Common Assumption of Cautious Rationality and Iterated Admissibility*

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Abstract. Iterated Admissibility (IA) is a powerful but conceptually puzzling solution concept for games in strategic form. Focusing on finite games, Brandenburger, Friedenberg and Keisler [17] provide epistemic foundations for IA within continuous and belief-complete type structures, in which players' interactive beliefs are represented by Lexicographic Conditional Probability Systems, the notion of Certainty is replaced by "Assumption", and the notion of Rationality requires—on top of lexicographic expected utility maximization—that every open event is deemed possible (although its probability may be infinitesimal). They show that each iteration n of the IA procedure characterizes the behavioral implications of Rationality and $(n-1)$-Mutual Assumption of Rationality, but Rationality and Common Assumption of Rationality is impossible. Dekel, Friedenberg and Siniscalchi [25] extend the result, allowing for all Lexicographic Probability Systems (LPS's). In this paper, we introduce novel notions of Cautiousness and Assumption for LPS's, whose preference-based foundations build on a weak "infinitely more likely than" relation between uncertain events. We weaken the notion of rationality of [17] and [25] to a notion of Cautious Rationality by requiring that only the payoff relevant events are deemed possible, and prove that IA characterizes the behavioral implications of Cautious Rationality and Common Assumption of Cautious Rationality in a canonical, belief-complete type structure.

Keywords: Iterated Admissibility, Weak dominance, Infinitely More Likely, Lexicographic Probability Systems, Rationality, Assumption.

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1 Introduction

Iterated Admissibility (henceforth IA, the iterated deletion of weakly dominated strategies) is an important and widely applied solution concept for games in strategic form.[1] In dynamic games with generic payoffs at terminal nodes, IA coincides with Pearce's [40] extensive-form rationalizability (Shimoji [43]), a prominent solution concept whose foundations are well understood (Battigalli and Siniscalchi [5]). Yet, while IA has an independent intuitive appeal, its theoretical foundations have proved to be elusive (see, e.g., Samuelson [41]). Thus, the decision-theoretic principles and the hypotheses about strategic reasoning that yield IA require careful scrutiny.

A recent literature—starting with the seminal contribution of Brandenburger, Friedenberg and Keisler ([17], henceforth BFK)—has tackled this issue building on two key ideas. The decision-theoretic aspects of the problem have been represented and solved through the *Lexicographic Expected Utility* theory of [9]. Lexicographic Expected Utility preferences are represented by *Lexicographic Probability Systems* (henceforth LPS's), i.e., lists of probabilistic conjectures in a priority order, each of which becomes relevant when the previous ones fail to identify a unique best alternative. In games with complete information, opponents' strategies constitute the payoff-relevant uncertainty. In order to come up with an educated conjecture about opponents' choices, a player naturally starts reasoning about opponents' beliefs and strategies. BFK modeled this aspect with the tools of *Epistemic Game Theory*, the formal, mathematical analysis of how players reason about each other in games.[2]

Inspired by BFK, we adopt Lexicographic Expected Utility and Epistemic Game Theory for our epistemic foundation of IA in finite games. However, we start from partially different basic principles. Specifically, we provide notions of *Rationality*, *Cautiousness* and *Assumption* that, appropriately combined, justify the choice of iteratively admissible strategies in the following sense: IA characterizes the behavioral implications of Cautious Rationality and Common Assumption of Cautious Rationality. Before moving to a detailed discussion of BFK and of the related literature, we briefly introduce the main features of our approach.

1. We model players' beliefs as LPS's. In line with recent findings and developments in the field, we do not require the LPS's to be *mutually singular*, i.e., we do not require the different conjectures in the LPS to have (essentially) disjoint supports.

2. We define a simple notion of Cautiousness that, together with Lexicographic Expected Utility maximization, justifies the choice of admissible strategies.

3. We use a monotone notion of "infinitely more likely than" with the following interpretation: A player deems an event infinitely more likely than another if she prefers to bet on the first rather than on the second regardless of the winning prizes for the two bets.

[1] For instance, IA has been applied in voting (Moulin [39]) and money-burning games (Ben Porath and Dekel [8]).

[2] See Dekel and Siniscalchi [24] for a recent survey.

4. We define a notion of Assumption of an event with the following preference-based foundation: Every payoff-relevant implication of the event is deemed infinitely more likely than the complement of the event. Under our notion of "infinitely more likely than", this implies that the whole event is deemed infinitely more likely than its complement too, despite the absence of mutual singularity.

5. We show that in a canonical lexicographic type structure—hence, absent extraneous restrictions on players' hierarchies of beliefs—there are states consistent with Cautious Rationality and Common Assumption of Cautious Rationality (henceforth $\mathrm{R^cCAR^c}$), and the behavioral implication of these epistemic conditions is that players choose within the (nonempty) set of iteratively admissible strategies.

The remainder of the Introduction is structured as follows. First, we illustrate the issues pertaining to the justification of IA and how Lexicographic Expected Utility helps to address them. Second, we summarize the seminal contribution of BFK, which inspired our work. Third, we highlight three issues of their construction that we find problematic, and we discuss how they have been separately addressed in the related literature. It turns out that our notions of Cautiousness and Assumption are weaker than all their counterparts in this literature. Thus, our hypotheses are the most general (so far) among those that justify IA. However, as we will discuss in Section 6, our results are robust to the introduction of additional restrictions to the notion of Assumption.

1.1 Iterated Admissibility and Lexicographic Expected Utility theory

Consider the iterated deletion of weakly dominated strategies in the following game.[3]

$Ann\backslash Bob$	L	C	R
T	$(4,1)$	$(4,1)$	$(0,1)$
U	$(0,1)$	$(0,1)$	$(4,1)$
M	$(3,1)$	$(2,1)$	$(2,1)$
D	$(9,1)$	$(0,1)$	$(0,1)$
B	$(0,0)$	$(4,1)$	$(0,1)$

Figure 1

Strategy L is weakly dominated by C and R, while B is weakly dominated by T. In the reduced game without L and B, D is (strictly) dominated by M. All the remaining strategies are iteratively admissible, including M, although the latter is not the unique best reply to any conjecture over C and R. Note that, in the reduced game, L is a (not unique) best reply to every conjecture over T, U and M. How can we justify the fact that M is iteratively admissible and L is not?

Strategy L displays the "inclusion-exclusion" problem, first identified by Samuelson [41]. A strategy is weakly dominated if and only if it is not a best reply to any fully mixed

[3] This is an elaboration of an example due to Pierpaolo Battigalli ([3]; see also the Introduction Section of BFK).

conjecture (see Section 3). Thus, L is eliminated in the first round because it is never optimal when Bob assigns positive probability to B (i.e., he "*includes*" B). On the other hand, if Bob is certain that Ann uses the same criterion, he must *exclude* the possibility that B is played. To justify the fact that L is not iteratively admissible, just sticking to fully mixed conjectures is not a viable solution: D would be erroneously rescued. This tension can be solved as follows.

Bob has a primary hypothesis under which Ann avoids weakly dominated strategies; thus, it assigns probability 0 to B. This conjecture leaves Bob indifferent between L, C and R, so he considers a secondary hypothesis. The secondary hypothesis does not exclude that Ann may play a weakly dominated strategy; thus it assigns positive probability to B. Since L is not a best reply to this conjecture, Bob does not choose L.

Strategy M can instead be justified under the same assumptions about Ann's strategic reasoning. Ann's primary hypothesis, under which Bob avoids weakly dominated strategies, assigns probability $\frac{1}{2}$ to C and R. Strategy M is a best reply to this conjecture, but not the only one. Therefore Ann considers a secondary hypothesis, which assigns probability $\frac{1}{2}$ to L and R. Strategy M is a best reply to this conjecture and a rational Ann may choose it. Note that both conjectures of Ann need to assign positive probability to R.

Blume et al. [9] introduce a model of choice under uncertainty, known as Lexicographic Expected Utility theory, which allows to formalize the beliefs and the choice criterion mentioned above. The beliefs of the decision maker are represented by LPS's, finite sequences of probability measures $(\mu_1, ..., \mu_n)$ over the space of uncertainty. The intended interpretation is the one above: μ_1 represents the decision maker's primary theory about the state of the world; μ_2 represents a secondary, alternative theory which the decision maker entertains but regards as "infinitely less plausible" than μ_1; and so on. In light of this interpretation, the decision maker first compares her alternatives according to the expected utility they yield under μ_1; in case a few alternatives yield the same expected utility, she compares them (and only them) under μ_2, and so on.

To identify which LPS's over strategies justify the iteratively admissible strategies and, more importantly, which hypotheses motivate them, it is necessary to analyze players' interactive beliefs. Next we summarize the approach of BFK.

1.2 The epistemic framework of BFK

BFK use lexicographic type structures—i.e., type structures in which beliefs are LPS's—as the analogue of standard type structures, i.e., type structures in which beliefs are probability measures. Type structures are a convenient modelling device, due to Harsanyi [30], to describe players' *hierarchies of beliefs*; that is, their beliefs about the play of the game (first-order beliefs), their beliefs about players' beliefs about play (second-order beliefs), and so on. Type structures enrich the standard description of a game, by providing a language that allows to express assumptions about players' mutual beliefs in rationality, and then derive implications about behavior.

Within the epistemic apparatus of lexicographic type structures, BFK introduce a notion of Rationality which incorporates the admissibility requirement, and they use a notion of Assumption in place of Certainty to solve the inclusion-exclusion problem. To illustrate these notions, we append to the game of Figure 1 a lexicographic type structure

$\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in\{1,2\}}$, where Ann is player 1 and Bob is player 2. First, for each player $i \in \{1,2\}$, there is an underlying *space of primitive uncertainty* S_{-i}, i.e., the strategy set of the co-player; so, for instance, $S_2 = \{L, C, R\}$ is the underlying space of uncertainty for player 1. Second, for each player $i \in \{1,2\}$, there is a set T_i of *types*. Here we describe an example with one possible type for Ann, viz. $T_1 = \{t_1'\}$, and two possible types for Bob, viz. $T_2 = \{t_2', t_2''\}$. Third, for each player $i \in \{1,2\}$, there is a *belief map* β_i which assigns to each type an LPS over the his/her underlying space of uncertainty and the co-player's types. In BFK, such LPS is required to be a Lexicographic Conditional Probability System (henceforth, LCPS), i.e., a mutually singular LPS. In this example, we associate Ann's type t_1' with a length-2 LCPS $\beta_1(t_1') = (\nu_1^1, \nu_1^2)$, where ν_1^1 and ν_1^2 are probability measures over $S_2 \times T_2$. Type t_2' is associated with a full support measure over $S_1 \times T_1$ (i.e., a length-1 LPS). Type t_2'' is associated with a measure over $S_1 \times T_1$ without full support. Figure 2 illustrates the probabilities assigned by the two measures in $\beta_1(t_1')$ to the elements of the set $S_2 \times T_2$. For instance, the strategy-type pair (L, t_2') has probability 0 under ν_1^1, and $\frac{1}{4}$ under ν_1^2.

$\beta_1(t_1')$	t_2'	t_2''
L	$0, \frac{1}{4}$	$0, \frac{1}{4}$
C	$\frac{1}{2}, 0$	$0, \frac{1}{10}$
R	$\frac{1}{2}, 0$	$0, \frac{2}{5}$

Figure 2

Player 1's first-order belief over $\{L, C, R\}$ is given by a length-2 LPS $\left(\mathrm{marg}_{S_2}\nu_1^1, \mathrm{marg}_{S_2}\nu_1^2\right)$, where, the symbol marg_{S_2} stands for the marginalization operator. A standard procedure, which will be discussed in Section 2, shows how it is possible to specify all higher-order beliefs. Figure 3 describes player 1's first-order belief induced by $\beta_1(t_1')$.

	L	C	R
$\mathrm{marg}_{S_2}\nu_1^1$	0	$\frac{1}{2}$	$\frac{1}{2}$
$\mathrm{marg}_{S_2}\nu_1^2$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{2}{5}$

Figure 3

Within the framework of (lexicographic) type structures, a *state* specifies a strategy-type pair for each player; e.g., in the type structure $\mathcal{T}$, a possible state is (U, L, t_1', t_2'), that is, an element of the set $S_1 \times S_2 \times T_1 \times T_2$.

BFK's notion of Rationality. In a type structure, Rationality is a property of strategy-type pairs. BFK define a strategy-type pair (s_i, t_i) to be *rational* if (1) s_i is a *lexicographic best reply* to the first-order belief induced by t_i, and (2) the LCPS $\beta_i(t_i)$ associated with t_i is of *full-support*. The support of an LPS is defined as the union of the supports of each component measure; so, for instance, the LCPS $\beta_1(t_1') = (\nu_1^1, \nu_1^2)$ is of full-support. Moreover, the reader can easily verify that strategies T, U and M are best replies to $\mathrm{marg}_{S_2}\nu_1^1$, but M does better than T and U against $\mathrm{marg}_{S_2}\nu_1^2$. To distinguish between our notion of Cautious Rationality and BFK's notion of Rationality, we will refer to the latter as "Open-minded Rationality" (R^{o}).[4]

[4] The motivation for this terminology is twofold. First, full-support aims to capture the idea that a player takes all strategies and types of the co-players into consideration. Second, full-support is equivalent to the requirement that every *open* set in the space of strategy-type pairs be assigned positive probability by at least one measure of the LPS.

So, the statement "Ann is open-minded rational" is represented in structure $\mathcal{T}$ by event $R_1^{\mathrm{o}} = \{(M, t_1')\}$. For Bob, the open-minded rational strategy-type pairs associate the full-support belief of type t_2' with the weakly dominant strategies C and R, but no pair (s_2, t_2'') can satisfy Open-minded Rationality because $\beta_2(t_2'')$ is not of full-support. Thus, $R_2^{\mathrm{o}} = \{(C, t_2'), (R, t_2')\}$.

BFK's notion of Assumption. In BFK, Assumption of an event E requires two things. First, it requires that E be deemed infinitely more likely than not-E, without necessarily ruling out the possibility that not-E occurs. In BFK, this means that the LCPS assigns probability 1 to E under the first $m \geq 1$ measures and probability 0 under the remaining measures. Under $\beta_1(t_1')$, R_2^{o} is deemed infinitely more likely than its complement, because $\nu_1^1(\{(C, t_2'), (R, t_2')\}) = 1$ and $\nu_1^2(\{(C, t_2'), (R, t_2')\}) = 0$. Second, Assumption requires that all the *parts* of E, i.e., the non-empty, *relatively open* subsets of E, be deemed infinitely more likely than not-E. This means that the LCPS assigns positive probability to each part under one of the first m measures. Under $\beta_1(t_1')$, both proper subsets of R_2^{o}, viz. $\{(C, t_2')\}$ and $\{(R, t_2')\}$, are deemed infinitely more likely than the complement of R_2^{o} (finite sets are endowed with the discrete topology, in which all subsets—including singletons—are open). Therefore, we say that type t_1' assumes R_2^{o}.

BFK's results. BFK provide an epistemic justification of each iteration of the IA (maximal) elimination procedure within lexicographic type structures with continuous and onto belief maps, henceforth, "belief-complete" structures. One of their main results states that, within belief-complete type structures, the strategies which survive $m+1$ steps of iterated elimination of inadmissible strategies in a finite game are those consistent with Open-minded Rationality and mth-mutual Assumption of Open-minded Rationality ($\mathrm{R^o}m\mathrm{AR^o}$) for every natural number m. They prove that belief-complete type structures exist, but their proof is not constructive. This contrasts with analogous work on other solution concepts, where a canonical, belief-complete type structure is constructed and type sets consists of all the collectively coherent hierarchies of beliefs (e.g., [4] and [5]). Since they do not explicitly state their results in terms of belief hierarchies, the meaning of key epistemic properties of beliefs, such as the full-support property, or the property that some event is assumed, is partially self-referential. These and other features of their approach are somewhat problematic. In what follows we illustrate them and also summarize how these issues have been addressed in the literature.

1.3 Issues with BFK's approach and related literature

Issue 1. BFK restrict attention to LCPS's. There are reasons to find this restriction uncompelling.

First, the finite-order beliefs implied by a type structure for LCPS's are LPS's but not necessarily LCPS's. As Figure 3 shows, player 1's first-order belief induced by $\mathcal{T}$ is not an LCPS. It is also possible that *none* of the finite-order beliefs induced by the LCPS of a type are LCPS's (we show this in our companion paper [20]). On the other hand, first-order beliefs with overlapping supports are actually needed to justify some iteratively admissible strategies, as can be verified for M in the example above. Finally, as noted by Lee [34], the difference between an LPS and LCPS may be merely cosmetic, as they could represent the same lexicographic preference relation.

Dekel et al. [25] provide further interesting arguments for why mutual singularity is not a compelling hypothesis (see, e.g., the "coin example" in their introduction). Then, they provide a characterization in terms of LPS's (not just LCPS's) of BFK's preference-based notion of Assumption. With this, they show that BFK's results about the epistemic characterization of IA hold through, including the impossibility result discussed in Issue 3 below. The preference-based notion of Assumption in BFK is based on the notion of "infinitely more likely than" of Blume et al. [9]. As we will discuss (cf. Appendix A; see also [9, p. 70]), such notion of "infinitely more likely than" has some unappealing properties, especially in absence of mutual singularity.

Issue 2. As we mentioned, Open-minded Rationality includes a full-support condition over strategies *and* types. Intuitively, this condition aims to capture the idea that a player does not exclude any belief hierarchy induced by the types of the opponent. Yet, the full-support condition crucially depends on the topology of the type space. Therefore, a strategy-type pair may be regarded as open-minded rational or not depending on the topology, even if the type induces the same hierarchy of beliefs (see Example 1 in Section 4.2).

In a related vein, BFK's notion of Assumption also depends on the topology on types, as observed by Dekel and Siniscalchi ([24, p. 691, footnote 99]). The recent work of Keisler and Lee [33] highlights the fact that two belief-complete type structures can induce the same hierarchies of beliefs, yet the topology on the type sets can be changed in an appropriate way so that the types may not assume the same set of events.

Issue 3. BFK considered the natural conjecture that, in all belief-complete type structures, the strategies that survive all rounds of IA are exactly the strategies that are played in states at which there is Open-minded Rationality and Common Assumption of Open-minded Rationality ($\mathrm{R^oCAR^o}$)—i.e., states at which there is $\mathrm{R^o}m\mathrm{AR^o}$ for every natural number m. However, they obtain a negative result: In every belief-complete *continuous* type structure (i.e., with continuous belief maps) $\mathrm{R^oCAR^o}$ is not possible at any state.

Keisler and Lee [33] show that the impossibility ceases to hold when continuity is dropped. This result is difficult to interpret and makes the original impossibility even more puzzling: What does a topological condition like continuity of the belief maps represent in terms of players' belief hierarchies? This question is closely related to the topological dependencies outlined in Issue 2.

Yang [44] and Catonini [18] propose a modification of BFK's notion of Assumption which yields a non-empty "common assumption of rationality" event under some conditions. In [18], the result is obtained in a canonical type structure for LCPS's (see [20] for its construction and properties). In [44], the result is obtained in the larger canonical type structure for LPS's, but the definition and the preference-based foundation of Assumption still crucially require the LPS to be a (full-support) LCPS. Therefore, it is impossible to assess whether the types associated with non-mutually singular LPS's assume a given event or not.

In our view, our notions of Cautiousness, "infinitely more likely than" and Assumption satisfactorily address these issues. Players are cautious whenever their first-order beliefs have full-support, and this is a condition that can be expressed in terms of belief hiearchies and primitives of the model.

The main departures of our preference-based notion of Assumption from BFK are two. (a) As a building-block, we use an "infinitely more likely than" relation between events—due to Lo [37]—which is weaker than the one in [9]. This allows to obtain a simple characterization of Assumption for all LPS's (as in [25]) and, at the same time, to provide a transparent comparison between Assumption and a weaker notion, namely "Weak Belief"—a notion which is well suited for the epistemic analysis of Permissibility (cf. Discussion Section).

(b) In our version of Assumption, players have a "cautious attitude" towards an assumed event E, i.e., they consider each payoff-relevant implication of E "infinitely more likely than" not-E. This is in line with the notion of Cautiousness, which coincides with Assumption if E is the whole space of uncertainty (e.g., $E = S_2 \times T_2$ for player 1 in the example above). The choice of payoff-relevant subsets of E can be expressed in terms of players' preferences over the primitive space of uncertainty. This makes the whole analysis invariant to the topology on types and allows to obtain a non-empty $\mathrm{R^cCAR^c}$ event in sufficiently rich (in terms of hierarchies) type structures. Interestingly, the same holds through in the canonical type structure for LCPS's constructed in [20], which constitutes a common ground for the comparison with BFK (see Section 6.4).

Therefore, we provide a characterization of IA using *expressible* epistemic assumptions about rationality and beliefs, that is, assumptions which are expressible in a language describing primitive terms (strategies) and terms derived from the primitives (beliefs about strategies, beliefs about strategies and beliefs of others, etc.)—cf. Battigalli et al. ([6]).

