exists a unique equilibrium with support $\mathcal{S}$;*
4. *If $w^c(\beta)$ is strictly concave, then no equilibria with $|\mathcal{S}| \geqslant 2$ exist.*

Part 1 of Proposition 2 states that static equilibria discussed in Section 5.3 always exist if the premium for being a highly-regarded analyst is sufficiently large. If this is not true, reporting a less probable state can be too unappealing for the quack, since the probability of guessing the state is low, and the premium for doing so is too small. To guarantee that there are incentives to report an improbable state, we need to ensure that the premium is high enough.

In turn, parts 3 and 4 suggest that the same dynamic incentive constraints that yield the equilibrium dynamics described in Proposition 1 can preclude the existence of dynamic equilibria. Part 3 of Proposition 2 provides a sufficient condition for dynamic equilibria to exist, which is the convexity of the terminal payoff function $w^c(\cdot)$ and both states being "reasonably probable".[19] The following example illustrates that the bound on prior probability p_0 that we provide is arbitrarily tight and cannot be improved.

Example. Suppose $w(\beta) = w^c(\beta) = \beta^2$, $T = 2$, and the expert's private information arrives at constant rate λ (so $F(t) = 1 - (1-\lambda)^t$). Let us try to construct an equilibrium with $\mathcal{S} = \{1, 2\}$. The quack's values from the respective reports G and B in the first period are

$$\frac{p_0^2 \cdot \lambda^2}{r^Q(G,1)^2} \cdot \left(1 + \frac{1}{p_0}\right),$$
$$\frac{(1-p_0)^2 \cdot \lambda^2}{r^Q(B,1)^2} \cdot \left(1 + \frac{1}{1-p_0}\right).$$

[18] A less demanding sufficient condition is $\lim_{\beta\to+\infty} w^c(\beta) \geqslant \frac{p_0}{1-p_0} \cdot w^c(1)$.

[19] An empirical inquiry by Bernhardt, Campello, and Kutsoati [2004] also proposed the convexity of payoffs in reputation as an explanation of the observed dynamics of the analysts' reports. In their case, the observed phenomenon was the strong anti-herding in predictions.

The quack's values from the reports G and B in the second period and no report (evaluated at $t = 1$) are, respectively,

$$\left(\frac{1-\lambda}{1-r^Q(G,1)-r^Q(B,1)}\right)^2 \cdot \left(1 + p_0 \cdot \left(\frac{\lambda}{r^Q(G,2)}\right)^2\right),$$
$$\left(\frac{1-\lambda}{1-r^Q(G,1)-r^Q(B,1)}\right)^2 \cdot \left(1 + (1-p_0) \cdot \left(\frac{\lambda}{r^Q(B,2)}\right)^2\right),$$
$$\left(\frac{1-\lambda}{1-r^Q(G,1)-r^Q(B,1)}\right)^2 \cdot \left(1 + \left(\frac{1-\lambda}{1-r^Q(G,2)-r^Q(B,2)}\right)^2\right).$$

By equalizing the three latter payoffs, we can explicitly calculate $r^Q(G,2)$ and $r^Q(B,2)$:

$$r^Q(G,2) = \frac{\sqrt{p_0} \cdot \lambda}{1-\lambda+\lambda \cdot \sqrt{p_0} + \lambda \cdot \sqrt{1-p_0}},$$
$$r^Q(B,2) = \frac{\sqrt{1-p_0} \cdot \lambda}{1-\lambda+\lambda \cdot \sqrt{p_0} + \lambda \cdot \sqrt{1-p_0}}.$$

At the same time, by Proposition 1, the resulting solution for this system should satisfy $\beta(m,1) \leqslant \beta_1$ for $m \in \{G, B\}$. Combining this with the quack's indifference between reporting G in the first and in the second period, we obtain that

$$\frac{1}{p_0} \geqslant p_0 \cdot \left(\frac{\lambda}{r^Q(G,2)}\right)^2.$$

Using the expression for $r^Q(G,2)$ obtained above, we get that for the equilibrium to exist, λ and p_0 mutually have to satisfy

$$\lambda p_0 + \lambda \cdot \sqrt{p_0 \cdot (1-p_0)} + (1-\lambda) \cdot \sqrt{p_0} \leqslant 1.$$

The region for λ and p_0 which are consistent with this restriction is depicted in Figure 3. As can be seen from the graph, when λ approaches 1, the set of admissible values for p_0 shrinks to $p_0 = \{\frac{1}{2}\}$. Therefore, the neighborhood of $p_0 = \frac{1}{2}$ where an equilibrium exists can be arbitrarily small. If p_0 is outside the admissible region, no dynamic equilibrium exists in this example. ■

Part 4 of Proposition 2, in turn, provides a necessary condition for dynamic equilibria to exist. It claims that if the terminal payoff function $w^c(\cdot)$ is strictly concave, then no dynamic equilibria exist. The root of the potential non-existence lies in the conflict between the quack's incentives to report instead of staying silent and the expert's incentives to report sooner rather than later. For the quack, any report is a lottery, with a chance to win reputation if he guesses the state and to lose reputation otherwise. This lottery has zero mean in terms of likelihoods: $\mathbb{E}_\omega[\beta^\omega(m,t)] = \beta(m,t)$. In turn, by Proposition 1, the expert's incentives imply $\beta(m,t) \leqslant \beta_t \leqslant \beta_{t+1} \leqslant \ldots \leqslant \beta_T$ for all (m,t). This means that staying silent yields a pointwise-higher reputation until T than any report, and higher expected reputation β after T.[20] In turn, any report not only harms reputation until the state revelation in comparison to staying silent, but also presents a risky low-mean lottery for the

[20]If no report was made, the revelation of the state does not affect the belief about the analyst: $\beta^\omega(\varnothing) = \beta_T$.

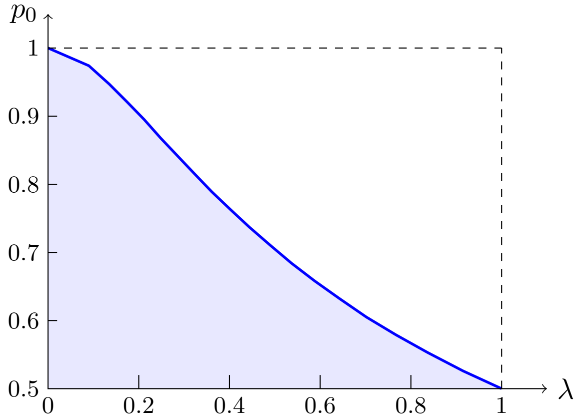

Figure 3: Admissible values for λ and p_0 (the shaded region).
Note: $w(\beta) = w^c(\beta) = \beta^2$, $T = 2$, $\mathcal{S} = \{1, 2\}$.

quack. The only scenario in which the quack wants to make a report under such conditions, is if he disproportionately values the upside of the report lottery—i.e., if $w^c(\cdot)$ is convex. We note that this rules out the existence of dynamic equilibria for the payoff function linear in belief, $w^c(\beta) = \frac{\beta}{1+\beta} = b$, used in the example in Section 5.3.

The convexity of reputational payoff can be interpreted as coming from either the analyst's inherent risk-seeking preferences, or coming from the environment, which only rewards the best and most reputable analysts. The latter aligns well with many expert markets being superstar markets, where the top tail of analysts, who are perceived to be the most competent, attract a disproportionately large part of the business and enjoy significantly higher payoffs. For example, Groysberg, Healy, and Maber [2011] show that this holds in financial analyst markets: all else equal, the "All-Star" recognition produces a premium to the analysts' compensation. Yin and Zhang [2014, p. 574] claim that "this seems to be a widely accepted fact: the popular press suggests that a large pay discrepancy exists between All-Star analysts and Non-All-Star analysts." In another application, Heckman and Moktan [2020] show that researchers from top Economics departments have easier access to publication in top-five Economics journals, which, in turn, significantly improves their career prospects, so the payoff from initial reputation is convex in the academic job market. In yet another context, Li and Martin [2019] demonstrate that for entrepreneurs on Kickstarter, the effect of positive past reputation on fundraising success is stronger than that of negative past reputation. Li [2007] establishes a theoretical result similar to ours—that payoffs to reputation must be convex in expert markets. Our result is, however, milder: Proposition 2 above does not say that payoffs must be convex, but rather claims that without this, a prediction market would collapse to a static equilibrium.

5.5 Comparison of Equilibria

In this section, we study which equilibria are "more preferred" by the observer. Each equilibrium has two dimensions relevant to the observer: informativeness of the reports in different periods, and the probability to observe a report in a given period. Sparser support $\mathcal{S}$ means that reports arrive less frequently in equilibrium, and it may take longer for a given piece of information to be disclosed by the expert, but it does not necessarily imply that less information is transmitted overall. The private signal observed by the expert at $t' \notin \mathcal{S}$ is not forgotten—its revelation is delayed until $t'' = \min\{t \in \mathcal{S} | t > t'\}$, but it *is* reported eventually. Therefore, if two supports have identical $\bar{t}$, the total amount of information reported *by the experts* is eventually the same. However, the impact of the support on the overall informativeness of reports in a given period is not straightforward. Equilibria with sparser support can, in principle, feature a larger delay between the expert receiving a signal and having a chance to report it but have less noise from the quacks. Making unambiguous ranking among equilibria is then not possible without explicitly modeling the utility function of the observer.

However, we show below that extending the reporting deadline $\bar{t}$ to a later date not only increases the amount of reported information by the expert, but also increases the report informativeness *in all periods.*[21]

Proposition 3. *Consider two equilibria with the respective supports* $\mathcal{S} = \{t_1, \ldots, t_k\}$*,* $\tilde{\mathcal{S}} = \{t_1, \ldots, t_k, t_{k+1}, \ldots, t_{k+n}\}$ *with* $k \geqslant 1$*. Then reports are more informative in the latter equilibrium for all* $t \in \tilde{\mathcal{S}}$*.*

The proposition says that expanding support to the right is unambiguously beneficial: it both allows more information to be transmitted by the informed expert (in case he observes his private information between t_{k+1} and t_{k+n}) *and* increases the informativeness of *all* reports (weakly for all $t \leqslant t_k$ and strictly for all $t > t_k$). The intuition behind the latter phenomenon is, simply speaking, that the more reporting options are available to the quack in a given equilibrium, the thinner he spreads over them. More specifically, extending the support to the right ceteris paribus implies that the reputation β_t of the silent analyst should improve at the new dates, as per Proposition 1. This renders the option of staying silent more attractive to the quack. The quack should be indifferent between staying silent and making a report, so to restore this indifference after expanding the support, we have to make reporting more appealing. This is achieved by prescribing pointwise lower $r^Q(m, t)$ in equilibrium, thereby improving report informativeness.

Figure 4 illustrates Proposition 3, as well as demonstrates the limits to making more general statements. It plots posterior beliefs $p(m, t)$ about the state following message m over t for all equilibria with different supports, corresponding to the seven dots/lines on the Figure. The prior is assumed to be symmetric, $p_0 = \frac{1}{2}$. Proposition 3 states that extending message support to the right increases informativeness in all periods in the support. One can infer this from Figure 4 by comparing the trajectory $p(m, t)$ in the equilibrium with $\mathcal{S} = \{1\}$ to that in the equilibrium with $\mathcal{S} = \{1, 2\}$ to that in the equilibrium with $\mathcal{S} = \{1, 2, 3\}$. On the other hand, Figure 4 suggests that no other comparisons can be readily made. Extending message support to the left allows the early news to be communicated in an informative way, but decreases the informativeness of the late messages—one can see this by comparing the equilibria with $\mathcal{S} = \{3\}$ to $\mathcal{S} = \{2, 3\}$ and $\mathcal{S} = \{1, 2, 3\}$. Further, it is not clear whether shifting support to the right is beneficial: the comparison of singleton equilibria ($\mathcal{S} = \{1\}$, $\mathcal{S} = \{2\}$, and $\mathcal{S} = \{3\}$) suggests that delaying the single reporting period is beneficial, since it both increases report informativeness and enables later private signals to be revealed by the experts. However, this comparison fails to extend to non-singleton equilibria: the

[21] Boleslavsky and Taylor [2024] make a similar point in a substantially different model. Frug [2018] presents a setting where the opposite is true and putting an analyst under time pressure improves the quality of communication due to endogenous learning choices (and no reputational concerns).

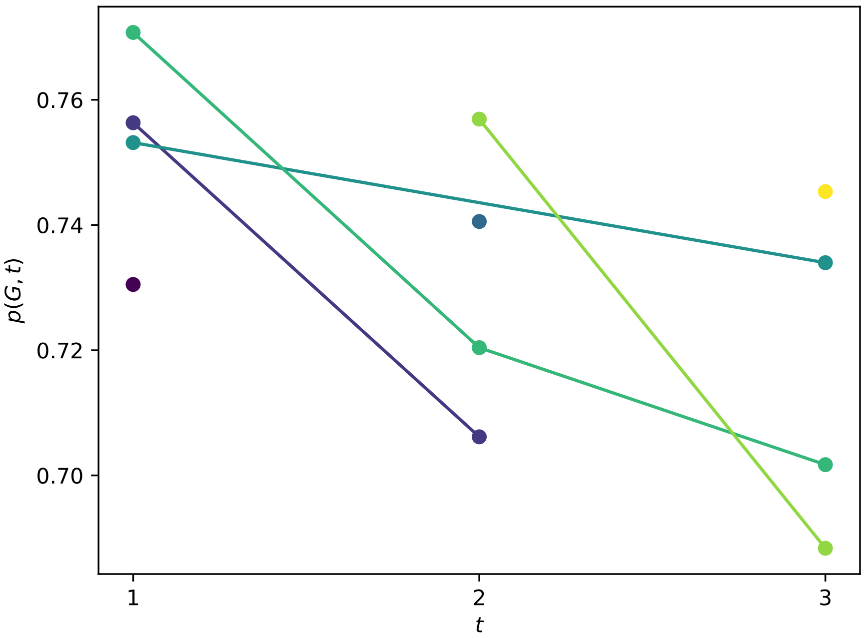

Figure 4: Report informativeness depending on support.
Note: the figure assumes $T = 3$, $p_0 = 1/2$, $b_0 = 1/2$, $w(\beta) = \beta^2$, $w^c(\beta) = 100 \cdot w(\beta)$, $F(t) = 1 - 0.5^t$. The seven dots/lines plot posterior beliefs $p(m,t)$ about the state following message m over t for all seven equilibria with different supports ($p(G,t) = p(B,t)$ because $p_0 = 1/2$) Each isolated dot represents a static equilibrium; each line connecting two or three dots represents a dynamic equilibrium where reports occur only in the corresponding periods.

equilibrium with $\mathcal{S} = \{2,3\}$ has better informativeness in the first reporting period but worse in the second when compared to $\mathcal{S} = \{1,2\}$. This example suggests that the observer's most preferred equilibrium would depend on how she trades off informativeness against the delay and the amount of signals.

6 Discussion and Extensions

In this section, we discuss various assumptions made in the main model and argue about their (un)importance for our results.

Imperfect Private Signals. Our model assumes that the expert's private signal reveals the state perfectly or, equivalently, that the observer sees the state perfectly at time T. Suppose instead that at time t^*, the expert receives signal η about the state with precision $\pi := P(\eta = G|\omega = G) = P(\eta = B|\omega = B)$, $\pi \in \left(\frac{1}{2}, 1\right)$. As long as $\pi > \frac{1}{2}$, the expert's signal is still informative about the state, so his report about the state is more likely to be supported by the ex post evidence than the quack's random guess. Proposition 4 below shows that all results continue to hold in case $\pi < 1$ if $w^c(\cdot)$ is either convex, or has bounded derivative and the private signal is sufficiently precise.

