Nash-2 equilibrium: profit maximization under uncertainty *

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Abstract

The paper examines an interaction of boundedly rational agents that are able to calculate their benefits after reaction of an opponent to their own deviations from the current strategy. Accounting for strategic aspects of interaction among players can be implemented as a generalization of the Nash equilibrium concept. This is a possible compromise behavior: not absolutely myopic as Nash concept and not as wise as supergame approach. This leads to a farsighted equilibrium concept that we call a Nash-2 equilibrium. We prove the existence of Nash-2 equilibrium for almost every 2-person game and discuss the problem of possible multiplicity of such equilibria. For a number of models (Bertrand duopoly with homogeneous and heterogeneous product, Cournot duopoly, Tullock contest) the Nash-2 equilibrium sets are obtained and treated as tacit collusion or strong competition depending additional security considerations. For n-person games the idea of selective farsightedness is introduced by means of reflection network among players. Examples demonstrate that the reflection network topology fundamentally affects possible equilibria.

JEL classification: C72, D03, D43, D70, L13.

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1 Introduction

Profit maximization principle underlies most reasoning about rational behavior of agents. However, making an individual choice is necessarily based on expectations about how other players act. This is a great source of uncertainty in a prediction of equilibrium outcome. Nash equilibrium (NE) concept adopts the idea that each player unilaterally maximizes her own profit at the current game position under fixed opponent strategies. Effect produced by actions on the opponents' strategies is modelled by means of multi-stage game. The appropriate interpretation for static setting is that players are sophisticated enough to make correct predictions about the once and for all made choice of other participants and such beliefs are consistent in equilibrium.

Alternative approach to decision making has found an expression in modeling iterated strategic thinking process [5]. Multistep player's reasoning about possible consequent responses of opponents and her own further actions motivate a development of various bounded rationality concepts. A number of similar models of agents' cognitive hierarchy, or adhering to other terminology k-level rationality, or smart$_n$ players, are developed in [12] [11], [33]. Some empirical studies support k-level rationality approach ([11], [26]). The important point of hierarchical models is that each player assumes that other players have a lower level of rationality. It means that players of level-0 are strategically naive, while k-level players ($k > 0$) best respond on some beliefs about how their opponents are distributed by lower levels of rationality.

The reasonable level of rationality is an open question, moreover, it may be changed in the course of playing the next round of a game: "players iteratively adjust their depths of reasoning in response to each others' choices, then both the choices of all participants and their expressed depths of iterated reasoning should become closer to each other over time" [16].

The smallest but higher than Nash level of rationality is two. In this case a player, for instance, takes into account opponents' best responses (see [19] for cooperative equilibrium, and [4] for equilibrium in double best responses). But earlier well known example of such an approach is the method of conjectural variations (CV) in modelling oligopoly (since [6]). This approach accounts interaction among agents by means of explicit including a reaction function in the model. Despite existing criticism [17] pointing out an ambiguity of using formally static CV models for dynamic modeling, CV approach has proved to be a good substitute for repeated oligopolistic game [10], [13]. Matching belief about the behavior of the opponents with their actual best responses solves the problem of CV consistency [9]. The crucial point in the CV approach is that a player estimates *in his mind*, before she acts, the possible nearest reaction of other participants to the changes she produces. Definitely, this can be treated as a simple 2-level version of a general iterated strategic thinking process. The first level here is a desire to maximize her own profit. The second level is an attempt to predict opponents' responses, which in turn influence the final outcome. The generalization of the idea of reaction function is the concept of Markov perfect equilibrium [28].

So, the problem of prediction and belief consistency is solved in several ways: by ignoring any possible reactions (Nash approach: analysis of responses is transferred to fully dynamic setting), by imposing condition of coinciding real responses with expected ones (CV approach), or by constructing models with hierarchy of possibilities to predict behavior of opponents (cognitive hierarchy models). An important feature of all these methods is that they select the unique, the most plausible reaction and propose relatively small sets of equilibria (we consciously avoid the phrase "the unique equilibrium" since, in general, this is not always the case).

This paper offers an alternative principle of accounting responses: a player doesn't try to select any certain response of opponent and recognize that any action that increase opponent's utility is possible. This leads to multiplicity of equilibrium prediction, but it can be explained by natural limitation of making correct forecast of opponent behavior. However, such limited rationality approach is not necessarily bad for agents. As it will be shown in the paper, it often allows them to strategically support collusion or some intermediate between collusive and competitive outcomes. One extra level of farsightness in comparison with one-shot Nash rationality even under large uncertainty on other agent behavior may play a role of tacit communication between agents.

We develop an equilibrium concept that we call Nash-2 equilibrium. The main part of the results covers 2-person games. In Section 2 we formulate the definition of Nash-2 equilibrium in terms of modified notion of profitable deviation accounting profitable opponent responses. We introduce a natural division of Nash-2 profiles to secure and risky which reflects the various toughness of competition between agents, the degree of toughness can govern the final choice. We examine a relation of the concept to close existing ones: sequentially stable set [15] and equilibrium in secure strategies [20] with its modification.

Section 3 clarifies the problem of existence of Nash-2 equilibrium. We show that pure Nash-2 equilibrium exists in considerably wider class of two-person games than Nash equilibrium, and for any game with bounded payoff functions it can be obtained by a small perturbation of payoffs. One more important point about Nash-2 equilibria is multiplicity of predicted outcomes in most games. The problem of selection the unique equilibrium profile from the wide set can be solved in several ways in dependence of concrete game framework. The starting point (status quo, or a priory expectations) significantly influence the equilibrium realized. An approach that we offer is to construct the measure of feasibility on the set of Nash-2 equilibrium under the assumption that initially all game profiles are equiprobable. Several examples of how it is constructed are given.

Section 4 contains detailed analysis of several basic microeconomics models with Nash-2 approach: quantity and price competition, and rent-seeking model of Tullock. For the basic model of Cournot competition equilibria obtained can be divided into two types: ones are extension of Stackelberg leadership equilibria, and others are intermediate profiles with various degree of competition toughness. The results for Bertrand competition with imperfect substitutes and for Tullock contest demonstrate great potential for tacit collusion between agents.

In Section 5 we introduce the formal notion of reflection network for n players and formulate the definition of Nash-2 equilibrium for n-person games. The intuition of how this network can arise and is interpreted is given. We illustrate the idea with the models of Bertrand competition and Prisoner's dilemma and provide solutions of them for various types of reflection network. The crucial point is that accounting of reflection structure among players considerably affects equilibrium.

We conclude the paper by a brief discussion on the connection of presented approach with the problem of decision making under various types of uncertainty: about probabilities of the state of the world, about the depth of iterated thinking, about timing in the model.

2 Nash-2 equilibrium for 2-person games

2.1 Definition

Consider a 2-person non-cooperative game in the normal form

$$G = (i \in \{1, 2\}; s_i \in S_i; u_i : S_1 \times S_2 \to R),$$

where s_i, S_i and u_i are the strategies, the set of all available strategies and the payoff function, respectively, of player i, $i = 1, 2$. Henceforth, in this paper we will deal only with pure strategies.

Definition 1. A deviation s_i' of player i at profile $s = (s_i, s_{-i})$ is *profitable* if $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$.

When a player takes Nash logic she supposes that no reaction of other player will follow after her deviation and so such a deviation will increase her payoff irreversibly and in a certain way. Under assumption of iterated thinking a player expects some reaction of the opponent. We propose that if the agent doesn't know the depth of opponent's iteration than she has no grounds to make an unambiguous prediction of other player behavior. The only reasonable guess is that the opponent will not act to her own detriment. Such an uncertainty leads to additional requirement for deviation to be profitable in view of possible responses and, as a result, more cautious behavior.