It should be mentioned that Dekel et al. [25] put forward two variants of BFK's preference-based notion of Assumption which admit a simpler characterization in terms of LPS's. Both variants are based on two extensions of the notion of "weak dominance" to infinite state spaces. Their approach is therefore different form ours, despite the fact that one of their variants of Assumption admits an LPS-based characterization which is similar to our version.[5] However, those alternative notions of Assumption in [25] also depend on the topology on types, so the aforementioned drawbacks (cf. Issues 2-3) still apply. Further comments on this issue and on the related literature are deferred to the Discussion Section.

1.4 Structure of the paper

The remainder of this paper is structured as follows. Section 2 gives some preliminary technical concepts and notations that will be used throughout. Section 3 provides formal definitions of LPS's, type structures and hierarchies of lexicographic beliefs. Most of the results in this section are proved in our companion paper ([20]). We record only the properties that will be important for the statement of our results. Section 4 introduces the underlying game-theoretic framework, and the notions of Cautious Rationality and Assumption. In Section 5, we state and prove the main result. Section 6 concludes with a discussion and further comments on the related literature. Appendix A illustrates the

[5] Specifically, this is the case for the notion of TWD-Assumption ([25, Definition 4.4])—see the Discussion Section. In the Supplemental Appendix, we show that TWD-Assumption can be given an alternative preference-based foundation in terms of the notion of "infinitely more likely than" we use in this paper.

notion of "infinitely more likely than" and the preference-based foundation of Assumption and Cautiousness. Appendix B collects all the proofs omitted from the main text.

2 Preliminaries

We begin with some definitions and the basic notation that will be used throughout the paper.[6] A measurable space is a pair (X, Σ_X), where X is a set and Σ_X is a σ-field, the elements of which are called *events*. When it is clear from the context which σ-field on X we are considering, we suppress reference to Σ_X and simply write X to denote a measurable space. All the sets considered in this paper are assumed to be metrizable topological spaces, and they are endowed with the Borel σ-field. A *Polish* space is a topological space which is homeomorphic to a complete, separable metrizable space. A *Lusin* space is a topological space which is the continuous, injective image of a complete, separable metrizable space. Clearly, a Polish space is also Lusin. [7]

If $(X_n)_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint topological spaces, then the set $X = \cup_{n\in\mathbb{N}} X_n$ is endowed with the *direct sum topology*.[8] The set X is Lusin (resp. Polish) provided each X_n is Lusin (resp. Polish). Fix a countable collection of pairwise disjoint topological spaces $(Y_n)_{n\in\mathbb{N}}$, and let $Y = \cup_{n\in\mathbb{N}} Y_n$. For a given indexed family of mappings $(f_n)_{n\in\mathbb{N}}$, where $f_n : X_n \to Y_n$, let $f : X \to Y$ be the function defined as

$$f(x) = f_n(x), \; x \in X_n.$$

Following the terminology in [26], the map $f : X \to Y$ is called the *combination* of the functions $(f_n)_{n\in\mathbb{N}}$, and is often denoted by $\cup_{n\in\mathbb{N}} f_n$.

We consider any product, finite or countable, of topological spaces as a topological space with the product topology. As such, a countable product of Lusin (resp. Polish) spaces is also Lusin (resp. Polish). Furthermore, given topological spaces X and Y, we denote by Proj_X the canonical projection from $X \times Y$ onto X; in view of our assumption, the map Proj_X is continuous. Finally, for a measurable space X, we denote by Id_X the identity map on X, that is, $Id_X(x) = x$ for all $x \in X$.

3 Hierarchies of lexicographic beliefs and lexicographic type structures

3.1 Lexicographic probability systems

Given a topological space X, we denote by $\mathcal{M}(X)$ the set of Borel probability measures on X. The set $\mathcal{M}(X)$ is endowed with the *weak**-topology. Thus, if X is Lusin (resp.

[6] A more detailed presentation of the following concepts, as well as related mathematical results, can be found in [11], [22], [26]. In the remainder of the paper, we shall make use of the results mentioned in this section, sometimes without referring to them explicitly.

[7] If X is a Lusin topological space, and Σ_X is the corresponding Borel σ-field, then the measurable space (X, Σ_X) is Standard Borel ([22, Proposition 8.6.13]).

[8] In this topology, a set $O \subseteq X$ is open if and only if $O \cap X_n$ is open in X_n for all $n \in \mathbb{N}$. The assumption that the spaces X_n are pairwise disjoint is without any loss of generality, since they can be replaced by a homeomorphic copy, if needed (see [26, p. 75]).

Polish), then $\mathcal{M}(X)$ is also Lusin (resp. Polish). We denote by $\mathcal{N}(X)$ (resp. $\mathcal{N}_n(X)$) the set of all finite (resp. length-n) sequences of Borel probability measures on X, that is,

$$\begin{aligned}\mathcal{N}(X) &= \cup_{n\in\mathbb{N}}\mathcal{N}_n(X)\\ &= \cup_{n\in\mathbb{N}}(\mathcal{M}(X))^n.\end{aligned}$$

Each $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$ is called **lexicographic probability system** (LPS). We say that $\overline{\mu}$ is a **mutually singular** LPS, or a **lexicographic conditional probability system** (LCPS), if there are Borel sets $E_1, ..., E_n$ in X such that, for every $l \leq n$, $\mu_l(E_l) = 1$ and $\mu_l(E_m) = 0$ for $m \neq l$. Write $\mathcal{L}(X)$ (resp. $\mathcal{L}_n(X)$) for the set of LCPS's (resp. length-n LCPS's). Both topological spaces $\mathcal{N}(X)$ and $\mathcal{L}(X)$ are Lusin provided X is Lusin. In particular, if X is Polish, so are $\mathcal{N}(X)$ and $\mathcal{L}(X)$.[9]

For every Borel probability measure μ on a topological space X, the support of μ, denoted by $\mathrm{Supp}\mu$, is the smallest closed subset $C \subseteq X$ such that $\mu(C) = 1$. The support of an LPS $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$ is thus defined as $\mathrm{Supp}\overline{\mu} = \cup_{l\leq n}\mathrm{Supp}\mu_l$. So, an LPS $\overline{\mu} = (\mu_1, ..., \mu_n) \in \mathcal{N}(X)$ is of **full-support** if $\cup_{l\leq n}\mathrm{Supp}\mu_l = X$. We write $\mathcal{N}_n^+(X)$ for the set of all full-support, length-n LPS's and $\mathcal{N}^+(X)$ (resp. $\mathcal{L}^+(X)$) for the set of full-support LPS's (resp. full-support LCPS's).

Suppose we are given topological spaces X and Y, and a Borel map $f : X \to Y$. The map $\widetilde{f} : \mathcal{M}(X) \to \mathcal{M}(Y)$, defined by

$$\widetilde{f}(\mu)(E) = \mu\left(f^{-1}(E)\right), \mu \in \mathcal{M}(X), E \in \Sigma_Y,$$

is called the image (or pushforward) measure map of f. For each $n \in \mathbb{N}$, the map $\widehat{f}_{(n)} : \mathcal{N}_n(X) \to \mathcal{N}_n(Y)$ is defined by

$$(\mu_1, ..., \mu_n) \mapsto \widehat{f}_{(n)}((\mu_1, ..., \mu_n)) = \left(\widetilde{f}(\mu_k)\right)_{k\leq n}.$$

Thus the map $\widehat{f} : \mathcal{N}(X) \to \mathcal{N}(Y)$ defined by

$$\widehat{f}(\overline{\mu}) = \widehat{f}_{(n)}(\overline{\mu}), \overline{\mu} \in \mathcal{N}_n(X),$$

is called the **image LPS map of** f. In other words, the map $\widehat{f}$ is the combination of the functions $\left(\widehat{f}_{(n)}\right)_{n\in\mathbb{N}}$, and it is Borel measurable.

In particular, if X and Y are Lusin spaces, then the marginal measure of $\mu \in \mathcal{M}(X \times Y)$ on X is defined by $\mathrm{marg}_X\mu = \widetilde{\mathrm{Proj}}_X(\mu)$. Consequently, the marginal of $\overline{\mu} \in \mathcal{N}(X \times Y)$ on X is defined by $\overline{\mathrm{marg}}_X\overline{\mu} = \widehat{\mathrm{Proj}}_X(\overline{\mu})$, and the function $\widehat{\mathrm{Proj}}_X : \mathcal{N}(X \times Y) \to \mathcal{N}(X)$ is continuous and surjective.

3.2 Hierarchies of lexicographic beliefs

In this section, we review the formal construction of the *canonical hierarchic space*, that is, the space of all hierarchies of lexicographic beliefs displaying *Coherence* and *common*

[9] We refer the reader to our companion paper [20] for a proof of those results.

Full Belief of Coherence. (Precise definitions will be given below.) For an in-depth, formal analysis and details of the construction, which parallels the architecture originally developed by Brandeburger and Dekel [15], the reader can consult [20]. Here, we emphasize only the properties which will be important for our results.

Fix a two-player set I;[10] given a player $i \in I$, we denote by $-i$ the other player in I. For each $i \in I$, let S_{-i} be a non-empty space—called *space of primitive uncertainty*—describing aspects of the strategic interaction that player i is uncertain about. Throughout this paper, S_{-i} will represent player $-i$'s strategy set: Player i does not know which strategy player $-i$ is going to choose. Yet, other interpretations are also possible; for instance, S_{-i} may include player $-i$'s set of payoff functions, among which the true one is not known to player i. We assume that for each $i \in I$, S_i is a Lusin space with cardinality $|S_i| \geq 2$.[11]

Each player $i \in I$ is endowed with a lexicographic belief on S_{-i}; such belief is called first-order (lexicographic) belief. However, first-order beliefs do not exhaust all the uncertainty faced by each player: Player i realizes that player $-i$ has a first-order belief on S_i as well, and this belief is unknown to him. Thus, player i's second-order beliefs are represented by an LPS over S_{-i} and the space of $-i$'s first-order beliefs. Continuing in this fashion, one is naturally led to consider infinite hierarchies of lexicographic beliefs.

Formally, for each $i \in I$ define inductively the collection of spaces $\left(X_i^k\right)_{k\geq 0}$ as

$$X_i^0 = S_{-i}, \tag{3.1}$$

$$X_i^{k+1} = X_i^k \times \mathcal{N}\left(X_{-i}^k\right);\ k \geq 0. \tag{3.2}$$

An element $h_i^{k+1} = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ..., \overline{\mu}_i^{k+1}\right) \in \prod_{l=0}^{k} \mathcal{N}\left(X_i^l\right)$ is a $(k+1)$-order belief hierarchy, and $\overline{\mu}_i^{k+1} = \left(\mu_i^{k+1,1}, ..., \mu_i^{k+1,n}\right) \in \mathcal{N}\left(X_i^k\right)$ denotes i's $(k+1)$-order LPS, with $\mu_i^{k+1,m} \in \mathcal{M}\left(X_i^k\right)$ being the m-level of the $(k+1)$-order LPS. It is easily seen that, according to our notation,

$$X_i^{k+1} = X_i^0 \times \prod_{l=0}^{k} \mathcal{N}\left(X_{-i}^l\right).$$

The set of all possible, infinite hierarchies of LPS's for player i is $H_i^0 = \prod_{k=0}^{\infty} \mathcal{N}\left(X_i^k\right)$. The space H_i^0 is endowed with the product topology, so that H_i^0 is a Lusin space.

The notion of coherence for hierarchies of beliefs (defined below) says that beliefs at different orders cannot contradict each other. To state this formally, let $\mathrm{Proj}_{X_i^{k-1}} : X_i^k \to X_i^{k-1}$ denote the coordinate projection, for all $k \geq 1$. Recall that the marginal of $\overline{\mu}_i^{k+1} \in \mathcal{N}\left(X_i^k\right)$ over X_i^{k-1}, viz. $\overline{\mathrm{marg}}_{X_i^{k-1}} \overline{\mu}_i^{k+1}$, is defined as the image LPS of $\overline{\mu}_i^{k+1}$ under $\mathrm{Proj}_{X_i^{k-1}}$, namely $\widehat{\mathrm{Proj}}_{X_i^{k-1}}\left(\overline{\mu}_i^{k+1}\right)$.

Definition 1 *A hierarchy of beliefs $h_i = \left(\overline{\mu}_i^1, \overline{\mu}_i^2, ...\right) \in H_i^0$ is* ***coherent*** *if*

$$\overline{\mathrm{marg}}_{X_i^{k-1}} \overline{\mu}_i^{k+1} = \overline{\mu}_i^k,\ \forall k \geq 1.$$

[10] The analysis can be trivially extended to more than two players.

[11] This assumption is made mainly to avoid trivial cases and streamline the exposition. All the results in this paper remain true under the weaker assumption that $|S_i| \geq 2$ for *at least one* player $i \in I$.

This definition of coherence is a simple generalization of the notion of coherence as in [38] or [15]; the two notions coincide if each $\overline{\mu}_i^k$ is a standard probability measure (i.e., a length-1 LPS). Note that a hierarchy of beliefs satisfying this coherence requirement consists of an infinite sequence of LPS's of the same length.[12]

For each player $i \in I$, the space of all coherent hierarchies of beliefs is denoted by H_i^1. Standard arguments (cf. [20]) show that each H_i^1 is a closed subset of H_i^0, hence Lusin. By a version of the Kolmogorov Extension Theorem for LPS's (cf. [15, Lemma 1]), it can be shown that a coherent hierarchy for a player can be summarized by a single LPS over the cartesian product of his own space of primitive uncertainty and opponents' hierarchies. So we record the following result (cf. [15, Proposition 1]).

Proposition 1 *For each $i \in I$, there exists a homeomorphism $f_i : H_i^1 \to \mathcal{N}\left(S_{-i} \times H_{-i}^0\right)$ such that*

$$\overline{\mathrm{marg}}_{X_i^{k-1}} f_i\left(\left(\overline{\mu}_i^1, \overline{\mu}_i^2, ...,\right)\right) = \overline{\mu}_i^k, \ \forall k \geq 1.$$

The homeomorphism just described implies that a player i's coherent hierarchy of LPS's determines his LPS over player $-i$'s hierarchies of beliefs. However, even if player i's hierarchy $h_i \in H_i^1$ is coherent, $f_i(h_i)$ could deem possible an incoherent hierarchy of the other player, that is, player i may believe (in an appropriate sense defined below) that player $-i$'s hierarchy is not coherent. We consider the case in which there is *common Full Belief of Coherence*.

Formally, we say that player i, endowed with a coherent hierarchy h_i, **fully believes** an event $E \subseteq S_{-i} \times H_{-i}^0$ if $f_i(h_i)(E) = \overline{1}$, where $\overline{1}$ denotes a finite sequences of 1s; that is to say, every probability measure of the LPS $f_i(h_i) \in \mathcal{N}\left(S_{-i} \times H_{-i}^0\right)$ assigns probability 1 to E.[13] **Common Full Belief** of coherence is imposed by defining inductively, for each $i \in I$, the following sets:

$$\begin{aligned} H_i^{l+1} &= \left\{h_i \in H_i^1 \left| f_i(h_i)\left(S_{-i} \times H_{-i}^l\right) = \overline{1}\right.\right\}, \ l \geq 1, \\ H_i &= \cap_{l \geq 1} H_i^l. \end{aligned}$$

The set $\Pi_{i \in I} H_i$ is naturally interpreted as the set of players' hierarchies such that each player fully believes that the other player's hierarchy is coherent, fully believes that the other player fully believes that his hierarchy is coherent, and so on. The following Proposition shows that a homeomorphism result, analogous to the one provided by Proposition 1, also holds for each space of hierarchies H_i.

Proposition 2 *The restriction of f_i to H_i induces a homeomorphism $\overline{f}_i$ from H_i onto $\mathcal{N}(S_{-i} \times H_{-i})$.*

[12] As we shall see below, from any type in a lexicographic type structure we can derive a corresponding coherent hierarchy with the property of all orders of beliefs being of the same length. In Section 6.5, we will compare the notion of coherence in Definition 1 with the alternative notion due to Lee ([35]).

[13] For a preference-based characterization of the notion of Full Belief, see Appendix A. Full Belief coincides with what Asheim and Søvik [2] call "certain belief". BFK put forward a notion of "belief" which coincides with Full Belief under the additional assumption that the LPS (actually, an LCPS) is of full-support ([17, Proposition A.1]).

Hereafter, we shall refer to the set $H = \Pi_{i\in I} H_i$ as the *canonical hierarchic space*.

We conclude this section with a few remarks concerning the topological structure of the canonical hierarchic spaces H. First note that H is a Lusin space; in particular, H is Polish provided each space S_i is Polish. Note however that H *is not compact*, even if the underlying spaces of primitive uncertainty are compact (e.g., finite, as we shall assume in Section 5). To see this, observe that $\mathcal{M}(X)$ is compact if X is compact, and this in turn implies that the space $\mathcal{N}_n(X)$ is also compact for some *finite* $n \in \mathbb{N}$. But the same conclusion does not hold for the space $\mathcal{N}(X)$.[14] By contrast, the canonical hierarchic spaces of both standard beliefs and conditional beliefs turn out the be compact metrizable if each space S_i is compact metrizable.

3.3 Lexicographic type structures

The following definition is a natural generalization of the standard definition of epistemic type structures with beliefs represented by probability measures, i.e., length-1 LPS (cf. [31]).

Definition 2 *An $(S_i)_{i\in I}$-**based lexicographic type structure** is a structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$, where*

1. *for each $i \in I$, T_i is a Lusin space;*
2. *for each $i \in I$, the function $\beta_i : T_i \to \mathcal{N}(S_{-i} \times T_{-i})$ is measurable.*

*We call each space T_i **type space** and we call each β_i **belief map**.[15] Members of type spaces, viz. $t_i \in T_i$, are called **types**. Say $t_i \in T_i$ is a **mutually singular** type if $\beta_i(t_i) \in \mathcal{L}(S_{-i} \times T_{-i})$. Say $t_i \in T_i$ is a **full-support** type if $\beta_i(t_i) \in \mathcal{N}^+(S_{-i} \times T_{-i})$. Each element $(s_i, t_i)_{i\in I} \in S \times T$ is called **state (of the world)**.*

A lexicographic type structure—or type structure, for short—formalizes Harsanyi's implicit approach to model hierarchies of beliefs. But clearly the canonical hierarchic space $H = \Pi_{i\in I} H_i$ gives rise to an $(S_i)_{i\in I}$-based type structure $\mathcal{T}_u = \langle S_i, T_i, \beta_i \rangle_{i\in I}$, by setting $T_i = H_i$ and $\beta_i = \overline{f}_i$ for each $i \in I$. Hereafter, we shall refer to $\mathcal{T}_u = \langle S_i, H_i, \overline{f}_i \rangle_{i\in I}$ as the **canonical (lexicographic) type structure**.

The formalism of lexicographic type strucures was first introduced by BFK ([17, Section 7]) under the additional requirement that each belief is represented by an LCPS. In what follows, we will say that a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ is an **LCPS type structure** if the range of each belief map β_i is contained in $\mathcal{L}(S_{-i} \times T_{-i})$.

Definition 3 *An $(S_i)_{i\in I}$-based lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ is*

[14] This is an instance of a well-known mathematical fact (see [26, Theorem 2.2.3]): If $(X_\theta)_{\theta\in\Theta}$ is an indexed family of non-empty compact spaces $|X_\theta| > 1$ for all $\theta \in \Theta$, then the direct sum $\cup_{\theta\in\Theta} X_\theta$ is compact if and only if the right-directed set Θ is finite.

[15] Observe that some authors ([4], [31]) use the terminology "type space" for what is called "type structure" here.

- ***finite*** *if the cardinality of each type space* T_i *is finite;*
- ***compact*** *if each type space* T_i *is compact;*
- ***belief-complete*** *if each belief map* β_i *is onto;*
- ***continuous*** *if each belief map* β_i *is continuous.*

Analogous definitions hold if $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ *is an LCPS type structure.*

The idea of belief-completeness was introduced by Brandenburger [14] and adapted to the present context. Note that each type space in a belief-complete type structure has the cardinality of the continuum. The structure $\mathcal{T}_u$ is a particular instance of a belief-complete and continuous type structure. But there exist also belief-complete and continuous type structures which are different from $\mathcal{T}_u$.[16] While finite type structures are trivially compact and continuous (but not belief-complete), the argument given at the end of the previous section shows that a belief-complete, compact and continuous lexicographic type structure cannot exist in the current framework.