Proposition 4. *Propositions 1, 2, and 3 are true for $\pi < 1$ if either of the following holds:*

1. *$w^c(\cdot)$ is convex;*
2. *$w^c(\cdot)$ is continuously differentiable and there exist $0 < \underline{d} \leqslant \bar{d} < +\infty$ such that $\frac{dw^c(\beta)}{d\beta} \in [\underline{d}, \bar{d}]$ for all β, and $\pi > \frac{\bar{d}}{\underline{d}+\bar{d}}$.*

The intuition behind Proposition 4 can be seen as follows. When describing the intuition behind Proposition 1, we have mentioned that in order to provide the incentives for the informed expert to reveal his private information immediately instead of waiting for a later date, the premium for guessing the state correctly should be decreasing in t on $\mathcal{S}$. In general, this premium is a multiple of $w^c(\beta^m(m,t)) - w^c(\beta^{-m}(m,t))$, but with $\pi = 1$ it collapses to $w^c(\beta^m(m,t))$, since $\beta^{-m}(m,t) = 0$. Proposition 4 provides alternative conditions under which the decreasing premium is equivalent to decreasing $\beta^m(m,t)$ in case $\pi < 1$. If $w^c(\cdot)$ is convex, it holds due to $\beta^m(m,t)$ and $\beta^{-m}(m,t)$ being scalar multiples of each other. The second condition relaxes convexity to just bounded derivative of $w^c(\cdot)$ but the idea is the same: if $\frac{dw^c(\beta)}{d\beta}$ is bounded so that $w^c(\cdot)$ is not too concave globally, and the signal is precise enough, we can establish the desired monotonicity of $\beta^m(m,t)$.

Flow and Terminal Payoffs. Our model accounts for both flow payoffs that the analyst receives before the state revelation, and the terminal payoffs the analyst receives after the state is revealed. This is different from the literature on communication with career concerns surveyed in Section 3, most of which only focuses on one or the other. Importantly, both payoff types matter for our analysis. Terminal payoffs provide incentives to the quacks to report at all and incentives to the experts to report early. Flow payoffs, in turn, generate the dynamics of informativeness captured in our main result: the choice between reporting now or later is affected by the flow reputation that the analyst receives in the interim. That said, we do not put bounds on how important either payoff type is to the analyst: so long as there is *some* flow payoff from reputation, and *some* terminal payoff after a report can be assessed, our results apply.

It is immediate that terminal reputation is important for analysts, who are often evaluated based on the accuracy of their predictions (see, e.g., Hong and Kubik [2003]). However, there is also evidence that suggests that it is not simply "being right" that affects analysts' payoffs, but "being seen" is a significant factor by itself. Kuperman, Athavale, and Eisner [2003] suggest that financial analysts gain a celebrity status by appearing on TV, rather than being right. "*In the end, the success of the security analysis industry may reflect more what people think analysts are worth than what value analysts can really add. We all know that investors chase the celebrities who appear on the covers of Business Week, Money magazine, and so on*" [De Bondt, 1995, p.13]. Rees, Sharp, and Twedt [2015] present a rare empirical investigation of how media exposure affects financial analysts' payoffs and show that being cited in the media affects analysts' career outcomes even after controlling for the accuracy of their predictions. These effects are likely attributable to the effects of exposure (which attracts new receivers and can be beneficial in and of itself), as opposed to reputation (among existing receivers). However, they demonstrate that accuracy is not the only determinant of an analyst's payoff, and flow payoff matters as well.

Richer Private Signals for Experts. Our analysis extends to the case when there are more than two states of the world. Assume $\omega \in \{\omega_1, \omega_2, \ldots, \omega_n\}$. Then with the modified version of (TE), which is $r^E_{\omega_i}(\omega_i, t) = 1$ for all $t \in \mathcal{S}$, Propositions 1 - 4 remain valid. The only required modification is the second part of Proposition 2, where the existence would be guaranteed only in the neighborhood of the uniform distribution among the states.

Another possible extension of the model would be allowing richer (non-binary) state-contingent distributions of the expert's private signal (while keeping the reports binary). If we modify (TE) and require that the expert reports any informative signal he gets as soon as possible, Propositions 1 - 4 continue to hold. This is because an uninformed quack is indifferent in such an equilibrium between all reports and staying silent, hence any amount of additional private information breaks this tie for an informed expert in favor of the corresponding report.

Equilibria with Silent Quacks. Making a report is a risky lottery for the quack: he wins big if guesses the state correctly, and he loses big if he does not. The quack only accepts this gamble if the payoffs from reputation are sufficiently convex (see Section 5.4). However, this raises the question of whether quacks may choose to abstain from reporting altogether in other circumstances.

Lemma 6 in the Appendix suggests that this is not possible, and quacks must always be present in the market. Otherwise, entry is too lucrative: if only experts make reports, then anyone who makes a report must be an expert. This conclusion relies on assumption (OP)—specifically, on the requirement that the observer does not revise belief $b(m, t) = 1$ in case the analyst's report turns out to be incorrect. We believe this refinement is reasonable, and it describes the only possible scenario when either the expert, or the observer have some noise in their signal about the state, as described above. However, in case of perfect signals considered in the main model and with some conditions on $w(\cdot)$ and $w^c(\cdot)$, we can find off-path beliefs that support an equilibrium in which the quack stays silent. We explore such equilibria in more detail in Supplementary Appendix (see Section B.1).

Informed Quacks. Our model assumes that quacks do not acquire any informative signals about the state. It would be natural to ask what happens if quacks also receive a (perfectly revealing) signal, but at a smaller rate than the experts do. Refinement (TE) automatically implies that an informed quack would also prefer to report the state as soon as he can. Therefore, the only indeterminate strategy is the one for uninformed quacks: should they always wait for a signal to arrive and be silent otherwise, or should they still try their luck and guess the state? Both options can arise in equilibrium.

Consider first a strategy profile in which both quacks and experts reveal their private signal truthfully as soon as it arrives, and stay silent otherwise. Sustaining it as an equilibrium would require that the uninformed quack prefers to stay silent, which may or may not be sustainable depending on the model primitives. Note that for given model primitives, if the quack's learning rate is low enough, then this type of *dynamic* equilibria is *not* sustainable. Indeed, silence in such equilibria yields a very low reputation for the quack, and, therefore, very low payoffs, both flow and terminal. At the same time, an uninformed report would give the quack an almost perfect reputation at least until T (and possibly after, if he guesses the state), making this option more appealing.

Another possibility is a noisy equilibrium, in which uninformed quacks are indifferent between sending an uninformed report and waiting for a private signal. The logic behind our results does not carry over immediately to the analysis of such equilibria, because the quack's indifference condition would include an additional option value of receiving a signal. However, generically, if the quack's learning rate is low enough, our monotonicity results continue to hold. Recall the discussion in Section 5.2, which argued that all of our monotonicity results are generically strict. It implies that if we increase the quack's learning rate away from zero by an amount that is small enough, then the results will continue to hold. This is because all beliefs are continuous in the quack's strategies and learning probabilities, and the quack's strategies themselves are continuous in the quack's learning probabilities.[22]

7 Conclusion

The paper presents a model of dynamic cheap talk in the presence of career concerns. We discover that competition between competent (experts) and incompetent (quacks) analysts imposes plenty of structure on equilibrium outcomes. In particular, we show that to incentivize the experts—whose reports drive the whole market—to make early predictions, it must be that early reports are perceived more favorably by the public than later reports. Perhaps more surprisingly, we discover that the presence of quacks in the market together with the monotonicity above generates an automatic penalty for any report: an analyst who makes a prediction will see his reputation plummeting, and he will only be redeemed if his prediction will turn out to be correct. This does not discourage the quacks from speaking up, but disciplines their incentives. Moreover, this reputation dynamics implies that for non-trivial equilibria to exist, analysts' payoffs must be sufficiently convex in reputation, which is the case if, e.g., the premium for being the top analyst in the field is very large.

[22]The quack's strategies constitute a unique solution to a system of indifference conditions which are continuous in the quack's learning probabilities.

These predictions are novel in the literature, and are driven by us explicitly modeling the dynamic payoff structure of the analysts. Our model accounts for both flow payoffs while the public is still uncertain about the correctness of the analyst's prediction, and terminal payoffs realized after the true state is revealed. The model can be extended in multiple directions, e.g., to account for competition among analysts, or for the arrival of public signal in the background. Richer private news processes for analysts can also add another strategic layer to the timing decision of the analyst's prediction. All of these are prospective avenues for future research.

Appendix

A.1 Supplementary Notation

Before we proceed to the proofs, we start with some preliminaries. Specifically, it is useful to explicitly derive the expressions for strategies and beliefs that did not appear in the main text. The expert's report probabilities can be rewritten for any $t \in \mathcal{S}$ as

$$\begin{aligned} r^E(G,t) &= \mathbb{E}_\eta\left[r^E_\eta(G,t)\right] = \lambda_{\mathcal{S}}(t)\cdot p_t, \\ r^E(B,t) &= \mathbb{E}_\eta\left[r^E_\eta(B,t)\right] = \lambda_{\mathcal{S}}(t)\cdot(1-p_t), \\ \mathbb{E}_\eta[r^E_\eta(m,t)|\omega] &= \lambda_{\mathcal{S}}(t)\cdot\mathbb{I}\left\{\omega=m\right\}, \end{aligned} \tag{6}$$

where $\mathbb{I}\{\cdot\}$ is an indicator function and $\lambda_{\mathcal{S}}(t)$ is the equilibrium hazard rate of a private signal arrival after the previous period in the support:

$$\lambda_{\mathcal{S}}(t) := \frac{F(t)-F(\tau)}{1-F(\tau)} \text{ for } \tau = \max\left\{\mathcal{S}|\tau<t\right\}.$$

Employing the Bayes' rule, we can obtain the updating rules for the observer's beliefs about the expert's type γ, and simplify them using (6). In case no report was made in period $t \in \mathcal{S}$, the belief is updated as

$$\beta_t = \beta_{t-1}\cdot\frac{1-r^E(G,t)-r^E(B,t)}{1-r^Q(G,t)-r^Q(B,t)} = \beta_{t-1}\cdot\frac{1-\lambda_{\mathcal{S}}(t)}{1-r^Q(G,t)-r^Q(B,t)}. \tag{7}$$

After reports (G,t) and (B,t) for $t\in\mathcal{S}$, we have, respectively:

$$\begin{aligned} \beta(G,t) &= \beta_{t-1}\cdot\frac{r^E(G,t)}{r^Q(G,t)} &= \beta_{t-1}\cdot\frac{p_0\cdot\lambda_{\mathcal{S}}(t)}{r^Q(G,t)}, \\ \beta(B,t) &= \beta_{t-1}\cdot\frac{r^E(B,t)}{r^Q(B,t)} &= \beta_{t-1}\cdot\frac{(1-p_0)\cdot\lambda_{\mathcal{S}}(t)}{r^Q(B,t)}, \end{aligned} \tag{8}$$

and then once the state is revealed to be ω, we have:

$$\beta^\omega(m,t) = \beta_{t-1}\cdot\frac{\mathbb{E}_\eta\left[r^E_\eta(m,t)\mid\omega\right]}{r^Q(m,t)} = \beta_{t-1}\cdot\frac{\lambda_{\mathcal{S}}(t)}{r^Q(m,t)}\cdot\mathbb{I}\left\{\omega=m\right\}. \tag{9}$$

It is also useful to introduce a shorthand notation for the analyst's expected payoff from making report m in future period $\tau \geqslant t$ at private history (h_t,η):

$$W^\gamma_{t,\eta}(m,\tau) := \mathbb{E}\left[\sum_{s=t}^{T-1} w\left(\beta(h_s)\right) + w^c\left(\beta(h_T)\right)\middle| t,\eta,(m,\tau)\right]. \tag{10}$$

With this notation we have that report (m,t) is optimal at t if and only if $V^\gamma_{t,\eta} = W^\gamma_{t,\eta}(m,t)$. Moreover, we use $W^\gamma_{t,\eta}(\varnothing)$ to denote the respective value from staying silent until the end of period T. For the quack we omit subscript η and use $W^Q_t(m,\tau)$ and $W^Q_t(\varnothing)$, because the quack never gets any private information.

Finally, we let $-\eta$ and $-m$ denote the "opposites" of η and m respectively: e.g., if $\eta = G$ then $-\eta = B$.

A.2 Report's Informativeness

The explicit expression for Blackwell experiment $\phi_t(m|\omega)$ induced by period-t reports, which is the probability to observe message m conditional on state being ω, is

$$\begin{aligned}\phi_t(m|\omega) &= \frac{(1-b_{t-1})\cdot r^Q(m,t) + b_{t-1}\cdot \mathbb{E}_\eta\left[r^E_\eta(m,t)|\omega\right]}{\sum\limits_{m'\in\{G,B\}}\Big[(1-b_{t-1})\cdot r^Q(m',t) + b_{t-1}\cdot \mathbb{E}_\eta\left[r^E_\eta(m',t)|\omega\right]\Big]} \\ &= \frac{(1-b_{t-1})\cdot r^Q(m,t) + b_{t-1}\cdot \lambda_{\mathcal{S}}(t)\cdot \mathbb{I}\{\omega=m\}}{(1-b_{t-1})\cdot[r^Q(G,t)+r^Q(B,t)] + b_{t-1}\cdot\lambda_{\mathcal{S}}(t)}.\end{aligned} \tag{11}$$

Then following report m at $t\in\mathcal{S}$, due to (9) the observer's belief about the state is updated as:

$$\rho(m,t) = \rho_{t-1}\cdot\frac{\phi_t(m|G)}{\phi_t(m|B)} = \rho_{t-1}\cdot\frac{1+\beta^G(m,t)}{1+\beta^B(m,t)}. \tag{12}$$

A.3 Auxiliary Results

Lemma 5. *In any equilibrium:*

1. *$\rho_t = \rho_0$ for all $t\in\mathcal{S}$, i.e., belief about the state remains constant in the absence of reports;*
2. *$\beta^\omega(\varnothing) = \beta_T$ for any $\omega\in\{G,B\}$, i.e., state revelation does not affect reputation if no report was made.*

Proof. We now use (6) to establish both claims. Belief about the state is updated as follows in case no report was made:

$$\begin{aligned}\rho_{t+1} &= \rho_t\cdot\frac{1-(1-b_{t-1})\cdot\left(r^Q(G,t)+r^Q(B,t)\right) - b_{t-1}\cdot\mathbb{E}_\eta\left[r^E_\eta(G,t)+r^E_\eta(B,t)|\omega=G\right]}{1-(1-b_{t-1})\cdot\left(r^Q(G,t)+r^Q(B,t)\right) - b_{t-1}\cdot\mathbb{E}_\eta\left[r^E_\eta(G,t)+r^E_\eta(B,t)|\omega=B\right]} \\ &= \rho_t\cdot\frac{1-(1-b_{t-1})\cdot\left(r^Q(G,t)+r^Q(B,t)\right) - b_{t-1}\cdot\lambda_{\mathcal{S}}(t)}{1-(1-b_{t-1})\cdot\left(r^Q(G,t)+r^Q(B,t)\right) - b_{t-1}\cdot\lambda_{\mathcal{S}}(t)} = \rho_t.\end{aligned}$$

To establish that $\beta^\omega(\varnothing)=\beta_T$, note that from Bayes' rule,

$$\begin{aligned}\beta^\omega(\varnothing) &= \beta_0\cdot\prod_{t=1}^T\left(\frac{1-\mathbb{E}_\eta\left[r^E_\eta(G,t)+r^E_\eta(B,t)|\omega\right]}{1-r^Q(G,t)-r^Q(B,t)}\right) \\ &= \beta_0\cdot\prod_{t=1}^T\left(\frac{1-\lambda_{\mathcal{S}}(t)}{1-r^Q(G,t)-r^Q(B,t)}\right) = \beta_0\cdot\prod_{t=1}^T\left(\frac{1-r^E(G,t)-r^E(B,t)}{1-r^Q(G,t)-r^Q(B,t)}\right) = \beta_T\end{aligned}$$

for any $\omega\in\{G,B\}$. □

Lemma 6. *In any equilibrium, any report (m,t) for $m\in\{G,B\}$ is made with positive probability by a quack if and only if it is ever made by an expert: $r^E(m,t)>0 \iff r^Q(m,t)>0$.*

Proof. We show that $r^E(m,t)>0$ if and only if $r^Q(m,t)>0$ for any (m,t) for any history with $b(h_t)\in(0,1)$. Together with the fact that $b_0\in(0,1)$, it means that on equilibrium path we never arrive at a [non-terminal] history with $b(h_t)\in\{0,1\}$, hence the statement is true for all histories on equilibrium path.