Definition 2. A deviation s'_i of player i at profile $s = (s_i, s_{-i})$ is *secure* if for any profitable deviation s'_{-i} of the opponent at intermediate profile (s'_i, s_{-i}) player i is not worse off:

$$u_i(s'_i, s'_{-i}) \geq u_i(s_i, s_{-i}).$$

Definition 3. A strategy profile is *a Nash-2 equilibrium* if no player has a profitable and secure deviation.

We will denote the set of Nash-2 equilibria by NE-2.

In other words, players do not realize some profitable deviations so far as they fail to remain gainful after some reasonable reaction of other player. Obviously, any Nash equilibrium is also a Nash-2 equilibrium. Moreover, more elegant intuitive division of Nash-2 profiles can be given. Profitable deviations (they may exist at Nash-2 equilibrium, but they are not secure) can be of two types. The first kind is harmful for the opponent (such deviations are referred to as threats [20]), while the second type is not.

Definition 4. A *threat* of player i to player $-i$ at strategy profile s is a strategy s'_i such that

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{and} \quad u_{-i}(s'_i, s_{-i}) < u_{-i}(s_i, s_{-i}).$$

The strategy profile s is said to *pose a threat* from player i to player $-i$. A strategy profile s is *secure* for player i if s poses no treats from other players to i.

So, the set of Nash-2 equilibria can be naturally divided into two sets: secure profiles and risky outcomes. Secure part itself forms the set of equilibrium in secure strategies (intuitive formulation is contained in [20]). Strictly speaking, the definition of equilibrium in secure strategies is formulated as follows.

Definition 5. A strategy profile is an *equilibrium in secure strategies* (EinSS) if

i) it is secure,

ii) no player has a profitable and secure deviation.

The interpretation of such a division is the following. Secure part can be regarded as a tough competition where agents avoid any possible threats, even non-credible. It often leads to the situations with low profits since players in such situations have nothing to lose [22].

On the other hand, risky situations (NE-2 \ EinSS) are characterised by the observation that agents have opportunities to harm one to another, but they do not actualize these threats because of possible credible "counter-sanctions". In a number of situations such a cautious (but not overly) behavior enables agents to hold on higher profits than in case when players also care about security. Detailed examples will be presented in Sect. 4.

2.2 Related concepts

2.2.1 Graph model of conflict resolution

The idea of accounting ambiguous responses to one's own unilateral improvements has been elaborated in the graph model of conflict resolution theory, a methodology for analyzing real-word conflicts [8]. These authors proposed a new theory for non-cooperative games allowing players to make moves and chains of counter-moves with some limited horizon, and to carry out non-myopic calculations. Their analysis focuses on 2×2 games and points out the importance of starting point, threat power, and abilities of players to think ahead for prediction of stable outcomes. Motivation of applicability of the new theory to modeling real-life situations presented in [8] and [27] is entirely suitable for the theory of Nash-2 equilibrium that we develop. Moreover, in contrast with graph model approach accounting only ordinal preferences Nash-2 equilibrium allows to make more accurate description of stable situations for models with a large (sometimes, infinite) number of possible game situations.

The most close concept to Nash-2 equilibrium within this non-myopic theory of conflict resolution is the sequential stability concept [15]. Let us reformulate the definition of sequentially stable state here in terms of two-person game in normal form introduced in Sect. 2.1. (we use the definition from [14])

Definition 6. For a two players $N = \{i, j\}$ and a conflict G, an outcome $s^{SEQ} \in S$ is *sequentially stable* for player i iff for every unilateral profitable deviation of player i to profile s_1 there exists a unilateral profitable deviation of player j from s_1 to s_2 such that $u_i(s) \geq u_i(s_2)$. A state is *sequentially stable* for the conflict iff it is sequentially stable for both players.

This definition differs from definition of Nash-2 equilibrium only in the strictness of the last inequality. Obviously, in 2-person game *if profile s is a Nash-2 equilibrium, then s is a sequentially stable state.*

Despite the similarity of these two concepts, the difference turns out to be truly significant for specific models. Striking example is the basic Bertrand model of price competition.

Example 1 (Bertrand competition with homogeneous product). Consider two firms producing a homogeneous product with equal marginal costs c. Let p_1 and p_2 be the prices proposed by firms 1 and 2, respectively. Consumers buy the product with lowest price, the demand being a linear function of the price $Q(p) = 1 - p$. Firms' profits are given by the following function

$$\pi_i(p_1, p_2) = \begin{cases} (p_i - c)(1 - p_i), & if\, p_i < p_{-i}, \\ (p_i - c)(1 - p_i)/2, & if\, p_i = p_{-i}, \\ 0, & if\, p_i > p_{-i}, \end{cases} \qquad i = 1, 2$$

Nash-2 concept states that an equilibrium might be with any price level $p = p_1 = p_2 \in [c, 1]$. In particular, NE-2 includes the collusive (monopoly) price level $p = \frac{1+c}{2}$. Indeed, without loss of generality assume that the firm 1 proposes a prise $p_1 > p_2$, and gets zero profit. Then this firm can undercut the firm 2 and set the prise $p_2 - \varepsilon$ with sufficiently small ε. Its deviation from the strategy p_1 to $p_2 - \varepsilon$ is profitable. Moreover, it is also secure as the worst that can happen with the firm 1 is that the firm 2 in turn undercuts it and the firm 1 comes back to zero profit. So, any situation with $p_1 \neq p_2$ is not a Nash-2 equilibrium.

On the contrary, in case of sequential stable state a possibility for the firm 1 to return to initial profit level immediately means that a situation with $p_1 > p_2$ occurs to be sequentially stable. So, according to Definition 6 *any* profile in Bertrand duopoly is sequentially stable.

This example demonstrates the crucial importance of allowing players to deviate from the initial state even if there is a slight possibility to come back to initial profit.

2.2.2 Equilibrium in secure strategies

One more game theoretical approach close to ours introduces a *security* as an additional motivation for players' behavior. Two second-stage-foreseeing concepts that have been proposed are bargaining set based on the notion of threats and counter-threats (for cooperative games, see [2]) and equilibrium in secure strategies (EinSS, see [20]). The idea of both concepts is that players worry not only about own first-stage payoffs and opponents' responses, but also about the absence of harmful actions ("threats") of the opponents. On grounds of security several attempts to introduce the concept equivalent to Nash-2 equilibrium (independently of our study) have been made: they are threatening-proof profile [21], equilibrium contained by counter-threats [23], and equilibrium in threats and counter-threats [24]. These are different names of the same as Nash-2 equilibrium concept, but motivated by security logic of decision making[1].

The key point of this paper is that Nash-2 equilibrium outcomes are supported by farsightness of agents and includes secure and non-secure situations, and they are regarded as various degrees of competition toughness among them.

The following relation for two-person games takes place [21]: *any NE is an equilibrium in secure strategies, and any equilibrium in secure strategies is a Nash-2 equilibrium*[2]. The converse is generally not true.

2.2.3 Equilibrium in double best responses

In Nash-2 definition of secure deviation each player accounts all profitable responses of the opponent. The restriction of the range of profitable opponent's deviations to the set of her best responses is developed in [19], [4], which introduce the similar concepts of equilibrium: cooperative equilibrium and equilibrium in double best responses. Let us reproduce the main definitions from [4].

Definition 7. *Best response* of player i to a situation $s = (s_i, s_{-i})$ is an action $BR_i(s_i, s_{-i}) = \arg\max_{s'_i \in S_i} u_i(s'_i, s_{-i})$. *Double best response* of player i to a situation $s = (s_i, s_{-i})$ is an action

$$DBR_i(s_i, s_{-i}) = \arg\max_{s'_i \in S_i} u_i(s'_i, BR_{-i}(s'_i, s_{-i})).$$

A profile s is *an equilibrium in double best responses* if for any player i $s_i = DBR_i(s)$.

In other words, in equilibrium in double best responses no player has a deviation which after opponent's best response on it leads the player who deviates the first to more profitable situation than the initial one. Such a concept provides an efficient equilibrium for some class of network formation games (see [4]), or for coordination games. However, the problem of consistency of such a concept arises by analogy with the criticism of conjectural variation approach.