3.4 From types to belief hierarchies

A type structure provides an implicit representation about players' uncertainty, in the sense that it does not describe hierarchies of beliefs directly. In this Section we show that it is possible to associate with the subjective belief of each type an explicit hierarchy of beliefs. To accomplish this task, we fix a given $(S_i)_{i\in I}$-based type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$, and we define, for each player $i \in I$, a *hierarchy description map* $d_i : T_i \to H_i^0$ associating with each $t_i \in T_i$ a corresponding hierarchy of LPS's. Each hierarchy description map is defined inductively (cf. [4]):

- (base step: $k = 1$) For each $i \in I$ and $t_i \in T_i$, define the first-order hierarchy description map $d_i^1 : T_i \to \mathcal{N}(S_{-i})$ by

$$d_i^1(t_i) = \overline{\mathrm{marg}}_{S_{-i}}(\beta_i(t_i)).$$

  For each $i \in I$, define $\psi_{-i}^1 : S_{-i} \times T_{-i} \to X_i^1 = S_{-i} \times \mathcal{N}(S_i)$ by

$$\psi_{-i}^1 = \left(\mathrm{Id}_{S_{-i}}, d_{-i}^1\right).$$

[16] A simple but elegant argument was first used by BFK ([17, Proposition 7.2]) to state the existence of a belief-complete type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ where each type space is Polish and each S_i is a finite, discrete space. Such an argument can be easily adapted to our framework as follows. Every Lusin space is analytic, so it is the image of the Baire space $\mathbb{N}^{\mathbb{N}}$ under a continuous map ([22, Corollary 8.2.8]; see also [32, p. 85])). For given spaces of primitive uncertainty $(S_i)_{i\in I}$, let $T_i = \mathbb{N}^{\mathbb{N}}$, for each $i \in I$. The above result implies the existence of continuous belief maps β_i from T_i onto $\mathcal{N}(S_{-i} \times T_{-i})$. These maps give us a belief-complete lexicographic type structure.

- (inductive step: $k+1$, $k \geq 1$) Suppose we have already defined, for each $i \in I$, the functions $d_i^k : T_i \to \mathcal{N}\left(X_i^{k-1}\right)$ and $\psi_{-i}^k : S_{-i} \times T_{-i} \to X_i^k = X_i^{k-1} \times \mathcal{N}\left(X_{-i}^{k-1}\right)$. For each $i \in I$ and $t_i \in T_i$, define $d_i^{k+1} : T_i \to \mathcal{N}\left(X_i^k\right)$ by
$$d_i^{k+1}(t_i) = \widehat{\psi}_{-i}^k(\beta_i(t_i));$$
the map $\psi_{-i}^{k+1} : S_{-i} \times T_{-i} \to X_i^{k+1}$ is defined by
$$\psi_{-i}^{k+1} = \left(\psi_{-i}^k, d_{-i}^{k+1}\right),$$
so that $\psi_{-i}^{k+1} = \left(\mathrm{Id}_{S_{-i}}, d_{-i}^1, ..., d_{-i}^k, d_{-i}^{k+1}\right)$.

For each $i \in I$, the hierarchy description map $d_i : T_i \to H_i^0$ is defined as $d_i(t_i) = (d_i^1(t_i), d_i^2(t_i), ...)$, $t_i \in T_i$; the map $\psi_{-i} : S_{-i} \times T_{-i} \to S_{-i} \times H_{-i}^0$ is defined in a natural way as $\psi_{-i} = \left(\mathrm{Id}_{S_{-i}}, d_{-i}\right)$.

In [20], it is shown that each d_i is a measurable function, and it is continuous if each belief map is continuous. An analogous conclusion holds for the map $\widehat{\psi}_{-i} = \left(\widehat{\mathrm{Id}_{S_{-i}}, d_{-i}}\right) : \mathcal{N}(S_{-i} \times T_{-i}) \to \mathcal{N}\left(S_{-i} \times H_{-i}^0\right)$.

3.5 Type morphisms and universality

In what follows, given a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$, we denote by T the Cartesian product of type spaces, that is, $T = \Pi_{i \in I} T_i$.

Definition 4 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}' = \langle S_i, T_i', \beta_i' \rangle_{i \in I}$ be two $(S_i)_{i \in I}$-based lexicographic type structures. For each $i \in I$, let $\varphi_i : T_i \to T_i'$ be a measurable map such that*
$$\beta_i' \circ \varphi_i = \left(\widehat{\mathrm{Id}_{S_{-i}}, \varphi_{-i}}\right) \circ \beta_i.$$
Then the function $(\varphi_i)_{i \in I} : T \to T'$ is called ***type morphism (from $\mathcal{T}$ to $\mathcal{T}'$)****.*

The morphism is called ***type isomorphism*** *if the map $(\varphi_i)_{i \in I}$ is a Borel isomorphism. Say $\mathcal{T}$ and $\mathcal{T}'$ are* ***isomorphic*** *if there is a type isomorphism between them.*

The notion of type morphism captures the idea that a type structure $\mathcal{T}$ is "contained in" another type structure $\mathcal{T}'$ if $\mathcal{T}$ can be mapped into $\mathcal{T}'$ in a way that preserves the beliefs associated with types. Condition (2) in the definition of type morphism expresses consistency between the function $\varphi_i : T_i \to T_i'$ and the induced function $\left(\widehat{\mathrm{Id}_{S_{-i}}, \varphi_{-i}}\right) : \mathcal{N}(S_{-i} \times T_{-i}) \to \mathcal{N}\left(S_{-i} \times T_{-i}'\right)$. That is, the following diagram commutes:

$$\begin{array}{ccc} T_i & \xrightarrow{\beta_i} & \mathcal{N}(S_{-i} \times T_{-i}) \\ \Big\downarrow \varphi_i & & \Big\downarrow (\widehat{Id_{S_{-i}}, \varphi_{-i}}) \\ T_i' & \xrightarrow{\beta_i'} & \mathcal{N}(S_{-i} \times T_{-i}') \end{array} \tag{3.3}$$

The notion of type morphism does not make any reference to hierarchies of LPS's. But, as one should expect, the important property of type morphisms is that they preserve the explicit description of lexicographic belief hierarchies.

Proposition 3 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}' = \langle S_i, T_i', \beta_i' \rangle_{i \in I}$ be two $(S_i)_{i \in I}$-based lexicographic type structures. If $(\varphi_i)_{i \in I} : T \to T'$ is a type morphism from $\mathcal{T}$ to $\mathcal{T}'$, then $d_i(t_i) = d_i(\varphi_i(t_i))$ for each $t_i \in T_i$, $i \in I$.*

Proposition 3 clarifies the sense in which a type structure $\mathcal{T}$ can be regarded as "substructure" of $\mathcal{T}'$. In words, Proposition 3 states that if $\mathcal{T}$ can be mapped into $\mathcal{T}'$ *via* type morphism, then every $(S_i)_{i \in I}$-based belief hierarchy that is generated by some type in $\mathcal{T}$ is also generated by its image in $\mathcal{T}'$. Heifetz and Samet [31, Proposition 5.1] provide the above result for the case of standard type structures. Proposition 3 is indeed a straightforward generalization of Heifetz and Samet's result, and its proof relies on standard arguments.[17]

We now ask: Is there a type structure into which any other type structure can be mapped? Alternatively put, since a type structure generates hierarchies of LPS's, does there exist a type structure that generates all hierarchies of beliefs? A type structure satisfying this requirement is called universal.

Definition 5 *An $(S_i)_{i \in I}$-based type structure $\mathcal{T}' = \langle S_i, T_i', \beta_i' \rangle_{i \in I}$ is **universal** if for every other $(S_i)_{i \in I}$-based type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ there is a unique type morphism from $\mathcal{T}'$ to $\mathcal{T}$.*[18]

Not surprisingly (and in line with standard results on hierarchies of beliefs—cf. [38],[4]), the canonical type structure $\mathcal{T}_u$ turns out to be universal, as stated in the following

Theorem 1 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ be an arbitrary $(S_i)_{i \in I}$-based lexicographic type structure, and, for each $i \in I$, let $d_i : T_i \to H_i^0$ be the hierarchy description map. Then, for each $i \in I$,*

1. *$d_i(T_i) \subseteq H_i$,*
2. *$(d_i)_{i \in I}$ is the unique type morphism from $\mathcal{T}$ to $\mathcal{T}_u = \langle S_i, H_i, \overline{f}_i \rangle_{i \in I}$.*

   Thus $\mathcal{T}_u$ is a universal lexicographic type structure, and it is unique up to type isomorphism.

Note that, since any two universal type structures are isomorphic, and $\mathcal{T}_u$ is belief-complete, as immediate consequence of Theorem 1 we get:

Corollary 1 *Every universal type structure is belief-complete.*

The reverse implication of Corollary 1 does not hold, i.e., a belief-complete type structure is not necessarily universal. We will see (Section 6.3) that this has an implication for the epistemic analysis of IA.

[17]The statement of Proposition 3 can be rephrased by saying that every type morphism is also a *hierarchy morphism*, i.e., a map between type structures which preserves the hierarchies of beliefs associated with types. See [29] for a general analysis on the relationship between hirerarchy and type morphisms.

[18]Within the framework of category theory, $(S_i)_{i \in I}$-based type structures for player set I, as objects, and type morphisms, as morphisms, form a category. The "universal type structure" is a terminal object in the category of type structures.

4 Cautiousness, Assumption and Iterated Admissibility

4.1 Iterated Admissibility

Consider a finite game $G = \langle I, (S_i, u_i)_{i\in I}\rangle$, where I is a two-player set and, for every $i \in I$, S_i is the set of strategies with $|S_i| \geq 2$ and $u_i : S \to \mathbb{R}$ is the payoff function. Each strategy set S_i is given the obvious topology, i.e., the discrete topology. Define the expected payoff function π_i by extending u_i on $\mathcal{M}(S_i) \times \mathcal{M}(S_{-i})$ in the usual way:

$$\pi_i(\sigma_i, \sigma_{-i}) = \sum_{(s_i,s_{-i})\in S_i\times S_{-i}} \sigma_i(s_i)\sigma_{-i}(s_{-i})u_i(s_i, s_{-i}).$$

The notion of admissible strategy is standard.

Definition 6 *Fix a set $X_i \times X_{-i} \subseteq S_i \times S_{-i}$. A strategy $s_i \in S_i$ is **admissible with respect to** $X_i \times X_{-i}$ if and only if there exists $\sigma_{-i} \in \mathcal{M}(S_{-i})$ such that* $\mathrm{Supp}\sigma_{-i} = X_{-i}$ *and $\pi_i(s_i, \sigma_{-i}) \geq \pi_i(s'_i, \sigma_{-i})$ for every $s'_i \in X_i$. If strategy $s_i \in S_i$ is admissible with respect to $S_i \times S_{-i}$, we simply say that s_i is admissible.*

Remark 1 *Fix a set $X_i \times X_{-i} \subseteq S_i \times S_{-i}$. A strategy $s_i \in S_i$ is **weakly dominated with respect to** $X_i \times X_{-i}$ if there exists $\sigma_i \in \mathcal{M}(S_i)$ with $\sigma_i(X_i) = 1$ such that $\pi_i(\sigma_i, s_{-i}) \geq \pi_i(s_i, s_{-i})$ for every $s_{-i} \in X_{-i}$ and $\pi_i(\sigma_i, s'_{-i}) > \pi_i(s_i, s'_{-i})$ for some $s'_{-i} \in X_{-i}$. A standard result ([40, Lemma 4]) states that a strategy $s_i \in S_i$ is not weakly dominated with respect to $X_i \times X_{-i}$ if and only if it is admissible with respect to $X_i \times X_{-i}$.*

The set of iteratively admissible strategies (henceforth IA set) is defined inductively.

Definition 7 *For each $i \in I$, set $S_i^0 = S_i$ and for every $m \in \mathbb{N}$, let S_i^m be the set of all $s_i \in S_i^{m-1}$ which are admissible w.r.to $S_i^{m-1} \times S_{-i}^{m-1}$. A strategy $s_i \in S_i^m$ is called m-**admissible**. A strategy $s_i \in S_i^\infty = \cap_{m=0}^\infty S_i^m$ is called **iteratively admissible**.*

Note that $S_i^m \supseteq S_i^{m+1} \neq \emptyset$ for all $m \in \mathbb{N}$. Moreover, since each strategy set S_i is finite, there exists $M \in \mathbb{N}$ such that $\prod_{i\in I} S_i^\infty = \prod_{i\in I} S_i^M$. Consequently, the IA set $\prod_{i\in I} S_i^\infty$ is non-empty.

4.2 Rationality and Cautiousness

For any two vectors $x = (x_l)_{l=1}^n, y = (y_l)_{l=1}^n \in \mathbb{R}^n$, we write $x \geq_L y$ if either (1) $x_l = y_l$ for every $l \leq n$, or (2) there exists $m \leq n$ such that $x_m > y_m$ and $x_l = y_l$ for every $l < m$. Append to the game G a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$.

Definition 8 . *A strategy $s_i \in S_i$ is* ***optimal under*** *$\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n) \in \mathcal{N}(S_{-i} \times T_{-i})$ if*

$$\left(\pi_i(s_i, \text{marg}_{S_{-i}}\mu_i^l)\right)_{l=1}^n \geq_L \left(\pi_i(s_i', \text{marg}_{S_{-i}}\mu_i^l)\right)_{l=1}^n, \ \forall s_i' \in S_i.$$

We say that s_i is a ***lexicographic best reply to*** *$\overline{\text{marg}}_{S_{-i}}\beta_i(t_i)$ on S_{-i} if it is optimal under $\beta_i(t_i)$.*

Definition 9 *A type $t_i \in T_i$ is* ***cautious (in $\mathcal{T}$)*** *if $\overline{\text{marg}}_{S_{-i}}\beta_i(t_i) \in \mathcal{N}^+(S_{-i})$.*

Thus, for strategy-type pairs we define the following notions.

Definition 10 *Fix a strategy-type pair $(s_i, t_i) \in S_i \times T_i$.*

1. *Say (s_i, t_i) is* ***rational (in $\mathcal{T}$)*** *if s_i is optimal under $\beta_i(t_i)$.*
2. *Say (s_i, t_i) is* ***cautiously rational (in $\mathcal{T}$)*** *if it is rational and t_i is cautious. Let R_i^c be the set of all cautiously rational $(s_i, t_i) \in S_i \times T_i$.*
3. *Say (s_i, t_i) is* ***open-minded rational (in $\mathcal{T}$)*** *if it is rational and $\beta_i(t_i) \in \mathcal{N}^+(S_{-i} \times T_{-i})$.*

Open-minded Rationality is the notion of rationality employed by BFK, Dekel et al. [25], and Yang [44], and it includes a full-support requirement on types. BFK show ([17, Lemma 7.2]) that if a strategy-type pair (s_i, t_i) is open-minded rational, then s_i is admissibile. The following result states that an analogous conclusion holds for the weaker notion of Cautious Rationality.

Proposition 4 *If strategy-type pair $(s_i, t_i) \in S_i \times T_i$ is cautiously rational, then s_i is admissible.*

Proof: By definition, if $(s_i, t_i) \in R_i^c$, then s_i is a lexicographic best reply to $\overline{\text{marg}}_{S_{-i}}\beta_i(t_i) \in \mathcal{N}^+(S_{-i})$, where $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^k) \in \mathcal{N}(S_{-i} \times T_{-i})$. By [10, Proposition 1], to every $\overline{\text{marg}}_{S_{-i}}(\mu_i^1, ..., \mu_i^k) \in \mathcal{N}^+(S_{-i})$ there corresponds a probability measure $\nu_i \in \mathcal{M}(S_{-i})$, with $\text{Supp}\nu_i = S_{-i}$, such that $\pi_i(s_i, \nu_i) \geq \pi_i(s_i', \nu_i)$ for every $s_i' \in S_i$. ■

Furthermore, Cautious Rationality has a convenient invariance property under type morphisms between type structures. The following results state this formally.

Lemma 1 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ be lexicographic type structures, such that there exists a type morphism $(\varphi_i)_{i \in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$. Fix a type $t_i \in T_i$. Thus*

(i) $\overline{\mathrm{marg}}_{S_{-i}} \beta_i (t_i) \in \mathcal{N}^+ (S_{-i})$ *if and only if* $\overline{\mathrm{marg}}_{S_{-i}} \beta_i^* (\varphi_i (t_i)) \in \mathcal{N}^+ (S_{-i})$.

(ii) *A strategy-type pair (s_i, t_i) is rational in $\mathcal{T}$ if and only if $(s_i, \varphi_i (t_i))$ is rational in $\mathcal{T}^*$.*

Proof: Part (i): Let O be a non-empty, open subset of S_{-i}. Fix a type $t_i \in T_i$, and let $\beta_i (t_i) = \left(\beta_i^1 (t_i), ..., \beta_i^n (t_i)\right)$ be the associated LPS. If $\overline{\mathrm{marg}}_{S_{-i}} \beta_i (t_i) \in \mathcal{N}^+ (S_{-i})$, then there is $l \leq n$ such that $\beta_i^l (t_i) (O \times T_{-i}) > 0$. It follows from the definition of type morphism that $\overline{\mathrm{marg}}_{S_{-i}} \beta_i^* (\varphi_i (t_i)) \in \mathcal{N}^+ (S_{-i})$, since

$$\begin{aligned} (\beta_i^*)^l (\varphi_i (t_i)) \left(O \times T_{-i}^*\right) &= \beta_i^l (t_i) \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1} \left(O \times T_{-i}^*\right)\right) \\ &= \beta_i^l (t_i) (O \times T_{-i}) > 0. \end{aligned}$$

An analogous argument shows that the reverse implication is also true.

Part (ii): Pick an arbitrary $t_i \in T_i$. It is enough to show that

$$\overline{\mathrm{marg}}_{S_{-i}} \beta_i (t_i) = \overline{\mathrm{marg}}_{S_{-i}} \beta_i^* (\varphi_i (t_i)) .$$

But this follows from the fact that a type morphism preserves first-order beliefs (Proposition 3). Hence (s_i, t_i) is rational in $\mathcal{T}$ if and only $(s_i, \varphi_i (t_i))$ is rational in $\mathcal{T}^*$, as required. ■

Corollary 2 *A strategy-type pair (s_i, t_i) is cautiously rational in $\mathcal{T}$ if and only $(s_i, \varphi_i (t_i))$ is cautiously rational in $\mathcal{T}^*$.*

Note that an analogous invariance property does *not* hold for Open-minded Rationality. Consider for instance a finite type structure $\mathcal{T}$ and the canonical type structure $\mathcal{T}_u$.[19] For any full-support type t_i in $\mathcal{T}$, the corresponding type $d_i (t_i)$ has finite support too, so it cannot be a full support type in $\mathcal{T}_u$. Thus, open-mindedness does not represent a condition on the hierarchy of beliefs, whereas cautiousness captures full-support of first-order beliefs. Moreover, open-mindedness depends crucially on the topology of the type spaces. As a consequence, the invariance does not hold even when the two type structures are isomorphic (but not homeomorphic). The following example elaborates on this point further.

[19] As is shown in [20], a full-support type in $\mathcal{T}_u$ corresponds to a *full-support hierarchy*, i.e., a hierarchy with the property of all orders of beliefs being of full-support.

Example 1 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ be a symmetric type structure in which $T_i = \mathbb{Q} \cap [0,1]$. The set $\mathbb{Q} \cap [0,1]$ is endowed with the relative topology inherited from the Euclidean topology on $[0,1]$, and this makes each T_i a Lusin space, but not Polish. We can construct an isomorphic type structure $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ as follows. Let $T_i^* = \mathbb{Q} \cap [0,1]$ be given the discrete topology, so that T_i^* becomes a Polish space. So, T_i and T_i^* are Borel isomorphic (they originate the same Borel σ-field, namely the power set), but not homeomorphic. For each $i \in I$, let $\varphi_i : T_i \to T_i^*$ be the identity map. Moreover, each belief map β_i^* satisfies $\beta_i^* = \widehat{(\mathrm{Id}_{S_i}, \varphi_i)} \circ \beta_i \circ (\varphi_i)^{-1}$. It is easy to check that $(\varphi_i)_{i \in I}$ is a type isomorphism. Fix $t_i \in T_i$ such that $\beta_i(t_i) \in \mathcal{N}_1^+(S_{-i} \times T_{-i})$ and $\beta_i(t_i)\left(\left\{\left(s'_{-i}, 0\right)\right\}\right) = 0$ for some $s'_{-i} \in S_{-i}$. The set $\left\{\left(s'_{-i}, 0\right)\right\}$ is closed in T_i, but (cl)open in T_i^*. It turns out that $\beta_i^*(\varphi_i(t_i))\left(\left\{\left(s'_{-i}, 0\right)\right\}\right) = 0$, hence $\varphi_i(t_i) \in T_i^*$ is not a full-support type.*

4.3 Assumption

In this Section, we introduce our notion of Assumption, which is given a preference-based treatment in Appendix A. Here, for its operational convenience, we state the definition of Assumption in terms of LPS's.

Definition 11 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and a non-empty event $E \subseteq S_{-i} \times T_{-i}$. Fix also $t_i \in T_i$ with $\beta_i(t_i) = (\mu^1, ..., \mu^n)$. We say that E is* ***assumed under*** *$\beta_i(t_i)$* ***at level*** *$m \leq n$ if the following conditions hold:*

(i) *$\mu^l(E) = 1$ for all $l \leq m$;*

(ii) *for every elementary cylinder $C = \{s_{-i}\} \times T_{-i}$, if $E \cap C \neq \emptyset$ then $\mu^k(E \cap C) > 0$ for some $k \leq m$.*

We say that E is ***assumed under*** *$\beta_i(t_i)$ if it is assumed under $\beta_i(t_i)$ at some level $m \leq n$.*

We say that $t_i \in T_i$ ***assumes*** *E if E is assumed under $\beta_i(t_i)$.*

The notion of Assumption captures the idea that event E and its payoff-relevant components, viz. $E \cap C \neq \emptyset$, are infinitely more likely than not-E. Condition (i) in Definition 11 simply states that the player is (almost fully) confident in E. That is, the player thinks that E is infinitely more likely to occur than not-E. Condition (ii) adds a cautious attitude towards the event. That is, the player entertains the hypothesis that every payoff-relevant implication of E is infinitely more likely to occur than not-E. This is the same attitude that cautiousness reflects towards the whole space $S_{-i} \times T_{-i}$; note, indeed, that a type $t_i \in T_i$ is cautious if and only if t_i assumes $S_{-i} \times T_{-i}$ (cf. Remark A.1 in Appendix A).