$r^E(m,t)>0$ implies $r^Q(m,t)>0$. Suppose by contradiction that $r^Q(m,t)=0$. Then $b(m,t)=b^\omega(m,t)=1$, meaning that $W^Q_t(m,t)$ attains maximum among all continuation payoffs (feasible or not). At the same time, belief consistency implies that if a quack plays an action with positive probability in equilibrium, the observer's belief following this action must be $b<1$. This equilibrium action thus yields less than the maximal payoff, and (m,t) is then a strictly profitable deviation for the quack—a contradiction.

$r^Q(m,t) > 0$ implies $r^E(m,t) > 0$. Suppose that $r^E(m,t) = 0$. Since $r^Q(m,t) > 0$, we have $b(m,t) = b^\omega(m,t) = 0$, hence $W_t^Q(m,t)$ attains minimum among all continuation payoffs. However, since belief about the analyst's type is a martingale from the observer's point of view, we have either $b(-m,t) > 0$ or $b_{t+1} > 0$. Thus, at least one of these strategies (reporting $-m$ or staying silent at t) strictly dominates the strategy of reporting (m,t) for the quack, so $r^Q(m,t) = 0$. □

A.4 Proofs of the Main Results

Proof of Proposition 1. The proof proceeds in several steps.

Step 1. Suppose the expert has private information $\eta = G$ at time t and has not yet made any report. He chooses either to make a report (m,τ) with $\tau \geqslant t$ to maximize $W_{t,G}^E(m,\tau)$, given by

$$W_{t,G}^E(m,\tau) = \sum_{s=t}^{\tau-1} w(\beta_s) + \sum_{s=\tau}^{T-1} w(\beta(m,\tau)) + w^c\left(\beta^G(m,\tau)\right),$$

or not to make any report at all. A quack is, in equilibrium, indifferent between all these options at time t. His continuation value from report $W_t^Q(m,\tau)$, which can similarly be written as

$$W_t^Q(m,\tau) = \sum_{s=t}^{\tau-1} w(\beta_s) + \sum_{s=\tau}^{T-1} w(\beta(m,\tau)) + p_0 \cdot w^c\left(\beta^G(m,\tau)\right) + (1-p_0)\cdot w^c\left(\beta^B(m,\tau)\right),$$

is constant over all (m,τ) for a given t. Observe that

$$W_{t,G}^E(m,\tau) = W_t^Q(m,\tau) + (1-p_0)\cdot\left(w^c\left(\beta^G(m,\tau)\right) - w^c\left(\beta^B(m,\tau)\right)\right). \tag{13}$$

If the expert decides not to make any report at all, his value is the same as the quack's value. If the expert decides to make report G, which corresponds to his private signal, at any time τ, due to (9) we get $W_{t,G}^E(G,\tau) > W_t^Q(G,\tau)$. Similarly, if the expert reports B, which is the opposite to his signal, we get $W_{t,G}^E(B,\tau) < W_t^Q(B,\tau)$. Therefore, it is optimal for the expert to report that the state is G, the only actual choice he makes is when. From (13) the optimization problem of the expert with $\eta = G$ is equivalent to maximizing $w^c\left(\beta^G(G,\tau)\right) - w^c\left(\beta^B(G,\tau)\right)$ over all $\tau \in \{\mathcal{S} | \tau \geqslant t\}$. Similarly, the expert with $\eta = B$ chooses report (B,τ) which maximizes $w^c\left(\beta^B(B,\tau)\right) - w^c\left(\beta^G(B,\tau)\right)$. (TE) implies that an informed expert must weakly prefer to report instantly. Therefore, it must be that (G,t) maximizes $w^c\left(\beta^G(G,\tau)\right) - w^c\left(\beta^B(G,\tau)\right)$, and (B,t) maximizes $w^c\left(\beta^B(B,\tau)\right) - w^c\left(\beta^G(B,\tau)\right)$ across all reports with $\tau \in \{\mathcal{S} \,|\, \tau \geqslant t\}$. This holds for any $t \in \mathcal{S}$, implying that in any equilibrium, both these differences must be weakly decreasing in $t \in \mathcal{S}$.

Step 2. Further, since $\beta^B(G,t) = \beta^G(B,t) = 0$, it is immediate that $\beta^G(G,t)$ and $\beta^B(B,t)$ are both (weakly) decreasing in $t \in \mathcal{S}$. To show the first part of the proposition, recall from (12) that for all $t \in \mathcal{S}$, we have

$$\frac{\phi_t(G|G)}{\phi_t(G|B)} = \frac{1+\beta^G(G,t)}{1+\beta^B(G,t)} = 1 + \beta^G(G,t), \tag{14}$$

which is then a decreasing function of t on $\mathcal{S}$. The same holds for message $m = B$, meaning that earlier messages are more informative about the state.

Step 3. We next show that in any equilibrium, it must be that $\beta_{t_1} \geqslant \beta(m,t_1)$. Assume the contrary, i.e., there exists $m \in \{G,B\}$ such that $\beta_{t_1} < \beta(m,t_1)$. The quack's values from reports (m,t_1) and (m,t_2)

evaluated at t_1 are

$$W_{t_1}^Q(m,t_1) = (T-t_1)\cdot w\left(\beta(m,t_1)\right) + p_0 \cdot w^c\left(\beta^G(m,t_1)\right) + (1-p_0)\cdot w^c\left(\beta^B(m,t_1)\right),$$
$$W_{t_1}^Q(m,t_2) = (t_2-t_1)\cdot w\left(\beta_{t_1}\right) + (T-t_2)\cdot w\left(\beta(m,t_2)\right) + p_0 \cdot w^c\left(\beta^G(m,t_2)\right) + (1-p_0)\cdot w^c\left(\beta^B(m,t_2)\right).$$

As $w(\cdot)$ is strictly increasing, and $\beta_{t_1} < \beta(m,t_1)$, the requirement that quack is indifferent between the two reports, $W_{t_1}^Q(m,t_1) = W_{t_1}^Q(m,t_2)$, implies that

$$(T-t_2)\cdot w\left(\beta(m,t_1)\right) + p_0 \cdot w^c\left(\beta^G(m,t_1)\right) + (1-p_0)\cdot w^c\left(\beta^B(m,t_1)\right) <$$
$$< (T-t_2)\cdot w\left(\beta(m,t_2)\right) + p_0 \cdot w^c\left(\beta^G(m,t_2)\right) + (1-p_0)\cdot w^c\left(\beta^B(m,t_2)\right).$$

Consequently, it must be that either $\beta(m,t_1) < \beta(m,t_2)$, or $\beta^G(m,t_1) < \beta^G(m,t_2)$, or $\beta^B(m,t_1) < \beta^B(m,t_2)$. However, (8) and (9) imply that $\beta^{-m}(m,t) = 0$ and $\beta^m(m,t)$ differs from $\beta(m,t)$ by a constant factor for any $t \in \mathcal{S}$. Therefore, in particular, it must be that $\beta^m(m,t_1) < \beta^m(m,t_2)$, which contradicts $\beta^m(m,t)$ being weakly decreasing on $\mathcal{S}$.

Step 4. Finally, we show how the second and the third claims in the proposition follow from the previous steps. We have shown that $\beta_{t_1} \geqslant \beta(m,t_1)$ for any $m \in \{G,B\}$. Then using the same logic as in Step 3 above, one can show that the quack's indifference between reports (m,t_1) and (m,t_2) requires that $\beta(m,t_1) \geqslant \beta(m,t_2)$ for both $m \in \{G,B\}$. The martingale property of beliefs together with the resulting inequalities $\beta_{t_1} \geqslant \beta(m,t_1) \geqslant \beta(m,t_2)$ for $m \in \{G,B\}$ imply $\beta_{t_2} \geqslant \beta_{t_1}$. By iterating this argument, we conclude that $\beta(m,t) \leqslant \beta_t$ and β_t is weakly increasing for $t \in \mathcal{S}$. This proves the second and third parts of Proposition 1 and concludes the proof. □

Proof of Proposition 2. Condition (TE) pins down the strategy of the expert. Therefore, an equilibrium with a given support $\mathcal{S}$ exists if and only if there exists a quack's strategy $\{r^Q(m,t)\}_{m\in\{G,B\},t\in\mathcal{S}}$ such that the two following conditions are satisfied. First, the quack should be indifferent between any report (m,t) for $m \in \{G,B\}$, $t \in \mathcal{S}$ and staying silent. Second, the expert should find it optimal to abide by (TE).

Part 1. Fix some $t \in \mathcal{S}$. Due to Lemma 5, the quack's indifference conditions in this case are given by $W_t^Q(m,t) = W_t^Q(\varnothing)$ for $m \in \{G,B\}$, which can be rewritten as

$$\begin{aligned}(T-t)\cdot w\left(\beta(G,t)\right) + p_0\cdot w^c\left(\beta^G(G,t)\right) &= (T-t)\cdot w\left(\beta(B,t)\right) + (1-p_0)\cdot w^c\left(\beta^B(B,t)\right),\\ (T-t)\cdot w\left(\beta(G,t)\right) + p_0\cdot w^c\left(\beta^G(G,t)\right) &= (T-t)\cdot w\left(\beta_t\right) + w^c\left(\beta_t\right).\end{aligned} \tag{15}$$

Using the belief updating rules (7)–(9), the first equation reduces to

$$(T-t)\cdot w\left(\beta_0\cdot\frac{p_0\cdot\lambda_{\mathcal{S}}(t)}{r^Q(G,t)}\right) + p_0\cdot w^c\left(\beta_0\cdot\frac{\lambda_{\mathcal{S}}(t)}{r^Q(G,t)}\right) =$$
$$= (T-t)\cdot w\left(\beta_0\cdot\frac{(1-p_0)\cdot\lambda_{\mathcal{S}}(t)}{r^Q(B,t)}\right) + (1-p_0)\cdot w^c\left(\beta_0\cdot\frac{\lambda_{\mathcal{S}}(t)}{r^Q(B,t)}\right). \tag{16}$$

Since $w(\cdot)$ and $w^c(\cdot)$ are continuous, both sides of the equality are continuous in $r^Q(G,t), r^Q(B,t)$. Further, fix some $c \in (0,1]$ and let $r^Q(G,t) + r^Q(B,t) = c$. Under the assumption in the proposition, the respective side of (16) is unbounded when either $r^Q(G,t)$ or $r^Q(B,t)$ approach 0. Then by the intermediate value theorem there exist unique $r^Q(G,t), r^Q(B,t)$ that solve the equation above. Note also that $r^Q(G,t), r^Q(B,t)$

if considered as functions of c are both strictly increasing in c.[23]

Next, consider the second equation in (15), which can be rewritten as

$$(T-t)\cdot w\left(\beta_0\cdot\frac{p_0\cdot\lambda_{\mathcal{S}}(t)}{r^Q(G,t)}\right)+p_0\cdot w^c\left(\beta_0\cdot\frac{\lambda_{\mathcal{S}}(t)}{r^Q(G,t)}\right)= \\ =(T-t)\cdot w\left(\beta_0\cdot\frac{1-\lambda_{\mathcal{S}}(t)}{1-r^Q(G,t)-r^Q(B,t)}\right)+w^c\left(\beta_0\cdot\frac{1-\lambda_{\mathcal{S}}(t)}{1-r^Q(G,t)-r^Q(B,t)}\right). \quad (17)$$

Consider (17) as an equation in $c = r^Q(G,t) + r^Q(B,t)$ where we treat $r^Q(G,t)$ and $r^Q(B,t)$ as (strictly increasing) functions of c from (16). Again, by the intermediate value theorem, there exists a unique value of c that solves (17). Therefore, $r^Q(G,t)$ and $r^Q(B,t)$ that solve (16) for this particular c solve system (15).

It is left to verify the incentive constraints of the expert. The uninformed expert is also indifferent between reporting and not at t, which does not contradict (TE). The payoff of the informed expert from reporting his private signal truthfully is strictly higher than the value of the respective report for the quack. Therefore, the expert strictly prefers to report truthfully than to stay silent. Reporting opposite to the private signal gives the informed expert the value which is strictly less than the value of the quack. Therefore, (TE) is satisfied. This proves part 1 of the proposition.

Part 2. It is enough to consider $w^c(\beta) = \beta$. Let $r^Q(G,t) = p_0\cdot\lambda_{\mathcal{S}}(t)$ and $r^Q(B,t) = (1-p_0)\cdot\lambda_{\mathcal{S}}(t)$ for $t\in\mathcal{S}$. In this case $\beta_t = \beta(m,t) = \beta_0$ for all $t\in\mathcal{S}$, i.e., the reputation stays constant before T irrespective of the analyst's action. Due to the linearity of $w^c(\cdot)$ we also get that $p_0\cdot w^c(\beta^G(G,t)) = (1-p_0)\cdot w^c(\beta^B(B,t)) = \beta_0$, therefore the quack is indifferent between all reports and no report. Incentives of the informed expert are also satisfied: $\beta_0 = \beta(m,t_1) \leqslant \beta_{t_1} = \beta_0$, which is true. The uniqueness of this profile follows from the following argument. If we assume that $r^Q(G,t_1) < p_0\cdot\lambda_{\mathcal{S}}(t_1)$, then $r^Q(B,t_1) < (1-p_0)\cdot\lambda_{\mathcal{S}}(t_1)$. This increases the value of a report and decreases b_{t_1}. Therefore, $r^Q(G,t_2) < p_0\cdot\lambda_{\mathcal{S}}(t_2)$ and $r^Q(B,t_2) < (1-p_0)\cdot\lambda_{\mathcal{S}}(t_2)$. This argument unravels further and we eventually get $r^Q(G,t) < p_0\cdot\lambda_{\mathcal{S}}(t)$ and $r^Q(B,t) < (1-p_0)\cdot\lambda_{\mathcal{S}}(t)$ for $t\in\mathcal{S}$. Therefore, in this profile the value of a report for the quack is strictly higher and the value of staying silent is strictly lower than in the aforementioned equilibrium. This contradicts these values to be equal. The case with $r^Q(G,t_1) > p_0\cdot\lambda_{\mathcal{S}}(t_1)$ is analogous.