Though for some games an equilibrium in double best responses coincides with Nash-2 equilibrium (for instance, in 2-person Prisoner's dilemma), in general it is equivalent neither to Nash-2 equilibrium, nor to equilibrium in secure strategies.

[1] We will accurately refer to existence results during the further exposition in case of some intersection.

[2] Authors formulated this result in terms of threatening-proof profile.

Example 2. Consider the following game which does not have pure Nash equilibrium.

	L	C	R
T	(1.5,3)	(0,0)	(4.5,0)
B	(2,0)	(0,1)	(2.5,2)

(T,L) is a Nash-2 equilibrium, but it is not an equilibrium in secure strategies and not an equilibrium in double best responses. (B,C) is a Nash-2 equilibrium, an equilibrium in secure strategies and an equilibrium in double best responses. (B,R) is a Nash-2 equilibrium, an equilibrium in double best responses, but it is not an equilibrium in secure strategies.

This example shows how all three concepts could predict the worst equilibrium with payoffs (0,1), but only Nash-2 equilibrium concept catches the best for player 2 outcome (T,L).

2.2.4 Explicit and tacit collusion

As it has been already stated in the paper, Nash-2 equilibrium is often a suitable explanation for the phenomenon of tacit collusion between two player. Naturally, the question on the relation of Nash-2 equilibrium and explicit collusion (cooperative behavior) arises. In the example 2 three profiles (T,L),(T,R), and (B,R) can be chosen during cooperation; nevertheless, only (T,L) and (B,R) is supported by non-cooperative concept of Nash-2 equilibrium. It is to be mentioned that they are risky outcomes.

In the general case of two-person game, if explicit collusion is a Nash-2 equilibrium then it is in NE-2 \ EinSS, or more strictly:

Theorem 1. *If a collusion outcome is not a Nash equilibrium then it is not a secure profile.*

Proof. Let $(s_1^c, s_2^c) = \arg\max_{s_1,s_2}(u_1(s_1,s_2) + u_2(s_1,s_2))$. Assume that it is secure. It means that it poses no threats from one to another. The two cases are possible: there no profitable deviations and for any profitable deviation $s_1 \to s_i'$ of player i the another player is not worse off $u_{-i}(s_i', s_{-i})$.

In the first case we deal with NE. In the second case $u_i(s_i', s_{-i}) + u_{-i}(s_i', s_{-i}) > u_1(s_1^c, s_2^c) + u_2(s_1^c, s_2^c)$ and this contrary to the fact that (s_1^c, s_2^c) is collusive outcome. □

3 Existence and multiplicity of Nash-2 equilibrium in 2-person games

3.1 Existence for finite 2-person games

An important advantage of Nash-2 equilibrium concept is that it exists in most games and fails to exist only in "degenerate" cases. Let us start with finite games and formulate this idea accurately. For this purpose we introduce the notion of secure cycle.

Definition 8. A path of profiles $\{(s_i^t, s_{-i}^t)\}_{t=1,\ldots,T}$ is called a *secure path* of length T if each its arc $(s_i^t, s_{-i}^t) \to (s_i^{t+1}, s_{-i}^{t+1}) = (s_i^{t+1}, s_{-i}^t)$ is a secure profitable deviation from s_i^t to s_i^{t+1} for some player i. This path is called a *secure cycle* if it is closed: $(s_i^1, s_{-i}^1) = (s_i^T, s_{-i}^T)$, minimum of such a T is called a *length of cycle.*

Using this notion one can easily check the following theorem providing the criterion for the absence of Nash-2 equilibrium in some game.

Proposition 1. *The finite 2-person game in normal form does not have a Nash-2 equilibrium if and only if it contains at least one secure cycle of finite length, and there is a finite secure path from any profile to some secure cycle.*

Proof. Assume that no profile is a Nash-2 equilibrium, then from any profile a profitable secure deviation exists at least for one player. Without loss of generality, assume that player 1 deviates at odd steps while player 2 deviates at even ones. For any secure path starting from (s_1^1, s_2^1) the following inequalities holds

$$u_1(s_1^{2t+1}, s_2^{2t+1}) \geq u_1(s_1^{2t+3}, s_2^{2t+3}), \quad t = 0, 1, \ldots,$$

$$u_1(s_1^{2t+1}, s_2^{2t+1}) > u_1(s_1^{2t+2}, s_2^{2t+2}), \quad t = 0, 1, \ldots,$$

$$u_2(s_1^{2t}, s_2^{2t}) \geq u_2(s_1^{2t+2}, s_2^{2t+2}), \quad t = 1, \ldots,$$

$$u_2(s_1^{2t}, s_2^{2t}) > u_2(s_1^{2t+1}, s_2^{2t+1}), \quad t = 1, \ldots.$$

Since the game is finite, at some moment $\Theta < \infty$ (and not exceeding the number of possible game profiles) this path necessarily starts to reach the same situations again and forms a cycle of length $T \leq \Theta$. Moreover, in order for this to be possible it is necessary and sufficient that all non-strict inequalities above become equalities for all profiles forming the secure cycle. □

An important observation that secure cycles are very special: all nodes where player 1 deviates should have *the same* payoff for this player

$$u_1(s_1^{2t+1}, s_2^{2t+1}) = u_1(s_1^{2t+3}, s_2^{2t+3})\ \forall t,$$

the same is true for even nodes and player 2: $(u_2(s_1^{2t}, s_2^{2t}) = u_2(s_1^{2t+2}, s_2^{2t+2}))$.

Corollary 1. *Whenever a game does not have NE-2, any perturbation of payoffs that breaks at least one equality for payoffs in secure cycle yields NE-2 existence.*

Example 3 (Heads or Tails).

	R	L
T	1	-1
B	-1	1

	R	L
T	1.01	-1
B	-1	1

Left matrix corresponds to a well-known zero-sum game "Heads or Tails". In this game all profiles form a secure cycle and Nash-2 equilibrium does not exist. But a small perturbation of just one payoff (see the right matrix) immediately yields that (B,L) becomes a Nash-2 equilibrium.

Every 2-person game with n strategies for player 1 and m strategies for player 2 can be associated with a point in $\mathbb{R}^{2nm}$ (ordered payoffs for each pair of strategies are coordinates of this point). So, we can define the measure on the set of games as a measure of corresponding subset in Euclidean space.

The minimum length of secure cycle is four, and at least two equalities on payoffs should take place for a game not to have a Nash-2 equilibrium. So, the dimension of the subset of all such games does not exceed $2nm - 2$, and this subset has measure 0 in $\mathbb{R}^{2nm}$. So, the following theorem holds.

Theorem 2. *Nash-2 equilibrium exists in almost every 2-person finite game.*

Note that Theorem 2 demonstrates the existence of NE-2 but not the optimal algorithm of finding it in arbitrary game.

3.2 Existence for 2-person games with infinite number of strategies

Logic underlying discrete games can be easily extended to the case of infinite number of strategies. Loosely speaking, we need the boundedness of utility function and some condition ensuring the sequence of utilities in secure path to grows up to the limit value not too slowly.

One way is to define an ε-equilibrium and claim the existence of ε-equilibrium for 2-person games with some condition on the limit of utilities. This approach is realized in [23, Propositions 2 and 8].

We develop other ideology for games with infinite strategy sets (continuous or discontinuous 2-person games). The only reason why the logic of Sect. 3.1 may fail is that if players are permitted to use hardly different strategies they may ensure very slow but infinite growth of profits. In order to exclude such possibility we consider the games in which a deviation is costly. We now assume that a player have to pay some fixed cost $d \geq 0$ for any unilateral changing of her strategy, we call d a *cost of deviation*.

Then, definitions 1 and 2 can be rewrite as following.