Both attitudes can be properly formalized with the same preference-based notion of "infinitely more likely than". The notion of "infinitely more likely than" we adopt in this paper is (strictly) *weaker* than the one in Blume et al. [9], and it allows to keep the two different attitudes separated. Moreover, it is monotone and can be intuitively interpreted in terms of bets. Finally, it features other desirable properties, also in absence of mutual singularity and in presence of significant Savage-null events. We compare our approach with the one based on [9] in Appendix A.

We now discuss some important properties of Assumption.

Lemma 2 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. For type $t_i \in T_i$, let $\beta_i(t_i) = \overline{\mu}_i$.*

1. *If E and F are non-empty events in $S_{-i} \times T_{-i}$ which are assumed under $\overline{\mu}_i$ at level m, then $\mathrm{Proj}_{S_{-i}}(E) = \mathrm{Proj}_{S_{-i}}(F)$. If E and F are assumed under $\overline{\mu}_i$, then either $\mathrm{Proj}_{S_{-i}}(E) \subseteq \mathrm{Proj}_{S_{-i}}(F)$ or $\mathrm{Proj}_{S_{-i}}(F) \subseteq \mathrm{Proj}_{S_{-i}}(E)$.*
2. *Fix non-empty events $E_1, E_2, \ldots$ in $S_{-i} \times T_{-i}$. Suppose that, for each k, E_k is assumed under $\overline{\mu}_i$. Thus $\cap_k E_k$ and $\cup_k E_k$ are assumed under $\overline{\mu}_i$.*

Assumption satisfies one direction of conjunction as well as one direction of disjunction (Lemma 2.2). The failure of the other direction of conjunction reveals that, although the "infinitely more likely than" relation is monotone (cf. Property P2 in Appendix A), Assumption fails to satisfy monotonicity. That is, if E is assumed under $\overline{\mu}_i$, the same conclusion need not follow for an event F satisfying $E \subseteq F$. The reason why this can occur is that, if $E \subseteq F$, there may be some payoff-relevant components of $F \diagdown E$ which are not deemed infinitely more likely than not-F.[20] This can be illustrated by the following simple example.

Example 2 *Let $S_{-i} = \left\{s^1_{-i}, s^2_{-i}, s^3_{-i}\right\}$ and $T_{-i} = \left\{t^*_{-i}\right\}$. Consider the LPS $\overline{\mu}_i = (\mu^1_i, \mu^2_i) \in \mathcal{N}(S_{-i} \times T_{-i})$ with $\mu^1_i\left(\left\{\left(s^1_{-i}, t^*_{-i}\right)\right\}\right) = 1$ and $\mu^2_i\left(\left\{\left(s^2_{-i}, t^*_{-i}\right)\right\}\right) = \mu^2_i\left(\left\{\left(s^3_{-i}, t^*_{-i}\right)\right\}\right) = \frac{1}{2}$. Consider the events $E = \left\{s^1_{-i}\right\} \times T_{-i}$ and $F = \left\{s^1_{-i}, s^2_{-i}\right\} \times T_{-i}$. Clearly, $E \subseteq F$; however, E is assumed under $\overline{\mu}_i$ at level 1, while F is not assumed (indeed, $\mu^1_i(F) = 1$ and $\mu^2_i(F) = \frac{1}{2}$, and, with $m = 1$, Condition (ii) of Definition 11 is not satisfied for $C = \left\{s^2_{-i}\right\} \times T_{-i}$).*

For each player $i \in I$, let $\mathbf{A}_i : \Sigma_{S_{-i} \times T_{-i}} \to \Sigma_{S_i \times T_i}$ be the operator defined by

$$\mathbf{A}_i(E_{-i}) = \{(s_i, t_i) \in S_i \times T_i \,|\, t_i \text{ assumes } E_{-i}\}, \; E_{-i} \in \Sigma_{S_{-i} \times T_{-i}}.$$

Corollary D.1 in the Supplemental Appendix shows that the set $\mathbf{A}_i(E_{-i})$ is Borel in $S_i \times T_i$ for every event $E_{-i} \subseteq S_{-i} \times T_{-i}$; so the operator $\mathbf{A}_i : \Sigma_{S_{-i} \times T_{-i}} \to \Sigma_{S_i \times T_i}$ is well-defined.

The Assumption operator $\mathbf{A}_i$ has invariance properties under type morphisms between type structures which are analogous to the ones of (Cautious) Rationality (cf. Lemma 1 and Corollary 2).[21]

[20] So, in our version of Assumption, non-monotonicity hinges *only* on the "cautious attitude" towards the event (namely, Condition (ii) of Definition 11). In a related vein, the notion of *Strong Belief* ([5]) shares with Assumption a similar feature. Indeed, Strong Belief is based on a monotone likelihood relation between uncertain events (conditional probability-one belief), but it does not satisfy monotonicity (cf. [5, Section 3.2]; we thank Pierpaolo Battigalli for this observation). By contrast, in BFK's version of Assumption, non-monotonicity is *also* a consequence of a non-monotonicity property of the "infinitely more likely than" relation of Blume et al. [9] (see Appendix A, Proposition A.3).

[21] We thank an anonymous referee for suggesting to us the result stated in Lemma 3.

Lemma 3 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ and $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i\in I}$ be lexicographic type structures such that there exists a type morphism $(\varphi_i)_{i\in I} : T \to T^*$ from $\mathcal{T}$ to $\mathcal{T}^*$. Let $E_{-i} \subseteq S_{-i} \times T_{-i}$ and $E_{-i}^* \subseteq S_{-i} \times T_{-i}^*$ be non-empty events satisfying the following conditions:*

1) $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i}) \subseteq E_{-i}^*$;

2) $\mathrm{Proj}_{S_{-i}}(E_{-i}) = \mathrm{Proj}_{S_{-i}}\left(E_{-i}^*\right)$.

Then $\left(\mathrm{Id}_{S_i}, \varphi_i\right)(\mathbf{A}_i(E_{-i})) \subseteq \mathbf{A}_i\left(E_{-i}^*\right)$.

Corollary 3 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ and $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i\in I}$ be isomorphic type structures, and let $(\varphi_i)_{i\in I} : T \to T^*$ be the corresponding type isomorphism. Fix $t_i \in T_i$. Thus t_i assumes an event $E_{-i} \subseteq S_{-i} \times T_{-i}$ if and only if $\varphi_i(t_i)$ assumes $\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})$.*

Lemma 3 will play an instrumental role in the proof of our main results (Theorem 2 and Lemma 5). The implications of Corollary 3 will be discussed extensively in Section 6.

Our notion of Assumption is weaker than, and is implied by, the corresponding notions due to Dekel et al. [25] and, as for LCPS's, BFK and Yang [44]. A comparison between the notions of Assumption is deferred to the Discussion Section (Section 6.2). Here, we just stress that our version of Assumption is, in a precise sense, "topology-free": First, differently than in BFK and [44], it does not impose any full-support restriction on the LPS under which the event is assumed; second, Condition (ii) in Definition 11, differently from Condition (iii) in BFK and Dekel et al. [25] (cf. Section 6.2), does not depend on the topology of the type spaces. As mentioned earlier, also Cautiousness is topology-free. Therefore, all the results in the following sections do not depend on the topology of the type spaces.

5 Common Assumption of Cautious Rationality and the main result

We now provide an epistemic foundation of IA in "sufficiently rich" (i.e., belief-complete) type structures. In what follows, fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i\in I}$ and, for each player $i \in I$, let $R_i^1 = R_i^c$. For each $m > 1$, define R_i^m inductively by

$$R_i^{m+1} = R_i^m \cap \mathbf{A}_i\left(R_{-i}^m\right).$$

We write $R_i^0 = S_i \times T_i$ and $R_i^\infty = \cap_{m\in\mathbb{N}} R_i^m$ for each $i \in I$. If $(s_i, t_i)_{i\in I} \in \prod_{i\in I} R_i^{m+1}$, we say that there is **Cautious Rationality and mth-order Assumption of Cautious Rationality** ($\mathbf{R}^c m\mathbf{AR}^c$) at this state. If $(s_i, t_i)_{i\in I} \in \prod_{i\in I} R_i^\infty$, we say that there is **Cautious Rationality and Common Assumption of Cautious Rationality** ($\mathbf{R}^c\mathbf{CAR}^c$) at this state.

Note that, for each $m > 1$,

$$R_i^{m+1} = R_i^1 \cap \left(\cap_{l \le m} \mathbf{A}_i \left(R_{-i}^l\right)\right),$$

and each R_i^m is Borel in $S_i \times T_i$ (see Lemma D.5 in the Supplemental Appendix).

By an easy induction argument, using Corollary 2 and Corollary 3, we can claim:

Remark 2 *Let $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and $\mathcal{T}' = \langle S_i, T_i', \beta_i' \rangle_{i \in I}$ be $(S_i)_{i \in I}$-based type structures such that there exists a type isomorphism $(\varphi_i)_{i \in I} : T \to T'$ from $\mathcal{T}$ to $\mathcal{T}'$. Thus there is $R^c mAR^c$ (resp. $R^c CAR^c$) at state $(s_i, t_i)_{i \in I}$ if and only if there is $R^c mAR^c$ (resp. $R^c CAR^c$) at state $(s_i, \varphi_i(t_i))_{i \in I}$.*

We now state the main result of this paper.

Theorem 2 *Fix a belief-complete type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. The following statements hold:*

(i) *for each $m \ge 0$, $\prod_{i \in I} \mathrm{Proj}_{S_i}(R_i^m) = \prod_{i \in I} S_i^m$;*

(ii) *if $\mathcal{T}$ is universal, then $\prod_{i \in I} R_i^\infty \neq \emptyset$ and $\prod_{i \in I} \mathrm{Proj}_{S_i}(R_i^\infty) = \prod_{i \in I} S_i^\infty$.*

The proof of Theorem 2 will make use of the following results.

Lemma 4 *Fix a player $i \in I$. If $s_i \in S_i^m$, then there exists $\mu_{s_i} \in \mathcal{M}(S_{-i})$ such that* $\mathrm{Supp}\mu_{s_i} = S_{-i}^{m-1}$ *and*

$$\pi_i(s_i, \mu_{s_i}) \ge \pi_i(s_i', \mu_{s_i}), \forall s_i' \in S_i.$$

Lemma 5 *Fix the canonical type structure $\mathcal{T}_u = \langle S_i, H_i, \overline{f}_i \rangle_{i \in I}$. There exists a finite type structure $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ such that, for each $i \in I$ and each $m \ge 1$,*
(i) $\mathrm{Proj}_{S_i}(R_i^{*,m}) = S_i^m$,
(ii) $(\mathrm{Id}_{S_i}, d_i)(R_i^{*,m}) \subseteq R_i^m$.

For a proof of Lemma 4, see [17, Lemma E.1]. The proof of Lemma 5 is delegated to the next section.

Proof of Theorem 2: **Part (i)**: For $m = 0$ the statement is immediate. We show by induction that the statement holds for $m \ge 1$. One direction of the proof makes use of a selection argument; that is, for each $i \in I$ and each $m \ge 0$, there are maps $\rho_i^m : S_i \to T_i$ such that the maps $(\mathrm{Id}_{S_i}, \rho_i^m) : S_i \to S_i \times T_i$ satisfy $(\mathrm{Id}_{S_i}, \rho_i^m)(s_i) = (s_i, \rho_i^m(s_i)) \in R_i^m \setminus R_i^{m+1}$ provided $s_i \in S_i^m$.[22] (Of course, each map ρ_i^m is continuous, since strategy

[22] This implies that each map $(\mathrm{Id}_{S_i}, \rho_i^m)$ satisfies $\mathrm{Proj}_{S_i} \circ (\mathrm{Id}_{S_i}, \rho_i^m) = \mathrm{Id}_{S_i}$. Put differently, $(\mathrm{Id}_{S_i}, \rho_i^m)$ is a continuous selection of the correspondence $\mathrm{Proj}_{S_i}^{-1} : S_i \to 2^{R_i^m \setminus R_i^{m+1}}$.

sets are endowed with the discrete topology.) To this end, we first define, for each $i \in I$, the map $\rho_i^0 : S_i \to T_i$ as follows: fix some $t_i^0 \in T_i$ such that $\overline{\mathrm{marg}}_{S_{-i}}\beta_i(t_i^0) \notin \mathcal{N}^+(S_{-i})$ (such t_i^0 exists by belief-completeness), and let $\rho_i^0(s_i) = t_i^0$ for all $s_i \in S_i$. It follows that $(s_i, \rho_i^0(s_i)) \in R_i^0 \setminus R_i^1 \neq \emptyset$ for all $s_i \in S_i$, because $\rho_i^0(s_i) \in T_i$ is not cautious.

($m = 1$) Fix $i \in I$. Let $s_i \in \mathrm{Proj}_{S_i}(R_i^1)$, so that $(s_i, t_i) \in R_i^1$ for some $t_i \in T_i$. By Proposition 4, s_i is admissible, i.e., $s_i \in S_i^1$. This shows that $\mathrm{Proj}_{S_i}(R_i^1) \subseteq S_i^1$.

Conversely, let $s_i \in S_i^1$. So there is a probability measure $\mu_i^1 \in \mathcal{M}(S_{-i})$, with $\mathrm{Supp}\mu_i^1 = S_{-i}$, such that s_i is a (lexicographic) best reply to μ_i^1. Let $\iota : S_{-i} \to S_{-i} \times T_{-i}$ be the function defined by $\iota(s_{-i}) = \left(s_{-i}, \rho_{-i}^0(s_{-i})\right)$, $s_{-i} \in S_{-i}$, where $\rho_{-i}^0 : S_{-i} \to T_{-i}$ is the (constant) function we previously defined. The map ι is continuous and is such that $\iota(s_{-i}) \in R_{-i}^0 \setminus R_{-i}^1$ for all $s_{-i} \in S_{-i}$. Hence the pushforward measure $\widetilde{\iota}(\mu_i^1) \in \mathcal{M}(S_{-i} \times T_{-i})$ is well-defined, and satisfies $\widetilde{\iota}(\mu_i^1)\left(R_{-i}^1\right) = 0$; moreover $\overline{\mathrm{marg}}_{S_{-i}}\widetilde{\iota}(\mu_i^1) = \mu_i^1 \in \mathcal{N}_1^+(S_{-i})$. By belief-completeness, there is $t_{s_i}^1 \in T_i$ such that $\beta_i\left(t_{s_i}^1\right) = \widetilde{\iota}(\mu_i^1)$. Clearly, $\left(s_i, t_{s_i}^1\right) \in R_i^1 \setminus R_i^2$, and this shows that $S_i^1 \subseteq \mathrm{Proj}_{S_i}(R_i^1 \setminus R_i^2)$, and so $S_i^1 \subseteq \mathrm{Proj}_{S_i}(R_i^1)$.

By arbitrariness of $i \in I$, it follows that $\prod_{i \in I} \mathrm{Proj}_{S_i}(R_i^1) = \prod_{i \in I} S_i^1$. We can conclude the proof of the basis step by defining a profile of continuous maps $(\rho_i^1 : S_i \to T_i)_{i \in I}$ as follows: for each $i \in I$,

$$\rho_i^1(s_i) = \begin{cases} t_{s_i}^1, & \text{if } s_i \in S_i^1, \\ \rho_i^0(s_i), & \text{if } s_i \in S_i \setminus S_i^1. \end{cases}$$

It turns out that $(s_i, \rho_i^1(s_i)) \in R_i^1 \setminus R_i^2$ whenever $s_i \in S_i^1$, as required.

($m \geq 2$) Suppose that the statement has been shown to hold for all $l = 1, ..., m-1$, and that, for each $i \in I$ and $l \leq m-1$, we have shown the existence of continuous maps $\rho_i^l : S_i \to T_i$ satisfying $\left(s_i, \rho_i^l(s_i)\right) \in R_i^l \setminus R_i^{l+1}$ for all $s_i \in S_i^l$. We show that the statement is true for $l = m$.

Fix a player $i \in I$, and let $s_i \in \mathrm{Proj}_{S_i}(R_i^m)$, so that $(s_i, t_i) \in R_i^m$ for some $t_i \in T_i$. It follows from the definition of R_i^m that $(s_i, t_i) \in R_i^{m-1}$, so, by the induction hypothesis, $s_i \in S_i^m$. Also, R_{-i}^{m-1} is assumed under $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n)$ at some level $k \leq n$, hence

$$\cup_{l \leq k} \mathrm{Supp}\mathrm{marg}_{S_{-i}}\mu_i^l = \mathrm{Proj}_{S_{-i}}\left(R_{-i}^{m-1}\right) = S_{-i}^{m-1},$$

where the first equality follows from Lemma B.2 and the second equality follows from the induction hypothesis. So we can form a nested convex combination of the measures $\mathrm{marg}_{S_{-i}}\mu_i^l$, for $l = 1, ..., k$, to get a probability measure $\nu_i \in \mathcal{M}(S_{-i})$, with $\mathrm{Supp}\nu_i = S_{-i}^{m-1}$, such that s_i is a best reply to ν_i (see [10, Proposition 1]). Thus, s_i is admissible w.r.to $S_i \times S_{-i}^{m-1}$, and a fortiori w.r.to $S_i^{m-1} \times S_{-i}^{m-1}$. Hence $\mathrm{Proj}_{S_i}(R_i^m) \subseteq S_i^m$.

Conversely, let $s_i \in S_i^m$. By Lemma 4, it follows that, for all $l = 1, ..., m$, there is $\nu_i^l \in \mathcal{M}(S_{-i})$, with $\mathrm{Supp}\nu_i^l = S_{-i}^{l-1}$, for which s_i is a best reply among all strategies in S_i. We now show the existence of an LPS $\overline{\mu}_i = (\mu_i^1, ..., \mu_i^m) \in \mathcal{N}(S_{-i} \times T_{-i})$ such that

(a) $\mathrm{marg}_{S_{-i}}\mu_i^l = \nu_i^{m+1-l}$ for each $l = 1, ..., m$; and

(b) R_{-i}^{m-l} is assumed under $\overline{\mu}_i$ at level l for each $l = 1, ..., m-1$, while R_{-i}^m is not assumed.

To this end, we use the fact that we have already shown the existence of functions $\rho_i^0(\cdot)$,...,$\rho_i^{m-1}(\cdot)$ (induction hypothesis) for each $i \in I$. We construct probability measures $\mu_i^l \in \mathcal{M}(S_{-i} \times T_{-i})$ as follows:

$$\mu_i^l = \widetilde{\left(\mathrm{Id}_{S_{-i}}, \rho_{-i}^{m-l}\right)}\left(\nu_i^{m+1-l}\right), \forall l \in \{1, ..., m\}.$$

Let $\overline{\mu}_i \in \mathcal{N}(S_{-i} \times T_{-i})$ be the concatenation of those measures, i.e., $\overline{\mu}_i = (\mu_i^1, ..., \mu_i^m)$. It readily follows that $\overline{\mu}_i$ satisfies property (a), since $\mathrm{Proj}_{S_{-i}} \circ \left(\mathrm{Id}_{S_{-i}}, \rho_{-i}^{m-l}\right) = \mathrm{Id}_{S_{-i}}$ for all $l \in \{1, ..., m\}$. We now show that also property (b) holds. Using the properties of the functions $\rho_{-i}^{m-l}(\cdot)$ specified above, we get that, for all $l \in \{1, ..., m\}$,

$$\mu_i^l\left(R_{-i}^{m-l} \setminus R_{-i}^{m+1-l}\right) = \nu_i^{m+1-l}\left(\left(\mathrm{Id}_{S_{-i}}, \rho_{-i}^{m-l}\right)^{-1}\left(R_{-i}^{m-l} \setminus R_{-i}^{m+1-l}\right)\right) = \nu_i^{m+1-l}\left(S_{-i}^{m-l}\right) = 1. \tag{5.1}$$

Specifically, for $l = 1$ this yields

$$\begin{aligned} \mu_i^1\left(R_{-i}^{m-1} \setminus R_{-i}^m\right) &= 1, \\ \mu_i^1\left(R_{-i}^m\right) &= 0, \end{aligned}$$

hence condition (i) of Assumption is satisfied for R_{-i}^{m-1}, while R_{-i}^m cannot be assumed under $\overline{\mu}_i$. Moreover, note that

$$\mathrm{Suppmarg}_{S_{-i}}\mu_i^1 = \mathrm{Supp}\nu_i^m = S_{-i}^{m-1}.$$

By the induction hypothesis, $S_{-i}^{m-1} = \mathrm{Proj}_{S_{-i}} R_i^{m-1}$, hence Condition (ii)' of Lemma B.2 holds. Analogous arguments show that all the conditions of Assumption are satisfied for R_{-i}^{m-l} at level l for each $l = 2, ..., m-1$. This shows that property (b) is satisfied.[23]

It now follows from belief-completeness that there is $t_{s_i}^m \in T_i$ such that $\beta_i\left(t_{s_i}^m\right) = (\mu_i^1, ..., \mu_i^m)$; this implies $\left(s_i, t_{s_i}^m\right) \in R_i^m \setminus R_i^{m+1}$, hence $s_i \in \mathrm{Proj}_{S_i}\left(R_i^m \setminus R_i^{m+1}\right)$, and a fortiori $s_i \in \mathrm{Proj}_{S_i}\left(R_i^m\right)$.