Part 3. Fix some $|\mathcal{S}|\geqslant 2$. The quack's indifference conditions are given by $W^Q_{t_1}(m,t) = W^Q_{t_1}(\varnothing)$.

Note that these conditions can be considered as a system in $r^Q(G,t_1)$ and $r^Q(B,t_1)$. Each of these report probabilities determines the value the quack gets from the respective report and also pins down β_{t_1}. This, in turn, uniquely pins down $r^Q(G,t_2)$ and $r^Q(B,t_2)$ and so on. Therefore, we basically need to find $r^Q(G,t_1)$, $r^Q(B,t_1)$ such that the values of reports at $t=t_1$ are equal, and this value is also equal to the value of no report. Finally, for the resulting profile to constitute an equilibrium, we then have to verify that $\beta(m,t_1)\leqslant\beta_{t_1}$ for $m\in\{G,B\}$.

First, we prove the claim for $p_0=\frac{1}{2}$. Consider $r^Q(G,t_1) = r^Q(B,t_1) = \frac{1}{2}\cdot\lambda_{\mathcal{S}}(t_1)$. Then to sustain the quack's indifference between all reports we should have $r^Q(G,t) = r^Q(B,t) = \frac{1}{2}\cdot\lambda_{\mathcal{S}}(t)$ for all $t\in\mathcal{S}$. for this strategy profile we have

$$W^Q_{t_1}(m,t) = (T-t_1)\cdot w(\beta_0) + \frac{1}{2}\cdot w^c(2\beta_0),$$

and

$$W^Q_{t_1}(\varnothing) = (T-t_1)\cdot w(\beta_0) + w^c(\beta_0).$$

[23] If c increases, at least $r^Q(G,t)$ or $r^Q(B,t)$ has to increase as well. Therefore, the value of any report has to fall. Finally, due to (16) both $r^Q(G,t)$ and $r^Q(B,t)$ then have to increase.

Because $w^c(\cdot)$ is *strictly* convex, we have $W^Q_{t_1}(m,t) > W^Q_{t_1}(\varnothing)$. At the same time, with $r^Q(G,t_1) = r^Q(B,t_1) \to \frac{1}{2}$ the value of any report is strictly smaller than the value of no report. Moreover, if we gradually increase $r^Q(G,t_1) = r^Q(B,t_1)$, the value of a report continuously decreases, while the value of no report continuously increases. Therefore by the intermediate value theorem there exists a unique value for $r^Q(G,t_1) = r^Q(B,t_1)$ such that the quack is indifferent between all reports and no report. Additionally, because $r^Q(G,t_1) = r^Q(B,t_1) > \frac{1}{2}\cdot\lambda_{\mathcal{S}}(t_1)$, we have $\beta(m,t_1) \leqslant \beta_{t_1}$ for $m \in \{G,B\}$ and therefore this profile constitutes an equilibrium.

We next establish the existence of an equilibrium with any support in some neighborhood of $p_0 = \frac{1}{2}$. Consider the indifference condition between reports G and B at t_1:

$$(T-t_1)\cdot w\left(\beta_0\cdot\frac{p_0\cdot\lambda_{\mathcal{S}}(t_1)}{r^Q(G,t_1)}\right) + p_0\cdot w^c\left(\beta_0\cdot\frac{\lambda_{\mathcal{S}}(t_1)}{r^Q(G,t_1)}\right) = \\ = (T-t_1)\cdot w\left(\beta_0\cdot\frac{(1-p_0)\cdot\lambda_{\mathcal{S}}(t_1)}{r^Q(B,t_1)}\right) + (1-p_0)\cdot w^c\left(\beta_0\cdot\frac{\lambda_{\mathcal{S}}(t_1)}{r^Q(B,t_1)}\right). \quad (18)$$

When $p_0 = \frac{1}{2}$ for $r^Q(G,t_1) = p_0\cdot\lambda_{\mathcal{S}}(t_1)$ the solution to (18) is $r^Q(B,t_1) = (1-p_0)\cdot\lambda_{\mathcal{S}}(t_1)$. If $p_0 > \frac{1}{2}$, because $w^c(\cdot)$ is strictly convex, when $r^Q(G,t_1) = p_0\cdot\lambda_{\mathcal{S}}(t_1)$, we have that $r^Q(B,t_1) > (1-p_0)\cdot\lambda_{\mathcal{S}}(t_1)$.

Fix $r^Q(G,t_1) = p_0\cdot\lambda_{\mathcal{S}}(t_1)$. Then $r^Q(B,t_1)$, the solution to (18), is continuous in p_0. Due to the indifference conditions of the quack, the report probabilities $r^Q(m,t)$ in further periods and all beliefs b_t are also continuous functions of p_0. Therefore, because with $p_0 = \frac{1}{2}$ we had that the value of a report was strictly higher than the value of no report when $r^Q(G,t_1) = \frac{1}{2}\cdot\lambda_{\mathcal{S}}(t_1)$ and $r^Q(B,t_1) = \frac{1}{2}\cdot\lambda_{\mathcal{S}}(t_1)$, the inequality remains valid in some neighborhood of $p_0 = \frac{1}{2}$ for $r^Q(G,t_1) = p_0\cdot\lambda_{\mathcal{S}}(t_1)$ and $r^Q(B,t_1)$ which solves (18).

Therefore we can start with $r^Q(G,t_1) = p_0\cdot\lambda_{\mathcal{S}}(t_1)$ and the respective value of $r^Q(B,t_1)$ which constitutes a solution to (18) and gradually increase both to maintain (18) until $r^Q(G,t_1)+r^Q(B,t_1) = 1$. The respective report probabilities in further periods will (continuously) grow as well, and therefore the value of no report will increase. Clearly, when $r^Q(G,t_1) + r^Q(B,t_1) \to 1$ the value of no report becomes strictly higher than the value of any report. Therefore by the intermediate value theorem there exist a unique pair of $r^Q(G,t)$ and $r^Q(B,t)$ which solves the system of indifference conditions for the quack. Similarly, because $r^Q(G,t_1) \geqslant \frac{1}{2}\cdot\lambda_{\mathcal{S}}(t_1)$ and $r^Q(B,t_1) \geqslant \frac{1}{2}\cdot\lambda_{\mathcal{S}}(t_1)$ we have $\beta(m,t_1) \leqslant \beta_{t_1}$ for $m \in \{G,B\}$ this profile constitutes an equilibrium.

Finally, note the support was taken arbitrarily, and the number of possible supports is also finite. For any $\mathcal{S}$ we can calculate the neighborhood of $p_0 = \frac{1}{2}$ for which an equilibrium with support $\mathcal{S}$ exists. Therefore, we can take $\bar{p}_0$ to be the minimal value of p_0 for which an equilibrium exists for all supports.

Part 4. Consider $\bar{t}$, the last point in support $\mathcal{S}$. The quack's indifference condition at $\bar{t}$ is given by (15). If $w^c(\cdot)$ is strictly concave, the second equality in (15) together with (8) and (9) imply $\beta(G,\bar{t}) \geqslant \beta_{\bar{t}}$. This contradicts the equilibrium characterization in Proposition 1, meaning that even if there exists a solution to the system of quack's indifference conditions, this profile is not consistent with the informed expert's incentives. We conclude that no equilibrium exists in this case. □

Proof of Proposition 3. First, notice that the claim trivially holds for $t \in \{t_{k+1},\ldots,t_{k+n}\}$. Therefore, we next establish the claim for $t \in \mathcal{S}$. Denote by $W^Q_t(m,\tau)$ and $\tilde{W}^Q_t(m,\tau)$ the quack's respective values of making a report and by β and $\tilde{\beta}$ the beliefs corresponding to strategy profiles $\{r^\gamma_\eta(m,t)\}$ and $\{\tilde{r}^\gamma_\eta(m,t)\}$.

We begin by showing that $W^Q_{t_1}(m,t_1) \leqslant \tilde{W}^Q_{t_1}(m,t_1)$ for both $m \in \{G,B\}$. Assume the contrary: that

$W^Q_{t_1}(m, t_1) > \tilde{W}^Q_{t_1}(m, t_1)$ for both $m \in \{G, B\}$.[24] Because the expert's strategy is identical in both profiles, this directly implies $r^Q(m, t_1) < \tilde{r}^Q(m, t_1)$ for both $m \in \{G, B\}$. From (7) it then follows that $\beta_{t_1} < \tilde{\beta}_{t_1}$. This, in turn, implies that $r^Q(G, t_2) < \tilde{r}^Q(G, t_2)$ and $r^Q(B, t_2) < \tilde{r}^Q(B, t_2)$, because $W^Q_{t_1}(m, t_1) = W^Q_{t_1}(m, t_2)$ and $\tilde{W}^Q_{t_1}(m, t_1) = \tilde{W}^Q_{t_1}(m, t_2)$ for $m \in \{G, B\}$. Iterating this logic further, we obtain that $\beta_t < \tilde{\beta}_t$ for all $t \in \mathcal{S}$. Additionally, by Proposition 1, we have $\tilde{\beta}_{t_k} \leqslant \tilde{\beta}_{t_{k+1}} \leqslant \ldots \leqslant \tilde{\beta}_{t_{k+n}}$. Therefore, $W^Q_{t_1}(\varnothing) \leqslant \tilde{W}^Q_{t_1}(\varnothing)$. Making no report is on path in both equilibria, thus $W^Q_{t_1}(G, t_1) = W^Q_{t_1}(\varnothing)$ and $\tilde{W}^Q_{t_1}(G, t_1) = \tilde{W}^Q_{t_1}(\varnothing)$. Consequently, $W^Q_{t_1}(m, t_1) \leqslant \tilde{W}^Q_{t_1}(m, t_1)$, which gives us a contradiction with the initial assumption.

Condition $W^Q_{t_1}(m, t) \leqslant \tilde{W}^Q_{t_1}(m, t)$ directly implies that $r^Q(m, t_1) \geqslant \tilde{r}^Q(m, t_1)$ because the expert's strategy is identical in both equilibria. As in the argument above, this inequality unravels further and implies $r^Q(m, t) \geqslant \tilde{r}^Q(m, t)$ for all $t \in \mathcal{S}$. By (11), this implies that $\tilde{\phi}_t$ is more informative than ϕ_t for all $t \in \mathcal{S}$. □

A.5 Proof of Proposition 4.

We first state some preliminaries and then describe how the proof of each statement changes in case $\pi < 1$. Let $p_0^* := \pi \cdot p_0 + (1 - \pi) \cdot (1 - p_0)$ denote the total probability of signal realization $\eta = G$ for the expert. Then

$$
\begin{aligned}
\beta(G, t) &= \beta_{t-1} \cdot \frac{r^E(G, t)}{r^Q(G, t)} = \beta_{t-1} \cdot \frac{p_0^* \cdot \lambda_{\mathcal{S}}(t)}{r^Q(G, t)}, \\
\beta(B, t) &= \beta_{t-1} \cdot \frac{r^E(B, t)}{r^Q(B, t)} = \beta_{t-1} \cdot \frac{(1 - p_0^*) \cdot \lambda_{\mathcal{S}}(t)}{r^Q(B, t)},
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
\beta^m(m, t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta\left[r^E_\eta(m, t) \mid \omega = m\right]}{r^Q(m, t)} = \beta_{t-1} \cdot \frac{\pi \cdot \lambda_{\mathcal{S}}(t)}{r^Q(m, t)}, \\
\beta^{-m}(m, t) &= \beta_{t-1} \cdot \frac{\mathbb{E}_\eta\left[r^E_\eta(m, t) \mid \omega = -m\right]}{r^Q(m, t)} = \beta_{t-1} \cdot \frac{(1 - \pi) \cdot \lambda_{\mathcal{S}}(t)}{r^Q(m, t)}.
\end{aligned} \tag{20}
$$

Lemma 6. The proof of Lemma 6 applies without further modifications to the case $\pi < 1$.

Proposition 1. The proof of Proposition 1 requires some adaptations as follows. In Step 1, equation (13) becomes

$$
W^E_{t,G}(m, \tau) = W^Q_t(m, \tau) + \left(\frac{\pi p_0}{p_0^*} - p_0\right) \cdot \left(w^c\left(\beta^G(m, \tau)\right) - w^c\left(\beta^B(m, \tau)\right)\right),
$$

meaning the informed expert with $\eta = G$ still chooses τ to maximize $w^c\left(\beta^G(G, \tau)\right) - w^c\left(\beta^B(G, \tau)\right)$ and the logic in Step 1 applies.

Step 2 of the proof shows that $w^c\left(\beta^m(m, \tau)\right) - w^c\left(\beta^{-m}(m, \tau)\right)$ is (strictly) decreasing on $\mathcal{S}$ if and only if $\beta^m(m, t)$ is (strictly) decreasing on $\mathcal{S}$. We demonstrate this separately for the cases stated in Proposition 4.

Case 1: $\pi < 1$ and $w^c(\cdot)$ is convex.

From the belief updating rules (20), we have $\beta^{-m}(m, t) = \frac{1-\pi}{\pi} \cdot \beta^m(m, t)$, where $\frac{1-\pi}{\pi} \in (0, 1)$ because $\pi > \frac{1}{2}$. Therefore

$$
w^c\left(\beta^m(m, \tau)\right) - w^c\left(\beta^{-m}(m, \tau)\right) = w^c\left(\beta^m(m, t)\right) - w^c\left(\frac{1 - \pi}{\pi} \cdot \beta^m(m, t)\right). \tag{21}
$$

[24] Because $W^Q_{t_1}(G, t_1) = W^Q_{t_1}(B, t_1)$ and $\tilde{W}^Q_{t_1}(G, t_1) = \tilde{W}^Q_{t_1}(B, t_1)$, we have that either $W^Q_{t_1}(m, t_1) > \tilde{W}^Q_{t_1}(m, t_1)$ for both $m \in \{G, B\}$, or $W^Q_{t_1}(m, t_1) < \tilde{W}^Q_{t_1}(m, t_1)$ for both $m \in \{G, B\}$.

Because $w^c(x)$ is convex, the monotonicity of $w^c(\beta^m(m,\tau)) - w^c(\beta^{-m}(m,\tau))$ is then equivalent to the monotonicity of $\beta^m(m,\tau)$.

Case 2: $\pi \in \left(\frac{\bar{d}}{\underline{d}+\bar{d}}, 1\right)$ and $\frac{dw^c(\beta)}{d\beta} \in [\underline{d}, \bar{d}]$.