Definition 9 ($1^\star$). A deviation s_i' of player i at profile $s = (s_i, s_{-i})$ is *profitable* if $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}) + d$.

Definition 10 ($2^\star$). A deviation s_i' of player i at profile $s = (s_i, s_{-i})$ is *secure* if for any profitable deviation s_{-i}' of the opponent at intermediate profile (s_i', s_{-i}) player i is not worse off:

$$u_i(s_i', s_{-i}') \geq u_i(s_i, s_{-i}) + d.$$

Note that these definitions coincide with definitions 1 and 2, respectively, if $d = 0$. The definition of Nash-2 equilibrium remains the same.

Note that introducing any $d > 0$ guarantees that the game does not contains any secure cycle. Similarly to Theorem 2, the following theorem holds.

Theorem 3. *Nash-2 equilibrium exists in every 2-person game with strictly positive cost of deviation and utility functions bounded from above.*

It is to be stressed that we do not require the continuity of utilities or compactness of action sets. Moreover, for most games Nash-2 equilibrium exists even in case of zero cost of deviations. Examples in Sect. 4 will completely demonstrate this.

3.3 Selection among multiple equilibria profiles

The reverse side of existence is multiplicity of predicted outcomes. This problem can be resolved in several ways in dependence of concrete game framework.

In the case of tough competition between firms one can choose, for instance, an equilibrium in secure strategies as the most attractive. In Hotelling linear city model EinSS concept provides the unique equilibrium corresponding to dumping pricing [22].

The totally different approach is to choose the collusion outcome like in Bertrand or Cournot model or, at least, Pareto efficient profiles in the set of Nash-2 profiles.

If players join the game successively, one after another, then Nash-2 equilibrium is an operative explanation why Stackelberg leadership outcome remains stable. In games with such a history of events Stackelberg equilibrium can be selected as a concrete Nash-2 profile. An example is Cournot duopoly.

Alternative way of solving the problem is to introduce the measure on the set of Nash-2 equilibria that reflects the probability with which a concrete equilibrium can be realized. This can be done in different ways, and we present here one of them.

We suppose that originally players randomly get into any game profile s with equal probabilities $\nu_0(s) = \frac{\mu(s)}{\mu(S_1 \times S_2)}$, where $\mu(A)$ is a measure of the set A. If the profile s is not a Nash-2 equilibrium then a secure path from this profile to some Nash-2 profile exists. Denote the subset of NE-2 that can be achieved from the profile s by any secure path by NE-2_s. For simplicity we assume that when a player learns the whole range of reachable from s Nash-2 profiles she chooses each of them also with equal probabilities. (Naturally, more complicated method is to assign a probability proportional to the number of secure paths from s to concrete Nash-2 equilibrium.) So, the final probability of each Nash-2 profile to be realised is

$$\nu(s) = \frac{\mu(s)}{\mu(S_1 \times S_2)} + \sum_{\tilde{s}: s \in \text{NE-}2_{\tilde{s}}} \frac{\mu(\tilde{s})}{\mu(\text{NE-}2_{\tilde{s}})\mu(S_1 \times S_2)}, \quad \forall s \in \text{NE-2}.$$

These probabilities form the *measure of feasibility* on the set NE-2.

If a Nash-2 profile s is not reachable from any point of $S_1 \times S_2$ (we will call it isolated), then $\nu(s) = \nu_0(s)$.

For the sake of visualization in the case of discrete action sets let us construct a directed graph Γ by the following rule. The nodes of Γ are game profiles. The directed link from node s to node s' exists if there is a secure path from s to s', and there are no secure paths starting at s'.

In this graph the nodes with zero outdegree $deg^+(s)$ are Nash-2 equilibria. The links demonstrate how not Nash-2 profiles transmit their initial probabilities to Nash-2 profiles by means of secure paths. Here for all $s \in$ NE-2 the number of all profiles from which a secure path to s exists equals to the indegree $deg^-(s)$ of s in Γ. In particular, if $\forall s \in \Gamma$ $deg^+(s) \leq 1$, then

$$\nu(s) = \frac{1}{|S_1| \cdot |S_2|} \left(1 + deg^-(s)\right), \quad \forall s \in \text{NE-2},$$

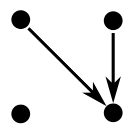
$|A|$ is the number of elements in the set A.

Let us give several examples.

Example 4.

In this example two situations in this game are Nash-2 equilibria, players get zero profits in both. Indeed, strategy profile (B,R) is a NE and Nash-2 equilibrium, and profile (B,L) is a Nash-2 equilibrium, but not a NE.

	L	R
T	1	-1
B	0	0



The graph Γ is shown on the right. As one can see, (B,L) is an isolated Nash-2 profile, thus $\nu(B,L) = 1/4$.

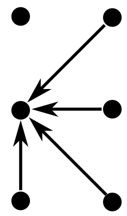
$deg^-(B,R) = 2$. Thereby, $\nu(B,R) = \frac{1}{4}(1+2) = 3/4$.

The probability of NE to be realized is considerably greater than for the another profile.

Example 5.

In this example Nash equilibrium fails to exist.

	L	R
T	(2/3, 1/3)	(-1, 2)
C	(1/2, 1/2)	(1, 0)
B	(1, 0)	(0, 1)



NE-2 set consists of two strategy profiles (C,L) and (T,L) with profits (1/2, 1/2) and (2/3, 1/3), respectively. (T,L) is an isolated Nash-2 equilibrium, thus $\nu(T,L) = 1/6$. $deg^-(C,L) = 4$. Thereby, $\nu(C,L) = \frac{1}{6}(1+4) = 5/6$.

Hence, though at first sight two Nash-2 profiles are similar, it is much more plausible that (C,L) will occur.

Example 6 (Bertrand model with homogeneous product).

Consider the simplest model of price competition, as in Example 1. In this case there is a secure path from each profile (p_1, p_2), $p_1 \neq p_2$, $p_1, p_2 \in [c, 1]$, to Nash-2 profile (p, p) with $p \in [c, \min(p_1, p_2)]$. Figure 1 reflects the structure of possible secure paths in this game.

Explicit calculations yield (see Figure 2)

$$\nu(p,p) = \frac{2}{1-c}\left(\ln\frac{1-c}{p-c} - \frac{1-p}{1-c}\right), \quad \forall p \in [c, 1].$$

One can think about this measure function in the sense that the probability to come into the ϵ-neighbourhood of the prices (p,p) is $\int_{p-\varepsilon}^{p+\varepsilon} \nu(x)dx$. Note that the probability of low prices close to marginal cost is appreciably greater

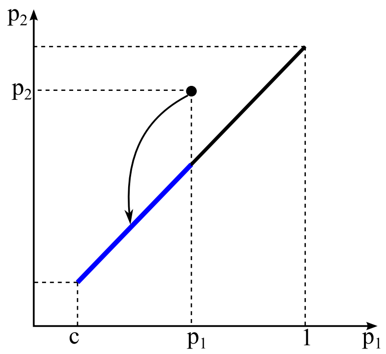


Figure 1: The structure of secure paths in Bertrand model

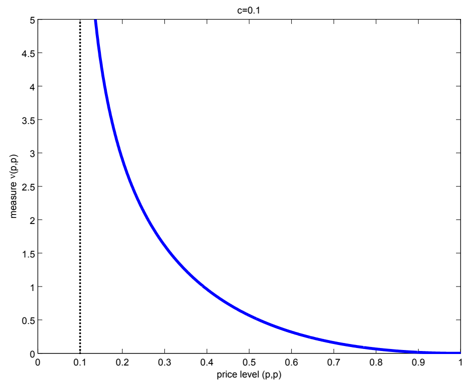


Figure 2: The measure of feasibility on the set NE-2 in Bertrand model with $c = 0.1$

than that for high prices. It is caused by the high appeal of undercutting an opponent. However, the collusion price level $(\frac{1+c}{2}, \frac{1+c}{2})$ also has a positive measure of feasibility.