By arbitrariness of $i \in I$, it follows that $\prod_{i \in I} \mathrm{Proj}_{S_i}\left(R_i^m\right) = \prod_{i \in I} S_i^m$. To conclude the proof of the inductive step, it remains to define a profile of continuous maps $(\rho_i^m : S_i \to T_i)_{i \in I}$. This is done by letting, for each $i \in I$,

$$\rho_i^m(s_i) = \begin{cases} t_{s_i}^m, & \text{if } s_i \in S_i^m, \\ \rho_i^0(s_i), & \text{if } s_i \in S_i \setminus S_i^m. \end{cases}$$

Clearly, each map $\rho_i^m : S_i \to T_i$ satisfies $(s_i, \rho_i^m(s_i)) \in R_i^m \setminus R_i^{m+1}$ whenever $s_i \in S_i^m$.

Part (ii): We can assume, without loss of generality, that $\mathcal{T}$ is the canonical type structure (Corollary 1). Then, by Lemma 5, there exists a finite type structure $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ such that, for each $i \in I$ and each $m \geq 1$,

(a) $\mathrm{Proj}_{S_i}\left(R_i^{*,m}\right) = S_i^m$,

(b) $(\mathrm{Id}_{S_i}, d_i)\left(R_i^{*,m}\right) \subseteq R_i^m$.

Since $(R_i^{*,m})_{m \in \mathbb{N}}$ is a weakly decreasing sequence of finite sets, there exists $N \in \mathbb{N}$ such that $R_i^{*,N} = R_i^{*,\infty}$. So, it follows from (a) that $\mathrm{Proj}_{S_i}\left(R_i^{*,\infty}\right) = S_i^\infty$. Then, for every $s_i \in S_i^\infty$, there exists $t_i \in T_i^*$ such that $(s_i, t_i) \in R_i^{*,m}$ for all $m \in \mathbb{N}$. It thus follows from (b) that $(\mathrm{Id}_{S_i}, d_i)((s_i, t_i)) \in R_i^m$ for all $m \in \mathbb{N}$. Hence $(\mathrm{Id}_{S_i}, d_i)((s_i, t_i)) \in R_i^\infty$. Consequently $S_i^\infty \subseteq \mathrm{Proj}_{S_i}\left(R_i^\infty\right) \neq \emptyset$. By Part (i) of the theorem, $\mathrm{Proj}_{S_i}\left(R_i^\infty\right) \subseteq S_i^\infty$. The conclusion follows. ■

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[23] Moreover, since the sets $\left(R_{-i}^l\right)_{l \in \{1,...,m\}}$ are monotonically (weakly) decreasing, then Eq. (5.1) yields $\mu_i^l\left(R_{-i}^{m-1}\right) = 0$ for all $l \in \{2, ..., m\}$, and this shows that condition (iii) of BFK-Assumption (see Definition 12 below) is satisfied for R_{-i}^{m-1}.

5.1 Construction of the finite type structure $\mathcal{T}^*$ and the proof of Lemma 5

Let M be the smallest natural number such that $\prod_{i\in I} S_i^{\infty} = \prod_{i\in I} S_i^M$. By Lemma 4, it follows that, for every $n \in \{1, ..., M+1\}$ and $s_i \in S_i^n$, there exists $\mu_{s_i}^n \in \mathcal{M}(S_{-i})$ such that $\text{Supp}\mu_{s_i}^n = S_{-i}^{n-1}$ and

$$\pi_i(s_i, \mu_{s_i}^n) \geq \pi_i(s_i', \mu_{s_i}^n), \forall s_i' \in S_i.$$

We use this result to construct a finite type structure $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i\in I}$ as follows.

For each $i \in I$ and $k \in \{0, 1, ..., M+1\}$, define sets T_i^k as follows:

$$T_i^k = S_i^k \times \{k\};$$

in other words, each T_i^k is a homeomorphic copy of S_i^k, but all the T_i^k's are pairwise disjoint sets. In particular, note that, since $S_i^M = S_i^{M+1}$, then both T_i^M and T_i^{M+1} are homeomorphic copies of S_i^M.

For each $i \in I$, type spaces T_i^* are defined by

$$T_i^* = \bigcup_{k\in\{0,1,...,M+1\}} T_i^k.$$

In what follows, we will denote a type $t_i \in T_i^k$ by (s_i, k).

Next, for each $i \in I$, belief maps $\beta_i^* : T_i^* \to \mathcal{N}(S_{-i} \times T_{-i}^*)$ are defined by an inductive procedure which specifies, for all $k \in \{0, 1, ..., M+1\}$, the properties of the LPS's associated with each $(s_i, k) \in T_i^k$.

($k = 0$) For each $i \in I$ and $s_i \in S_i^0$, let $\beta_i^*((s_i, 0))$ be any probability measure on $S_{-i} \times T_{-i}^*$ such that

$$\text{Suppmarg}_{S_{-i}}\beta_i^*((s_i, 0)) \neq S_{-i}.$$

($k = 1$) For each $i \in I$ and $s_i \in S_i^1$, let $\beta_i^*((s_i, 1))$ be any probability measure $\nu_{s_i}^1$ on $S_{-i} \times T_{-i}^*$ such that

$$\nu_{s_i}^1 (\{(s_{-i}, (s_{-i}, 0))\}) = \mu_{s_i}^1 (\{s_{-i}\}), \forall s_{-i} \in S_{-i},$$

which implies $\nu_{s_i}^1 \left(S_{-i} \times T_{-i}^0\right) = 1$.

($1 < k \leq M$) Suppose we have already defined, for each $i \in I$ and $s_i \in S_i^{k-1}$, the LPS $\beta_i^*((s_i, k-1)) \in \mathcal{N}_{k-1}(S_{-i} \times T_{-i}^*)$. Thus, for each $i \in I$ and $s_i \in S_i^k$, define $\beta_i^*((s_i, k)) \in \mathcal{N}_k(S_{-i} \times T_{-i}^*)$ by

$$\beta_i^*((s_i, k)) = \left(\nu_{s_i}^k, \beta_i^*((s_i, k-1))\right),$$

where $\nu_{s_i}^k \in \mathcal{M}(S_{-i} \times T_{-i}^*)$ satisfies

$$\nu_{s_i}^k (\{(s_{-i}, (s_{-i}, k-1))\}) = \mu_{s_i}^k (\{s_{-i}\}), \forall s_{-i} \in S_{-i}^{k-1},$$

so that $\nu_{s_i}^k \left(S_{-i} \times T_{-i}^{k-1}\right) = 1$.

($k = M+1$) For each $i \in I$ and $s_i \in S_i^M = S_i^{M+1}$, define $\beta_i^*((s_i, M+1)) \in \mathcal{N}_{M+1}(S_{-i} \times T_{-i}^*)$ by

$$\beta_i^*((s_i, M+1)) = \left(\nu_{s_i}^{M+1}, \beta_i^*((s_i, M))\right),$$

where $\nu_{s_i}^{M+1} \in \mathcal{M}(S_{-i} \times T_{-i}^*)$ satisfies

$$\nu_{s_i}^{M+1}\left(\{(s_{-i}, (s_{-i}, M+1))\}\right) = \mu_{s_i}^{M+1}\left(\{s_{-i}\}\right), \forall s_{-i} \in S_{-i}^M.$$

so that $\nu_{s_i}^{M+1}\left(S_{-i} \times T_{-i}^{M+1}\right) = 1$.

Observe that for all $k \in \{1, ..., M+1\}$:

- $\beta_i^*((s_i, k)) = \left(\nu_{s_i}^k, \nu_{s_i}^{k-1}, ..., \nu_{s_i}^1\right)$;
- $\mathrm{marg}_{S_{-i}} \nu_{s_i}^k = \mu_{s_i}^k$;
- $\mathrm{Supp}\nu_{s_i}^k \subseteq S_{-i} \times T_{-i}^{k-1}$ for $k < M+1$ and $\mathrm{Supp}\nu_{s_i}^{M+1} \subseteq S_{-i} \times T_{-i}^{M+1}$;
- for every $t_i \in T_i^k$, $\beta_i^*(t_i)$ is a length-k LPS, and for every $t_i \in T_i^0$, $\beta_i^*(t_i)$ is a length-1 LPS.

For each $i \in I$, define the following sets:

$$\begin{aligned}
\Delta_{S_i^k \times T_i^k} &= \{(s_i, t_i) \in S_i \times T_i^* | t_i = (s_i, k)\}, \forall k \in \{0, ..., M\}, \\
\Delta_{S_i^M \times T_i^{M+1}} &= \{(s_i, t_i) \in S_i \times T_i^* | t_i = (s_i, M+1)\}.
\end{aligned}$$

That is, each set $\Delta_{S_i^k \times T_i^k}$ is homeomorphic to the diagonal of $S_i^k \times S_i^k$;[24] thus, each measure of an LPS $\beta_i^*((s_i, k)) = \left(\nu_{s_i}^k, \nu_{s_i}^{k-1}, ..., \nu_{s_i}^1\right)$ is concentrated on those "diagonal" sets, namely

$$\begin{aligned}
\mathrm{Supp}\nu_{s_i}^k &= \Delta_{S_{-i}^{k-1} \times T_{-i}^{k-1}}, \forall k \in \{1, ..., M\}, \\
\mathrm{Supp}\nu_{s_i}^{M+1} &= \Delta_{S_{-i}^M \times T_{-i}^{M+1}}.
\end{aligned}$$

The remainder of this Section is devoted to show that $\mathcal{T}^*$ satisfies the requirements of Lemma 5. To this end, we first record, for future reference, two properties of the type structure $\mathcal{T}^*$.

Claim 1 *For all $k \in \{1, ..., M+1\}$, each type $(s_i, k) \in T_i^k$ is cautious.*

[24] The diagonal of $S_i^k \times S_i^k$ is the set

$$\left\{(s_i, s_i') \in S_i^k \times S_i^k \,|\, s_i = s_i'\right\},$$

which is homeomorphic to $\Delta_{S_i^k \times T_i^k}$ under the coordinate projection $\mathrm{Proj}_{S_i^k \times S_i^k} : \Delta_{S_i^k \times T_i^k} \to S_i^k \times S_i^k$. Furthermore, the diagonal of $S_i^k \times S_i^k$ is homeomorphic to S_i^k.

Proof: For all $k \in \{1, ..., M+1\}$, each type $(s_i, k) \in T_i^k$ is associated with the LPS $\beta_i^*((s_i, k)) = \left(\nu_{s_i}^k, \nu_{s_i}^{k-1}, ..., \nu_{s_i}^1\right)$, with $\nu_{s_i}^1$ satisfying

$$\mathrm{Suppmarg}_{S_{-i}} \nu_{s_i}^1 = \mathrm{Supp}\mu_{s_i}^1 = S_{-i}.$$

$\square$

The following property of $\mathcal{T}^*$ is not crucial for the proof of the main result; however, it shows that such finite type structure can be used to characterize IA under stronger notions of Assumption. Specifically, Assumption could be strenghtened with Condition (iii) in the definition of "BFK-Assumption" for LCPS—see Section 6.2, Definition 12.

Claim 2 *The type structure $\mathcal{T}^*$ is an LCPS type structure. Furthermore, for each $t_i \in T_i^*$, the induced hierarchy $d_i(t_i) = (d_i^1(t_i), d_i^2(t_i), ...)$ is such that*

$$d_i^k(t_i) \in \mathcal{L}\left(X_i^{k-1}\right), \forall k \geq 2.$$

Proof: For $k = 0, 1$, every type $(s_i, k) \in T_i^k$ is associated with a probability measure, hence the result is trivially true. So, pick any $(s_i, k) \in T_i^k$ with $k \geq 2$. Then $\beta_i^*((s_i, k)) = \left(\nu_{s_i}^k, \nu_{s_i}^{k-1}, ..., \nu_{s_i}^1\right)$ has the property that $\mathrm{Supp}\nu_{s_i}^l \subseteq S_{-i} \times T_{-i}^{l-1}$ for all $l < k$, while $\mathrm{Supp}\nu_{s_i}^k \subseteq S_{-i} \times T_{-i}^k$ if $k = M+1$, and $\mathrm{Supp}\nu_{s_i}^k \subseteq S_{-i} \times T_{-i}^{k-1}$ otherwise. Since all the sets T_i^k's are pairwise disjoint, then $\beta_i^*((s_i, k))$ is mutually singular.

To prove the second statement, it is enough to show that $d_i^2(t_i) \in \mathcal{L}(S_{-i} \times \mathcal{N}(S_i))$. This will imply $d_i^k(t_i) \in \mathcal{L}\left(X_i^{k-1}\right)$ for all $k \geq 3$ (see [20]). As above, pick any $(s_i, k) \in T_i^k$ for which $k \geq 2$, and recall that $\beta_i^*((s_i, k)) = \left(\nu_{s_i}^k, \nu_{s_i}^{k-1}, ..., \nu_{s_i}^1\right)$. Thus

$$\begin{aligned} d_i^2((s_i, k)) &= \widehat{\psi}_{-i}^1(\beta_i^*((s_i, k))) \\ &= \left(\widetilde{\psi}_{-i}^1\left(\nu_{s_i}^k\right), \widetilde{\psi}_{-i}^1\left(\nu_{s_i}^{k-1}\right), ..., \widetilde{\psi}_{-i}^1\left(\nu_{s_i}^1\right)\right), \end{aligned}$$

where $\psi_{-i}^1 = \left(\mathrm{Id}_{S_{-i}}, d_{-i}^1\right)$. Let $(E_l)_{l=1}^k$ be Borel sets in $S_{-i} \times \mathcal{N}(S_i)$ such that $E_l = S_{-i} \times \mathcal{N}_l^+(S_i)$ for all $l = 1, ..., k$. Clearly, the sets $(E_l)_{l=1}^k$ are pairwise disjoint. By Claim 1 and by construction, it turns out that, for all $l = 2, ..., k-1$,

$$\begin{aligned} \widetilde{\psi}_{-i}^1\left(\nu_i^l\right)(E_{l-1}) &= \nu_{s_i}^l\left(\left(\mathrm{Id}_{S_{-i}}, d_{-i}^1\right)^{-1}\left(S_{-i} \times \mathcal{N}_{l-1}^+(S_i)\right)\right) \\ &= \nu_{s_i}^l\left(S_{-i} \times T_{-i}^{l-1}\right) \\ &= 1, \end{aligned}$$

where the last equality follows from the fact that $\mathrm{Supp}\nu_{s_i}^l \subseteq S_{-i} \times T_{-i}^{l-1}$. An analogous argument for $l = k$ shows that $\widetilde{\psi}_{-i}^1\left(\nu_{s_i}^k\right)(E_{k-1}) = 1$ if $k \leq M$, while $\widetilde{\psi}_{-i}^1\left(\nu_{s_i}^{M+1}\right)(E_{M+1}) = 1$. To complete the proof, observe that $\widetilde{\psi}_{-i}^1\left(\nu_{s_i}^1\right)(S_{-i} \times (\mathcal{N}(S_i) \backslash \mathcal{N}^+(S_i))) = 1$ because $\nu_{s_i}^1\left(S_{-i} \times T_{-i}^0\right) = 1$ and types in T_{-i}^0 are not cautious. This shows that $d_i^2(t_i) \in \mathcal{L}(S_{-i} \times \mathcal{N}(S_i))$, as required. $\square$

The following result establishes the main properties of the sets of states (i.e., strategy-type profiles) $(s_i, t_i)_{i \in I} \in S \times T^*$ consistent with $\mathrm{R}^{\mathrm{c}}m\mathrm{AR}^{\mathrm{c}}$.

Claim 3 *Fix the type structure $\mathcal{T}^*$. For each player $i \in I$, the following properties hold:*

1. *for all $k \in \{1, ..., M\}$,*

$$\beta_i^*((s_i, k)) \left(R_{-i}^{*,k}\right) = \overline{0}, \ s_i \in S_i^k;$$

2. *for all $k \in \{0, 1, ..., M\}$,*

$$\Delta_{S_i^k \times T_i^k} \subseteq R_i^{*,k} \diagdown R_i^{*,k+1};$$

3. $\Delta_{S_i^M \times T_i^{M+1}} \subseteq R_i^{*,M+1}$*;*

4. *for all $l \geq 1$, $R_i^{*,M+1} = R_i^{*,M+l}$.*

The proof of Claim 3, although simple, is provided in Appendix B because some of the details are rather tedious. Here, we just remark that, in parts 2-3 of Claim 3, the sets $\left(\Delta_{S_i^k \times T_i^k}\right)_{k=0,...,M}$ and $\Delta_{S_i^M \times T_i^{M+1}}$ are included in, but not necessarily equal to, $\left(R_i^{*,k} \diagdown R_i^{*,k+1}\right)_{k=0,...,M}$ and $R_i^{*,M+1}$, respectively. The following is an example in which strict inclusion occurs. Suppose there are *distinct* $s_i, s_i' \in S_i^1$ such that the measures $\mu_{s_i}^1$ and $\mu_{s_i'}^1$—which exist by Lemma 4—satisfy $\mu_{s_i}^1 = \mu_{s_i'}^1$. Then, clearly $(s_i, (s_i', 1)) \in R_i^{*,1} \diagdown R_i^{*,2}$, but $(s_i, (s_i', 1)) \notin \Delta_{S_i^1 \times T_i^1}$.

We now show that R^{c}CAR^c holds in $\mathcal{T}^*$, and epistemically justifies the IA set.

Claim 4 *Fix the type structure $\mathcal{T}^*$. Thus, for each player $i \in I$, the following statements hold true:*

(1) *for each $k \geq 0$,* $\mathrm{Proj}_{S_i}\left(R_i^{*,k}\right) = S_i^k$*;*

(2) $\mathrm{Proj}_{S_i}\left(R_i^{*,\infty}\right) = S_i^M$*;*

(3) *for each $k \in \{0, 1, ..., M\}$,* $\mathrm{Proj}_{S_i}\left(R_i^{*,k}\right) = \mathrm{Proj}_{S_i}\left(R_i^{*,k} \setminus R_i^{*,k+1}\right)$.

Proof: Part (1): The inclusion $\mathrm{Proj}_{S_i}\left(R_i^{*,k}\right) \subseteq S_i^k$ follows from the same arguments as those in the proof of part (i) of Theorem 2. On the other hand, it follows from Claim 3 that $\Delta_{S_i^k \times T_i^k} \subseteq R_i^{*,k}$ for all $k \geq 0$. Hence $S_i^k \subseteq \mathrm{Proj}_{S_i}\left(R_i^{*,k}\right)$ for all $k \geq 0$.

Part (2): Claim 3.4 entails $R_i^{*,\infty} = \cap_{m \geq M+1} R_i^{*,m} = R_i^{*,M+1} \neq \emptyset$ for each $i \in I$. So, it readily follows from part (1) that $\mathrm{Proj}_{S_i}\left(R_i^{*,\infty}\right) = S_i^M$.

Part (3): Claim 3.2 yields

$$\mathrm{Proj}_{S_i}\left(R_i^{*,k}\right) = S_i^k \subseteq \mathrm{Proj}_{S_i}\left(R_i^{*,k} \setminus R_i^{*,k+1}\right), \forall k \in \{0, 1, ..., M\}.$$

The opposite inclusion trivially holds. $\square$

Note that Claim 4.(3) shows that the sets $\left(R_i^{*,k}\right)_{k\geq 0}$ stop shrinking at step $k = M+1$ (indeed, $R_i^{*,M+1} = R_i^{*,M+l}$ for all $l \geq 1$, by Claim 3.4). By way of constrast, in a belief-complete type structure, these sets keep shrinking forever. This can be easily seen in one direction of the proof of part (i) of Theorem 2, in which it is shown that $R_i^m \setminus R_i^{m+1} \neq \emptyset$ for all $m \geq 0$.[25]

Finally, we show that

Claim 5 *Fix type structure $\mathcal{T}^*$ and the canonical type structure $\mathcal{T}_u$. Thus, for each player $i \in I$,*

$$(\mathrm{Id}_{S_i}, d_i)\left(R_i^{*,k}\right) \subseteq R_i^k,\ \forall k \geq 1.$$

Proof: By induction on k.

($k = 1$) If $(s_i, t_i) \in R_i^{*,1}$, then $(s_i, d_i(t_i)) \in R_i^1$ by Lemma 1.