Note that (21) still applies. Similarly to the previous case, take any $t' < t''$ with $t', t'' \in \mathcal{S}$ and let $x' := \beta^m(m, t')$, $x'' := \beta^m(m, t'')$. Suppose $w^c(x') - w^c\left(\frac{1-\pi}{\pi} \cdot x'\right) \geqslant w^c(x'') - w^c\left(\frac{1-\pi}{\pi} \cdot x''\right)$. Then we show $x' \geqslant x''$. Assume the converse—that $x' < x''$. Then

$$
\begin{aligned}
0 \leqslant &\left(w^c(x') - w^c\left(\frac{1-\pi}{\pi} \cdot x'\right)\right) - \left(w^c(x'') - w^c\left(\frac{1-\pi}{\pi} \cdot x''\right)\right) \leqslant \\
&\leqslant -\underline{d} \cdot (x'' - x') + \bar{d} \cdot \left(\frac{1-\pi}{\pi} \cdot x'' - \frac{1-\pi}{\pi} \cdot x'\right) = (x'' - x_1) \cdot \left(\bar{d} \cdot \frac{1-\pi}{\pi} - \underline{d}\right),
\end{aligned} \tag{22}
$$

where the latter inequality follows from the bounds on the derivative of $w^c(\cdot)$. The assumption $\pi > \frac{\bar{d}}{\underline{d}+\bar{d}}$ implies $\bar{d} \cdot \frac{1-\pi}{\pi} - \underline{d} < 0$, which directly contradicts (22). We conclude that if $w^c(x') - w^c\left(\frac{1-\pi}{\pi} \cdot x'\right) \geqslant w^c(x'') - w^c\left(\frac{1-\pi}{\pi} \cdot x''\right)$, then $x' \geqslant x''$.

Conversely, if $x' \geqslant x''$, then

$$
0 \leqslant (x' - x'') \cdot \left(\underline{d} - \bar{d} \cdot \frac{1-\pi}{\pi}\right) \leqslant \left(w^c(x') - w^c\left(\frac{1-\pi}{\pi} \cdot x'\right)\right) - \left(w^c(x'') - w^c\left(\frac{1-\pi}{\pi} \cdot x''\right)\right),
$$

which gives the result.

Step 3 follows by replacing (8) and (9) with (19) and (20), respectively, and noticing that $\beta^{-m}(m,t)$ differs from $\beta(m,t)$ by a constant factor in case $\pi < 1$. Step 4 follows without any modifications.

Proposition 2. In comparison to the proof for $\pi = 1$, (15) adjusts to

$$
\begin{aligned}
(T-t) \cdot w(\beta(G,t)) + p_0 \cdot w^c\left(\beta^G(G,t)\right) + (1-p_0) \cdot w^c\left(\beta^B(G,t)\right) = \\
= (T-t) \cdot w(\beta(B,t)) + p_0 \cdot w^c\left(\beta^G(G,t)\right) + (1-p_0) \cdot w^c\left(\beta^B(B,t)\right), \\
(T-t) \cdot w(\beta(G,t)) + p_0 \cdot w^c\left(\beta^G(G,t)\right) + (1-p_0) \cdot w^c\left(\beta^B(G,t)\right) = (T-t) \cdot w(\beta_t) + w^c(\beta_t).
\end{aligned} \tag{23}
$$

All further steps in part 1, part 2, and part 3 are identical.

For part 4, similar logic implies that if $w^c(\cdot)$ is strictly concave, then $\beta(m,\bar{t}) \geqslant \beta_{\bar{t}}$ for both m, meaning no equilibrium with $|\mathcal{S}| \geqslant 2$ exists as it violates Proposition 1. This proves that if Proposition 1 holds then Proposition 2 holds when $\pi < 1$, meaning it holds under the conditions stated in Proposition 4.

Proposition 3. The proof of Proposition 3 applies without further modifications to the case $\pi < 1$. □

References

V. V. Acharya, P. DeMarzo, and I. Kremer. Endogenous information flows and the clustering of announcements. *American Economic Review*, 101(7):2955–2979, December 2011.

C. Aghamolla and B.-J. An. Voluntary disclosure with evolving news. *Journal of Financial Economics*, 140 (1):21–53, 2021.

A. Andina-Díaz and J. A. García-Martínez. Reputation and news suppression in the media industry. *Games and Economic Behavior*, 123:240–271, 2020.

R. Aumann and S. Hart. Long cheap talk. *Econometrica*, 71(6):1619–1660, November 2003.

D. Bernhardt, M. Campello, and E. Kutsoati. Analyst compensation and forecast bias. Working Paper, 2004.

D. Bernhardt, M. Campello, and E. Kutsoati. Who herds? *Journal of Financial Economics*, 80(3):657–675, 2006.

D. Bernhardt, C. Wan, and Z. Xiao. The reluctant analyst. *Journal of Accounting Research*, 54(4):987–1040, 2016.

A. Beyer, D. A. Cohen, T. Z. Lys, and B. R. Walther. The financial reporting environment: Review of the recent literature. *Journal of Accounting and Economics*, 50(2-3):296–343, 2010.

R. Boleslavsky and C. R. Taylor. Make it 'til you fake it. *Journal of Economic Theory*, 217:105812, 2024.

R. A. Cooper, T. E. Day, and C. M. Lewis. Following the leader: a study of individual analystsâĂŹ earnings forecasts. *Journal of Financial Economics*, 61(3):383–416, 2001.

V. Crawford and J. Sobel. Strategic information transmission. *Econometrica*, 50(6):1431–1451, November 1982.

A. Dasgupta and A. Prat. Information aggregation in financial markets with career concerns. *Journal of Economic Theory*, 143(1):83–113, November 2008.

W. F. De Bondt. Investor psychology and the dynamics of security prices. *Behavioral Finance and Decision Theory in Investment Management*, (7):7–12, 1995.

M. Dewatripont, I. Jewitt, and J. Tirole. The economics of career concerns, part i: Comparing information structures. *Review of Economic Studies*, 66(1):183–198, January 1999.

A. Dixit. R0 for covid research: An early estimate and policy implications. Working paper, 2020.

M. S. Drake, J. R. M. Jr, B. J. Twedt, and J. D. Warren. Social media analysts and sell-side analyst research. *Review of Accounting Studies*, 28(2):385–420, 2023.

M. Drugov and M. Troya-Martinez. Vague lies and lax standards of proof: On the law and economics of advice. *Journal of Economics & Management Strategy*, 28(2):298–315, 2019.

W. Dziuda and C. Salas. Communication with detectable deceit. Working paper, 2019.

J. Ely and J. Välimäki. Bad reputation. *Quarterly Journal of Economics*, 118(3):785–814, August 2003.

M. Farrell, T. C. Green, R. Jame, and S. Markov. The democratization of investment research and the informativeness of retail investor trading. *Journal of Financial Economics*, 145(2):616–641, 2022.

A. Frug. Strategic gradual learning and information transmission. *Journal of Economic Theory*, 177:594–615, 2018.

J. R. Graham. Herding among investment newsletters: theory and evidence. *Journal of Finance*, 54(1): 237–268, February 1999.

G. Gratton, R. Holden, and A. Kolotilin. When to drop a bombshell. *Review of Economic Studies*, 85(4): 2139–2172, October 2017.

S. Grenadier, A. Malenko, and N. Malenko. Timing decisions in organizations: Communication and authority in a dynamic environment. *American Economic Review*, 106(9):2552–2581, September 2016.

B. Groysberg, P. M. Healy, and D. A. Maber. What drives sell-side analyst compensation at high-status investment banks? *Journal of Accounting Research*, 49(4):969–1000, 2011.

I. Guttman. The timing of analysts' earnings forecasts. *Accounting Review*, 85(2):513–545, March 2010.

I. Guttman, I. Kremer, and A. Skrzypacz. Not only what but also when: A theory of dynamic voluntary disclosure. *American Economic Review*, 104(8):2400–2420, August 2014.

J. J. Heckman and S. Moktan. Publishing and promotion in economics: The tyranny of the top five. *Journal of Economic Literature*, 58(2):419–470, 2020.

B. Holmström. Managerial incentive problems: A dynamic perspective. *Review of Economic Studies*, 66(1): 169–182, January 1999.

H. Hong and J. D. Kubik. Analyzing the analysts: Career concerns and biased earnings forecasts. *The Journal of Finance*, 58(1):313–351, 2003.

H. Hong, J. D. Kubik, and A. Solomon. Security analysts' career concerns and herding of earnings forecasts. *RAND Journal of Economics*, 31(1):121–144, Spring 2000.

S. Kang and J. W. Kim. The fragility of experts: A moderated-mediation model of expertise, expert identity threat, and pverprecision. *Academy of Management Journal*, 65(2):577–605, 2022.

S. Keskek, S. Tse, and J. W. Tucker. Analyst information production and the timing of annual earnings forecasts. *Review of Accounting Studies*, 19(4):1504–1531, December 2014.

J. C. Kuperman, M. Athavale, and A. Eisner. Financial analysts in the media: Evolving roles and recent trends. *American Business Review*, 21(2):74, 2003.

O. A. Lamont. Macroeconomic forecasts and microeconomic forecasters. *Journal of Economic Behavior & Organization*, 48(3):265–280, July 2002.

E. Li and J. S. Martin. Capital formation and financial intermediation: The role of entrepreneur reputation formation. *Journal of Corporate Finance*, 59:185–201, 2019.

W. Li. Changing one's mind when the facts change: incentives of experts and the design of reporting protocols. *The Review of Economic Studies*, 74(4):1175–1194, 2007.

C. Margaria and A. Smolin. Dynamic communication with biased senders. *Games and Economic Behavior*, 110:330–339, 2018.

B. Mariano. Market power and reputational concerns in the ratings industry. *Journal of Banking & Finance*, 36(6):1616–1626, June 2012.

M. J. Osborne and A. Rubinstein. *Bargaining and Markets*. Academic Press Limited, 1990. ISBN 0-12-528632-5.

M. Ottaviani and P. N. Sørensen. Professional advice. *Journal of Economic Theory*, 126(1):120–142, January 2006a.

M. Ottaviani and P. N. Sørensen. The strategy of professional forecasting. *Journal of Financial Economics*, 81(2):441–466, August 2006b.

M. Ottaviani and P. N. Sørensen. Reputational cheap talk. *RAND Journal of Economics*, 37(1):155–175, March 2006c.

F. Pavesi and M. Scotti. Good lies. *European Economic Review*, 141:1039–1065, January 2022.

A. Prat. The wrong kind of transparency. *American Economic Review*, 95(3):862–877, June 2005.

C. Prendergast and L. Stole. Impetuous youngsters and jaded old-timers: Acquiring a reputation for learning. *Journal of Political Economy*, 104(6):1105–1134, December 1996.

L. Rees, N. Sharp, and B. Twedt. Who's heard on the street? determinants and consequences of financial analyst coverage in the business press. *Review of Accounting Studies*, 20(1):173–209, 2015.

D. S. Scharfstein and J. C. Stein. Herd behavior and investment. *American Economic Review*, 80(3):465–479, June 1990.

S. Shahanaghi. Reputation and misreporting in news media. Working paper, 2025.

P. K. Shroff, R. Venkataraman, and B. Xin. Timeliness of analysts' forecasts: The information content of delayed forecasts. *Contemporary Accounting Research*, 31(1):202–229, 2014.

E. Starkov. Only time will tell: Credible dynamic signaling. *Journal of Mathematical Economics*, 109: 102894, 2023. doi: https://doi.org/10.1016/j.jmateco.2023.102894.

B. Trueman. Analyst forecasts and herding behavior. *Review of Financial Studies*, 7(1):97–124, January 1994.

A. Vong. Reputational cheap talk versus prior information. Working paper, 2025.

H. Yin and H. Zhang. Tournaments of financial analysts. *Review of Accounting Studies*, 19:573–605, 2014.

Supplementary Appendix

B.1 Ideal Equilibria

Informative equilibria with nontrivial supports need not exist with concave payoffs, as evidenced by Proposition 2. A question arises: are uninformative and static equilibria the only possible outcomes when analysts are too risk-averse? The answer is "not necessarily".

The key to answering this question is assumption (OP), which requires that once an analyst has gained perfect reputation, it persists forever—even if his prediction turned out to be wrong. The assumption is known under the name "Never Dissuaded Once Convinced" in the signaling and bargaining literature, see Starkov [2023] for an overview. This assumption can be justified by a perturbation of the model in which the expert's (or the observer's) signal about the state is incorrect with vanishing probability—and thus so are his predictions.

However, this is not the only possible perturbation of the model with perfect signals. One may alternatively think of a version of the model with the infinitesimal number of "behavioral" analysts who are not strategic in their reports and just voice their opinions at random times. Since their number is infinitesimal, Bayes' rule still prescribes that for any (m, t) such that $r^E_\eta(m, t) > 0 = r^Q(m, t) = r^E_\varnothing(m, t)$ with $\eta = m$, we should have $\beta(m, t) = +\infty$ ($b(m, t) = 1$). However, since an informed expert is never wrong, if a prediction turns out incorrect, this would imply that it was actually made by one of the few behavioral analysts, who may be competent or not. This perturbation could support any belief $\beta^{-m}(m, t) \in [0, +\infty]$.

In this section, we replace (OP) by an alternative assumption (OP') which prescribes the worst possible off-path belief after an incorrect prediction supposedly made by an expert, same as any other off-path history:

(OP) off the equilibrium path the beliefs are $\rho = \rho_0$ and $\beta = 0$, with the exception that an extreme belief $\beta = +\infty$ ($b = 1$) is not updated;

(OP') off the equilibrium path the beliefs are $\rho = \rho_0$ and $\beta = 0$.

The alternative assumption (OP') allows for the existence of *ideal equilibria*, in which only experts make reports, while quacks stay silent:

Definition. *An equilibrium that satisfies (OP') and (TE) is* ideal *if $\mathcal{S} \neq \varnothing$ and $r^Q(m, t) = 0$ for all (m, t).*

It is immediate that if an ideal equilibrium exists, it is preferred by the observer to any other equilibrium with the same support $\mathcal{S}$, and the full-support ($\mathcal{S} = \mathcal{T}$) ideal equilibrium is the observer's first-best outcome. However, as we show below, for an ideal equilibrium to exist, stringent conditions on payoffs are required in addition to the specific off-path beliefs imposed by (OP').

The silence of the quacks in an ideal equilibrium can be enforced by the worst possible terminal reputation if the analyst's report turned out incorrect: $\beta^{-m}(m, t) = 0$. For this threat to be sufficient, the quacks should be sufficiently afraid of the chance of getting stuck with bad reputation in the long run, relative to the gains from good reputation in the short term. In other words, the payoff from reputation $w(\cdot)$ must be sufficiently concave. As before, the exact existence conditions are not very insightful, but we can show separately a necessary condition and a sufficient condition for ideal equilibria to exist. The necessary condition in Proposition 7 says that payoff functions must be non-convex: if either $w(\cdot)$, or $w^c(\cdot)$ is convex, then ideal equilibria do not exist. When related to part 2 of Proposition 2, this tells us that ideal equilibria are, informally speaking, complementary to informative equilibria in the sense of existence. Proposition 8 then provides a sufficient condition for ideal equilibria to exist, which quantifies the sufficient degree of payoff concavity.

Proposition 7. *If either $w(\cdot)$, or $w^c(\cdot)$ is convex, then there exists no ideal equilibrium.*

Proposition 8. *An ideal equilibrium with any support exists if the following sufficient condition holds:*

$$T \cdot w\left(\beta_0 \cdot (1 - F(T))\right) + w^c\left(\beta_0 \cdot (1 - F(T))\right) \geqslant \lim_{\beta \to +\infty} \left(T \cdot w(\beta) + p_0 \cdot w^c(\beta)\right). \tag{24}$$

Proof of Propositions 7 and 8. In an ideal equilibrium, $r_G^E(G,t) = r_B^E(B,t) = 1$ and $r_\varnothing^E(m,t) = r^Q(m,t) = 0$ for $m \in \{G, B\}$ and $t \in \mathcal{S}$. The prescribed strategy is trivially optimal for the informed expert, since it yields the highest possible payoff for the remainder of the game: he gets $\bar{w} := \lim_{\beta \to +\infty} w(\beta)$ until T and $\bar{w}^c := \lim_{\beta \to +\infty} w^c(\beta)$ at T. The uninformed expert's preference for staying silent at any t is at least as strong as that of the quack (due to the option value of receiving news in the future and obtaining the maximal continuation payoff). Therefore, quack's preference for staying silent at all $t \in \mathcal{S}$ is a necessary and sufficient condition for an ideal equilibrium to exist.