4 Examples

Let us turn now to some applications of Nash-2 equilibrium concept to well-known microeconomics models. We will start with Cournot duopoly with homogeneous product, linear demand, and equal marginal costs, and demonstrate in terms of Nash-2 equilibrium whether the possibilities for collusion or more strong competition actually exist. Then we examine the model of price competition of firms producing imperfect substitutes. Finally we will discuss the computational solution of rent-seeking game (Tullock contest) and outline the difference between secure-but-strong-competitive and risky-but-collusive outcomes.

4.1 Cournot duopoly

Let two firms produce q_1 and q_2 units of homogeneous product, respectively, with equal constant marginal costs c per unit. We assume the equilibrium price $p(Q)$ to be a linear decreasing function $p(Q) = 1 - Q$ of total output $Q = q_1 + q_2$. The profit function of the firm $i = 1, 2$ is

$$\pi_i(q_1, q_2) = q_i \cdot (p(Q) - c) = q_i(1 - q_1 - q_2 - c).$$

In Nash equilibrium firms produce by one third of maximal total output which ensures positive prices on the market

$$q_1^* = q_2^* = \frac{1-c}{3}, \qquad \pi_1^* = \pi_2^* = \left(\frac{1-c}{3}\right)^2.$$

Theorem 4. *Nash-2 equilibria* (q_1, q_2) *are of two kinds:*
a) they belong to the set

$$\left\{\left(b; \frac{1-c-b}{2}\right) \cup \left(\frac{1-c-b}{2}; b\right) \mid \quad b \in \left[\frac{1-c}{3}; 1-c\right)\right\}.$$

b) they are

$$q_1 = q_2 \in (0, \ (1-c)/3)$$

including collusive outcome $(1-c)/4$, $(1-c)/4$.

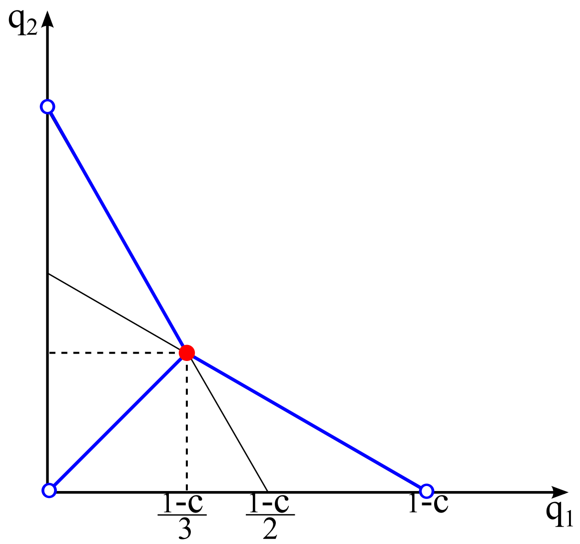


Figure 3: Bold point is NE, NE-2. Bold lines are NE-2.

One can easily check that the equilibrium set satisfying condition (a) consists of secure profiles. For $b \in \left(\frac{1-c}{3}; \frac{2(1-c)}{3}\right)$ such situations are fruitful for the firm that overproduces, and maximum is reached at $b = \frac{1-c}{2}$, and for any

b they are bad for the another firm (in comparison with Nash equilibrium profits). A special cases of secure Nash-2 equilibrium are Stackelberg outcomes $\left(\frac{1-c}{2}; \frac{1-c}{4}\right)$ if the firm 1 is a leader, and $\left(\frac{1-c}{4}; \frac{1-c}{2}\right)$ if the firm 2 is a leader.

The set NE-2\EinSS (condition (b)) includes collusive outcome. Hence, in Cournot duopoly collusion is strategically (tacitly) supported by the concept of Nash-2 equilibrium. The profiles with $q_1 = q_2 \in \left(\frac{1-c}{4}, \frac{1-c}{3}\right)$ cover all intermediate situations between Nash competition and cooperative behavior.

Proof. Reaction functions of both firms are

$$r_1(q_2) = (1 - c - q_2)/2, \qquad r_2(q_1) = (1 - c - q_1)/2.$$

Note that any decreasing of production of any player is profitable for her opponent. It immediately yields that if for one player it is profitable to decrease her price then another player (if she is not at her best response) has a profitable secure deviation. Therefore, such situations are not Nash-2 equilibria.

If one player (for definiteness, the firm 1) plays exactly her best response on the firm's 2 output, while the firm 2 produces more than it best response level, then such a situation is a Nash-2 equilibrium. Indeed, the firm 1 hasn't a profitable deviation, and any profitable deviation of the firm 2 decreases the output: $q_2 \to q_2 - \varepsilon$, for some $\varepsilon > 0$. However, if the firm 2 deviates then the firm 1 acquires a profitable deviation $q_1 \to q_1 + \varepsilon - \delta$ with enough small $0 < \delta < \varepsilon$ such that the deviation $q_2 - \varepsilon$ is not secure for the firm 2.

Now turn out to the case when both firms produce less than best response level: $q_1 \leq (1 - c - q_2)/2$ and $q_2 \leq (1 - c - q_1)/2$.

Assume first that $q_1 > q_2$ (the symmetric case is similar). Then the firm 2 has the profitable secure deviation from q_2 to $1 - c - q_1 - q_2 - \varepsilon$ with $0 < \varepsilon < q_1 - q_2$. After this q_1 becomes greater than new best response level and any profitable deviation of the firm 1 decreases q_1 which is acceptable for the firm 2.

The last possible situation is $q_1 = q_2 = q$. Let us show that (q, q), $q \leq (1 - c)/3$, is a Nash-2 equilibrium. Let us carry out the reasoning for the firm 1. Any profitable deviation of the firm 1 has a form $q_1 \to q + \varepsilon$ with $0 < \varepsilon < 1 - c - 3q$. After this the firm 2 has the profitable deviation from q_2 to $1 - c - 2q - \varepsilon - \delta$ which leads to breaking the security requirement for the firm 1 if $0 < \delta < \frac{q}{q+\varepsilon}(1 - c - 3q - \varepsilon)$. □

Nash-2 equilibrium provides a number of regimes with various degree of toughness from competitive till collusive. An explanation what outcome will be observed can be given on the base of the oligopolistic equilibrium [1] suitably generalizing conjectural variation approach and introducing an extra coefficient of competitive toughness.

4.2 Bertrand competition with differentiated product

Consider more general model of price competition between two firms producing imperfect substitutes with marginal costs equal c_1 and c_2, respectively. The coefficient of substitution is $\gamma \in [0, \infty)$. Firms' demand curves are

$$q_1 = 1 - p_1 - \gamma(p_1 - p_2), \quad q_2 = 1 - p_2 - \gamma(p_2 - p_1).$$

The firms' profits are

$$\pi_1(p_1, p_2) = (p_1 - c_1)(1 - p_1 - \gamma(p_1 - p_2)).$$

$$\pi_2(p_1, p_2) = (p_2 - c_2)(1 - p_2 - \gamma(p_2 - p_1)).$$

The case of $\gamma = 0$ corresponds to the monopoly. When $\gamma \to \infty$ the product becomes more and more homogeneous.

In Nash equilibrium prices are equal to

$$p_1^* = \frac{2 + 3\gamma + 2(1+\gamma)^2 c_1 + \gamma(1+\gamma)c_2}{(2+3\gamma)(2+\gamma)}$$

$$p_2^* = \frac{2 + 3\gamma + 2(1+\gamma)^2 c_2 + \gamma(1+\gamma)c_1}{(2+3\gamma)(2+\gamma)},$$

if $p_1^* \geq c_1$, $p_2^* \geq c_2$.

If marginal costs are equal $c_1 = c_2 = c$, then $p_1^* = p_2^* = \frac{1+(1+\gamma)c}{2+\gamma} > c$. As $\gamma \to \infty$ we face to classical Bertrand model.