($k \geq 2$) Suppose that the statement is true for $k-1$. Let $(s_i, t_i) \in R_i^{*,k}$. So $(s_i, t_i) \in R_i^{*,k-1}$ and, by the induction hypothesis, $(s_i, d_i(t_i)) \in R_i^{k-1}$. So we need to show that $(s_i, d_i(t_i)) \in \mathbf{A}_i\left(R_{-i}^{k-1}\right)$; this will imply $(s_i, d_i(t_i)) \in R_i^k$, as required.

Now note that

(a) $\left(\mathrm{Id}_{S_{-i}}, d_{-i}\right)\left(R_{-i}^{*,k-1}\right) \subseteq R_{-i}^{k-1}$,

(b) $\mathrm{Proj}_{S_{-i}}\left(R_{-i}^{*,k-1}\right) = \mathrm{Proj}_{S_{-i}}\left(R_{-i}^{k-1}\right)$.

Part (a) is the induction hypothesis, while part (b) follows from Claim 4.(1) and part (i) of Theorem 2. Since $(s_i, t_i) \in \mathbf{A}_i\left(R_{-i}^{*,k-1}\right)$, then Lemma 3 yields $(s_i, d_i(t_i)) \in \mathbf{A}_i\left(R_{-i}^{k-1}\right)$. □

We can conclude this Section by providing the proof of Lemma 5.

Proof of Lemma 5: Immediate from Claim 4.(1) and Claim 5. ■

6 Discussion

6.1 Weak Belief and Permissibility

A weaker concept than Assumption (and Full Belief) of an event is that of *Weak Belief*. Formally, an event E is weakly believed if E is "infinitely more likely than" not-E. As shown in Appendix A, this requires that $\mu^1(E) = 1$ for a given LPS $(\mu^1, ..., \mu^n)$. Using this condition on LPS's, Brandenburger [13] introduced the solution concept of *Permissibility*, and showed its equivalence to the Dekel-Fudenberg procedure where one round of elimination of inadmissible strategies is followed by iterated elimination of strictly dominated strategies ([23]; see also [12], [7]).

[25] Moreover, it can be deduced from Theorem 2.(i) that $\mathrm{Proj}_{S_i}(R_i^m) = \mathrm{Proj}_{S_i}\left(R_i^m \setminus R_i^{m+1}\right)$ for all $m \geq 0$ (cf. [17, Lemma E.3]).

Permissibility in finite games can be given an epistemic foundation in our framework. To see this, fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. For each player $i \in I$, one can define an operator $\mathbf{WB}_i : \Sigma_{S_{-i} \times T_{-i}} \to \Sigma_{S_i \times T_i}$ as follows:

$$\mathbf{WB}_i (E_{-i}) = \{(s_i, t_i) \in S_i \times T_i \,|t_i \text{ weakly believes } E_{-i}\}, \, E_{-i} \in \Sigma_{S_{-i} \times T_{-i}}.$$

Note that, differently from the Assumption operator $\mathbf{A}_i$, $\mathbf{WB}_i$ is *monotone*. This is so because Weak Belief in E_{-i} does not impose any cautious attitude toward E_{-i}, i.e., payoff-relevant components of E_{-i} are *not* required to be "infinitely more likely than" not-E_{-i}. In [21], we show that the notions of Cautious Rationality and Weak Belief can be appropriately combined to justify the choice of permissible strategies in the following sense: Permissibility characterizes the behavioral implications of Cautious Rationality and Common Weak Belief of Cautious Rationality in a universal type structure. This gives an analogue of Theorem 2, thus providing an affirmative answer to a question raised by BFK ([17, p. 333]).

It should be noted that we cannot replace the notion of "infinitely more likely than" with the one of Blume et al. [9] in the definition of Weak Belief. As we discuss in Appendix A, the latter notion is *not* monotone. As such, if the operator $\mathbf{WB}_i$ were based on a notion of Weak Belief in terms of the "infinitely more likely than" relation of [9], then $\mathbf{WB}_i$ would not be monotone as well, hence not well suited for an epistemic analysis of Permissibility.

6.2 Alternative notions of Assumption: Comparison with Dekel et al. [25]

As mentioned earlier, the concept of Assumption was first introduced by BFK ([17, Section 5]) for the case in which beliefs are represented by full-support LCPS's. The recent contribution of Dekel et al. [25] covers the general case with unrestricted LPS's. To facilitate the comparison with our notion of Assumption, we record the LPS-based definition of "BFK-Assumption" ([25, Definition 3.2]).

Definition 12 *Fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ and a non-empty event $E \subseteq S_{-i} \times T_{-i}$. Fix also $t_i \in T_i$ with $\beta_i(t_i) = (\mu^1, ..., \mu^n)$. Say that E is* ***BFK-assumed under*** *$\beta_i(t_i)$* ***at level*** *$m \leq n$ if the following conditions hold:*

(i) $\mu^l(E) = 1$ *for all $l \leq m$;*

(ii) $E \subseteq \cup_{l \leq m} \mathrm{Supp}\mu^l$; *and*

(iii) *for each $l > m$, there exists $(\alpha_1^l, ..., \alpha_m^l) \in \mathbb{R}^m$ such that, for each Borel $F \subseteq E$,*

$$\mu^l(F) = \sum_{j=1}^{m} \alpha_j^l \mu^j(F).$$

Condition (i) and Condition (iii) of Definition 12 say that E is "infinitely more likely than" (in the sense of [9]) not-E (see Definition A.6 and Proposition A.4 in Appendix A). If the LPS $\beta_i(t_i)$ in Definition 12 is, in fact, an LCPS (as in BFK, [18] and [44]), then Condition (iii) is equivalent to the following condition:

(iii)' $\mu^l(E) = 0$ for all $l > m$.

Dekel et al. [25] also provide two weaker notions of BFK-Assumption. The first one, called "PWD-Assumption", turns out to be equivalent to BFK-Assumption for LCPS's.

The second one, called "TWD-Assumption", requires that only Conditions (i)-(ii) of Definition 12 hold. Note that, if Condition (i) holds, then Condition (ii) is equivalent to the following condition:

(ii)' for every open set $O \subseteq S_{-i} \times T_{-i}$, if $E \cap O \neq \emptyset$ then $\mu^l(E \cap O) > 0$ for some $l \leq m$.

Condition (ii) of our notion of Assumption (Definition 11) is weaker than Condition (ii)'. This is so because, *technically*, every elementary cylinder, viz. $C = \{s_{-i}\} \times T_{-i}$, is a (cl)open set. The weakening of Condition (ii)' is crucial for our main result, in particular for Theorem 2.(ii).[26] Therefore, it is straightforward to show that our notion of Assumption is weaker than all their counterparts mentioned above.

The difference between our notion of Assumption and the other ones is sharper in terms of preference-based foundations. We thoroughly discuss these aspects in Appendix A and Supplemental Appendix C. Here, we point out that we can include Condition (iii) of Definition 12 in our definition of Assumption and obtain a new notion with a similar preference-based foundation as in [25]. The main result of the paper (Theorem 2) would continue to hold under this stronger notion of Assumption. Sections C and E of the Supplemental Appendix discuss the required modifications.

6.3 Belief-completeness vs terminality

Part (i) of Theorem 2 is an analogue of Theorem 6.2 in Dekel et al. [25]. In the Supplemental Appendix, it is shown that, for any finite non-degenerate game, there exists an associated belief-complete type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$ in which the set of states $\prod_{i \in I} R_i^\infty$ is empty. The reason why this can occur is simple: While a belief-complete type structure induces all beliefs about types, it need not induce all possible hierarchies of beliefs.

To elaborate, fix a type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. Following the terminology in [27], we say that $\mathcal{T}$ is **finitely terminal** if, for each hierarchy $h_i = (\overline{\mu}_i^1, \overline{\mu}_i^2, \ldots) \in H_i$ there is $t_i \in T_i$ such that $\left(d_i^1(t_i), \ldots, d_i^k(t_i)\right) = \left(\overline{\mu}_i^1, \ldots \overline{\mu}_i^k\right)$ for all $k \in \mathbb{N}$. We say that $\mathcal{T}$ is **terminal** if, for each hierachy $h_i \in H_i$ there is $t_i \in T_i$ such that $d_i(t_i) = h_i$. In words, a type structure is finitely-terminal if it induces all finite-order beliefs. When a type structure induces all possible hierarchies of beliefs (e.g., the canonical one), the IA strategies are consistent with R$^{\mathrm{c}}$CAR$^{\mathrm{c}}$ and so there is some state at which there is R$^{\mathrm{c}}$CAR$^{\mathrm{c}}$

[26] As shown by Dekel et al. [25], the negative result in BFK is retained if the notion of BFK-Assumption is replaced by TWD-Assumption.

(Theorem 2.(ii)). But, for a given belief-complete type structure that does not induce all hierarchies of beliefs, the same conclusion need not follow.

How do terminal type structures relate to belief-complete type structures? In the context of ordinary probabilities (i.e., Subjective Expected Utility preferences) Friedenberg ([27, Theorem 3.1]) shows that a belief-complete type structure is terminal provided each type space is compact and each belief map is continuous.[27] In the lexicographic case, however, there is no analogue of the aforementioned result. It is shown in [20] that a belief-complete type structure is finitely terminal. Such result is, in some sense, tight: Belief-completeness is insufficient to establish terminality, even though the continuity requirement on the belief maps is met. As already remarked (see Section 3.3), a belief-complete, lexicographic type structure cannot be compact and continuous; as such, Friedenberg's result (cf. [27, Theorem 3.1]) cannot be extended to the lexicographic framework.

That said, it should be emphasized that the so-called "BFK's impossibility result" ([17, Theorem 10.1] and [25, Theorem 6.3]) does not hinge on the terminality property: As shown by Keisler and Lee [33], BFK's analysis of IA depends on topological features of belief-complete type structures which are unrelated to belief hierarchies. By contrast, our message is in line with analogous works on other solution concepts, such as Iterated (Strict) Dominance: Friedenberg and Keisler [28] show that, for any non-degenerate finite game, there exists an associated belief-complete, standard type structure in which no strategy is consistent with Rationality and Common Belief of Rationality. They also show that this arises due to the lack of terminality of belief-complete type structures. Therefore, our negative result is an analogue of Friedenberg and Keisler's result in the lexicographic framework.

6.4 Mutual singularity and comparison with BFK

The proof of part (ii) of Theorem 2 relies on showing the existence of a finite type structure $\mathcal{T}^* = \langle S_i, T_i^*, \beta_i^* \rangle_{i \in I}$ with some desirable properties for the application of Lemma 5. The properties of $\mathcal{T}^*$ stated in Claim 2 are of particular interest. In [20], we identify a class Υ of lexicographic type structures, the so-called "strongly LCPS type structures", and we show the existence of a canonical, belief-complete and continuous LCPS type structure, viz. $\mathcal{T}_u^{MS} = \langle S_i, \Lambda_i, g_i \rangle_{i \in I}$, which is universal *within* this class. That is, every type structure in Υ can be mapped into $\mathcal{T}_u^{MS}$ by the unique type morphism $(d_i)_{i \in I}$, as in Theorem 1. Claim 2 establishes that $\mathcal{T}^* \in \Upsilon$.

The following is a version of Theorem 2 for the case in which beliefs are represented by LCPS.

Theorem 3 *Fix a finite game $\langle I, (S_i, u_i)_{i \in I} \rangle$ and an associated belief-complete type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$. The following statements hold true:*

[27] The reverse implication is not true: A terminal type structure need not be belief-complete, unless the type structure is *belief-non-redundant* ([27, Proposition 4.1]), i.e., if distinct types induce distinct hierarchies of beliefs. This definition of belief-non-redundancy naturally extends to the case of lexicographic type structures. Note, however, that this notion pertains to hierarchies of LPS's, not necessarily to hierarchies of lexicographic *preferences*. Multiple LPS's may represent the same lexicographic preference relation. See [36] for a detailed analysis of this issue.

(i) *if $\mathcal{T}$ is an LCPS type structure, then, for each $m \geq 0$, $\prod_{i\in I} \mathrm{Proj}_{S_i}(R_i^m) = \prod_{i\in I} S_i^m$;*

(ii) *if $\mathcal{T}$ is isomorphic to $\mathcal{T}_u^{MS}$, then $\prod_{i\in I} R_i^\infty \neq \emptyset$ and $\prod_{i\in I} \mathrm{Proj}_{S_i}(R_i^\infty) = \prod_{i\in I} S_i^\infty$.*

Part (i) of Theorem 3 is an analogue of Theorem 9.1 in BFK, and its proof follows from arguments similar to those in the proof of Theorem 2.(i).[28] This result, if compared to Theorem 2.(i), may be surprising at first glance, since it only allows the players to hold a restricted set of LPS's representing their beliefs. However, as it is shown in [20], *every* belief-complete LCPS type structure is finitely terminal. In other words, the type structure $\mathcal{T}$ in the statement of Theorem 3.(i) has the same descriptive power as that of any other belief-complete (but possibly not LCPS) type structure, as long as finite-order epistemic conditions—such as Theorem 3.(i)—are concerned.

Part (ii) of Theorem 3 follows from the same proof as the one we provided for Theorem 2.(ii), but with $\mathcal{T}_u$ replaced by $\mathcal{T}_u^{MS}$. This last result is our key point of difference with BFK's negative result ([17, Theorem 10.1]).

6.5 Hierarchies of lexicographic minimal beliefs and coherence

In this paper, we have adopted a notion of coherence for hierarchies of LPS's (see Definition 1) which is weaker than the one in the recent work by Lee [35]. The starting point in Lee's analysis is that multiple LPS's may represent the same Lexicographic Expected Utility preference relation (cf. [9, p. 66]). Specifically, Lee restricts attention to *minimal beliefs*, that is, minimal-length representations of lexicographic preferences—for instance, $(\mu, \mu) \in \mathcal{N}(X)$ represents the same preference relation on X as the minimal LPS (μ). His notion of coherence allows a $(k+1)$-order belief $\overline{\mu}_i^{k+1}$ to be a longer LPS than k-order belief $\overline{\mu}_i^k$. Using such notion of coherence, Lee provides a "bottom-up" construction (*à la* Mertens and Zamir [38]) of the space of hierarchies of minimal beliefs in which some hierarchies cannot be generated by any type structure. The reason why this occurs is that, while the length of all k-order beliefs $\overline{\mu}_i^k$ is finite for all $k \in \mathbb{N}$, this may not be the case for $k \to \infty$. Consequently, there are hierarchies that cannot be summarized by a single LPS, which must necessarily have a finite length (cf. Proposition 1). Lee uses this fact to provide an epistemic justification of IA under the notion of Assumption as in Dekel et al. [25] (Definition 12), so to overturn "BFK's impossibility result".

In a companion paper [19], we provide an in-depth analysis on the relationship between Lee's approach and ours. Here, we just summarize the main findings and results, by referring the reader to [19] for details. First, by selecting only the hierarchies with an upper bound on the length of all finite-order beliefs, we show that a construction of a "canonical" type structure for hierarchies of minimal beliefs is possible, along the lines outlined in this paper (cf. Section 3). The canonical space of hierarchies constructed in this way turns out to be *behaviorally equivalent* to the canonical hierarchic space $H = \Pi_{i\in I} H_i$. This is so because Lee's notion of coherence preserves coherence of *preferences* exactly in the same

[28] One direction of the proof, namely $\mathrm{Proj}_{S_i}(R_i^m) \subseteq S_i^m$ for all $m \geq 1$, is the same as that in the proof of Theorem 2.(i). For the other direction, we cannot rely on the selection argument used in the proof of Theorem 2.(i), since this does not yield mutually singular LPS's. The proof requires some minor modifications to the proof of Theorem 9.1 in [17]. Details are available on request.

way as the notion of coherence in Definition 1 does. This version of the canonical type structure satisfies a terminality property analogous to that in Theorem 1, although it is *not* belief-complete in the sense of Definition 3.[29] Indeed, under an appropriate notion of hierarchy morphism, every type structure can be mapped into it in a way that preserves the hierarchies of minimal beliefs. We show that a property analogous to Lemma 3 holds, and, with some minor necessary changes, all the main results stated here hold through.

We finally remark an important conceptual point: Our results are *insensitive* to the presence of redundancies in the representation of Lexicographic Expected Utility preferences. As shown in [36, Section 2.4], there are many minimal-length LPS's which are preference-equivalent, so the canonical type structure with minimal beliefs is not the most parsimonious representation of hierarchies of Lexicographic Expected Utility preferences. A costruction of a non-redundant, canonical space of hierarchies of Lexicographic Expected Utility preferences is still possible (see [34]), and an analogue of Theorem 2 holds for this version of the canonical type structure.

6.6 $\mathrm{R^cCAR^c}$ in arbitrary type structures

BFK introduce the concept of **self-admissible set** (**SAS**) as a suitable, weak-dominance analogue of best-reply set—a concept, due to Pearce [40], based on strict dominance.

To formally define the SAS concept, we need an additional definition. Fix a finite game $\langle I, (S_i, u_i)_{i\in I}\rangle$. Say that a strategy $s_i' \in S_i$ of player i **supports** $s_i \in S_i$, if there exists a mixed strategy σ_i with $s_i' \in \mathrm{Supp}\sigma_i$ and $\pi_i(\sigma_i, s_{-i}) = \pi_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$.

Definition 13 *Fix a finite game $\langle I, (S_i, u_i)_{i\in I}\rangle$. A set $Q = \prod_{i\in I} Q_i \subseteq S$ is an **SAS** if, for every player i,*
(a) each $s_i \in Q_i$ is admissible,
(b) each $s_i \in Q_i$ is admissible with respect to $S_i \times Q_{-i}$,
(c) for every $s_i \in Q_i$, if s_i' supports s_i then $s_i' \in Q_i$.

Every finite game admits an SAS—in particular, the IA set is an SAS. But, as shown by BFK, many games possess other SAS's, which are even disjoint from the IA set. A comprehensive analysis of the properties of SAS's in a wide class of games is given in [16].

The notion of $\mathrm{R^cCAR^c}$ in arbitrary type structures can be characterized in terms of SAS's as follows.

Theorem 4 *Fix a finite game $\langle I, (S_i, u_i)_{i\in I}\rangle$.*
(1) For each type structure $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$, $\prod_{i\in I} \mathrm{Proj}_{S_i}(R_i^\infty)$ is an SAS.
(2) For each SAS $Q = \prod_{i\in I} Q_i \subseteq S$, there exists a finite type structure $\mathcal{T} = \langle S_i, T_i, \beta_i\rangle_{i\in I}$ such that

$$\prod_{i\in I} \mathrm{Proj}_{S_i}(R_i^\infty) = \prod_{i\in I} Q_i.$$

[29]For instance, the LPS $\overline{f}_i(h_i) = (v, v) \in \mathcal{N}(S_{-i} \times H_{-i})$ is not minimal, so it is not represented in the canonical type structure with minimal beliefs.

Note that, for the specific case in which Q is the IA set, the finite type structure $\mathcal{T}^*$ in Section 5.1 satisfies part (2) of Theorem 4.

Theorem 4 can be proven using arguments which are similar to those in BFK or Dekel et al. [25]. We discuss the required modifications in the Supplemental Appendix.

Appendix A: Preference-based representation of Assumption

Fix a lexicographic type structure $\mathcal{T} = \langle S_i, T_i, \beta_i \rangle_{i \in I}$, where each strategy set S_i is finite. To ease notation, it will be convenient to set $\Omega = S_{-i} \times T_{-i}$ and to drop i's subscript from LPS's $\overline{\mu}_i$ on Ω.

An act on Ω is a Borel measurable function $f : \Omega \to [0,1]$. We denote by ACT(Ω) the set of all acts on Ω. A Decision Maker has preferences over elements of ACT(Ω). For $x \in [0,1]$, write $\overrightarrow{x}$ for the constant act associated with x, i.e., $\overrightarrow{x}(\omega) = x$ for all $\omega \in \Omega$. Each constant act is identified with the associated outcome in a natural way. In what follows, we assume that the outcome space $[0,1]$ is in utils, i.e., material consequences are replaced by their von Neumann-Morgenstern utility. Given a Borel set $E \subseteq \Omega$ and acts $f, g \in$ACT(Ω), define $(f_E, g_{\Omega \backslash E}) \in$ACT($\Omega$) as follows:

$$(f_E, g_{\Omega \backslash E})(\omega) = \begin{cases} f(\omega), & \omega \in E, \\ g(\omega), & \omega \in \Omega \diagdown E. \end{cases}$$

Let $\succsim$ be a preference relation on ACT(Ω) and write $\succ$ (resp. $\sim$) for strict preference (resp. indifference). The preference relation $\succsim$ satisfies the following axioms:

Axiom 1 Order: $\succsim$ is a complete, transitive, reflexive binary relation on ACT(Ω).

Axiom 2 Independence: For all $f, g, h \in$ACT(Ω) and $\alpha \in (0,1]$,

$$\begin{aligned} f &\succ g \text{ implies } \alpha f + (1-\alpha) h \succ \alpha g + (1-\alpha) h, \text{ and} \\ f &\sim g \text{ implies } \alpha f + (1-\alpha) h \sim \alpha g + (1-\alpha) h. \end{aligned}$$

Moreover, let $\succsim_E$ denote the *conditional preference* given E, that is, $f \succsim_E g$ if and only if $(f_E, h_{\Omega \backslash E}) \succsim (g_E, h_{\Omega \backslash E})$ for some $h \in$ACT(Ω). Standard results (see [9, p. 64] for a proof) show that, under Axioms 1 and 2, $(f_E, h_{\Omega \backslash E}) \succsim (g_E, h_{\Omega \backslash E})$ holds for all $h \in$ACT(Ω) if it holds for some h.