Since the analyst's reputation jumps to $\beta(m,t) = +\infty$ after any report in an ideal equilibrium, due to the martingale property of beliefs, it must decrease after no report at any $t \in \mathcal{S}$. Therefore, any delay is costly for the quack, so it is sufficient to verify that reporting at t_1, the first point in the support, is not optimal for him. The quack's payoffs generated by reports (G, t_1) and (B, t_1) are given by, respectively,

$$\begin{aligned} W_{t_1}^Q(G, t_1) &= (T - t_1) \cdot \bar{w} + p_0 \cdot \bar{w}^c, \\ W_{t_1}^Q(B, t_1) &= (T - t_1) \cdot \bar{w} + (1 - p_0) \cdot \bar{w}^c. \end{aligned}$$

We have $p_0 \geqslant \frac{1}{2}$, hence $W_{t_1}^Q(G, t_1) \geqslant W_{t_1}^Q(B, t_1)$. The flow payoff the quack receives in period $t \in \mathcal{S}$ from staying silent until t in an ideal equilibrium equals $w\left(\beta_0\left(1 - F(t)\right)\right)$. Therefore, the value from not making a report, as evaluated at t_1, equals

$$W_{t_1}^Q(\varnothing) = \sum_{k=1}^{|\mathcal{S}|} (t_{k+1} - t_k) \cdot w\left(\beta_0\left(1 - F(t_k)\right)\right) + w^c\left(\beta_0\left(1 - F(t_{|\mathcal{S}|})\right)\right),$$

where $\mathcal{S} = \left\{t_1, ..., t_{|\mathcal{S}|}\right\}$, $t_{|\mathcal{S}|+1} := T$. Staying silent is optimal if and only if $W_{t_1}^Q(G, t_1) \leqslant W_{t_1}^Q(\varnothing)$. Because all terms in $W_{t_1}^Q(\varnothing)$ are finite, this requires that both $\bar{w}$ and $\bar{w}^c$ to be finite. However, if either $w(\cdot)$ or $w^c(\cdot)$ is convex, because they are strictly increasing it must be that $\bar{w} = +\infty$. Then $W_{t_1}^Q(G, t_1) \leqslant W_{t_1}^Q(\varnothing)$ cannot be upheld, so no ideal equilibria exist when either $w(\cdot)$, or $w^c(\cdot)$ is convex. This proves Proposition 7.

To prove Proposition 8, note that $F(t_k)$ is increasing in k, hence $\beta_0(1 - F(t_k))$ is decreasing in k, so

$$W_{t_1}^Q(\varnothing) \geqslant (T - t_1) \cdot w\left(\beta_0 \cdot (1 - F(T))\right) + w^c\left(\beta_0 \cdot (1 - F(T))\right).$$

Further, due to the monotonicity of $w(\cdot)$, we have $w\left(\beta_0 \cdot (1 - F(T))\right) < \bar{w}$. Therefore, if (24) holds, it implies

$$(T - t_1) \cdot w\left(\beta_0 \cdot (1 - F(T))\right) + w^c\left(\beta_0 \cdot (1 - F(T))\right) \geqslant (T - t_1) \cdot \bar{w} + p_0 \cdot \bar{w}^c$$

for any t_1, and therefore $W_{t_1}^Q(\varnothing) \geqslant W_{t_1}^Q(G, t_1)$. As argued above, this condition guarantees that an ideal equilibrium with support $\mathcal{S}$ exists. The support was arbitrary, hence this completes the proof. □

B.2 General Structure of Equilibria

This appendix shows that (TE) can be replaced by a weaker set of refinements: message labeling, symmetry, and reticence, all defined below. We next show that Propositions 1, 2, 3 and Proposition 4 continue to hold under this modified set of refinements. For convenience, we do not make any restrictions on the private signal informativeness π in this section, so the proofs are valid as for $\pi = 1$, as for $\pi < 1$.

We first introduce the notion of informative/uninformative reports.

Definition. *A report* (m, t) *is* uninformative *if both of the following hold:*

$$\beta(m, t) = \beta_{t-1}, \tag{25}$$

$$\rho(m, t) = \rho_{t-1}. \tag{26}$$

A report (m, t) *is* informative *if it is not uninformative.*

Condition (25) implies that the report is uninformative about the analyst's type, while (26) implies that it contains no information about the state. Cheap talk models following Crawford and Sobel [1982] are known to always have an equilibrium in which all reports are uninformative. In a dynamic setting like ours, reports can be informative or uninformative in any given period. In our main analysis, equilibria with uninformative reports are ruled out by refinement (TE), which, as argued in Section 5.1, implies that report $m = G$ is always suggestive of the state being $\omega = G$ and vice versa for report $m = B$.

For this section we keep assumption (OP) and replace (TE) with two following weaker conditions:

(ML) Message Labeling: at least one of the following conditions holds:

- $r^E_G(G, t) > r^E_G(B, t)$,
- $r^E_B(B, t) > r^E_B(G, t)$,
- $r^E_G(G, t) = r^E_G(B, t)$ and $r^E_B(G, t) = r^E_B(B, t)$.

(SY) Symmetry: $r^E_G(G, t) = r^E_B(B, t)$.

Message labeling (ML) requires that report m is more indicative of state $\omega = m$ than the other report. This assumption is without loss, since at any history h_t we can assign message labels G and B to the two messages in such a way that (ML) is satisfied. In contrast, Symmetry (SY) is a non-trivial restriction. It requires that the expert treats states and messages equally—if he has evidence of state G, he sends report G with the same probability that he would have sent report B if he had evidence of state B. This assumption is made primarily for tractability.

With these two restrictions, due to the assumption that the analyst can send at most one report, uninformative reports in any equilibrium are organized in a specific structure. This is illustrated by the following proposition.

Proposition 9. *For any equilibrium there exists* $\bar{t}$ *such that*

1. *All on-path reports* (m, t) *with* $t > \bar{t}$ *are uninformative.*
2. *All on-path reports* (m, t) *with* $t \leqslant \bar{t}$ *are informative. Moreover:*
   - *at every* $t < \bar{t}$ *the expert does not make a report unless he has received the corresponding signal, i.e.,* $r^E_\varnothing(m, t) = 0$ *and* $r^E_\eta(m, t) = 0$ *whenever* $\eta \neq m$*;*

- *at $t = \bar{t}$ the informed expert always reports his signal, i.e., $r^E_\eta(\eta, \bar{t}) = 1$.*

Proposition 9 says that in our model all equilibria feature at most two phases: in the early phase, the reports are informative, while in the later phase the reports do not contain any relevant information about the state or the type of the analyst.

To understand Proposition 9, it is enough to note that by Lemma 6, which still remains valid with no modifications, $t \in \mathcal{S}$ only if an expert is willing to report at t. His comparative advantage, relative to a quack, is his ability to acquire private signals. Therefore, the expert is only willing to participate in uninformative communication if he has no option to exploit his [current or possibly future] information by sending an informative report—i.e., if $\bar{t}$ has passed. Conversely, whenever an option to make an informative report now or in the future is present ($t < \bar{t}$), the expert would not be willing to use his only chance to make an unfounded report (or report contrary to his private information).

It is worth noting that $\bar{t}$ does not have to be in the interior of the support, so one of the phases may be absent: if $\bar{t} = 0$ then all reports are uninformative, while if $\bar{t} = t_{|\mathcal{S}|}$ then all reports are informative in equilibrium. We shall refer to the latter type of equilibria as *globally informative.*

Definition. *A* globally informative equilibrium *is an equilibrium where all reports in the support are informative.*

Note that in any globally informative equilibrium with support $\mathcal{S}$, it must be that $\bar{t} = t_{|\mathcal{S}|}$. The next proposition shows that the phase beyond $\bar{t}$ can be safely ignored altogether, and without loss of generality we can restrict attention to globally informative equilibria.

Proposition 10. *For any equilibrium with support $\mathcal{S}$ and corresponding $\bar{t}$, there exists an globally informative equilibrium with the same $\bar{t}$ and support $\tilde{\mathcal{S}} = \mathcal{S} \cap \{t \leqslant \bar{t}\}$ such that the two equilibria are:*

1. *payoff-equivalent for all players,*
2. *strategy-equivalent on $\tilde{\mathcal{S}}$.*

Propositions 10 establishes that any equilibrium strategy profile with some $\bar{t}$ can be obtained from a respective globally informative equilibrium with the same $\bar{t}$ by allowing for some uninformative reports in periods $\{\bar{t}+1, \dots, T\}$.

We also focus on *reticent* equilibria, in which the expert never makes any unfounded reports.

Definition. *An equilibrium is* reticent *if $r^E_\varnothing(G, \bar{t}) = r^E_\varnothing(B, \bar{t}) = 0$.*

The expert in such equilibria only makes a prediction if he has received the corresponding evidence. Proposition 9 established that this must be true for all $t < \bar{t}$, so the restriction only applies to the last point of the support $t = \bar{t}$. The restriction is needed exactly so that all points in the support can be treated equally, without $\bar{t}$ requiring individual statements.

Proposition 9 and Proposition 10 together imply that Lemma 5 remains valid, which allows us to keep the same informativeness measure as with refinement (TE).

Corollary 11. *In any globally informative reticent equilibrium:*

1. *$\rho_t = \rho_0$ for all $t \in \mathcal{S}$, i.e., belief about the state remains constant in the absence of reports;*
2. *$\beta^\omega(\varnothing) = \beta_T$ for any $\omega \in \{G, B\}$, i.e., state revelation does not affect reputation if no report was made.*

We now proceed to our main characterization result.

Proposition 12. *Propositions 1, 2, 3, and 4 are valid for reticent equilibria if (TE) is substituted with (ML) and (SY).*

The intuition for Proposition 12 is straightforward. Propositions 9 and 10 together with reticence imply that the expert never makes unfounded predictions, or predictions that contradict his evidence: $r^E_\varnothing(m,t) = 0$ and $r^E_\eta(m,t) = 0$ whenever $\eta \neq m$ for all $t \in \mathcal{S}$. Therefore, if $t \in \mathcal{S}$ it has to be that $r^E_G(G,t) = r^E_B(B,t) > 0$. Recall that (TE) assumed $r^E_G(G,t) = r^E_B(B,t) = 1$. Therefore, the only difference with (TE) is that the expert may want to delay the report, but should still at least weakly prefer to report his private signal truthfully as soon as he can. It thus results in the same set of incentive constraints for the expert and therefore provides the same set of results. In the proof for Proposition 12 we outline all differences in the respective proofs in comparison to the proofs of Propositions 1, 2, 3, and 4.

B.3 Proofs for Propositions 9 and 10, Corollary 11, and Proposition 12

We now proceed to prove the results. We first need to introduce the following probabilities:

$$z_{t,\eta} = P\{\eta_t = \eta, \mu_{t-1} = \varnothing | \eta^E_{t^*} = \eta\}$$

for $\eta \in \{\varnothing, G, B]\}$. I.e., $z_{t,\eta}$ is the probability that the expert has information η at time t and has not made a report prior to t, conditional on expert's signal realization being $\eta^*_t = \eta$ (or unconditional if $\eta = \varnothing$). With these probabilities, (6) transforms to

$$\begin{aligned} r^E(m,t) = \mathbb{E}_\eta[r^E_\eta(m,t)] &= \frac{p^*_t \cdot z_{t,G} \cdot r^E_G(m,t) + (1-p^*_t) \cdot z_{t,B} \cdot r^E_B(m,t) + z_{t,\varnothing} \cdot r^E_\varnothing(m,t)}{p^*_t \cdot z_{t,G} + (1-p^*_t) \cdot z_{t,B} + z_{t,\varnothing}}, \\ \mathbb{E}_\eta[r^E_\eta(m,t)|\omega] &= \frac{\pi \cdot z_{t,\omega} \cdot r^E_\omega(m,t) + (1-\pi) \cdot z_{t,-\omega} \cdot r^E_{-\omega}(m,t) + z_{t,\varnothing} \cdot r^E_\varnothing(m,t)}{\pi \cdot z_{t,\omega} + (1-\pi) \cdot z_{t,-\omega} + z_{t,\varnothing}}, \end{aligned} \tag{27}$$

where $p^*_t := p_t \cdot \pi + (1-p_t) \cdot (1-\pi)$. The first equalities in (7), (8) and (9) still remain valid.

To shorten the proof of Proposition 9 we next establish two auxiliary lemmas, both of which are valid for all $\pi \in (\frac{1}{2}, 1]$.

Lemma 13. *Fix any equilibrium and period t such that reports (G,t) and (B,t) are both on equilibrium path. If the expert with private signal $\eta = G$ (strictly) prefers report $m' \in \{G,B\}$ over $m'' \in \{G,B\}$ at t, then the expert with private signal $\eta = B$ (strictly) prefers m'' over m', and vice versa*

Proof. If both (G,t) and (B,t) are on path, then the quack should be indifferent between the two reports: $W^Q_t(G,t) = W^Q_t(B,t)$, meaning

$$\begin{aligned} &(T-t) \cdot w(\beta(G,t)) + p_0 \cdot w^c(\beta^G(G,t)) + (1-p_0) \cdot w^c(\beta^B(G,t)) = \\ &\qquad = (T-t) \cdot w(\beta(B,t)) + p_0 \cdot w^c(\beta^G(B,t)) + (1-p_0) \cdot w^c(\beta^B(B,t)). \end{aligned} \tag{28}$$

As assumed, the expert with $\eta = G$ prefers report (m',t) to (m'',t): $W^E_{t,G}(m',t) \geqslant W^E_{t,G}(m'',t)$, or

$$\begin{aligned} &(T-t) \cdot w(\beta(m',t)) + \frac{1}{p^*_0} \cdot \Big[p_0 \cdot \pi \cdot w^c(\beta^G(m',t)) + (1-p_0) \cdot (1-\pi) \cdot w^c(\beta^B(m',t))\Big] \geqslant \\ &\qquad \geqslant (T-t) \cdot w(\beta(m'',t)) + \frac{1}{p^*_0} \cdot \Big[p_0 \cdot \pi \cdot w^c(\beta^G(m'',t)) + (1-p_0) \cdot (1-\pi) \cdot w^c(\beta^B(m'',t))\Big]. \end{aligned}$$

Multiplying the inequality above by p_0^*, subtracting it from (28), and dividing the result by $(1-p_0^*)$, we get

$$(T-t)\cdot w(\beta(m',t))+\frac{1}{1-p_0^*}\cdot\Big[p_0\cdot(1-\pi)\cdot w^c(\beta^G(m',t))+(1-p_0)\cdot\pi\cdot w^c(\beta^B(m',t))\Big]\leqslant$$
$$\leqslant(T-t)\cdot w(\beta(m'',t))+\frac{1}{1-p_0^*}\cdot\Big[p_0\cdot(1-\pi)\cdot w^c(\beta^G(m'',t))+(1-p_0)\cdot\pi\cdot w^c(\beta^B(m'',t))\Big],$$

which exactly means that expert with $\eta=B$ prefers report (m'',t) over (m',t). Further, this preference is strict if and only if the preference of the expert with $\eta=G$ is strict. □

Lemma 14. *Fix any equilibrium and period t such that reports (G,t) and (B,t) are both on equilibrium path. If the expert with private signal $\eta\in\{G,B\}$ is indifferent between reports (G,t) and (B,t), then both reports must induce the same observer's beliefs: $\beta(G,t)=\beta(B,t)$ and $\beta^\omega(G,t)=\beta^\omega(B,t)$ for both $\omega\in\{G,B\}$.*

Proof. If both reports (G,t) and (B,t) are on path, the quack should be indifferent between these reports, as captured by condition (28). In turn, the indifference of the expert with information $\eta=G$ (case $\eta=B$ is analogous) is represented by the following condition: $W^E_{t,G}(G,t)=W^E_{t,G}(B,t)$ or, equivalently,

$$(T-t)\cdot w(\beta(G,t))+\frac{1}{p_0^*}\cdot\Big[p_0\cdot\pi\cdot w^c(\beta^G(G,t))+(1-p_0)\cdot(1-\pi)\cdot w^c(\beta^B(G,t))\Big]=$$
$$=(T-t)\cdot w(\beta(B,t))+\frac{1}{p_0^*}\cdot\Big[p_0\cdot\pi\cdot w^c(\beta^G(B,t))+(1-p_0)\cdot(1-\pi)\cdot w^c(\beta^B(B,t))\Big].$$

Subtracting this equality from (28) and rearranging, we get

$$w^c\left(\beta^G(G,t)\right)-w^c\left(\beta^B(G,t)\right)=w^c\left(\beta^G(B,t)\right)-w^c\left(\beta^B(B,t)\right).$$

Therefore, if $\beta^G(G,t)\geqslant\beta^G(B,t)$, then it must be that $\beta^B(G,t)\geqslant\beta^B(B,t)$, and if $\beta^G(G,t)\leqslant\beta^G(B,t)$, then $\beta^B(G,t)\leqslant\beta^B(B,t)$.