Let us describe the conditions which the set of Nash-2 profiles (p_1, p_2) meets.

Note firstly that two following conditions mean that markup and demand at equilibrium should be non-negative

$$p_1 \geq c_1, \quad p_2 \geq c_2, \qquad a)$$

$$q_1(p_1, p_2) \geq 0, \quad q_2(p_1, p_2) \geq 0. \qquad b)$$

The next condition states that only prices exceeding best response level can be a Nash-2 equilibrium, i.e.

$$p_1 \geq \frac{1 + \gamma p_2 + c_1(1+\gamma)}{2(1+\gamma)}, \quad p_2 \geq \frac{1 + \gamma p_1 + c_2(1+\gamma)}{2(1+\gamma)}. \qquad c)$$

One more claim is that in Nash-2 equilibrium the firms get not less than their maxmin benefits

$$\pi_1(p_1, p_2) \geq \frac{(1 - c_1(1+\gamma))^2}{4(1+\gamma)}, \quad \pi_2(p_1, p_2) \geq \frac{(1 - c_2(1+\gamma))^2}{4(1+\gamma)}. \qquad d)$$

The last conditions directly state the absence of secure profitable deviations

$$\begin{cases} \left(\frac{1-c_1}{2} - \frac{\gamma(1+\gamma)(p_2-c_2)}{2(1+2\gamma)}\right)\left(\frac{1+2\gamma+\gamma^2 c_2-(1+\gamma)^2 c_1}{2(1+\gamma)} + \frac{3}{2}(p_2-c_2)\right) \leq \pi_1(p_1,p_2), \\ \left(\frac{1-c_2}{2} - \frac{\gamma(1+\gamma)(p_1-c_1)}{2(1+2\gamma)}\right)\left(\frac{1+2\gamma+\gamma^2 c_1-(1+\gamma)^2 c_2}{2(1+\gamma)} + \frac{3}{2}(p_1-c_1)\right) \leq \pi_2(p_1,p_2). \end{cases} \qquad e)$$

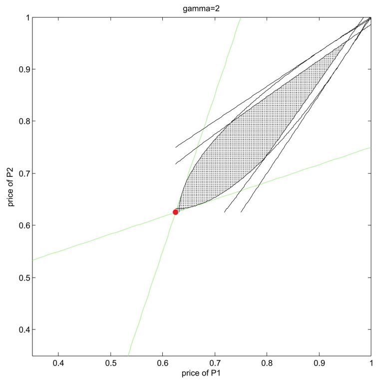


Figure 4: $c_1 = c_2 = 0.5$, $\gamma = 2$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

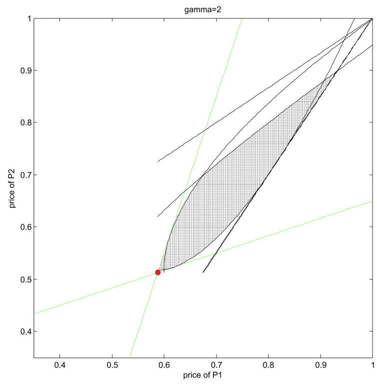


Figure 5: $c_1 = 0.5$, $c_2 = 0.3$, $\gamma = 2$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

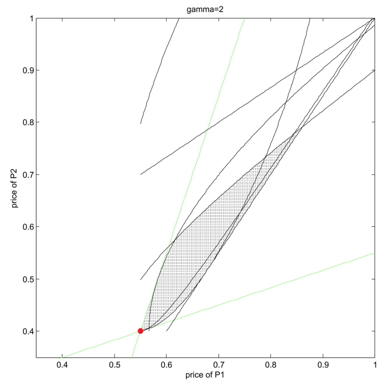


Figure 6: $c_1 = 0.5$, $c_2 = 0.1$, $\gamma = 2$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

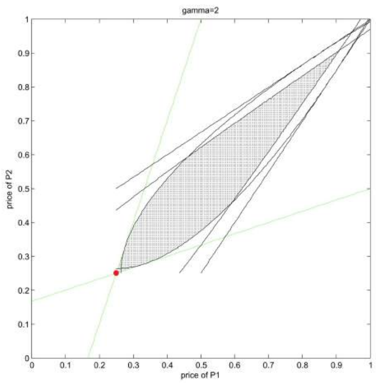


Figure 7: $c_1 = c_2 = 0$, $\gamma = 2$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

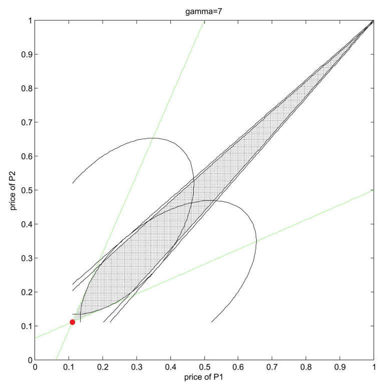


Figure 8: $c_1 = c_2 = 0$, $\gamma = 7$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

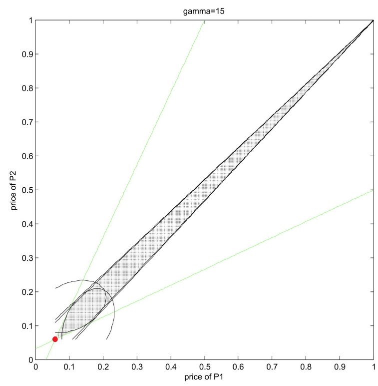


Figure 9: $c_1 = c_2 = 0$, $\gamma = 15$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

Proof. Let us start with the observation that, in contrast to Cournot duopoly, any increasing price of any firm is profitable for her opponent. From this fact it follows that if one firm assign a price that is less than best response level then it has a profitable and secure deviation. It provides condition (c).

Condition (d) immediately follows from the fact that if a firm gets less than minmax value then it has a profitable secure deviation to the strategy which ensures it.

Now look at the residual area and establish which situations are Nash-2 equilibria. In this area firms propose prices more than at best response level (condition (c)). Let us look on the situation by the firm 1. Any profitable deviation of the firm 1 decreases the price p_1 up to some $\tilde{p}_1^\varepsilon = p_1 - \varepsilon$ with $\varepsilon \in \left(0;\, 2\left(p_1 - \frac{1+\gamma p_2 + c_1(1+\gamma)}{2(1+\gamma)}\right)\right)$. The most harmful response of the firm 2 is maximal decreasing the price: $p_2 \to \tilde{p}_2^\varepsilon = 2 \cdot \frac{1+\gamma(p_1-\varepsilon)+c_2(1+\gamma)}{2(1+\gamma)} - p_2 + \delta$ with $\delta = +0$.

If (p_1, p_2) is Nash-2 profile then for any ε firm 1 should get worse:

$$\pi_1(\tilde{p}_1^\varepsilon, \tilde{p}_2^\varepsilon) < \pi_1(p_1, p_2),$$

or, equivalently, $\max_\varepsilon \pi_1(\tilde{p}_1^\varepsilon, \tilde{p}_2^\varepsilon) < \pi_1(p_1, p_2)$.

Explicit calculation of this maximum provides condition (e). □

As we can observe the set of Nash-2 equilibria becomes more asymmetric as the difference between marginal costs increases (see Fig. 4 – 6). On the other hand it becomes narrower and elongate as $\gamma \to \infty$ (see Fig. 7 – 9) and this asymmetry ceases to play an important role.

Note that in the case of $c_1 = c_2 = c$ the collusion profile $p_1 = p_2 = (1+c)/2$ is inside Nash-2 set. Nevertheless, another outcomes on the Pareto frontier of the set of profits at Nash-2 profiles exist.

4.3 Tullock contest

In rent-seeking modeling most papers focus on the manipulation efforts of firms to gain monopolistic advantages in the market. Tullock contest [29] is a widespread way to examine the processes of political lobbying for government benefits or subsidies, or to impose regulations on competitors in order to increase market share.