Throughout, we maintain the assumption that $\overline{\mu}$ is a Lexicographic Expected Utility representation of $\succsim$, i.e., $\succsim = \succsim^{\overline{\mu}}$. (This makes sense, since each Lexicographic Expected Utility representation satisfies Axioms 1 and 2.) In what follows, we call $C \subseteq \Omega$ an **elementary cylinder** if $C = \{s_{-i}\} \times T_{-i}$, for some $s_{-i} \in S_{-i}$. Given s_{-i} and event E, we say that $E_{s_{-i}}$ is a **relevant part** of the event E if $E_{s_{-i}} = C \cap E \neq \emptyset$ for the elementary cylinder $C = \{s_{-i}\} \times T_{-i}$. Clearly, every non-empty event E can be written as a finite, disjoint union of all its relevant parts.

Definition A.1 *Say that $\succsim^{\overline{\mu}}$ exhibits **cautiousness** if, for every elementary cylinder $C = \{s_{-i}\} \times T_{-i}$, there are $f, g \in$ACT(Ω) such that $f \succ_C^{\overline{\mu}} g$.*

Recall that an event $E \subseteq \Omega$ is **Savage-null** under $\succsim$ if $f \sim_E g$ for all $f, g \in$ACT(Ω). Say that E is **non-null** under $\succsim$ if it is not Savage-null under $\succsim$. Say that event $E \subseteq \Omega$ is **fully believed** under $\succsim$ if $f \sim_{\Omega \setminus E} g$ for all $f, g \in$ACT(Ω). We thus have:

Proposition A.1 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. An event $E \subseteq \Omega$ is Savage-null under $\succsim^{\overline{\mu}}$ if and only if $\mu^l(E) = 0$ for all $l \leq n$.*

Proof: If $\mu^l(E) = 0$ for all $l \leq n$, then obviously $f \sim_E^{\overline{\mu}} g$ for all $f, g \in$ACT(Ω). On the other hand, if $E \subseteq \Omega$ is Savage-null under $\succsim^{\overline{\mu}}$, then $\overrightarrow{1} \sim_E^{\overline{\mu}} \overrightarrow{0}$. That is,

$$\left(\int_E d\mu^l + \int_{\Omega \setminus E} f d\mu^l \right)_{l=1}^n = \left(0 + \int_{\Omega \setminus E} f d\mu^l \right)_{l=1}^n, \; \forall f \in \text{ACT}(\Omega),$$

which implies $\mu^l(E) = 0$ for all $l \leq n$. ■

Corollary A.1 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. A non-empty event $E \subseteq \Omega$ is fully believed under $\succsim^{\overline{\mu}}$ if and only if $\mu^l(E) = 1$ for all $l \leq n$.*

The set of *binary acts* (*bets*) on Ω is the set of all acts of the form $(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega \setminus E})$, for $x, y \in [0,1]$ and event $E \subseteq \Omega$. As the rankings of binary acts reveal the Decision Maker's underlying beliefs or likelihoods, we introduce an "infinitely more likely than" relation between events which is based on bets.

Definition A.2 *Fix events $E, F \subseteq \Omega$. Say that E is* ***more likely*** *than F if for all $x, y \in [0,1]$ with $x > y$,*

$$(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega \setminus E}) \succsim^{\overline{\mu}} (\overrightarrow{x}_F, \overrightarrow{y}_{\Omega \setminus F}).$$

Say that E is deemed ***infinitely more likely*** *than F, and write $E \gg^{\overline{\mu}} F$, if for all $x, y, z \in [0,1]$ with $x > y$,*

$$(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega \setminus E}) \succ^{\overline{\mu}} (\overrightarrow{z}_F, \overrightarrow{y}_{\Omega \setminus F}).$$

In words, E is more likely than F if the Decision Maker prefers to bet on E rather than on F given the same prizes for the two bets; this choice-theoretic notion is due to Savage ([42, p. 31]). On the other hand, E is infinitely more likely than F if betting on E is *strictly* preferable to betting on F, and strict preference persists no matter how bigger the prize z for winning the F bet is. This notion of "infinitely more likely than" is due to Lo ([37, Definition 1]).[30]

Note that, according to Definition A.2, if $E \gg^{\overline{\mu}} F$, then E is non-null under $\succsim^{\overline{\mu}}$, while F may, but *need not*, be Savage-null under $\succsim^{\overline{\mu}}$. When $\succsim^{\overline{\mu}}$ has a Subjective Expected Utility representation, $E \gg^{\bar{\mu}} F$ implies that F is Savage-null.

The likelihood relation $\gg^{\bar{\mu}}$ possesses some natural properties. First, it is irreflexive, asymmetric and transitive. Moreover, if $E \gg^{\bar{\mu}} F$, then

(P1) E is infinitely more likely than any Borel subset of F; and

[30] Lo introduces such definition for a wide class of preferences, including the Lexicographic Expected Utility model.

(P2) any Borel superset of E is infinitely more likely than F.

We will refer to (P2) as the monotonicity property of $\gg^{\overline{\mu}}$.

The next step is to characterize the likelihood order $\gg^{\overline{\mu}}$ between pairwise *disjoint* events in terms of LPS's representing $\succsim^{\overline{\mu}}$. To this end, we need additional notation. Given an LPS $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and non-empty event $E \subseteq \Omega$, let

$$\mathcal{I}_E = \inf \left\{ l \in \{1, ..., n\} \left| \mu^l(E) > 0 \right. \right\}.$$

Proposition A.2 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and disjoint events $E, F \subseteq \Omega$ with $E \neq \emptyset$.*

1. *E is more likely than F if and only if*

$$\left(\mu^l(E)\right)_{l=1}^n \geq_L \left(\mu^l(F)\right)_{l=1}^n.$$

2. *$E \gg^{\overline{\mu}} F$ if and only if $\mathcal{I}_E < \mathcal{I}_F$.*

Proof: Part 1: Let $x, y \in [0,1]$ with $x > y$. The statement follows from the following chain of logically equivalent relations.

$$\begin{aligned}
(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\setminus E}) \quad & \succsim^{\overline{\mu}} (\overrightarrow{x}_F, \overrightarrow{y}_{\Omega\setminus F}) \\
\Longleftrightarrow \quad & \left(\int_E x d\mu^l + \int_{\Omega\setminus E} y d\mu^l\right)_{l=1}^n \geq_L \left(\int_F x d\mu^l + \int_{\Omega\setminus F} y d\mu^l\right)_{l=1}^n \\
\Longleftrightarrow \quad & \left(x\mu^l(E) + y\mu^l(\Omega\setminus E)\right)_{l=1}^n \geq_L \left(x\mu^l(F) + y\mu^l(\Omega\setminus F)\right)_{l=1}^n \\
\Longleftrightarrow \quad & \left(x\mu^l(E) + y - y\mu^l(E)\right)_{l=1}^n \geq_L \left(x\mu^l(F) + y - y\mu^l(F)\right)_{l=1}^n \\
\Longleftrightarrow \quad & \left((x-y)\mu^l(E)\right)_{l=1}^n \geq_L \left((x-y)\mu^l(F)\right)_{l=1}^n \\
\Longleftrightarrow \quad & \left(\mu^l(E)\right)_{l=1}^n \geq_L \left(\mu^l(F)\right)_{l=1}^n.
\end{aligned}$$

Part 2: The statement is clearly true if F is Savage-null under $\succsim^{\overline{\mu}}$, so that, by Proposition A.1, $\mathcal{I}_F = \inf \emptyset = +\infty$. So, in what follows, let F be non-null under $\succsim^{\overline{\mu}}$.

(Necessity) Arguing contrapositively, suppose that $\mathcal{I}_E \geq \mathcal{I}_F$. We consider two cases:

(a) $\mathcal{I}_E > \mathcal{I}_F$. Let $x = z = 1$ and $y = 0$. We clearly have $(\overrightarrow{1}_F, \overrightarrow{0}_{\Omega\setminus F}) \succ^{\overline{\mu}} (\overrightarrow{1}_E, \overrightarrow{0}_{\Omega\setminus E})$, so $E \gg^{\overline{\mu}} F$ fails.

(b) $\mathcal{I}_E = \mathcal{I}_F$. In this case, observe that $\mu^{\mathcal{I}_F}(E), \mu^{\mathcal{I}_F}(F) \in (0,1)$. Let $x = \mu^{\mathcal{I}_F}(F)$, $z = 1$ and $y = 0$. For all $l < \mathcal{I}_F$, it holds that

$$\int (\overrightarrow{x}_E, \overrightarrow{0}_{\Omega\setminus E}) d\mu^l = \int (\overrightarrow{1}_F, \overrightarrow{0}_{\Omega\setminus F}) d\mu^l = 0,$$

while

$$\int (\overrightarrow{1}_F, \overrightarrow{0}_{\Omega\setminus F}) d\mu^{\mathcal{I}_F} = \mu^{\mathcal{I}_F}(F) > \mu^{\mathcal{I}_F}(F) \cdot \mu^{\mathcal{I}_F}(E) = \int (\overrightarrow{x}_E, \overrightarrow{0}_{\Omega\setminus E}) d\mu^{\mathcal{I}_F},$$

where the strict inequality follows from the observation above. Again, this shows that $E \gg^{\overline{\mu}} F$ fails.

(Sufficiency) Let $\mathcal{I}_E < \mathcal{I}_F$, and pick any $x, y, z \in [0,1]$ with $x > y$. For all $l < \mathcal{I}_E$, it holds that

$$\int (\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\backslash E}) d\mu^l = \int (\overrightarrow{z}_F, \overrightarrow{y}_{\Omega\backslash F}) d\mu^l = y.$$

Next note that, since $(x-y)\,\mu^{\mathcal{I}_E}(E) > 0$, then

$$x\mu^{\mathcal{I}_E}(E) + y\mu^{\mathcal{I}_E}(\Omega\backslash E) > y\mu^{\mathcal{I}_E}(\Omega\backslash F) = y,$$

that is,

$$\int (\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\backslash E}) d\mu^{\mathcal{I}_E} > \int (\overrightarrow{z}_F, \overrightarrow{y}_{\Omega\backslash F}) d\mu^{\mathcal{I}_E}.$$

This shows that $(\overrightarrow{x}_E, \overrightarrow{y}_{\Omega\setminus E}) \succ^{\overline{\mu}} (\overrightarrow{z}_F, \overrightarrow{y}_{\Omega\setminus F})$, as required. ■

We now introduce a new notion of belief, which we call **Weak Belief**, and we state and prove a characterization result in terms of LPS's for such notion.

Definition A.3 *Fix a non-empty event $E \subseteq \Omega$. Say that E is* ***weakly believed under*** $\succsim^{\overline{\mu}}$ *if* $E \gg^{\bar{\mu}} \Omega\setminus E$.

Theorem A.1 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and a non-empty event $E \subseteq \Omega$. Thus E is weakly believed under $\succsim^{\overline{\mu}}$ if and only if $\mu^1(E) = 1$.*

Proof: We have $\mu^1(E) = 1$ if and only if $\mathcal{I}_E = 1$ and $\mathcal{I}_{\Omega\setminus E} > 1$. Thus the result follows immediately from Proposition A.2. ■

Next the notion of Assumption in terms of the likelihood order $\gg^{\bar{\mu}}$.

Definition A.4 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. A non-empty event $E \subseteq \Omega$ is* **assumed under** $\succsim^{\overline{\mu}}$ *if it satisfies the following condition:*
(*) for every relevant part $E_{s_{-i}}$ of E, $E_{s_{-i}} \gg^{\bar{\mu}} \Omega \setminus E$.

That is, the event E is assumed under $\succsim^{\overline{\mu}}$ if every relevant part of E is deemed infinitely more likely than $\Omega \setminus E$. Since E can be written as a finite, disjoint union of all its relevant parts, it follows from the the monotonicity of $\gg^{\bar{\mu}}$ (Property P2) that E is weakly believed, i.e., $E \gg^{\bar{\mu}} \Omega \setminus E$. However, the opposite is not true. Indeed, the notion of Assumption is stronger that the notion of Weak Belief, as it captures cautious behavior. Note indeed the following

Remark A.1 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. The preference relation $\succsim^{\overline{\mu}}$ exhibits cautiousness if and only if Ω is assumed under $\succsim^{\overline{\mu}}$.*

We now state and prove a characterization result for the notion of Assumption. For the reader's convenience, we restate the LPS-based definition of Assumption given in the main text, but in terms of relevant parts.

Definition A.5 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$. A non-empty event $E \subseteq \Omega$ is* **assumed under $\overline{\mu}$ at level** *$m \leq n$ if satisfies the following conditions:*

(i) *$\mu^l(E) = 1$ for all $l \leq m$;*

(ii) *for every relevant part $E_{s_{-i}}$ of E, $\mu^k\left(E_{s_{-i}}\right) > 0$ for some $k \leq m$.*

We say that $E \subseteq \Omega$ is **assumed under** $\overline{\mu}$ *if it is assumed under $\overline{\mu}$ at some level $m \leq n$.*

Theorem A.2 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and a non-empty event $E \subseteq \Omega$. Thus E is assumed under $\succsim^{\overline{\mu}}$ if and only if E is assumed under $\overline{\mu}$.*

Proof: The proof is immediate if $\Omega \setminus E$ is Savage-null under $\succsim^{\overline{\mu}}$, so, in what follows, let $\Omega \setminus E$ be non-null under $\succsim^{\overline{\mu}}$.

(Necessity) Let $m = \mathcal{I}_{\Omega \setminus E} - 1$. Then, for every $k \leq m$, $\mu^k(\Omega \setminus E) = 0$, hence $\mu^k(E) = 1$. Thus, m satisfies condition (i) of Definition A.5. Since every relevant part $E_{s_{-i}}$ of E satisfies $E_{s_{-i}} \gg^{\bar{\mu}} \Omega \setminus E$, then Proposition A.2 yields $\mathcal{I}_{E_{s_{-i}}} < \mathcal{I}_{\Omega \setminus E}$. Thus, condition (ii) of Definition A.5 is satisfied.

(Sufficiency) If E is assumed under $\overline{\mu}$ at level m, then condition (i) of Definition A.5 implies $m + 1 = \mathcal{I}_{\Omega \setminus E}$. By this, condition (ii) yields that each $E_{s_{-i}}$ satisfies $\mathcal{I}_{E_{s_{-i}}} < \mathcal{I}_{\Omega \setminus E}$, hence, by Proposition A.2, $E_{s_{-i}} \gg \Omega \setminus E$. ■

We conclude this section with a brief comparison between the notion of "infinitely more likely than" in Definition A.2 and the one of Blume et al. [9]. (A more detailed analysis is provided in the Supplemental Appendix C.) Specifically, Blume et al. [9] examine a partial order $\gg_S^{\overline{\mu}}$ on events of Ω which is stronger than $\gg^{\bar{\mu}}$.

Definition A.6 *Fix disjoint events $E, F \subseteq \Omega$ with $E \neq \emptyset$. Thus $E \gg_S^{\overline{\mu}} F$ if*

1 *E is non-null under $\succsim^{\overline{\mu}}$, and*

2 *for all $f, g \in \mathrm{ACT}(\Omega)$, $f \succ_E^{\overline{\mu}} g$ implies $f \succ_{E \cup F}^{\overline{\mu}} g$.*

Condition 2 in Definition A.6 states that, when comparing any two acts f and g that give the same consequences in states not belonging to $E \cup F$, if $f \succ_E^{\overline{\mu}} g$, then the consequences in F "do not matter" for the strict preference $f \succ^{\overline{\mu}} g$.[31] In particular, if $F = \Omega \setminus E$, then Condition 2 corresponds to "Strict Determination", which is stated as axiom in BFK.

It is easy to check that if $E \gg_S^{\overline{\mu}} F$ then $E \gg^{\bar{\mu}} F$. The reverse implication is true provided both E and F are singleton sets—cf. Lemma C.3 in the Supplemental Appendix. The key difference is represented by the following property (for a proof, see Proposition C.1.(ii) in the Supplemental Appendix):

Proposition A.3 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and non-empty, pairwise disjoint events $E, F \subseteq \Omega$, with E non-null under $\succsim^{\overline{\mu}}$. The following property holds:*

[31] The definition of the partial order $\gg_S^{\overline{\mu}}$ is taken from [2, p. 65]. Definition 5.1 in Blume et al. [9, pp. 70-71] states that $E \gg_S^{\overline{\mu}} F$ if condition 2 in Definition A.6 is replaced by the following condition:

$$f \succ_E g \text{ implies } \left(f_{\Omega \setminus F}, h_F\right) \succ_{E \cup F} \left(g_{\Omega \setminus F}, h'_F\right)$$

for all $h, h' \in \mathrm{ACT}(\Omega)$. (Condition 1 is automatically satisfied in [9, Definition 5.1], since the Authors consider a finite state space without Savave-null events.) It is easy to check the equivalence between the two definitions.

(**) *Let $E_1 \subseteq \Omega$ be a non-empty event such that $E_1 \subseteq E$ and E_1 is non-null under $\succsim^{\overline{\mu}}$. Thus, if $E \gg_S^{\overline{\mu}} F$, then $E_1 \gg_S^{\overline{\mu}} F$.*

In words, Proposition A.3 states that $\gg_S^{\overline{\mu}}$ requires that each *non-null* Borel subset $E_1 \subseteq E$ be infinitely more likely than F. The cost is a non-monotonicity property of the order $\gg_S^{\overline{\mu}}$. That is, if $E, F, G \subseteq \Omega$ are non-empty, pairwise disjoint events with $E \gg_S^{\overline{\mu}} F$, it may *not* be the case that $E \cup G \gg_S^{\overline{\mu}} F$. Yet, it can be easily shown that $E \cup G \gg_S^{\overline{\mu}} F$ provided G is Savage-null under $\succsim^{\overline{\mu}}$. In other words, the union of E with a non-null event can reduce the likelihood ranking of an event, while the union with a Savage-null event, paradoxically, cannot.

Moreover, consider the following example, which is a variant of an example in [9, p. 70].

Example A.1 *Let $\Omega = S_{-i} \times T_{-i}$ be such that $S_{-i} = \left\{s_{-i}^1, s_{-i}^2, s_{-i}^3\right\}$ and $T_{-i} = \left\{t_{-i}^*\right\}$. Consider the LPS $\overline{\mu} = (\mu^1, \mu^2) \in \mathcal{N}(\Omega)$ with $\mu^1\left(\left\{\left(s_{-i}^1, t_{-i}^*\right)\right\}\right) = \mu^1\left(\left\{\left(s_{-i}^2, t_{-i}^*\right)\right\}\right) = \frac{1}{2}$ and $\mu^2\left(\left\{\left(s_{-i}^1, t_{-i}^*\right)\right\}\right) = \mu^2\left(\left\{\left(s_{-i}^3, t_{-i}^*\right)\right\}\right) = \frac{1}{2}$. Consider the event $E = \left\{s_{-i}^1, s_{-i}^2\right\} \times T_{-i}$ and take binary acts*

$$\begin{aligned} f &= \left(\overrightarrow{1}_{\left\{\left(s_{-i}^1, t_{-i}^*\right)\right\}}, \overrightarrow{0}_{\left\{s_{-i}^2, s_{-i}^3\right\} \times T_{-i}}\right), \\ g &= \left(\overrightarrow{1}_{\left\{\left(s_{-i}^3, t_{-i}^*\right)\right\}}, \overrightarrow{\left(\frac{1}{2}\right)}_{\left\{s_{-i}^1, s_{-i}^2\right\} \times T_{-i}}\right). \end{aligned}$$

Note that $E \gg_S^{\overline{\mu}} \Omega \diagdown E$ fails, because $f \succ_E^{\overline{\mu}} g$ while $g \succ^{\overline{\mu}} f$. However, $E = E_{s_{-i}^1} \cup E_{s_{-i}^2}$, with $E_{s_{-i}^1} \gg_S^{\overline{\mu}} \Omega \diagdown E$ and $E_{s_{-i}^2} \gg_S^{\overline{\mu}} \Omega \diagdown E$.

Example A.1 shows that even if all the (relevant) parts of an event E are infinitely more likely than not-E (in the sense of Blume et al. [9]), then E need not be infinitely more likely than its complement. This problem was one of the motivations for the introduction of LCPS's (see Blume et al. [9, pp. 70-71]).