At the same time, $\beta^G(m,t)$ and $\beta^B(m,t)$ average out to $\beta(m,t)$ (with weights that are independent of m). Therefore, if $\beta^G(G,t)\geqslant\beta^G(B,t)$ and $\beta^B(G,t)\geqslant\beta^B(B,t)$, then $\beta(G,t)\geqslant\beta(B,t)$, meaning the quack's indifference (28) can be sustained only if $\beta^G(G,t)=\beta^G(B,t)$, $\beta^B(G,t)=\beta^B(B,t)$, and $\beta(G,t)=\beta(B,t)$. The other case—$\beta^G(G,t)\leqslant\beta^G(B,t)$ and $\beta^B(G,t)\leqslant\beta^B(B,t)$—leads to the same conclusion. □

Proof of Proposition 9. The proof is valid for all $\pi\in(\frac{1}{2},1]$. The proof proceeds in several steps.

Step 1. We next consider $t_{|\mathcal{S}|}$, the last point of the support, and show that either all reports are uninformative at $t_{|\mathcal{S}|}$, or $r^E_G(G,t_{|\mathcal{S}|})=r^E_B(B,t_{|\mathcal{S}|})=1$. The quack is indifferent between making either report and staying silent if the latter is on path at $t_{|\mathcal{S}|}$. Next, the uninformed expert is indifferent between all these options as well: he neither knows the state at $t_{|\mathcal{S}|}$, nor has an option to communicate it in further periods, as $t_{|\mathcal{S}|}$ is the last point of the support. Note that at $t_{|\mathcal{S}|}$, staying silent can be interpreted as a report, because the analyst cannot make any reports beyond $t_{|\mathcal{S}|}$. Therefore, Lemmas 13 and 14 at $t_{|\mathcal{S}|}$ can be applied not only to reports, but also to the option of staying silent.

Note also that $t_{|\mathcal{S}|}\in\mathcal{S}$ implies at least one report $m\in\{G,B\}$ is on the equilibrium path at $t_{|\mathcal{S}|}$. If exactly one report m is on path and not reporting is off path, then $(m,t_{|\mathcal{S}|})$ is trivially uninformative.

If not reporting is on path, (SY) and (ML) together imply that if $t_{|\mathcal{S}|}\in\mathcal{S}_m\backslash\mathcal{S}_{-m}$ for some m, then $r^E_\eta\left(m,t_{|\mathcal{S}|}\right)=0$ for $\eta\in\{G,B\}$. So by (12) and (27), $\rho\left(m,t_{|\mathcal{S}|}\right)=\rho_{t_{|\mathcal{S}|-1}}$. Since $-m$ is off-path, this implies that $\rho_{t_{|\mathcal{S}|}}=\rho_{t_{|\mathcal{S}|-1}}$. From $r^E_\eta\left(m,t_{|\mathcal{S}|}\right)=0$ for both $\eta\in\{G,B\}$, it follows that the informed expert at least weakly prefers $\varnothing$ over m at $t_{|\mathcal{S}|}$. Therefore, by Lemma 13, the informed expert should be indifferent between staying silent and report m for any $\eta \in \{G, B\}$. Consequently, Lemma 14 implies that $\beta(m,t) = \beta_{t_{|\mathcal{S}|}}$ and, since $-m$ is off-path, that $\beta(m,t) = \beta_{t_{|\mathcal{S}|-1}}$. We thus conclude that $(m, t_{|\mathcal{S}|})$ is indeed an uninformative report.

Hence it only remains to show the statement of this step for the case when both reports $m \in \{G, B\}$ are on the equilibrium path at $t_{|\mathcal{S}|}$. We proceed to show it case-by-case, for the exhaustive set of cases according to the preference of the informed expert with $\eta = G$. Note that if the expert strictly prefers to stay silent in the respective case, then (OP) immediately implies that doing so is on the equilibrium path.

Case 1: The expert with $\eta = G$ strictly prefers to make report G over other options. Then $r^E_G(G, t_{|\mathcal{S}|}) = 1$, and by (SY) we get $r^E_B(B, t_{|\mathcal{S}|}) = 1$, which supports the claim.

Case 2: The expert with $\eta = G$ strictly prefers to make report B over other options. Then $r^E_G(B, t_{|\mathcal{S}|}) = 1$ and by (SY) we get $r^E_G(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 0$. Since $r^E_B(G, t_{|\mathcal{S}|}) \geqslant 0$, this then violates (ML).

Case 3: The expert with $\eta = G$ strictly prefers to stay silent over other options. Then $r^E_G(G, t_{|\mathcal{S}|}) = r^E_G(B, t_{|\mathcal{S}|}) = 0$ and by (SY), $r^E_B(B, t_{|\mathcal{S}|}) = 0$. By Lemma 13, the expert with $\eta = B$ must strictly prefer to report either B, or G over staying silent. Therefore, $r^E_B(G, t_{|\mathcal{S}|}) = 1$ which, in turn, violates (ML).

Case 4: The expert with $\eta = G$ is indifferent between reporting G and B, and strictly prefers these options to staying silent, which is on the equilibrium path. If the latter is on path, then by Lemma 13, the expert with $\eta = B$ strictly prefers to stay silent. Then $r^E_B(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 0$. By (SY) we get $r^E_G(G, t_{|\mathcal{S}|}) = 0$, so $r^E_G(B, t_{|\mathcal{S}|}) = 1$, which then violates (ML).

Case 5: The expert with $\eta = G$ is indifferent between reporting G and staying silent and strictly prefers these options to reporting B. Then by Lemma 13, the expert with $\eta = B$ strictly prefers to report B. Then $r^E_B(B, t_{|\mathcal{S}|}) = 1$, and by (SY) we get $r^E_G(G, t_{|\mathcal{S}|}) = 1$, which supports the claim.

Case 6: The expert with $\eta = G$ is indifferent between reporting B and staying silent and strictly prefers these options to reporting G. Then by (SY) we get $r^E_G(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 0$. By Lemma 13, the expert with $\eta = B$ strictly prefers to report G, and therefore $r^E_B(G, t_{|\mathcal{S}|}) = 1$ and $r^E_B(B, t_{|\mathcal{S}|}) = 0$, which, in turn, violates (ML) because $r^E_G(B, t_{|\mathcal{S}|}) \geqslant 0$.

Case 7: The expert with $\eta = G$ is indifferent between all $m \in \{G, B\}$ and staying silent if the latter is on the equilibrium path.[25] By Lemma 13, the expert with $\eta = B$ is indifferent between the same options. By Lemma 14, all options must induce the same beliefs: $\beta^G(G, t_{|\mathcal{S}|}) = \beta^G(B, t_{|\mathcal{S}|}) = \beta^G(\varnothing)$ and $\beta^B(G, t_{|\mathcal{S}|}) = \beta^B(B, t_{|\mathcal{S}|}) = \beta^B(\varnothing)$ (with the latter equality in each pair only being relevant if $m = \varnothing$ is on path at $t_{|\mathcal{S}|}$). We next show that these two groups of beliefs should be equal to each other. Suppose not and $\beta^G(m, t_{|\mathcal{S}|}) > \beta^B(m, t_{|\mathcal{S}|})$ for $m \in \{G, B\}$ and $\beta^G(\varnothing) > \beta^B(\varnothing)$ (the other case—when $\beta^G(m, t_{|\mathcal{S}|}) < \beta^B(m, t_{|\mathcal{S}|})$ for $m \in \{G, B\}$ and $\beta^G(\varnothing) < \beta^B(\varnothing)$—is analogous). Given (27), these inequalities amount to

$$\frac{\pi z_{t,G} r^E_G(m,t) + (1-\pi) z_{t,B} r^E_B(m,t) + z_{t,\varnothing} r^E_\varnothing(m,t)}{\pi z_{t,G} + (1-\pi) z_{t,B} + z_{t,\varnothing}} > \frac{\pi z_{t,B} r^E_B(m,t) + (1-\pi) z_{t,G} r^E_G(m,t) + z_{t,\varnothing} r^E_\varnothing(m,t)}{\pi z_{t,B} + (1-\pi) z_{t,G} + z_{t,\varnothing}}$$

for all $m \in \{G, B, \varnothing\}$. Here with some abuse of notation we let $r^E_\eta(\varnothing, t) := 1 - r^E_\eta(G, t) - r^E_\eta(B, t)$. However, if we add up all the inequalities above for $m \in \{G, B, \varnothing\}$, we get that $1 > 1$, which is a clear contradiction.

[25] If $m = \varnothing$ is off path, the argument still applies so long as one omits all beliefs related to making no report.

Therefore, $\beta^G(m, t_{|\mathcal{S}|}) = \beta^B(m, t_{|\mathcal{S}|}) = \beta(m, t_{|\mathcal{S}|})$ for $m \in \{G, B\}$ and $\beta^G(\varnothing) = \beta^B(\varnothing) = \beta_{t_{|\mathcal{S}|}}$. We then have $\rho(m, t_{|\mathcal{S}|}) = \rho_{t_{|\mathcal{S}|-1}}$, which establishes one condition for $\left(m, t_{|\mathcal{S}|}\right)$ to be a uninformative report. Moreover, revelation of state does not affect the analyst's reputation irrespective of whether he made a report at $t_{|\mathcal{S}|}$ or not. Then due to the martingale property of beliefs, for the expert to be indifferent between either report and staying silent, it must be that

$$\beta(G, t_{|\mathcal{S}|}) = \beta(B, t_{|\mathcal{S}|}) = \beta_{t_{|\mathcal{S}|}} = \beta_{t_{|\mathcal{S}|-1}},$$

which concludes the proof that either report is uninformative at $t_{|\mathcal{S}|}$.

Step 2. We next show that if $r^E_G(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 1$ then $V^E_{t_{|\mathcal{S}|},\eta} > V^Q_{t_{|\mathcal{S}|}}$ for $\eta \in \{G, B\}$. Since both $m \in \{G, B\}$ are optimal for the quack at $t_{|\mathcal{S}|}$, we have

$$\begin{aligned} V^E_{t_{|\mathcal{S}|},G} - V^Q_{t_{|\mathcal{S}|}} &= W^E_{t_{|\mathcal{S}|},G}(G, t_{|\mathcal{S}|}) - W^Q_{t_{|\mathcal{S}|}}(G, t_{|\mathcal{S}|}) = \\ &= \frac{(2\pi - 1) \cdot p_0 \cdot (1 - p_0)}{p^*_0} \cdot \left(w^c(\beta^G(G, t_{|\mathcal{S}|})) - w^c(\beta^B(G, t_{|\mathcal{S}|}))\right), \\ V^E_{t_{|\mathcal{S}|},B} - V^Q_{t_{|\mathcal{S}|}} &= W^E_{t_{|\mathcal{S}|},B}(B, t_{|\mathcal{S}|}) - W^Q_{t_{|\mathcal{S}|}}(B, t_{|\mathcal{S}|}) = \\ &= \frac{(2\pi - 1) \cdot p_0 \cdot (1 - p_0)}{1 - p^*_0} \cdot \left(w^c(\beta^B(B, t_{|\mathcal{S}|})) - w^c(\beta^G(B, t_{|\mathcal{S}|}))\right). \end{aligned} \tag{29}$$

Therefore, to establish the claim we need to verify that $\beta^G(G, t_{|\mathcal{S}|}) > \beta^B(G, t_{|\mathcal{S}|})$ and $\beta^B(B, t_{|\mathcal{S}|}) > \beta^G(B, t_{|\mathcal{S}|})$. Given $r^E_G(G, t_{|\mathcal{S}|}) = r^E_B(B, t_{|\mathcal{S}|}) = 1$ and (27) these inequalities are equivalent to

$$\begin{aligned} \frac{\pi \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(G, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}} &> \frac{(1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(G, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}}, \\ \frac{\pi \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(B, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}} &> \frac{(1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(B, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}}. \end{aligned}$$

Because $\pi > \frac{1}{2}$, the numerators on the LHS of the two inequalities are strictly larger than the numerators on the respective RHS. Therefore, at least one of these inequalities is always satisfied. Without loss, assume it is the first one. Then suppose by way of contradiction that the second inequality is violated. Then

$$\begin{aligned} \frac{\pi \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(G, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}} &> \frac{(1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(G, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}}, \\ \frac{\pi \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(B, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}} &\leqslant \frac{(1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing} \cdot r^E_\varnothing(B, t_{|\mathcal{S}|})}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}}. \end{aligned} \tag{30}$$

Then if we subtract the latter from the former, we get

$$\begin{aligned} &\frac{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing} \cdot (r^E_\varnothing(G, t_{|\mathcal{S}|}) + r^E_\varnothing(B, t_{|\mathcal{S}|}))}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}} > \\ &\qquad > \frac{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing} \cdot (r^E_\varnothing(G, t_{|\mathcal{S}|}) + r^E_\varnothing(B, t_{|\mathcal{S}|}))}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1 - \pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}}. \end{aligned}$$

If $r^E_\varnothing(G, t_{|\mathcal{S}|}) + r^E_\varnothing(B, t_{|\mathcal{S}|}) = 1$ then we get a contradiction because then the inequality reduces to $1 > 1$. On the other hand, if $r^E_\varnothing(G, t_{|\mathcal{S}|}) + r^E_\varnothing(B, t_{|\mathcal{S}|}) < 1$, then the inequality above reduces to

$$\frac{1}{\pi \cdot z_{t_{|\mathcal{S}|},G} + (1-\pi) \cdot z_{t_{|\mathcal{S}|},B} + z_{t_{|\mathcal{S}|},\varnothing}} < \frac{1}{\pi \cdot z_{t_{|\mathcal{S}|},B} + (1-\pi) \cdot z_{t_{|\mathcal{S}|},G} + z_{t_{|\mathcal{S}|},\varnothing}},$$

which clearly violates the second inequality in (30).