The contest success function translates the efforts $x = (x_1, x_2)$ of the players into the probabilities p_i that player i will obtain the resource R.

$$p_i(x_i, x_{-i}) = \frac{x_i^\alpha}{x_i^\alpha + x_{-i}^\alpha}, \quad x \neq 0, i = 1, 2.$$

If $x = (0,\, 0)$ then $p_i = p_{-i} = 1/2$.

The payoff function of each player is:

$$u_i(x_i, x_{-i}) = Rp_i(x_i, x_{-i}) - x_i.$$

Without loss of generality assume $R = 1$, $x_i \in [0,\, 1]$.

The players' behavior essentially depends on the value α. It can be treated as a responsiveness of the utility function to increasing the effort. When $\alpha \leq 2$ Nash equilibrium exists and equilibrium efforts equal to $\alpha/4$. In [25] the equilibrium in secure strategies in Tullock model was obtained for all α. Here we present the computer solution for the whole set of Nash-2 equilibrium (see Fig. 10 – 12).

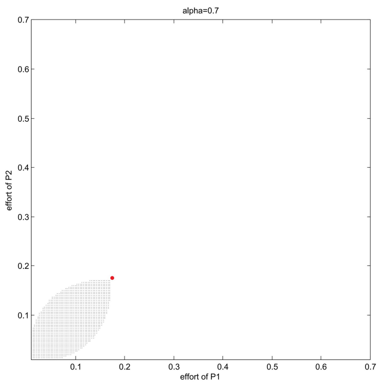


Figure 10: $\alpha = 0.7$. Bold point is NE, EinSS, NE-2. Shaded area is NE-2.

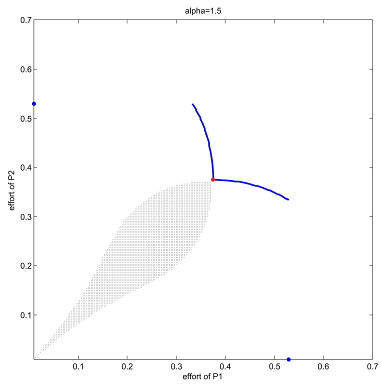


Figure 11: $\alpha = 1.5$. Bold central point is NE, EinSS, NE-2. Bold curve and points on the axes are EinSS, NE-2. Shaded area is NE-2.

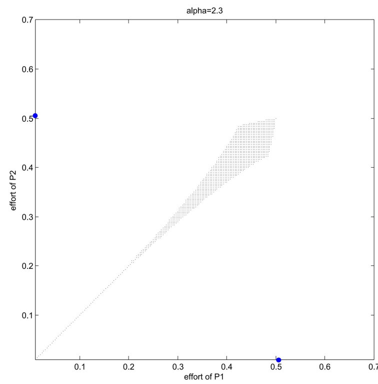


Figure 12: $\alpha = 2.3$. Bold points are EinSS, NE-2. Shaded area is NE-2.

Note that all equilibria in secure strategies, and in particular Nash equilibrium, are Pareto dominated by some Nash-2 profiles (see Fig. 13)

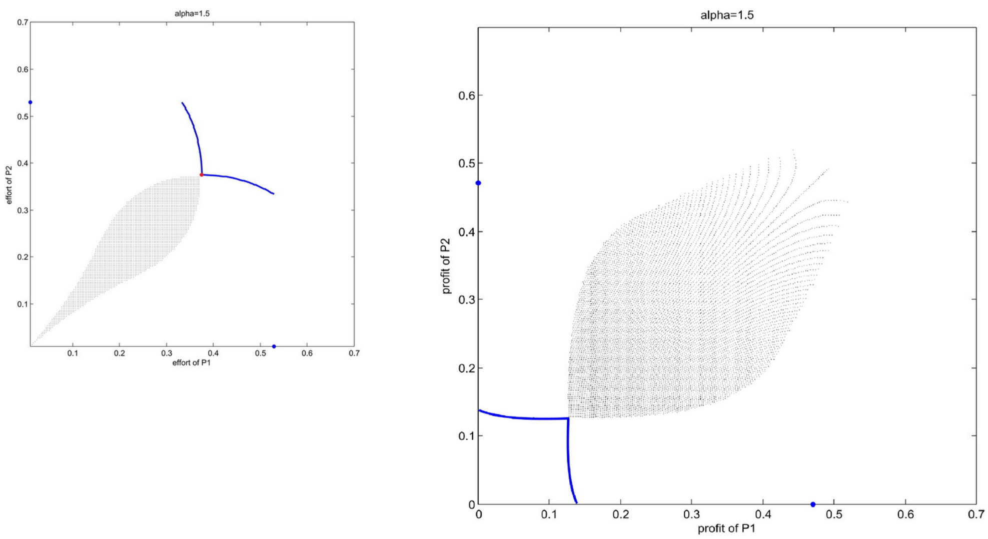


Figure 13: $\alpha = 1.5$. Efforts and profits. Bold curves and points on the axes are profits at equilibrium in secure strategies (and Nash-2 equilibrium), shaded area is the set of profits at risky Nash-2 profiles.

The set NE-2 \ EinSS seems to be intuitively clear: farsighted players in some sense are engaged in tacit collaboration and make smaller efforts to reach the same probability of obtaining the resource. This is what we mean by tacit collusion.

Remark 1 (on the efficiency of equilibrium obtained). By the measure of solution efficiency in rent-seeking game a *rent dissipation* is often used. If $R = 1$ then it is the sum of agent efforts at equilibrium. If $\alpha \geq 2$ rent dissipation in Nash equilibrium is equal to $\alpha/2$. The paper [3] shows that when $\alpha > 2$ and the strategy space is continuous full rent dissipation occurs in symmetric mixed-strategy equilibrium. It follows from our simulations that there are a wide range of risky Nash-2 equilibria that are more efficient than Nash pure or mixed strategy equilibrium for any α. To be exact, for $\alpha \leq 2$ any risky Nash-2 equilibrium together with *some* secure Nash-2 equilibrium (for accurate characterization of secure part see [25]) are more efficient. For $\alpha > 2$ some part of risky Nash-2 equilibria (in which $x_1 + x_2 < 1$) and only "monopolistic" (when only one player makes positive efforts) secure Nash-2 equilibria are more efficient. However, it is to be noted that sometimes zero efforts for one participant of the contest are not allowed by rules of the contest (for instance, if in this case the tender will not take place), then only risky Nash-2 profiles ensure smaller rent dissipation than mixed-strategy Nash equilibrium.

5 Extension to n-person games

The idea of an extension for n players is inspired by spatial economics notion of direct and indirect competitors [18]. In a game with large number of players it is natural to assume that each player divides her opponents into direct competitors whose reaction she worries about and tries to predict, and indirect competitors whose strategy is believed to be fixed as in Nash equilibrium concept. Such a selective farsightness looks more plausible than total ignorance of reactions or perfect prediction of future behavior of all other competitors. If we connect direct competitors by a directed link we get the network structure (so-called, reflection network) on the set of players. A player when decide whether to deviate or not accounts possible unilateral profitable responses of her neighbours in the reflection network, including simultaneous but non-cooperative responses of several other players.

The set of Nash-2 equilibrium crucially depends on the topology of reflection network. We illustrate this idea with the models of Bertrand competition and Prisoner's dilemma [30] with n participants and provide solutions of them for various types of reflection network.

5.1 Reflection network

Consider an n-person non-cooperative game in the normal form

$$G = (i \in I = \{1, \ldots, n\}; \quad s_i \in S_i; \quad u_i : S_1 \times \ldots \times S_n \to \mathbb{R}),$$

where s_i, S_i and u_i are the strategy, the set of all available strategies and the payoff function, respectively, of player i, $i = 1, \ldots, n$.