The likelihood order $\gg_S^{\overline{\mu}}$ between disjoint events can be given the following LPS-based characterization (for a proof, see Proposition C.2 in the Supplemental Appendix):

Proposition A.4 *Fix $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and disjoint events $E, F \subseteq \Omega$ with $E \neq \emptyset$. Thus $E \gg_S^{\overline{\mu}} F$ if and only if the following conditions hold:*

(1.1) $\mathcal{I}_E < \mathcal{I}_F$;

(1.2) for all $l \geq \mathcal{I}_F$, there exists $\left(\alpha_1^l, ..., \alpha_{\mathcal{I}_F - 1}^l\right) \in \mathbb{R}^{\mathcal{I}_F - 1}$ such that, for each Borel $G \subseteq E$,

$$\mu^l(G) = \sum_{j=1}^{\mathcal{I}_F - 1} \alpha_j^l \mu^j(G).$$

Note: for the specific case in which $F = \Omega \diagdown E$ and is non-null under $\succsim^{\overline{\mu}}$, Conditions (1.1) and (1.2) are equivalent to Conditions (i) and (iii) of BFK-Assumption (Definition 12) with $m = \mathcal{I}_{\Omega \diagdown E} - 1$.

Proposition A.4 clarifies the sense in which $E \gg_S^{\overline{\mu}} \Omega \diagdown E$ is not sufficient to characterize the notion of BFK-Assumption. As shown by Proposition A.3, $E \gg_S^{\overline{\mu}} \Omega \diagdown E$ implies that only the non-null Borel subsets of E are "infinitely more likely than" $\Omega \diagdown E$.[32] In case of a full-support LPS $\overline{\mu}$ on a finite space Ω, an event E is BFK-assumed under $\succsim^{\overline{\mu}}$ if and only if $E \gg_S^{\overline{\mu}} \Omega \diagdown E$.[33] However, as far as an infinite space Ω is concerned, it is important to specify a class of Borel subsets of E which must be non-null in order to capture IA (cf. BFK's Supplemental Appendix). BFK impose the requirement that every relatively open subset of E be non-null. (See their "Nontriviality" axiom.) In Section C of the Supplemental Appendix, we show that the requirement of the relevant parts as non-null Borel subsets gives an LPS-based version of BFK-Assumption which satisfies Condition (ii) of our notion of Assumption (Definition 11) in place of Condition (ii) of Definition 12.

Appendix B: Omitted proofs

It will be useful to single out an alternative characterization of Assumption:

Lemma B.1 *Fix an LPS $\overline{\mu} = (\mu^1, ..., \mu^n) \in \mathcal{N}(\Omega)$ and a non-empty event $E \subseteq \Omega$. Thus E is assumed under $\overline{\mu}$ if and only if there exists $m \leq n$ such that $\overline{\mu}$ satisfies Condition (i) of Definition 11 plus the following condition:*

(ii)' $E \subseteq \left(\cup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}} \mu^l\right) \times T_{-i}$.

Proof: Suppose E is assumed under $\overline{\mu}$ at level m. We show that $\overline{\mu}$ satisfies Condition (ii)'. Consider the clopen cylinder

$$O = \left(S_{-i} \diagdown \left(\cup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}} \mu^l\right)\right) \times T_{-i}.$$

We claim that $O \cap E = \emptyset$, so that condition (ii)' holds. To see this, suppose instead that $O \cap E \neq \emptyset$. Thus, by Condition (ii) of Definition 11, there exists $U = \{s_{-i}\} \times T_{-i} \subseteq O$ such that $U \cap E \neq \emptyset$ and $\mu^k (U \cap E) > 0$ for some $k \leq m$. As such, $\mu^k (O \cap E) > 0$ for some $k \leq m$. This implies that $\mathrm{marg}_{S_{-i}} \mu^k \left(S_{-i} \diagdown \left(\cup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}} \mu^l\right)\right) = \mu^k (O) > 0$ for some $k \leq m$, and so

$$O \cap \left(\left(\cup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}} \mu^l\right) \times T_{-i}\right) \neq \emptyset,$$

a contradiction.

Conversely, suppose that Conditions (i) and (ii)' hold. It is immediate to show that Condition (ii) of Definition 11 holds. Indeed, by Condition (ii)', for each $s_{-i} \in \cup_{l \leq m} \mathrm{Suppmarg}_{S_{-i}} \mu^l$ the corresponding cylinder set $U = \{s_{-i}\} \times T_{-i}$ satisfies the required properties. ■

For later use, we also find it convenient to state the following

[32] It noteworthy that this result is *insensitive* to the cardinality of the space of uncertainty Ω, as it is stated only in purely decision-theoretic terms.

[33] This is so because the full-support condition on $\overline{\mu}$ guarantees that all non-empty subsets of Ω (a finite, discrete space) are non-null.

Lemma B.2 *Let $t_i \in T_i$ assume the event $E \subseteq S_{-i} \times T_{-i}$ at level m, where $\beta_i(t_i) = (\mu_i^1, ..., \mu_i^n)$. Thus*

$$\cup_{l \leq m} \text{Suppmarg}_{S_{-i}} \mu_i^l = \text{Proj}_{S_{-i}}(E).$$

Proof: The set containment "$\supseteq$" follows from condition (ii)' of Lemma B.1. Indeed

$$\begin{aligned} \text{Proj}_{S_{-i}}(E) &\subseteq \text{Proj}_{S_{-i}}\left(\left(\cup_{l \leq m} \text{Suppmarg}_{S_{-i}} \mu_i^l\right) \times T_{-i}\right) \\ &= \cup_{l \leq m} \text{Suppmarg}_{S_{-i}} \mu_i^l. \end{aligned}$$

Conversely, fix $s_{-i} \notin \text{Proj}_{S_{-i}}(E)$. Thus $(\{s_{-i}\} \times T_{-i}) \cap E = \emptyset$. By Condition (i) of Definition 11, $\mu_i^l(E) = 1$ for each $l \leq m$, so $\mu_i^l(\{s_{-i}\} \times T_{-i}) = \text{marg}_{S_{-i}} \mu_i^l(\{s_{-i}\}) = 0$. This implies $s_{-i} \notin \text{Suppmarg}_{S_{-i}} \mu_i^l$. ■

Remark B.1 *The result in Lemma B.2 can be equivalently stated as*

$$\text{Proj}_{S_{-i}}\left(\cup_{l \leq m} \text{Supp} \mu_i^l\right) = \text{Proj}_{S_{-i}}(E).$$

Proof of Lemma 2: (Part 1) Let $\overline{\mu}_i = (\mu_i^1, ..., \mu_i^n)$ and suppose that E and F are assumed under $\overline{\mu}_i$ at level m. Using Lemma B.2, we get that $\text{Proj}_{S_{-i}}(E) = \text{Proj}_{S_{-i}}(F)$. If instead F is assumed or under $\overline{\mu}_i$ at level $p > m$, then $\text{Proj}_{S_{-i}}(E) \subseteq \text{Proj}_{S_{-i}}(F)$ since $\cup_{l \leq m} \text{Suppmarg}_{S_{-i}} \mu_i^l \subseteq \cup_{l \leq p} \text{Suppmarg}_{S_{-i}} \mu_i^l$.

(Part 2) Let $\overline{\mu}_i = (\mu_i^1, ..., \mu_i^n)$ and suppose that, for each k, E_k is assumed under $\overline{\mu}_i$ at some level m_k. Let $m_K = \min\{m_k \,|k = 1, 2, ...\}$. Also, let E_K be the set which is assumed under $\overline{\mu}_i$ at level m_K. We show that $\cap_k E_k$ is assumed under $\overline{\mu}_i$ at level m_K. First note that, for each k, $\mu_i^l(E_k) = 1$ for all $l \leq m_K$. By the σ-additivity property of probability measures, it follows that $\mu_i^l(\cap_k E_k) = 1$ for all $l \leq m_K$. Finally, it follows from Lemma B.1 that

$$\cap_k E_k \subseteq E_K \subseteq \left(\cup_{l \leq m_K} \text{Suppmarg}_{S_{-i}} \mu_i^l\right) \times T_{-i}.$$

Using again Lemma B.1, condition (ii) of Definition 11 is established. The proof for $\cup_k E_k$ is similar. ■

Proof of Lemma 3: Let $(s_i, t_i) \in \mathbf{A}_i(E_{-i})$, and set $\varphi_i(t_i) = t_i^*$. We show that event E_{-i}^* is assumed under $\beta_i^*(t_i^*)$, that is, conditions (i) and (ii) of Definition 11 are satisfied.

Since event E_{-i} is assumed under $\beta_i(t_i) = \left(\beta_i^1(t_i), ..., \beta_i^n(t_i)\right)$, then there exists $m \leq n$ such that $\beta_i^l(t_i)(E_{-i}) = 1$ for all $l \leq m$. Next note that

$$E_{-i} \subseteq \left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right) \subseteq \left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(E_{-i}^*\right),$$

where the first set containment is obvious, while the second one follows from condition 1). Hence, by definition of type morphism, it follows that, for all $l \leq n$,

$$\begin{aligned} \beta_i^l(t_i)(E_{-i}) &\leq \beta_i^l(t_i)\left(\left(\text{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(E_{-i}^*\right)\right) \\ &= \beta_i^{*,l}(\varphi_i(t_i))\left(E_{-i}^*\right) \\ &= \beta_i^{*,l}(t_i^*)\left(E_{-i}^*\right), \end{aligned}$$

which implies $\beta_i^{*,l}(t_i^*)\left(E_{-i}^*\right) = 1$ for all $l \leq m$, so that condition (i) of Definition 11 is satisfied.

To show that Condition (ii) of Definition 11 is also satisfied, we proceed as follows. Consider an elementary cylinder $C = \{s_{-i}\} \times T_{-i}^*$ satisfying $E_{-i}^* \cap C \neq \emptyset$. It turns out that

$$\begin{aligned}
\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(C \cap E_{-i}^*\right) &= \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}(C)\right) \cap \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(E_{-i}^*\right)\right) \\
&= \left(\{s_{-i}\} \times \varphi_{-i}^{-1}\left(T_{-i}^*\right)\right) \cap \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(E_{-i}^*\right)\right) \\
&= \left(\{s_{-i}\} \times T_{-i}\right) \cap \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(E_{-i}^*\right)\right) \\
&\supseteq \left(\{s_{-i}\} \times T_{-i}\right) \cap \left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)(E_{-i})\right)\right) \\
&\supseteq \left(\{s_{-i}\} \times T_{-i}\right) \cap E_{-i},
\end{aligned}$$

where the fourth line follows from condition 1) of the Lemma. Since $(\{s_{-i}\} \times T_{-i}) \cap E_{-i} \neq \emptyset$ (by condition 2) of the Lemma) and E_{-i} is assumed under $\beta_i(t_i)$ at level $m \leq n$, then there exists $l \leq m$ such that $\beta_i^l(t_i)\left((\{s_{-i}\} \times T_{-i}) \cap E_{-i}\right) > 0$. But

$$\begin{aligned}
\beta_i^l(t_i)\left((\{s_{-i}\} \times T_{-i}) \cap E_{-i}\right) &\leq \beta_i^l(t_i)\left(\left(\mathrm{Id}_{S_{-i}}, \varphi_{-i}\right)^{-1}\left(C \cap E_{-i}^*\right)\right) \\
&= \beta_i^{*,l}(t_i^*)\left(C \cap E_{-i}^*\right),
\end{aligned}$$

and since C is an arbitrary elementary cilynder, this shows that Condition (ii) of Definition 11 is satisfied, thus concluding the proof. ■

Proof of Claim3: **(Part 1)** By induction on $k \in \{1, ..., M\}$.

($k = 1$) Pick any $s_i \in S_i^1$. Thus

$$\beta_i^*((s_i, 1))\left(R_{-i}^{*,1}\right) = \nu_{s_i}^1\left(R_{-i}^{*,1}\right) = 0,$$

because (i) $\mathrm{Supp}\nu_{s_i}^1 \subseteq \left(S_{-i} \times T_{-i}^0\right)$ by definition, and (ii) $R_{-i}^{*,1} \cap \left(S_{-i} \times T_{-i}^0\right) = \emptyset$, since each type in T_{-i}^0 is not cautious.

($k+1$) Suppose that the statements has been shown to hold for $k \leq M-1$, i.e., for each $i \in I$ and $s_i \in S_i^k$,

$$\beta_i^*((s_i, k))\left(R_{-i}^{*,k}\right) = \overline{0}.$$

This implies that, for each $i \in I$,

$$R_{-i}^{*,k+1} \cap \left(S_{-i} \times T_{-i}^k\right) = \emptyset. \tag{6.1}$$

Pick arbitrary $i \in I$ and $s_i \in S_i^{k+1}$. We need to show that every measure in the LPS $\beta_i^*((s_i, k+1)) = \left(\nu_{s_i}^{k+1}, \beta_i^*((s_i, k))\right)$ assigns probability 0 to the event $R_{-i}^{*,k+1}$. To this end, first note that $s_i \in S_i^k$, and since $R_{-i}^{*,k+1} \subseteq R_{-i}^{*,k}$, then, by the induction hypothesis, $\beta_i^*((s_i, k))\left(R_{-i}^{*,k+1}\right) = \overline{0}$. Furthermore, since $\mathrm{Supp}\nu_{s_i}^{k+1} \subseteq \left(S_{-i} \times T_{-i}^k\right)$, then (6.1) yields $\nu_{s_i}^{k+1}\left(R_{-i}^{*,k+1}\right) = 0$. This shows that $\beta_i^*((s_i, k+1))\left(R_{-i}^{*,k+1}\right) = \overline{0}$, as required.

(Part 2) We prove by induction on $k \in \{0, 1, ..., M\}$ that

$$s_i \in S_i^k \implies (s_i, (s_i, k)) \in R_i^{*,k} \setminus R_i^{*,k+1}.$$

This will imply the thesis.

($k=0$) Let $s_i \in S_i$, $i \in I$. It follows from the construction of $\mathcal{T}^*$ that type $(s_i, 0)$ is not cautious, so $(s_i,(s_i,0)) \notin R_i^{*,1}$.

($k+1$) Suppose that the statement has been shown to hold for $k \leq M$, i.e., for each $i \in I$,

$$s_i \in S_i^k \implies (s_i,(s_i,k)) \in R_i^{*,k} \diagdown R_i^{*,k+1}.$$

Fix player i and $s_i \in S_i^{k+1}$. Note that type $(s_i, k+1)$ is cautious (Claim 1), and recall that

$$\begin{aligned}\overline{\operatorname{marg}}_{S_{-i}} \beta_i^*((s_i,k+1) &= \overline{\operatorname{marg}}_{S_{-i}} \left(\left(\nu_{s_i}^{k+1}, \nu_{s_i}^{k}, ..., \nu_{s_i}^{1}\right)\right) \\ &= \left(\mu_{s_i}^{k+1}, \mu_{s_i}^{k}, ..., \mu_{s_i}^{1}\right).\end{aligned}$$

So, for each $s_i' \in S_i$ it holds that

$$\left(\pi_i(s_i,\mu_{s_i}^{k+1}), \pi_i(s_i,\mu_{s_i}^{k}), ..., \pi_i(s_i,\mu_{s_i}^{1})\right) \geq_L \left(\pi_i(s_i',\mu_{s_i}^{k+1}), \pi_i(s_i',\mu_{s_i}^{k}), ..., \pi_i(s_i',\mu_{s_i}^{1})\right),$$

where the (lexicographic) inequality follows from the fact that s_i is a best reply to $\mu_{s_i}^l$ for all $l = 1, ..., k+1$. This shows that $(s_i,(s_i,k+1)) \in R_i^{*,1}$. We now show that $(s_i,(s_i,k+1))$ satisfies two additional requirements, namely

(a) $(s_i,(s_i,k+1)) \in \cap_{l \leq k} \mathbf{A}_i \left(R_{-i}^{*,l}\right)$, and

(b) $(s_i,(s_i,k+1)) \notin \mathbf{A}_i \left(R_{-i}^{*,k+1}\right)$.

By property (a), we will conclude that $(s_i,(s_i,k+1)) \in R_i^{*,k+1} = R_i^{*,1} \cap \left(\cap_{l\leq k} \mathbf{A}_i \left(R_{-i}^{*,l}\right)\right)$; by property (b), we will conclude that $(s_i,(s_i,k+1)) \notin R_i^{*,k+2} = R_i^{*,k+1} \cap \mathbf{A}_i \left(R_{-i}^{*,k+1}\right)$. This will show $(s_i,(s_i,k+1)) \in R_i^{*,k+1} \diagdown R_i^{*,k+2}$, as desired.

For property (a): Recall that $\beta_i^*((s_i,k+1) = \left(\nu_{s_i}^{k+1}, \beta_i^*((s_i,k)\right)$. Hence, using the same argument as in the proof of part (i) of Theorem 2, we get that $\mathrm{Proj}_{S_{-i}} \left(R_{-i}^{*,l}\right) \subseteq S_{-i}^l$ for each $l = 1, ..., k$. By the induction hypothesis and $\nu_{s_i}^{k+1}(\Delta_{S_{-i}^k \times T_{-i}^k}) = 1$, it follows that $\nu_{s_i}^{k+1}(R_{-i}^{*,k}) = 1$ and $\mathrm{Suppmarg}_{S_{-i}} \nu_{s_i}^{k+1} = S_{-i}^k$. So $(s_i, k+1)$ assumes $R_{-i}^{*,k}$. Since $\nu_{s_i}^{k+1}(R_{-i}^{*,l}) = 1$ for each $l = 1, ..., k-1$, then, by the induction hypothesis, (s_i, k) assumes $R_{-i}^{*,l}$ at some level m'. Thus, with $m = m'+1$, Condition (i) of Assumption (Definition 11) and Condition (ii)' of Lemma B.1 are satisfied. Therefore $(s_i, k+1)$ also assumes $R_{-i}^{*,l}$ ($l \leq k$).

For property (b): By the induction hypothesis, it follows that $\Delta_{S_{-i}^k \times T_{-i}^k} \cap R_{-i}^{*,k+1} = \emptyset$. Since $\nu_{s_i}^{k+1}(\Delta_{S_{-i}^k \times T_{-i}^k}) = 1$, then $(s_i, k+1)$ does not assume $R_{-i}^{*,k+1}$.

(Part 3) We show by induction on $k \in \{0, 1, ..., M+1\}$ that $\Delta_{S_i^M \times T_i^{M+1}} \subseteq R_i^{*,k}$ for each $i \in I$. For $k = 0$ the statement is immediate. We use a separate argument for $k \geq 1$. The same argument as in the proof of Part 2 can be applied for $k = 1$. For $k \in \{2, ..., M+1\}$, suppose that $\Delta_{S_i^M \times T_i^{M+1}} \subseteq R_i^{*,k-1}$ holds true for each $i \in I$. We need to show that, for each $i \in I$, $\Delta_{S_i^M \times T_i^{M+1}} \subseteq R_i^{*,k}$. Since $R_i^{*,k} = R_i^{*,k-1} \cap \mathbf{A}_i \left(R_{-i}^{*,k-1}\right)$, then by the induction hypothesis it is enough to prove that $\Delta_{S_i^M \times T_i^{M+1}} \subseteq \mathbf{A}_i \left(R_{-i}^{*,k-1}\right)$. To do this, fix $i \in I$ and $s_i \in S_i^M$. Recall that $\beta_i^*((s_i, M+1)) = (\nu_{s_i}^{M+1}, \beta_i^*((s_i, M))$. Since

$\nu_{s_i}^{M+1}(\Delta_{S_{-i}^M \times T_{-i}^{M+1}}) = 1$, by the induction hypothesis it follows that $\nu_{s_i}^{M+1}(R_{-i}^{*,k-1}) = 1$. The conclusion follows from arguments analogous to those in the proof of Part 2.

(Part 4) Let $(s_i', t_i) \in R_i^{*,M+1}$. First note that $t_i \in T_i^{M+1}$, because $(S_i \times T_i^k) \cap \mathbf{A}_i\left(R_{-i}^{*,M}\right) = \emptyset$ (this follows from Part 1, according to which $\beta_i^*((s_i,k))\left(R_{-i}^{*,k}\right) = \overline{0}$ for all $k \in \{1,...,M\}$ and all $s_i \in S_i^k$). So, by construction of the belief maps in $\mathcal{T}^*$, $\beta_i^*(t_i) = (\nu_{s_i}^{M+1}, \nu_{s_i}^{M}, ..., \nu_{s_i}^{1})$ for some $s_i \in S_i^{M+1}$. We now claim that $R_{-i}^{*,M+1}$ is assumed under $\beta_i^*(t_i)$. To this end, recall that $\text{Supp}\nu_{s_i}^{M+1} = \Delta_{S_{-i}^M \times T_{-i}^{M+1}}$, so, by Part 3, it follows that $\nu_{s_i}^{M+1}(R_{-i}^{*,M+1}) = 1$. Since $\text{Suppmarg}_{S_{-i}}\nu_{s_i}^{M+1} = S_{-i}^{M+1}$, this in turn implies the claim. Therefore $(s_i', t_i) \in \mathbf{A}_i\left(R_{-i}^{*,M+1}\right)$, and so $(s_i', t_i) \in R_i^{*,M+2}$. Certainly $R_i^{*,M+2} \subseteq R_i^{*,M+1}$; with this, we can conclude that $R_i^{*,M+2} = R_i^{*,M+1}$. We can apply the same reasoning for every $l > 1$ and $(s_i', t_i) \in R_i^{*,M+l}$ to conclude that $(s_i', t_i) \in \mathbf{A}_i\left(R_{-i}^{*,M+l}\right)$; this yields $R_i^{*,M+l+1} = R_i^{*,M+l} = R_i^{*,M+1}$, as required. ■

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