Step 3. If all reports are uninformative at $t_{|\mathcal{S}|}$ then $V^E_{t_{|\mathcal{S}|},\eta} = V^Q_{t_{|\mathcal{S}|}}$ for any $\eta \in \{G, B\}$. Indeed, in Case 7 of Step 1 we established that if report m is uninformative then $\beta^G(m, t_{|\mathcal{S}|}) = \beta^B(m, t_{|\mathcal{S}|})$. And by (29) we therefore get the result.

Step 4. We next show that if for some $t_i \in \mathcal{S}$, all reports in all periods $t > t_i$ are uninformative, then either all reports are uninformative in period t_i or $r^E_G(G, t_i) = r^E_B(B, t_i) = 1$. Note that it is enough to verify that incentives of an analyst are the same at t_i as they were at $t_{|\mathcal{S}|}$ in Step 1.

Indeed, by Lemma 6, the quack is indifferent between making either report at t_i and staying silent (and making a report in any further period in $\mathcal{S}$ or not making a report by T if this option is on path). The uninformed expert is indifferent between making either report and staying silent as well. This is because if he makes a report, he gets the same value as the quack because they have the same information. If he stays silent at t_i, his reputation is frozen at β_{t_i} until and after the revelation of the state at T (see Case 7 of Step 1). Therefore, $W^E_{t_i,\varnothing}(\varnothing) = W^Q_{t_i}(\varnothing)$ as well, so the uninformed expert is indeed indifferent between all $m \in \{G, B, \varnothing\}$, same as the quack. Finally, the informed expert with $\eta_{t_i} \in \{G, B\}$ faces a choice between the two reports and staying silent, with the latter, again, freezing his reputation at β_{t_i} for the remainder of the game.

Therefore, by following the same argument as we used for $t_{|\mathcal{S}|}$ in Step 1, we get the result. Note that the statements and the arguments from Steps 2 and 3 then carry over to t_i as well.

Step 5. Using the inductive reasoning we then get that either all reports in all periods within support $\mathcal{S}$ are uninformative or there exists $\bar{t}$ such that all reports in all periods $t > \bar{t}$ are uninformative and $r^E_G(G, \bar{t}) = r^E_B(B, \bar{t}) = 1$. First, it follows from Step 3 that $V^E_{\bar{t},\varnothing} = V^Q_{\bar{t}}$. Therefore, it is enough to establish that $V^E_{t,\varnothing} > V^Q_t$ for $t < \bar{t}$. Recall from Step 2 that $V^E_{\bar{t},\eta} > V^Q_{\bar{t}}$ for $\eta \in \{G, B\}$. Take any $t < \bar{t}$ and consider the following strategy for the expert who is uninformed at t: wait until $\bar{t}$ and make report $(\eta_{\bar{t}}, \bar{t})$ if $\eta_{\bar{t}} \in \{G, B\}$, and make a random report if $\eta_{\bar{t}} = \varnothing$. In the former case, the expert gets a strictly higher value than the quack, while in the latter the two are the same. Since the expert faces a strictly positive probability of receiving the private signal between t and $\bar{t}$, this strategy gives the uninformed expert a payoff that is strictly higher than what the quack gets, $V^E_{t,\varnothing} > V^Q_t$, which completes the proof.

Step 6. We next show that $r^E_\varnothing(G, t) = r^E_\varnothing(B, t) = 0$ for $t < \bar{t}$. Suppose the contrary: for some $t < \bar{t}$, the uninformed expert weakly prefers, without loss, report (G, t) over staying silent. But then

$$V^E_{t,\varnothing} = W^E_{t,\varnothing}(G, t) = W^Q_t(G, t) = V^Q_t,$$

because the uninformed expert and the quack have the same information about ω. At the same time, Step 5 above shows that there exists a strategy that grants the t-uninformed expert a payoff strictly higher than the quack's: $V^E_{t,\varnothing} > V^Q_t$, which gives us a contradiction.

Step 7. Finally, we establish that $r^E_G(B,t) = r^E_B(G,t) = 0$ for $t \leqslant \bar{t}$. Suppose the contrary: that for some $t \leqslant \bar{t}$, the expert either prefers to report G when $\eta_t = B$, or to report B when $\eta_t = G$. If this preference is strict, then by (SY) we have $r^E_G(G,t) = r^E_B(B,t) = 0$, which violates (ML). Therefore, the expert should be indifferent between the two reports for both $\eta_t \in \{G, B\}$. It follows from the arguments in Steps 2 and 5 that $V^E_{t,\eta} > V^Q_t$ for any $\eta \in \{G, B\}$ and $t \leqslant \bar{t}$, which means that

$$V^E_{t,G} - V^Q_t = W^E_{t,G}(G,t) - W^Q_t(G,t) > 0,$$
$$V^E_{t,B} - V^Q_t = W^E_{t,B}(G,t) - W^Q_t(G,t) > 0.$$

We have seen in Step 2 that these inequalities are equivalent to

$$\beta^G(G,t) > \beta^B(G,t),$$
$$\beta^B(G,t) > \beta^G(G,t),$$

so the two are clearly at a contradiction. □

Proof of Proposition 10. The proof is valid for all $\pi \in \left(\frac{1}{2}, 1\right]$. Let $\{r^\gamma_\eta(m,t)\}$ be an equilibrium strategy profile. Consider a new strategy profile $\{\tilde{r}^\gamma_\eta(m,t)\}$ such that $\tilde{r}^E_\eta(m,t) = r^E_\eta(m,t)$, $\tilde{r}^Q(m,t) = r^Q(m,t)$ for all $t \leqslant \bar{t}$ and $\tilde{r}^E_\eta(m,t) = \tilde{r}^Q(m,t) = 0$ for all $t > \bar{t}$. As strategies coincide on $\tilde{\mathcal{S}}$ and all reports (m,t) with $t > \bar{t}$ are uninformative in the original equilibrium, the following are true:

1. beliefs $\beta(m,t)$ and $\beta^\omega(m,t)$ induced by the two strategy profiles coincide for all $\omega, m \in \{G, B\}$, $t \in \tilde{\mathcal{S}}$;

2. belief sequences β_t induced by the two strategy profiles coincide for all $t \in \mathcal{T}$.

The latter statement also exploits the fact that $\mathcal{S}\backslash\tilde{\mathcal{S}}$ is nonempty (otherwise the proposition statement trivially holds), so it must be that $r^Q(G,\bar{t}) + r^Q(B,\bar{t}) < 1$ and $r^E(G,\bar{t}) + r^E(B,\bar{t}) < 1$.

The first statement above implies that any report (m,t) with $t \leqslant \bar{t}$ yields the same payoff under either strategy profile. The second statement claims that not making a report in any period yields the same payoffs as well. Strategy of reporting nothing yields the same payoff under $\{\tilde{r}^\gamma_\eta(m,t)\}$ as any report (m,t) with $t > \bar{t}$ under $\{r^\gamma_\eta(m,t)\}$, since all such reports are uninformative. Finally, any report (m,t) with $t \notin \mathcal{S}$ yields the same payoff under either strategy profile due to (OP). Everything said directly implies that if $r^\gamma_\eta(m,t)$ is a best response for type-γ analyst to strategy profile $\{r^\gamma_\eta(m,t)\}$ then $\tilde{r}^\gamma_\eta(m,t)$ is a best response to strategy profile $\{\tilde{r}^\gamma_\eta(m,t)\}$ and yields the same payoff. □

Proof of Corollary 11. Given Proposition 9 and reticence, we can calculate $z_{t,\eta}$ recursively:

$$z_{t,\eta} = z_{t-1,\eta} \cdot \left(1 - r^E_\eta(\eta, t-1)\right) + F(t) - F(t-1),$$
$$z_{t,\varnothing} = 1 - F(t),$$

with $z_{0,G} = z_{0,B} = 0$ and $z_{0,\varnothing} = 1$. Therefore, due to (SY) we have that $z_t := z_{t,G} = z_{t,B}$. We can thus simplify (27) taking reticence into account:

$$\begin{aligned} r^E(m,t) = \mathbb{E}_\eta[r^E_\eta(m,t)] &= \frac{p^*_t \cdot z_t \cdot r^E_G(m,t) + (1-p^*_t) \cdot z_t \cdot r^E_B(m,t)}{z_t + z_{t,\varnothing}}, \\ \mathbb{E}_\eta[r^E_\eta(m,t)|\omega] &= \frac{\pi \cdot z_t \cdot r^E_\omega(m,t) + (1-\pi) \cdot z_t \cdot r^E_{-\omega}(m,t)}{z_t + z_{t,\varnothing}}, \end{aligned} \tag{31}$$

We next establish the first part of the claim. Belief update about the state in case no report was made can be written as follows:

$$\rho_{t+1} = \rho_t \cdot \frac{1-(1-b_{t-1})\cdot\left(r^Q(G,t)+r^Q(B,t)\right) - b_{t-1}\cdot\mathbb{E}_\eta\left[r^E_\eta(G,t)+r^E_\eta(B,t)|\omega=G\right]}{1-(1-b_{t-1})\cdot\left(r^Q(G,t)+r^Q(B,t)\right) - b_{t-1}\cdot\mathbb{E}_\eta\left[r^E_\eta(G,t)+r^E_\eta(B,t)|\omega=B\right]}.$$

Then due to (SY) and (31) we get that the nominator is equal to the denominator and thus $\rho_{t+1} = \rho_t = \rho_0$.

To establish the second claim, we need to verify that

$$\frac{1-r^E(G,\bar{t})-r^E(B,\bar{t})}{1-r^Q(G,\bar{t})-r^Q(G,\bar{t})} = \frac{1-\mathbb{E}_\eta[r^E_\eta(G,\bar{t})+r^E_\eta(B,\bar{t})|\omega]}{1-r^Q(G,\bar{t})-r^Q(G,\bar{t})}.$$

Given (SY) and (31) we can explicitly confirm that it is indeed true. □

Proof of Proposition 12. For each of Propositions 1, 2, 3 we outline all relevant changes in the proofs as for case $\pi = 1$, as for case $\pi < 1$. Therefore, the result analogous to Proposition 4 is proved automatically.

Proposition 1. The only change to the proof is within Step 1. Lemma 6 implies that if $t \in \mathcal{S}$, the expert should make a prediction at t. Proposition 9 together with reticence imply that $r^E_\varnothing(G,t) = r^E_\varnothing(B,t) = 0$ and $r^E_B(G,t) = r^E_G(B,t) = 0$ for all $t \in \mathcal{S}$. Therefore, it implies that $r^E_G(G,t) = r^E_B(B,t) > 0$. In other words, the expert should weakly prefer to report his private information as soon as possible. Therefore, in period t the expert with private information η maximizes $w^c\left(\beta^G(m,\tau)\right) - w^c\left(\beta^B(m,\tau)\right)$ over all $\tau \in \{\mathcal{S}|\tau \geqslant t\}$. By Proposition 9 reporting $m \neq \eta$ or not making a report by $\bar{t}$ are not optimal. Therefore, the expert with $\eta = G$ chooses report (G,τ) to maximize $w^c\left(\beta^G(G,\tau)\right) - w^c\left(\beta^B(G,\tau)\right)$, and the expert with $\eta = B$ chooses report (B,τ) to maximize $w^c\left(\beta^B(B,\tau)\right) - w^c\left(\beta^G(B,\tau)\right)$. This implies that $w^c\left(\beta^G(G,t)\right) - w^c\left(\beta^B(G,t)\right)$ and $w^c\left(\beta^B(B,t)\right) - w^c\left(\beta^G(B,t)\right)$ are both decreasing in t on $\mathcal{S}$.

Steps 2,3,4 for the respective proofs of Proposition 1 and Proposition 4 apply without any modifications.

Proposition 2. Parts 1, 2, and 3 proceed without any changes. The proof of part 4 remains valid because by Proposition 9 we have $r^E_G(G,\bar{t}) = r^E_B(B,\bar{t}) = 1$, therefore (TE) is automatically satisfied in the last point of the support.

Proposition 3. The proof proceeds through the same steps until the very last paragraph. If both equilibria with supports $\mathcal{S}$ and $\tilde{\mathcal{S}}$ satisfy (TE), the very same argument applies. Therefore, assume that there is at least one equilibrium, say with support $\mathcal{S}$, and $t \in \mathcal{S}$ such that $r^E_G(G,t) = r^E_G(G,t) < 1$.

We next show that for any such equilibrium, we can construct an equilibrium which satisfies (TE) with the same beliefs after any history. Hence the argument above would apply to such equilibria as well because informativeness depends only on beliefs. Without loss, we can assume that $|\mathcal{S}| = k \geqslant 2$ because the last point of Proposition 9 implies that equilibria with $|\mathcal{S}| = 1$ always satisfy (TE). From Proposition 1 / Proposition 4 in any reticent equilibrium with $|\mathcal{S}| \geqslant 2$, it must be that $\beta(m,t_1) \leqslant \beta_{t_1}$. By the martingale property there are two possible cases: either $\beta(m,t_1) = \beta_{t_1}$ for both $m \in \{G,B\}$, or $\beta(m,t_1) < \beta_{t_1}$ for some $m \in \{G,B\}$.

Consider the latter case when $\beta(m,t_1) < \beta_{t_1}$ for some $m \in \{G,B\}$. Then, since $|\mathcal{S}| \geqslant 2$, in order to sustain the quack's indifference between reports, $\beta(m,t)$, $\beta^m(m,t)$, and $\beta^{-m}(m,t)$ should all be strictly decreasing for $t \in \mathcal{S}$. Therefore, the respective premium $w^c\left(\beta^m(m,\tau)\right) - w^c\left(\beta^{-m}(m,\tau)\right)$ for guessing the state correctly with report m is strictly decreasing for $t \in \mathcal{S}$ as well (see Steps 2 and 3 in the proof of Proposition 1 / Proposition 4). Step 1 of Proposition 1 / Proposition 4 and (SY) then imply that $r^E_G(G,t) = r^E_B(B,t) = 1$ for all $t \in \mathcal{S}$ which concludes the argument.

If $\beta(m, t_1) = \beta_{t_1}$ for both $m \in \{G, B\}$, then to sustain the quack's indifference between the reports within the support, we should have $\beta(m, t) = \beta_t = \beta_0$ for all $t \in \mathcal{S}$. Consider a modified strategy profile with $\hat{r}^E_G(G, t) = \hat{r}^E_B(B, t) = 1$ and $\hat{r}^Q(G, t) = p_0 \cdot \lambda_{\mathcal{S}}(t), \hat{r}^Q(B, t) = (1 - p_0) \cdot \lambda_{\mathcal{S}}(t)$ for $t \in \mathcal{S}$. The original and the modified strategy profiles generate the same beliefs after any history. Indeed, in the modified profile we still have $\beta(m, t) = \beta_t = \beta_0$ for $t \in \mathcal{S}$ and $\beta^m(m, t)$, $\beta^{-m}(m, t)$ being scalar multiples of β_0. Therefore, the new profile constitutes an equilibrium as well and satisfies (TE). This completes the proof. □