Let us define the **reflection network** g by the following rule:

- nodes are players i in I;
- links $g_{ij} = 1$ from player i to j exists iff player i takes into account possible profitable deviations of player j.
- $g_{ij} = 0$, otherwise.

Note that the reflection network is a *directed* graph. One can think about the reflection network in the following terms:

- Agents follow up some control sample of firms (direct competitors), taking strategies of other firms as given.
- Agents follow up geographically close competitors (spatial competition approach).
- Agents take into account reactions of only those firms, whose utility functions are exactly known to them.

The definition of profitable deviation of player i is the same as Def. 1, where s_{-i} is the strategy profile of all prayers except i.

Denote by $N_i(g)$ the set of neighbours j of player i in the graph g, such that $g_{ij} = 1$.

Definition 11 (secure deviation)**.** A deviation s'_i of player i at profile $s = (s_i, s_{-i})$ is secure if for any subset $J \subseteq N_i(g)$ and any profitable deviation s'_j of every player $j \in J$ at intermediate profile (s'_i, s_{-i}) even in case of simultaneous deviations of all players from J player i is not worse off, i.e.

$$u_i(s'_i, s'_J, s_{-iJ}) \geq u_i(s).$$

Here we assume that all players act independently (non-cooperative), but they are able to deviate simultaneously, so that player i should take this possibility into consideration.

If $N_i(g) = \emptyset$ then player i does not worry about any possible reactions, and so *every* her deviation is secure by definition. We will call this situation fully myopic behavior.

Definition 12 (NE-2)**.** A strategy profile is a Nash-2 equilibrium if no player has a profitable secure deviation.

It is easy to see that *every Nash equilibrium profile is also a Nash-2 equilibrium* irrespectively of the architecture of the reflection network. Moreover, in the case of empty reflection network they are coincide by definition. It is only in this sense we may regard Nash equilibrium as fully myopic concept.

5.2 Examples

5.2.1 Bertrand competition with homogeneous product

Consider the simplest model of price competition of n firms, concentrating at one point and producing homogeneous goods. Assume that they have equal marginal costs, the demand is linear,

$$\pi_i(p_1, \dots, p_n) = \begin{cases} (p_i - c)(1 - p_i)/K, & \text{if } p_i = \min\{p_j\}, \\ 0, & \text{if } p_i > p_j \text{ for some } j \neq i, \end{cases}$$

where K is the number of firms setting the minimum price p_i.

Nash solution yields the unique equilibrium, firms getting zero profits and equally sharing the market. Nevertheless, in case of non-trivial reflection network equilibrium set occurs considerably wider, and in some cases of reflection network Bertrand paradox is resolved.

Really, if each firm takes into account possible deviations of at least one other firm, or, in graph terminology, if for all nodes i in the network g their out-degree is greater or equal to 1 (see Fig. 14), then any price level greater than marginal costs is also a Nash-2 equilibrium, together with Nash equilibrium prices.

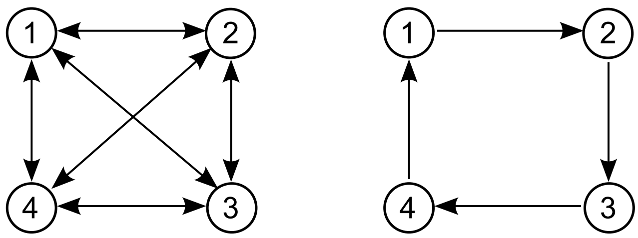


Figure 14: Complete and cycle reflection networks with 4 players, out-degree of every player is not less than 1

Nevertheless if *at least one* firm is fully myopic (see Fig. 15), then the only Nash-2 equilibrium coincides with Nash solution.

It is due to the threshold structure of demand: every infinitesimal decreasing price relative to common price level leads to the immediate winning

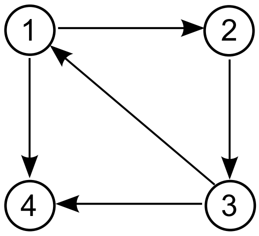


Figure 15: The reflection networks with 4 players, player 4 is fully myopic

all the market. Thus, this model is extremely sensitive to such a myopic deviations of any firm, and even one firm acting in a fully myopic way can break insecure tacit cooperation. There is no ability for cooperation among only some part of firms.

5.2.2 Prisoner's dilemma

Consider the model of n-player prisoner's dilemma as it introduced in [30]. Each player has two possible strategies: to cooperate with the community or to defect. The utility function is

$$u_i = \begin{cases} bA/n - c, & \text{if player } i \text{ cooperates,} \\ bA/n, & \text{if player } i \text{ defects,} \end{cases}$$

where A is a number of cooperators in the game, each of them brings profit b to the society, but pays the cost c. The total profit is equally divided to all n players irrespective of their real contribution. Unilateral defection is preferred for each individual $c > \frac{b}{n}$, nevertheless, full cooperation is more preferred for each player than common defection $b > c > 0$.

According to Nash rationality, cooperation is unlikely to emerge in the n-player prisoner's dilemma, and the same result is predicted by the evolutionary game theory [7].

But in the case of a non-empty reflection network cooperation is possible and, moreover, the number of cooperators depends both on the architecture of network and the relation between b and c.

First, observe that for any player who defects switching to cooperation is never a profitable deviation. Let us find the conditions under which the reverse deviation is insecure.

Assume that initially A players cooperate. Consider any cooperator i, assume that she reflects about n_i other cooperators. Her defection (which is always profitable) is a non-secure deviation if

$$\frac{bA}{n} - c > \frac{b(A-1-n_i)}{n}, \quad \text{and} \quad \frac{bA}{n} - c > 0,$$

that yields

$$n_i > n^* = \frac{cn}{b} - 1, \quad A > \frac{cn}{b}.$$

In particular, this means that a player reflecting about relatively small number of agents never cooperates. Therefore, in Nash-2 equilibrium any subset of players with sufficient number of links with other cooperators (more than n^*) in the reflection network is able to cooperate while all other defect if the number of cooperators is enough to provide positive profits for cooperators.

If these profits are very small, then for cooperation we need a complete reflection network among cooperators. However, if cooperative strategy leads to material losses then nothing will force players to cooperate.

So, for supporting cooperative behavior it is important not only to provide a balance between the value of individual return and the cooperation cost, but also to *ensure close contacts between cooperators*, as in the civil society.

Remark 2. The examples above demonstrate how significant is to take into account the agent reflection about possible behavior of the opponents. No matter what considerations (spatial or some others) underlie the reflection network, it fundamentally affects possible equilibria.

6 Conclusion

The problem that we discuss closely related to the equilibrium analysis of rational expectations under uncertainty. Classical Bayesian approach requires some certain information on probabilities of "the state of the world". The correctness of prediction and consistency with agents' behavior is an urgent field for discussion. However, process of assignment of these probabilities is often ambiguous.

Moreover, we argue that some aspects of rationality lie beyond utility function expression, namely, in a quasi-social structure of links among non-anonymous competitors. This factors together with initial point of consideration and natural limitations of iteration thinking depth lead us to the equilibrium concept providing multiple predictions. We assume that this multiplicity is a natural expression of great variety of real-life agents' behavior.

One more source of uncertainty is a problem of appropriate timing because of a lack of information on duration of interaction and intermediate moments of updating strategies. Using n-stage games is sometimes a rather rough way to treat such situations.

It is to be mentioned that the idea of constructing a measure on the set of Nash-2 equilibrium seems to have some analogy with mixed-strategy solution. But one should be very cautious with such conclusions as this distributions have different origin. Introducing any measure of feasibility on NE-2 essentially depends on some extra suggestions about agents' expectation and decision principles and thus may be defined in various ways.

Certainly, a lot of related issues is to be clarified in the future research. The most interesting part concerns games with many players and detecting some patterns of reflection networks. Nevertheless, we suppose that even simplest examples presented in this paper fully confirm that this approach is of interest.

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